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Homotopy commutative algebra and 2-nilpotent Lie algebra

Michel Dubois-Violette and Todor Popov

Abstract The homotopy transfer theorem due to Tornike Kadeishvili induces the structure of a homotopy commutative algebra, or C_∞ -algebra, on the cohomology of the free 2-nilpotent Lie algebra. The latter C_∞ -algebra is shown to be generated in degree one by the binary and the ternary operations.

1 Introduction

Every Universal Enveloping Algebra (UEA) $U\mathfrak{g}$ of a finite dimensional positively graded Lie algebra \mathfrak{g} belongs to the class of Artin-Schelter regular algebras (see e.g. [4]). As every finitely generated graded connected algebra, $U\mathfrak{g}$ has a free minimal resolution which is canonically built from the data of its Yoneda algebra $\mathcal{E} := \text{Ext}_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K})$. By construction the Yoneda algebra \mathcal{E} is isomorphic (as algebra) to the cohomology of the Lie algebra (with coefficients in the trivial representation provided by the ground field \mathbb{K})

$$\mathcal{E} = \text{Ext}_{U\mathfrak{g}}^\bullet(\mathbb{K}, \mathbb{K}) \cong H^\bullet(\mathfrak{g}, \mathbb{K}) \quad (1)$$

equipped with wedge product between cohomological classes in $H^\bullet(\mathfrak{g}, \mathbb{K})$.

The homotopy transfer theorem of Tornike Kadeishvili [7] implies that the Yoneda algebra $\mathcal{E} = \text{Ext}_{U\mathfrak{g}}^\bullet(\mathbb{K}, \mathbb{K})$ has the structure of homotopy associative algebra, or A_∞ -algebra. Since $\mathcal{E} \cong H^\bullet(\mathfrak{g}, \mathbb{K})$ is the cohomology of the exterior algebra

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$\wedge \mathfrak{g}^*$ which is graded-commutative, it has the structure of homotopy commutative and associative algebra, or C_∞ -algebra.

Throughout the text \mathfrak{g} will be the free 2-nilpotent graded Lie algebra, with degree one generators in the finite dimensional vector space V over a field \mathbb{K} of characteristic 0,

$$\mathfrak{g} = V \oplus \wedge^2 V .$$

The UEA $U(V \oplus \wedge^2 V)$ arises naturally in physics in the universal Fock-like space of the parastatistics algebra introduced by H.S. Green [5](see also [3]). Here we will concentrate on the case when V is an ordinary (even) vector space V , when the algebra $U\mathfrak{g}$ is the parafermionic algebra.

The aim of this note is to describe the Yoneda algebra \mathcal{E} of the UEA $U\mathfrak{g}$, i.e., the cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ with its C_∞ -structure induced by the isomorphism (1) through the homotopy transfer.

The cohomology space $H^\bullet(\mathfrak{g}, \mathbb{K})$ has a natural $GL(V)$ -action. The decomposition of the $GL(V)$ -module $H^\bullet(\mathfrak{g}, \mathbb{K})$ into irreducible Schur modules V_λ is known since the work of Józefiak and Weyman [6]; it contains all $GL(V)$ -modules with self-conjugated Young diagrams $\lambda = \lambda'$ once and exactly once. The decomposition of $\mathcal{E} = H^\bullet(\mathfrak{g}, \mathbb{K})$ into Schur modules provides a powerful tool to handle its C_∞ -algebra structure.

2 Artin-Schelter regularity

Let \mathfrak{g} be the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \oplus \wedge^2 V$ generated by the finite dimensional vector space V having Lie bracket

$$[x, y] := \begin{cases} x \wedge y & x, y \in V \\ 0 & \text{otherwise} \end{cases} . \quad (2)$$

We denote the Universal Enveloping Algebra $U\mathfrak{g}$ by PS and will refer to it as *parastatistics algebra* (by some abuse¹). The parastatistics algebra $PS(V)$ generated in V is graded

$$PS(V) := U\mathfrak{g} = U(V \oplus \wedge^2 V) = T(V)/([V, V], V) .$$

We shall write simply PS when the space of generators V is clear from the context.

Artin and Schelter [1] introduced a class of regular algebras sharing some “good” homological properties with the polynomial algebra $\mathbb{K}[V]$. These algebras were dubbed Artin-Schelter regular algebras (AS-regular algebra for short).

Definition 1. (AS-regular algebras) A connected graded algebra $\mathcal{A} = \mathbb{K} \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots$ is called Artin-Schelter regular of dimension d if

- (i) \mathcal{A} has finite global dimension d ,
- (ii) \mathcal{A} has finite Gelfand-Kirillov dimension,

¹ Strictly speaking $PS(V)$ is the creation parastatistics algebra, closed by creation operators alone.

(iii) \mathcal{A} is Gorenstein, i.e., $\text{Ext}_{\mathcal{A}}^i(\mathbb{K}, \mathcal{A}) = \delta^{i,d} \mathbb{K}$.

A general theorem claims that the UEA of a finite dimensional positively graded Lie algebra is an AS-regular algebra of global dimension equal to the dimension of the Lie algebra [4]. Hence the parastatistics algebra PS is AS-regular of global dimension $d = \frac{\dim V(\dim V + 1)}{2}$. In particular the finite global dimension of PS implies that the ground field \mathbb{K} has a minimal resolution P_\bullet by projective left PS -modules P_n

$$P_\bullet : \quad 0 \rightarrow P_d \rightarrow \cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{\varepsilon} \mathbb{K} \rightarrow 0. \quad (3)$$

Here \mathbb{K} is a trivial left PS -module, the action being defined by the projection ε onto $PS_0 = \mathbb{K}$. Since PS is positively graded and, in the category of positively graded modules over connected locally finite graded algebras, projective module is the same as free module [2], we have $P_n \cong PS \otimes E_n$ where E_n are finite dimensional vector spaces.

The minimal projective resolution is unique (up to an isomorphism). Minimality implies that the complex $\mathbb{K} \otimes_{PS} P_\bullet$ has “zero differentials” hence

$$H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet = E_n.$$

One can calculate the derived functor $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ using the resolution P_\bullet , it yields

$$\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) = E_n. \quad (4)$$

The data of a minimal resolution of \mathbb{K} by free PS -modules provides an easy way to find $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$. Conversely if the spaces $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ are known, then one can construct a minimal free resolution of \mathbb{K} .

The Gorenstein property guarantees that when applying the functor $\text{Hom}_{PS}(-, PS)$ to the minimal free resolution P_\bullet we get another minimal free resolution $P^\bullet := \text{Hom}_{PS}(P_\bullet, PS)$ of \mathbb{K} by right PS -modules

$$P^\bullet : \quad 0 \leftarrow \mathbb{K} \leftarrow P_d' \leftarrow \cdots \leftarrow P_n' \leftarrow \cdots \leftarrow P_2' \leftarrow P_1' \leftarrow P_0' \leftarrow 0 \quad (5)$$

with $P_n' \cong E_n^* \otimes PS$. Note that by construction $E_n^* = \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})$, thus one has vector space isomorphisms [2]

$$E_n \cong E_n^* \cong \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K}). \quad (6)$$

The Gorenstein property is an analog of the Poincaré duality, it implies $E_{d-n}^* \cong E_n$. The finite global dimension d of PS and the Gorenstein condition imply that its Yoneda algebra

$$\mathcal{E}^\bullet := \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K}) \cong \bigoplus_{n=0}^d E_n^*$$

is Frobenius [10]. More on Gorenstein property you can find in the first author's lecture “Poincaré duality for Koszul algebras” in the present volume.

3 Homology and cohomology of \mathfrak{g}

A non-minimal projective (in fact free) resolution of \mathbb{K} , $C(\mathfrak{g}) \xrightarrow{\varepsilon} \mathbb{K}$ is given by the standard Chevalley-Eilenberg chain complex $C_\bullet(\mathfrak{g}) = (U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g}, d_p)$ with differential maps

$$\begin{aligned} d_p(u \otimes x_1 \wedge \dots \wedge x_p) &= \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_p \\ &+ \sum_{i < j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_p \end{aligned} \quad (7)$$

The resolution $C_\bullet(\mathfrak{g})$ calculates the homologies of the derived complex $\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})$

$$E_n = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})) = H_n(\mathfrak{g}, \mathbb{K}),$$

coinciding with the homologies $H_n(\mathfrak{g}, \mathbb{K})$ of the Lie algebra \mathfrak{g} with trivial coefficients. The derived complex $\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})$ is the chain complex with degrees $\wedge^\bullet \mathfrak{g} = \mathbb{K} \otimes_{PS} PS \otimes \wedge^\bullet \mathfrak{g}$ and differentials $\partial_p := id \otimes_{PS} d_p : \wedge^p \mathfrak{g} \rightarrow \wedge^{p-1} \mathfrak{g}$. One has

$$\wedge^p \mathfrak{g} = \wedge^p (V \oplus \wedge^2 V) = \bigoplus_{s+r=p} \wedge^s (\wedge^2 V) \otimes \wedge^r (V) \quad (8)$$

and differentials $\partial_{p=r+s} : \wedge^s (\wedge^2 V) \otimes \wedge^r (V) \rightarrow \wedge^{s+1} (\wedge^2 V) \otimes \wedge^{r-2} (V)$ are given by

$$\begin{aligned} \partial_p : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge e_r &\mapsto \\ \sum_{i < j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_r. \end{aligned}$$

The differential ∂ is induced by the Lie bracket $[\cdot, \cdot] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$, it identifies a pair of degree 1 generators $e_i, e_j \in$ with one degree 2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j]$. The differential ∂_p is the extension of $\partial_2 := -[\cdot, \cdot]$ as coderivation on $\wedge^p \mathfrak{g}$.

The dual cochain complex $\text{Hom}_{PS}(C(\mathfrak{g}), \mathbb{K}) = (\wedge^\bullet \mathfrak{g}^*, \delta)$ calculates cohomology²

$$E_n^* = \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K}) \cong H^n(\text{Hom}_{PS}(C(\mathfrak{g}), \mathbb{K})) = H^n(\mathfrak{g}, \mathbb{K}). \quad (9)$$

The coboundary map $\delta^p : \wedge^p \mathfrak{g}^* \rightarrow \wedge^{p+1} \mathfrak{g}^*$ is transposed to the differential ∂_{p+1}

$$\begin{aligned} \delta^p : e_{i_1 j_1}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{l_1}^* \wedge \dots \wedge e_{l_r}^* &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1}^* \wedge \dots \wedge \hat{e}_{i_k j_k}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{i_k}^* \wedge e_{j_k}^* \wedge e_{l_1}^* \wedge \dots \wedge \dots \wedge e_{l_r}^*, \end{aligned} \quad (10)$$

it is (up to a conventional sign) the extension as derivation of the dualization of the Lie bracket $\delta^1 := [\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$. Thus the algebra $(\wedge^\bullet \mathfrak{g}^*, \delta)$ equipped with δ is a (graded-)commutative DGA.

² In the presence of metric one has $\delta := \partial^*$ (see below)

4 Homology of \mathfrak{g} as a $GL(V)$ -module

An irreducible polynomial $GL(V)$ -module V_λ is called Schur module, it has a basis labelled by semistandard Young tableaux which are fillings of the Young diagram λ with the numbers of the set $\{1, \dots, \dim V\}$. The action of the linear group $GL(V)$ on the space V of the generators of the Lie algebra \mathfrak{g} induces a $GL(V)$ -action on the UEA $PS = U\mathfrak{g} \cong S(V \oplus \wedge^2 V)$ and on the space $\wedge^\bullet \mathfrak{g} \cong \wedge^\bullet (V \oplus \wedge^2 V)$.

In the presence of metric g one has an identification $V \xrightarrow{g} V^*$, and $\wedge^\bullet \mathfrak{g} \xrightarrow{g} \wedge^\bullet \mathfrak{g}^*$. The adjoint operator $\partial_p^* : \wedge^p \mathfrak{g} \rightarrow \wedge^{p+1} \mathfrak{g}$ is defined by $g(\partial_p^* v, w) = g(v, \partial_{p+1} w)$. It turns out that the action of ∂_p^* always takes the form (similar to the action of δ^p)

$$\begin{aligned} \partial_p^* : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \dots \wedge e_{l_r} \mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1} \wedge \dots \wedge \hat{e}_{i_k j_k} \wedge \dots \wedge e_{i_s j_s} \otimes e_{i_k} \wedge e_{j_k} \wedge e_{l_1} \wedge \dots \wedge e_{l_r}, \end{aligned} \quad (11)$$

It is obvious that the maps ∂ and ∂^* both commute with the $GL(V)$ -action. The Laplacian $\Delta = \oplus_{p \geq 0} \Delta_p$ of the pair (\mathfrak{g}, g) is defined to be the self-adjoint operator

$$\Delta_p = \partial_{p+1} \partial_{p+1}^* + \partial_p^* \partial_p \in \text{End}(\wedge^p \mathfrak{g}).$$

Its kernel is a complete set of representatives for the homology classes in $H_p(\mathfrak{g}, \mathbb{K})$

$$\ker \Delta_p \cong H_p(\mathfrak{g}, \mathbb{K}).$$

The decomposition of the $GL(V)$ -module $H_n(\mathfrak{g}, \mathbb{K})$ into irreducible polynomial representations V_λ is given by the following theorem;

Theorem 1 (Józefiak and Weyman [6], Sigg [11]). *The homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \wedge^2 V$ decomposes into irreducible $GL(V)$ -modules*

$$H_n(\mathfrak{g}, \mathbb{K}) = H_n(\wedge^\bullet \mathfrak{g}, \partial) \cong \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda \quad (12)$$

where the sum is over self-conjugate Young diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

The data $H_n(\mathfrak{g}, \mathbb{K}) = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ encodes the minimal free resolution P_\bullet (cf. (3)).

The Euler characteristics of P_\bullet implies an identity about the $GL(V)$ -characters

$$\text{ch } PS(V) \cdot \text{ch} \left(\bigoplus_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} V_\lambda \right) = 1.$$

The character of a Schur module V_λ is the Schur function, $\text{ch } V_\lambda = s_\lambda(x)$. Due to the Poincaré-Birkhoff-Witt theorem $\text{ch } PS(V) = \text{ch } S(V \oplus \wedge^2 V)$ thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i < j} \frac{1}{(1-x_i x_j)} \sum_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) = 1. \quad (13)$$

But the latter identity is nothing but rewriting of the Littlewood identity [6]. The moral is that the Littlewood identity reflects a homological property of the algebra PS , namely the above particular structure of the minimal projective (free) resolution of \mathbb{K} by PS -modules.

5 Homotopy algebras A_∞ and C_∞

Definition 2. (A_∞ -algebra) A homotopy associative algebra, or A_∞ -algebra, over \mathbb{K} is a \mathbb{Z} -graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)

$$m_n : A^{\otimes n} \rightarrow A, \quad \deg(m_n) = 2 - n \quad n \geq 1$$

satisfying the Stasheff identities **SI**(\mathbf{n}) for $n \geq 1$

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad \mathbf{SI}(\mathbf{n})$$

where the sum runs over all decompositions $n = r + s + t$.

Here we assume the Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. We define the shuffle product $Sh_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \rightarrow A^{\otimes p+q}$ throughout the expression

$$(a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma) a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all (p, q) -shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$.

Definition 3. (C_∞ -algebra [7]) A homotopy commutative algebra, or C_∞ -algebra, is an A_∞ -algebra $\{A, m_n\}$ such that each operation m_n vanishes on non-trivial shuffles

$$m_n((a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_n)) = 0, \quad 1 \leq p \leq n-1. \quad (14)$$

In particular for m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞ -algebra such that $m_n = 0$ for $n \geq 3$ is a (super-)commutative DGA.

A morphism of two A_∞ -algebras A and B is a family of graded maps $f_n : A^{\otimes n} \rightarrow B$ for $n \geq 1$ with $\deg f_n = 1 - n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = \sum_{1 \leq r \leq n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_r})$$

where the sum is on all decompositions $i_1 + \dots + i_r = n$ and the sign on RHS is determined by $S = \sum_{k=1}^{r-1} (r-k)(i_k - 1)$. The morphism f is a *quasi-isomorphism* of A_∞ -algebras if f_1 is a quasi-isomorphism. It is strict if $f_i = 0$ for all $i \neq 1$. The identity morphism of A is the strict morphism f such that f_1 is the identity of A .

A morphism of C_∞ -algebras is a morphism of A_∞ -algebras vanishing on non-trivial shuffles $f_n((a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_n)) = 0, 1 \leq p \leq n-1$.

6 Homotopy Transfer Theorem

Lemma 1. *Every cochain complex (A, d) of vector spaces over a field \mathbb{K} has its cohomology $H^\bullet(A)$ as a deformation retract.*

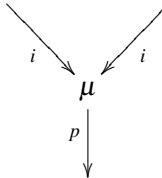
One can always choose a vector space decomposition of the cochain complex (A, d) such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$ where H^n is the cohomology and B^n is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \rightarrow A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n \oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \xrightleftharpoons[i]{p} H^n$ are chain homomorphisms (satisfying the additional conditions $hh = 0$, $hi = 0$ and $ph = 0$). With these choices done the complex $(H^\bullet(A), 0)$ is a deformation retract of (A, d)

$$h \circlearrowleft (A, d) \xrightleftharpoons[i]{p} (H^\bullet(A), 0) \quad , \quad pi = Id_{H^\bullet(A)} \quad , \quad ip - Id_A = dh + hd \quad .$$

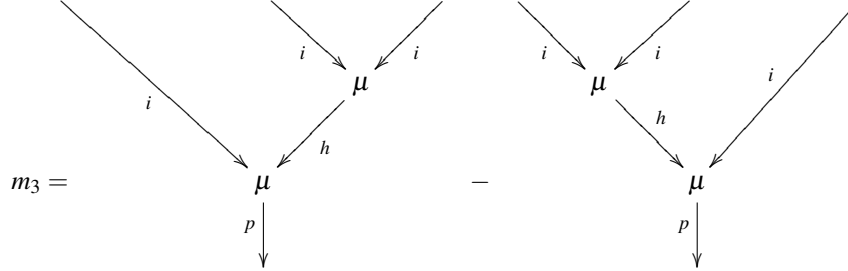
Let now (A, d, μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d . The cochain complexes (A, d) and its contraction $H^\bullet(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞ -structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$. For a friendly introduction to homotopy transfer theorems in much boarder context we send the reader to the textbook [9], see chapter 9.

Theorem 2 (Kadeishvili [7]). *Let (A, d, μ) be a (commutative) DGA over a field \mathbb{K} . There exists a A_∞ -algebra (C_∞ -algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$ -quasi-isomorphism $f_i : (\otimes^i H^\bullet(A), \{m_i\}) \rightarrow (A, \{d, \mu, 0, 0, \dots\})$ such that the inclusion $f_1 = i : H^\bullet(A) \rightarrow A$ is a cocycle-choosing homomorphism of cochain complexes. The differential m_1 on $H^\bullet(A)$ is zero ($m_1 = 0$) and m_2 is strictly associative operation induced by the multiplication on A . The resulting structure is unique up to quasi-isomorphism.*

Kontsevich and Soibelman [12] gave an explicit expressions for the higher operations of the induced A_∞ -structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion i , the root by the projection p the vertices by the product μ of the (commutative) DGA (A, d, μ) and the internal edges by the homotopy h . The C_∞ -structure implies additional symmetries on trees. We will make use of the graphic representation for the binary operation on $H^\bullet(A)$

$$m_2(x, y) := p\mu(i(x), i(y)) \quad \text{or} \quad m_2 =$$


and the ternary one $m_3(x, y, z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$ being the sum of two planar binary trees with three leaves



Theorem 3. *The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ of the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \wedge^2 V$ is a homotopy commutative algebra which is generated in degree 1 (i.e., in $H^1(\mathfrak{g}, \mathbb{K})$) by the operations m_2 and m_3 .*

Sketch of the proof. Let us choose a metric $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$ on the vector space V and an orthonormal basis $\langle e_i, e_j \rangle = \delta_{ij}$. The choice induces a metric on $\wedge^\bullet \mathfrak{g} \xrightarrow{g} \wedge^\bullet \mathfrak{g}^*$.

Due to the isomorphisms $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})$ (see eq. 6) and $V \cong V^*$ the theorem 1 implies the decomposition of $H^\bullet(\mathfrak{g}, \mathbb{K})$ into irreducible $GL(V)$ -modules

$$H^n(\mathfrak{g}, \mathbb{K}) \cong H^n(\wedge^\bullet \mathfrak{g}^*, \delta) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})(V^*) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over self-conjugate diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

In the presence of metric g the differential δ is identified with the adjoint of ∂ , $\delta \stackrel{g}{=} \partial^*$ while ∂ plays the role of a homotopy. In view of lemma 1 we have the cohomology $H^\bullet(\wedge^\bullet \mathfrak{g}^*, \delta^*)$ as deformation retract of the complex $(\wedge^\bullet \mathfrak{g}^*, \delta^*)$,

$$pi = Id_{H^\bullet(\wedge^\bullet \mathfrak{g}^*)}, \quad ip - Id_{\wedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta, \quad \delta^* \stackrel{g}{=} \partial.$$

Here the projection p identifies the subspace $\ker \delta \cap \ker \delta^*$ with $H^\bullet(\wedge^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\text{im} \delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\wedge^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

We apply the Kadeishvili homotopy transfer Theorem 2 for the commutative DGA $(\wedge^\bullet \mathfrak{g}^*, \mu, \delta^*)$ and its deformation retract $H^\bullet(\wedge^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g}, \mathbb{K})$ and conclude that the cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ is a C_∞ -algebra.

The Kontsevich and Soibelman tree representations of the operations m_n provide explicit expressions. Let us take μ to be the super-commutative product \wedge on the DGA $(\wedge^\bullet \mathfrak{g}^*, \delta^*)$. The projection p maps onto the Schur modules V_λ with $\lambda = \lambda'$.

The binary operation on the degree 1 generators $e_i \in H^1(\mathfrak{g}, \mathbb{K})$ is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \quad p(V_{(12)}) = 0.$$

Hence $H^\bullet(\mathfrak{g}, \mathbb{K})$ could not be generated in $H^1(\mathfrak{g}, \mathbb{K})$ as algebra with product m_2 .

The ternary operation m_3 restricted to $H^1(\mathfrak{g}, \mathbb{K})$ is nontrivial, indeed one has

$$\begin{aligned} m_3(e_i, e_j, e_k) &= p \{ e_i \wedge \partial(e_j \wedge e_k) - \partial(e_i \wedge e_j) \wedge e_k \} = p \{ e_{ij} \wedge e_k - e_i \wedge e_{jk} \} \\ &= p \{ (e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) - e_{ki} \wedge e_j \} = e_{ik} \wedge e_j \in H^2(\mathfrak{g}, \mathbb{K}) \end{aligned}$$

The completely antisymmetric combination in the brackets (...) spans the Schur module $V_{(1^3)}$, $p(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$ yields a Jacobi-type identity. The monomials $e_{ij} \wedge e_k$ modulo $V_{(1^3)}$ span a Schur module $V_{(2,1)} \in H^2(\mathfrak{g}, \mathbb{K})$ with basis in bijection with the semistandard Young tableaux $e_{ik} \wedge e_j \leftrightarrow \begin{smallmatrix} i & j \\ k & \end{smallmatrix}$ and $e_{ij} \wedge e_k \leftrightarrow \begin{smallmatrix} i & k \\ j & \end{smallmatrix}$.

We check the symmetry condition on ternary operation m_3 in C_∞ -algebra; indeed m_3 vanishes on the (signed) shuffles $Sh_{1,2}$ and $Sh_{2,1}$

$$m_3(e_i \sqcup e_j \otimes e_k) = m_3(e_i, e_j, e_k) - m_3(e_j, e_i, e_k) + m_3(e_j, e_k, e_i) = 0 = m_3(e_i \otimes e_j \sqcup e_k).$$

It is important that in the complexes $(\wedge^p \mathfrak{g}, \partial_p)$ and $(\wedge^p \mathfrak{g}^*, \delta^p)$ two different degrees are involved; one is the homological degree $p := r + s$ counting the number of \mathfrak{g} -generators, while the second is the tensor degree $t := 2s + r$ (also called weight). The differentials ∂ and δ preserve the tensor degree t but the spaces $H_n(\mathfrak{g}, \mathbb{K})$ and $H^n(\mathfrak{g}, \mathbb{K})$ are not homogeneous in t . The operation m_n is bigraded by homological and tensor gradings of bidegree $(p, t) = (2 - n, 0)$. The bi-grading impose the vanishing of many higher products.

On the level of Schur modules the ternary operation glues three fundamental $GL(V)$ -representations V_\square into a Schur module $V_{(2,1)}$. By iteration of the process of gluing boxes we generate all elementary hooks $V_k := V_{(k+1, 1^k)}$,

$$m_3(V_\square, V_\square, V_\square) = V_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}}, \quad m_3 \left(V_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}}, V_\square, V_\square \right) = V_{\begin{smallmatrix} \square & \square & \square \\ \square & & \end{smallmatrix}}, \dots, m_3(V_k, V_0, V_0) = V_{k+1}.$$

In our context the more convenient notation for Young diagrams is due to Frobenius: $\lambda := (a_1, \dots, a_r | b_1, \dots, b_r)$ stands for a diagram λ with a_i boxes in the i -th row on the right of the diagonal, and with b_i boxes in the i -th column below the diagonal and the rank $r = r(\lambda)$ is the number of boxes on the diagonal.

For self-dual diagrams $\lambda = \lambda'$, i.e., $a_i = b_i$ we set $V_{a_1, \dots, a_r} := V_{(a_1, \dots, a_r | a_1, \dots, a_r)}$ when $a_1 > a_2 > \dots > a_r \geq 0$ (and set the convention $V_{a_1, \dots, a_r} := 0$ otherwise). Any two elementary hooks V_{a_1} and V_{a_2} can be glued together by the binary operation m_2 , the decomposition of $m_2(V_{a_1}, V_{a_2}) \cong m_2(V_{a_2}, V_{a_1})$ is given by

$$m_2(V_{a_1}, V_{a_2}) = V_{a_1, a_2} \oplus \left(\bigoplus_{i=1}^{a_2} V_{a_1+i, a_2-i} \right) \quad a_1 \geq a_2$$

where the “leading” term V_{a_1, a_2} has the diagram with minimal height. Hence any m_2 -bracketing of the hooks $V_{a_1}, V_{a_2}, \dots, V_{a_r}$ yields³ a sum of $GL(V)$ -modules

³ The operation m_2 is associative thus the result does not depend on the choice of the bracketing.

$$m_2(\dots m_2(m_2(V_{a_1}, V_{a_2}), V_{a_3}), \dots, V_{a_r}) = V_{a_1, \dots, a_r} \oplus \dots$$

whose module with minimal height is precisely V_{a_1, \dots, a_r} . We conclude that all elements in the C_∞ -algebra $H^\bullet(\mathfrak{g}, \mathbb{K})$ can be generated in $H^1(\mathfrak{g}, \mathbb{K})$ by m_2 and m_3 .

□

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