



HAL
open science

Existence and stability of solutions of general semilinear elliptic equations with measure data

Laurent Veron

► **To cite this version:**

Laurent Veron. Existence and stability of solutions of general semilinear elliptic equations with measure data. 2012. hal-00718911v1

HAL Id: hal-00718911

<https://hal.science/hal-00718911v1>

Preprint submitted on 18 Jul 2012 (v1), last revised 28 Aug 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Existence and stability of solutions of general semilinear elliptic equations with measure data

Laurent Véron

Laboratoire de Mathématiques et Physique Théorique
Université François-Rabelais, Tours, FRANCE

Abstract We study existence and stability for solutions of $-Lu + g(x, u) = \omega$ where L is a second order elliptic operator, g a Caratheodory function and ω a measure in $\overline{\Omega}$. We present a unified theory of the Dirichlet problem and the Poisson equation. We prove the stability of the problem with respect to weak convergence of the data.

2010 Mathematics Subject Classification. 35J61, 35J66, 28A20.

Key words: Elliptic operators, Borel measures, Marcinkiewicz spaces, Δ_2 condition.

1 Introduction

Let Ω be a smooth bounded domain of \mathbb{R}^N , L a uniformly elliptic second order differential operator in divergence form with Lipschitz continuous coefficients and g is a real valued Caratheodory function defined in $\Omega \times \mathbb{R}$. If ω is a Radon measure on $\overline{\Omega}$, we study existence and stability of solutions of the generalized equation

$$-Lu + g(x, u) = \omega \quad (1.1)$$

in $\overline{\Omega}$. Precise assumptions are made on the coefficients of L so that uniqueness holds. A fundamental contribution is made by Benilan and Brezis [6], [3] who study the case where $L = \Delta$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is nondecreasing and positive on \mathbb{R}_+ : if μ is a bounded measure in Ω and g satisfies the *subcriticality assumption*

$$\int_1^\infty (g(s) - g(-s)) s^{-2\frac{N-1}{N-2}} ds < \infty, \quad (1.2)$$

then there exists a unique function $u \in L^1(\Omega)$ such that $g \circ u \in L^1(\Omega)$ (where $g \circ u(x) = g(x, u(x))$) satisfying

$$\int_\Omega (-u\Delta\zeta + g \circ u \zeta) dx = \int_\Omega \zeta d\mu, \quad (1.3)$$

for any $\zeta \in C_0^2(\Omega)$.

The boundary value problem with measures is first investigated by Gmira and Véron [7]. By adapting the method introduced by Benilan and Brezis they obtain the existence and uniqueness of a weak solution of

$$\begin{aligned} -\Delta u + g(u) &= 0 && \text{in } \Omega \\ u &= \lambda && \text{in } \partial\Omega \end{aligned} \quad (1.4)$$

when λ is a Radon measure. They assume that g , always nondecreasing, satisfies the *boundary subcriticality assumption*

$$\int_1^\infty (g(s) - g(-s)) s^{-\frac{2N}{N-2}} ds < \infty, \quad (1.5)$$

and prove the existence and uniqueness of a weak solution to (1.4). For this problem, in the integral identity (1.3) the right hand-side is replaced by $-\int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda$.

In [13] Véron extends Benilan-Brezis results in replacing Δ by a general uniformly elliptic second order differential operator with smooth coefficients. If g is nondecreasing and satisfies, for some $\alpha \in [0, 1]$, the α -subcriticality assumption,

$$\int_1^\infty (g(s) - g(-s)) s^{-2\frac{N+\alpha-1}{N+\alpha-2}} ds < \infty, \quad (1.6)$$

then if μ belongs to $\mathfrak{M}_{\rho^\alpha}(\Omega)$, which means

$$\|\mu\|_{\mathfrak{M}_{\rho^\alpha}} := \int_{\Omega} \rho^\alpha d|\mu| < \infty, \quad (1.7)$$

where $\rho(x) := \text{dist}(x, \partial\Omega)$, there exists a unique $u \in L^1(\Omega)$ such that $g(u) \in L^1_\rho(\Omega)$ satisfying

$$\int_{\Omega} (-uL^*\zeta + g(u)\zeta) dx = \int_{\Omega} \zeta d\mu \quad \forall \zeta \in C_c^{1,L^*}(\overline{\Omega}). \quad (1.8)$$

where

$$C_c^{1,L^*}(\overline{\Omega}) = \{\zeta \in C^1(\overline{\Omega}) : \zeta = 0 \text{ on } \partial\Omega, L^*\zeta \in L^\infty(\Omega)\}, \quad (1.9)$$

where L^* is the adjoint operator to L . Furthermore he proves the weak stability of the problem in the sense that if u_n is a solution of

$$\begin{aligned} -Lu_n + g(u_n) &= \mu_n && \text{in } \Omega \\ u_n &= 0 && \text{in } \partial\Omega \end{aligned} \quad (1.10)$$

for a sequence of measure $\{\mu_n\}$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \zeta d\mu_n = \int_{\Omega} \zeta d\mu \quad (1.11)$$

for all $\zeta \in C(\overline{\Omega})$ such that $\sup_{\Omega} \rho^{-\alpha} |\zeta| < \infty$, then $u_n \rightarrow u$ where u satisfies (1.1). However, a careful observation of the existence and stability statements proved in [13, Th 3.7, Cor 3.8] shows that the result is slightly stronger than the one stated since it implies the following:

Let $\alpha \in [0, 1]$ and $g : \mathbb{R} \mapsto \mathbb{R}$ be continuous function which satisfies the α -subcriticality assumption (1.6). If $\{\mu_n\}$ is a sequence of Radon measures in $\overline{\Omega}$ such that

$$\int_{\Omega} \rho^\alpha d|\mu_n| \leq M \quad (1.12)$$

for some $M > 0$ and (1.11) holds for ζ such that $\rho^{-\alpha}\zeta \in C(\overline{\Omega})$, then the corresponding solution u_n of (1.10) converges to the solution u of (1.1). In particular, if $\alpha = 1$, it contains the case where there exists a Radon measure λ on $\partial\Omega$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \zeta d\mu_n = - \int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda \quad \forall \zeta \in C_c^1(\overline{\Omega}). \quad (1.13)$$

The case where the nonlinearity g depends on the $\rho(x)$ variable has investigated by Marcus [8]. If $g(x, r)\text{sign } r \leq \rho(x)^\beta \tilde{g}(|r|)\text{sign } r$ for some $\beta > -2$ and \tilde{g} satisfying a subcriticality assumption

$$\int_1^\infty (\tilde{g}(s) - \tilde{g}(-s)) s^{-\frac{2N+\beta-1}{N-1}} ds < \infty, \quad (1.14)$$

then there exists a weak solution to problem (1.4) for any Radon measure λ . Furthermore stability holds.

The subcriticality is a key hypothesis in all the previous results: essentially it means that the problem can be solved for any measure if it can be solved for a Dirac measure. The different integral assumptions are just the transcription that the fact that g of the fundamental solution of the associated linear equation is integrable for a suitable measure associated to the distance function ρ .

The aim of this article is twofold: 1- to unify the problems for measures in Ω and on $\partial\Omega$; 2- to present under the form of an integrability condition a classical sufficient condition of solvability which has the advantage of being a natural extension to the supercritical case the previous subcriticality assumptions and to provide new results results of existence and stability for (1.1) in the spirit of [13]. A function $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ belongs to the class $G_{h,\Psi}$ if it is a Caratheodory function and there exist $\tilde{g} : \mathbb{R} \mapsto \mathbb{R}$ continuous, nondecreasing and vanishing at 0 a locally integrable nonnegative function h defined in Ω and a nonnegative continuous nonincreasing function $\Psi : [0, \infty) \mapsto [0, \infty)$ such that

$$|g(x, r)| \leq h(x) |\tilde{g}(r)| \quad \forall (x, r) \in \Omega \times \mathbb{R}. \quad (1.15)$$

It satisfies the Ψ -integrability condition if,

$$-\int_0^\infty (\tilde{g}(s) - \tilde{g}(-s)) d\Psi(t) ds < \infty. \quad (1.16)$$

Let G and K be respectively the Green and Poisson kernels corresponding to the operator L in Ω and $\mathbb{G}[\cdot]$ and $\mathbb{K}[\cdot]$ the corresponding potential operators. The natural subcritical assumptions in the framework of Marcus's results (with h instead of ρ^β) for solving

$$\begin{aligned} -Lu + g(x, u) &= \mu && \text{in } \Omega \\ u &= \lambda && \text{in } \partial\Omega \end{aligned} \quad (1.17)$$

would be

$$\int_1^\infty (\mathbb{G}[|\mu|] + \mathbb{K}[|\lambda|]) h(x) \rho(x) dx < \infty. \quad (1.18)$$

However this type of condition is not satisfactory since it may not hold if μ and λ are merely integrable functions since the problem admits always weak solutions. More generally it does not define a clear class of measures for which we can solve problem (1.17). We introduce new classes of Radon measures whose Green and Poisson potentials belong to a weighted Marcinkiewicz

space-type space. Let Ψ be a continuous nonincreasing and nonnegative function defined on $[0, \infty)$ and m is a bounded positive Borel measure in Ω and denote

$$M_m^\Psi(\Omega) := \left\{ f \in \mathcal{B}(\Omega) : \exists C > 0 \text{ s.t. } \int_{\lambda_f(t)} dm(x) \leq C\Psi(t), \forall t > 0 \right\} \quad (1.19)$$

where $\mathcal{B}(\Omega)$ denotes the space of Borel functions in Ω and $\lambda_f(t) = \{x \in \Omega : |f(x)| > t\}$. The main results of this article are the two next statements:

Theorem A *Let g be an element of the class $G_{h,\Psi}$ with $\rho h \in L^1(\Omega)$. Then for any $\mu \in \mathfrak{M}_\rho(\Omega)$ and $\lambda \in \mathfrak{M}(\partial\Omega)$ such that $\mathbb{G}[|\mu|]$ and $\mathbb{K}[|\lambda|]$ belong to $M_{\rho h}^\Psi(\Omega)$, there exists a solution to problem (1.17). If $r \mapsto g(x, r)$ is nondecreasing for a.e. $x \in \Omega$, this solution is unique.*

Actually we shall introduce a unique formulation for the data (μ, λ) as a *unique* measure ω on $\overline{\Omega}$ which allows to replace (1.17) by (1.1), and a unique assumption on the *extended* Green operator $\overline{\mathbb{G}}[|\omega|]$. We prove in particular the following:

Theorem B *Assume the assumptions on h, Ψ and g of Theorem A are satisfied and $r \mapsto g(x, r)$ is nondecreasing. If $\{\omega_n\}$ is a sequence of measures in $\mathfrak{M}_\rho(\overline{\Omega})$ which converges to $\omega \in \mathfrak{M}_\rho(\overline{\Omega})$ in the sense that*

$$\int_{\overline{\Omega}} \zeta d\omega_n \rightarrow \int_{\overline{\Omega}} \zeta d\omega \quad (1.20)$$

for any ζ such that $\rho^{-1}\zeta \in C(\overline{\Omega})$. If $\{\overline{\mathbb{G}}[|\omega_n|]\}$ are bounded in $M_{\rho h}^\Psi(\Omega)$, then the corresponding solutions u_{ω_n} of problem (1.10) converges to the solution u_ω of problem (1.1). If g satisfies the Δ_2 conditions, the convergence remains valid if only the $\{\overline{\mathbb{G}}[|\omega_{s_n}|\}\}$ are bounded in $M_{\rho h}^\Psi(\Omega)$, where ω_{s_n} denotes the singular parts of ω_n .

2 Linear equations and measures

Since $\partial\Omega$ is C^2 , there exists $\delta_0 > 0$ such that, If $x \in \Omega$ is such that $\rho(x) \leq \delta_0$, there exists a unique $\sigma := \sigma(x) \in \partial\Omega$ such that $|x - \rho(x)| = \rho(x)$. For $\delta > 0$ we denote

$$\Omega_\delta := \{x \in \Omega : \rho(x) > \delta\}, \quad \Omega'_\delta := \{x \in \Omega : \rho(x) < \delta\}, \quad \Sigma_\delta := \{x \in \Omega : \rho(x) = \delta\}, \quad \Sigma := \Sigma_0 = \partial\Omega.$$

The mapping $x \mapsto (\rho(x), \sigma(x))$ is a C^1 diffeomorphism from $\overline{\Omega'_\delta}$ onto $[0, \delta_0] \times \Sigma$.

2.1 Weighted measures on $\overline{\Omega}$

We denote by $\mathfrak{M}(\Omega)$ the set of Radon measures in Ω . If $\alpha \in [0, 1]$, we denote by $\mathfrak{M}_{\rho^\alpha}(\Omega)$ the subset of $\mathfrak{M}(\Omega)$ of measures such that

$$\|\mu\|_{\mathfrak{M}_{\rho^\alpha}} := \int_{\Omega} \rho^\alpha d|\mu| < \infty. \quad (2.21)$$

We also set

$$C_\alpha(\overline{\Omega}) := \{\zeta \in C(\Omega) : \rho^{-\alpha}\zeta \in C(\overline{\Omega})\}, \quad (2.22)$$

with norm

$$\|\zeta\|_{C_\alpha} := \sup_{x \in \overline{\Omega}} \rho^{-\alpha}(x) |\zeta(x)|. \quad (2.23)$$

Thus, if $\mu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ and $\zeta \in C_\alpha(\overline{\Omega})$, there holds

$$\left| \int_{\Omega} \zeta d\mu \right| \leq \|\mu\|_{\mathfrak{M}_{\rho^\alpha}} \|\zeta\|_{C_\alpha}. \quad (2.24)$$

Furthermore, since

$$\int_{\Omega_{\delta_0}} \rho^\alpha d|\mu| + \sum_{n=1}^{\infty} \int_{\{2^{-n}\delta_0 < \rho \leq 2^{1-n}\delta_0\}} \rho^\alpha d|\mu| = \int_{\Omega} \rho^\alpha d|\mu| < \infty,$$

there holds

$$\lim_{\delta \rightarrow 0} \int_{\Omega'_\delta} \rho^\alpha d|\mu| = 0. \quad (2.25)$$

We say that a sequence $\{\mu_n\} \subset \mathfrak{M}_{\rho^\alpha}(\Omega)$ converges weakly to $\mu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ if, for any $\zeta \in C_\alpha(\overline{\Omega})$, there holds

$$\lim_{n \rightarrow \infty} \int_{\Omega} \zeta d\mu_n = \int_{\Omega} \zeta d\mu. \quad (2.26)$$

However, the left-hand side expression of (2.26) may exist but not being a Radon measure in Ω . Therefore we define a more general set of linear functionals on C_α

Definition 2.1 We denote by $\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ the set of continuous linear functionals ω on $C_\alpha(\overline{\Omega})$ such that there exists a sequence $\{\mu_n\} \subset \mathfrak{M}_{\rho^\alpha}(\Omega)$ which converges weakly to ω .

The natural norm in $\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ is

$$\|\omega\|_{\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})} = \sup\{|\omega(\zeta)| : \zeta \in C_\alpha(\overline{\Omega}), \|\zeta\|_{C_\alpha} \leq 1\}. \quad (2.27)$$

Proposition 2.2 If $\omega \in \mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$, its restriction to $C_c(\Omega)$ is a Radon measure, denoted by μ , which belongs to $\mathfrak{M}_{\rho^\alpha}(\Omega)$. Furthermore, there exists a Radon measure λ on $\partial\Omega$ such that

$$\omega(\zeta) - \int_{\Omega} \zeta d\mu = \int_{\partial\Omega} \psi \llcorner_{\partial\Omega} d\lambda \quad \forall \zeta \in C_\alpha(\overline{\Omega}) \text{ and } \psi = \rho^{-\alpha}\zeta \in C(\overline{\Omega}). \quad (2.28)$$

Proof. Since ω is continuous, there exists $C > 0$ such that

$$|\omega(\zeta)| \leq C \|\zeta\|_{C_\alpha} \quad \forall \zeta \in C_\alpha(\overline{\Omega}). \quad (2.29)$$

This holds in particular if $\zeta \in C_c(\Omega)$ and proves that the restriction of ω to $C_c(\Omega)$ is a Radon measure that we denote by μ (as well as the associated Borel measure in Ω) and

$$\omega(\zeta) = \int_{\Omega} \zeta d\mu \quad \forall \zeta \in C_c(\Omega).$$

Let $\{\mu_n\} \subset \mathfrak{M}_{\rho^\alpha}(\Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \zeta d\mu_n = \omega(\zeta) \quad \forall \zeta \in C_\alpha(\overline{\Omega}).$$

By the Banach-Steinhaus theorem there exists $C > 0$ such that $\|\mu_n\|_{\mathfrak{M}_{\rho^\alpha}} \leq C$ for all $n \in \mathbb{N}$. Since for $\zeta \in C_c(\Omega)$,

$$\omega(\zeta) - \int_{\Omega} \zeta d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} \zeta d(\mu_n - \mu)$$

and

$$\left| \int_{\Omega} \zeta d(\mu_n - \mu) \right| \leq 2C \|\zeta\|_{C_\alpha},$$

it follows that $\{\lambda_n\} := \{\rho^\alpha(\mu_n - \mu)\}$ is a sequence of Radon measures on Ω , bounded in $\mathfrak{M}_{\rho^\alpha}(\Omega)$ and such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \zeta d\lambda_n = 0 \quad \forall \zeta \in C_c(\Omega).$$

Therefore there exists a Radon measure λ with support in $\partial\Omega$ and a subsequence λ_{n_k} such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \psi d\lambda_{n_k} = \int_{\partial\Omega} \psi|_{\partial\Omega} d\lambda,$$

which implies (2.28). \square

Corollary 2.3 *The mapping $T : \mathfrak{M}_{\rho^\alpha}(\Omega) \times \mathfrak{M}(\partial\Omega) \mapsto \mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ defined by*

$$T[\mu, \lambda](\zeta) = \int_{\Omega} \zeta d\mu + \int_{\partial\Omega} \psi|_{\partial\Omega} d\lambda \quad \forall \zeta \in C_\alpha(\overline{\Omega}) \text{ and } \psi = \rho^{-\alpha}\zeta \in C(\overline{\Omega}). \quad (2.30)$$

is one to one. Furthermore

$$\max\{\|\mu\|_{\mathfrak{M}_{\rho^\alpha}(\Omega)}, \|\lambda\|_{\mathfrak{M}(\partial\Omega)}\} \leq \|T[\mu, \lambda]\|_{\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})} \leq \|\mu\|_{\mathfrak{M}_{\rho^\alpha}(\Omega)} + \|\lambda\|_{\mathfrak{M}(\partial\Omega)}. \quad (2.31)$$

Proof. The mapping T is onto from Proposition 2.2. The mapping T is one to one since if $T[\mu, \lambda] = 0$, then $\mu = 0$ and $\int_{\partial\Omega} \psi|_{\partial\Omega} d\lambda = 0$ for any $\psi \in C(\overline{\Omega})$. This implies $\lambda = 0$. The right-hand side inequality (2.31) is clear since $\sup |\psi|_{\partial\Omega} \leq \|\zeta\|_{C_\alpha}$. Because of (2.25)

$$\int_{\Omega} \rho^\alpha d|\mu| = \sup \left\{ \int_{\Omega} \zeta d\mu : \zeta \in C_c(\Omega), \|\zeta\|_{C_\alpha} \leq 1 \right\}$$

This implies

$$\|\mu\|_{\mathfrak{M}_{\rho^\alpha}(\Omega)} \leq \|T[\mu, \lambda]\|_{\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})}$$

If $\phi \in C(\partial\Omega)$ is such that $|\phi| \leq 1$ and Φ is its harmonic lifting in Ω , the function $\zeta = \rho^\alpha\Phi$ belongs to $C_\alpha(\overline{\Omega})$ and satisfies $\|\zeta\|_{C_\alpha} \leq 1$. Let $\{\eta_n\} \subset C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta_n \leq 1$, $\eta_n(x) = 0$ if $\rho(x) \geq 2/n$, $\eta_n(x) = 1$ if $\rho(x) \leq 1/n$. Then $\zeta_n = \eta_n\rho^\alpha\Phi$ belongs also to $C_\alpha(\overline{\Omega})$ and $\|\zeta_n\|_{C_\alpha} \leq 1$. Since

$$T[\mu, \lambda](\zeta_n) = \int_{\Omega} \zeta_n d\mu + \int_{\partial\Omega} \phi d\lambda$$

and $\int_{\Omega} \zeta_n d\mu \rightarrow 0$ as $n \rightarrow \infty$, we derive

$$\|T[\mu, \lambda]\|_{\mathfrak{M}_{\rho^\alpha}(\bar{\Omega})} \geq \int_{\partial\Omega} \phi d\lambda.$$

This ends to proof. \square

Remark. If λ is a Radon measure on $\partial\Omega$ and we can define its δ^α -lifting $\Lambda_{\delta^\alpha}[\lambda] \in \mathfrak{M}(\Omega)$ by

$$\int_{\Omega} \zeta d\lambda_{\delta^\alpha} = \delta^{-\alpha} \int_{\Omega} \zeta(\delta, \sigma) d\lambda(\sigma).$$

Clearly $\lambda_{\delta^\alpha} \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ and if $\zeta \in C_\alpha(\bar{\Omega})$ and $\ell_\alpha(\zeta) = -\lim_{\rho \rightarrow 0} \rho^{-\alpha} \zeta$, then $\ell_\alpha(\zeta) \in C(\partial\Omega)$, there holds

$$\lim_{\delta \rightarrow 0} \int_{\Omega} \zeta d\lambda_{\delta^\alpha} = \int_{\Sigma} \ell_\alpha(\zeta) d\lambda. \quad (2.32)$$

In the particular case where $\alpha = 1$ $\ell_\alpha(\zeta) = \zeta_{\mathbf{n}} := \lim_{\rho \rightarrow 0} \rho^{-1} \zeta$, and

$$\lim_{\delta \rightarrow 0} \int_{\Omega} \zeta d\lambda_\delta = - \int_{\Sigma} \zeta_{\mathbf{n}} d\lambda. \quad (2.33)$$

2.2 The linear operator

Let $x = (x_1, \dots, x_N)$ the coordinates in \mathbb{R}^N and Ω a bounded domain in \mathbb{R}^N . We consider the operator L in divergence form defined by

$$Lu := - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^N b_i \frac{\partial u}{\partial x_i} - \sum_{i=1}^N \frac{\partial}{\partial x_i} (c_i u) + du \quad (2.34)$$

where the a_{ij} , b_i and c_i are Lipschitz continuous and d is bounded and measurable in Ω . We assume that the ellipticity condition

$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq a \sum_{i=1}^N \xi_i^2 \quad \forall \xi \in \mathbb{R}^N \quad (2.35)$$

holds for almost x in Ω , for some $a > 0$. We also assume the positivity condition

$$\int_{\Omega} \left(dv + \frac{1}{2} \sum_{i=1}^N (b_i + c_i) \frac{\partial v}{\partial x_i} \right) dx \geq 0 \quad \forall v \in C_c^1(\Omega), v \geq 0 \quad (2.36)$$

Under these assumptions, the bilinear form

$$(u, v) \mapsto A_L(u, v) = \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^N \left(b_i \frac{\partial u}{\partial x_i} v + c_i \frac{\partial v}{\partial x_i} u \right) + duv \right) dx \quad (2.37)$$

is continuous and coercive on $W^{1,2}(\Omega)$. We define the adjoint operator L^* by

$$L^*u := - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^N c_i \frac{\partial u}{\partial x_i} - \sum_{i=1}^N \frac{\partial}{\partial x_i} (b_i u) + du \quad (2.38)$$

We denote by G and K the Green and Poisson kernels corresponding to the operator L in Ω . We recall the following equivalence statement [10], [2]

Proposition 2.4 *Assume Ω has a C^2 boundary and (2.36) holds. Then there exists a positive constant C such that*

$$CG_{-\Delta} \leq G \leq C^{-1}G_{-\Delta} \quad \text{in } \Omega \times \Omega \setminus D_\Omega \quad (2.39)$$

where $D_\Omega = \{x \in \Omega \times \Omega : x \neq y\}$ and

$$CK_{-\Delta} \leq K \leq C^{-1}K_{-\Delta} \quad \text{in } \Omega \times \partial\Omega. \quad (2.40)$$

2.3 Linear equation with measure data

If $m \in \mathfrak{M}_+(\Omega)$ is a bounded Borel measure and $\Psi : [0, \infty) \mapsto [0, \infty)$ is continuous and non-increasing, we define the subset $M_m^\Psi(\Omega)$ of the set $\mathcal{B}(\Omega)$ of Borel measurable functions by

$$M_m^\Psi(\Omega) := \left\{ f \in \mathcal{B}(\Omega) : \exists C > 0 \text{ s.t. } \int_{\lambda_f(t)} dm(x) \leq C\Psi(t), \forall t > 0 \right\} \quad (2.41)$$

where

$$\lambda_f(t) = \{x \in \Omega : |f(x)| > t\}. \quad (2.42)$$

Notice that $\Psi(t) \leq m(\Omega)$ for $t \geq 0$. Denote

$$\bar{\lambda}_f(t) = \{x \in \Omega : |f(x)| \geq t\}. \quad (2.43)$$

Since Ψ is continuous, (2.41) implies

$$\int_{\bar{\lambda}_f(t)} dm(x) \leq C\Psi(t), \forall t > 0.$$

If we modify Ψ in order to impose $\Psi(0) = m(\Omega)$, (2.41) is equivalent to

$$M_m^\Psi(\Omega) := \left\{ f \in \mathcal{B}(\Omega) : \exists C > 0 \text{ s.t. } \int_{\bar{\lambda}_f(t)} dm(x) \leq C\Psi(t), \forall t \geq 0 \right\} \quad (2.44)$$

We denote by $C_m^\Psi(f)$ the smallest constant C such that (2.41) holds. If $t \mapsto \Psi(t)/\Psi(2t)$ remains bounded on $[0, \infty)$, $M_m^\Psi(\Omega)$ is a vector space $f \mapsto C_m^\Psi(f)$ is a quasi-norm on the quotient space $M_m^\Psi(\Omega)/\mathcal{R}$ where \mathcal{R} is the equivalence relation $f_1 \mathcal{R} f_2 \iff f_1 - f_2 = 0$ m -a.e. in Ω . In general $M_m^\Psi(\Omega)$ is not a vector space

When $\Psi(t) = t^{-p}$ with $p \geq 1$ and $m(x) = \rho(x)^\alpha$, with $\alpha \in [0, 1]$, we denote by $M_{\rho^\alpha}^p(\Omega)$ the corresponding Marcinkiewicz space. The following results proved in [5] with $L = -\Delta$ are valid for a general operator L

Proposition 2.5 Let $\alpha \in [0, 1]$, $N \geq 2$. If $\mu \in \mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ and $N + \alpha - 2 > 0$,

$$\|\mathbb{G}[\mu]\|_{M_{\rho^\alpha}^{(N+\alpha)/(N+\alpha-2)}} \leq C \|\mu\|_{\mathfrak{M}_{\rho^\alpha}}, \quad (2.45)$$

$$\|\nabla \mathbb{G}[\mu]\|_{M_{\rho^\alpha}^{(N+\alpha)/(N+\alpha-1)}} \leq C \|\mu\|_{\mathfrak{M}_{\rho^\alpha}}. \quad (2.46)$$

Furthermore, for any $\gamma \in [0, 1]$ and $\lambda \in \mathfrak{M}(\partial\Omega)$,

$$\|\mathbb{K}[\lambda]\|_{M_{\rho^\gamma}^{(N+\gamma)/(N-1)}} \leq C \|\lambda\|_{\mathfrak{M}}. \quad (2.47)$$

We recall the following result proved in [13, Th 2.9]

Theorem 2.6 Let $\alpha \in [0, 1]$. For every $\mu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ and $\lambda \in \mathfrak{M}(\partial\Omega)$, there exists a unique $u := u_{\mu, \lambda} \in L^1(\Omega)$ satisfying

$$\begin{aligned} -Lu &= \mu & \text{in } \Omega \\ u &= \lambda & \text{in } \partial\Omega, \end{aligned} \quad (2.48)$$

in the following weak sense

$$-\int_{\Omega} u L^* \zeta dx = \int_{\Omega} \zeta d\mu - \int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda \quad \forall \zeta \in C_{c^1, L}(\overline{\Omega}). \quad (2.49)$$

Furthermore, if $\{(\mu_n, \lambda_n)\}$ is bounded in $\mathfrak{M}_{\rho^\alpha}(\Omega) \times \mathfrak{M}(\partial\Omega)$ and converges weakly with respect to $C_\alpha(\overline{\Omega}) \times C(\partial\Omega)$ to $(\mu, \lambda) \in \mathfrak{M}_{\rho^\alpha}(\Omega) \times \mathfrak{M}(\partial\Omega)$, then u_{μ_n, λ_n} converges to $u_{\mu, \lambda}$.

Remark. If we define the measure $\omega \in \mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ by $\omega = T[\mu, \lambda]$ (see (2.30)), then it can also be expressed by

$$\int_{\overline{\Omega}} \zeta d\omega := \int_{\Omega} \zeta d\mu - \int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda \quad \forall \zeta \in C_1(\overline{\Omega}), \quad (2.50)$$

since $\zeta \in C_1(\overline{\Omega})$ implies that $\zeta_{\mathbf{n}}$ exists on $\partial\Omega$ and is continuous. We define the *global* Green operator on $\overline{\Omega}$ by

$$\overline{\mathbb{G}}[\omega] := \mathbb{G}[\mu] + \mathbb{P}_L[\lambda]. \quad (2.51)$$

and (2.48) is replaced by the unique equation

$$-Lu = \omega \quad \text{in } \overline{\Omega}. \quad (2.52)$$

Then (2.45)-(2.47) with $\alpha = 1$ are equivalent to

$$\|\overline{\mathbb{G}}[\omega]\|_{M_{\rho}^{(N+1)/(N-1)}} \leq C \|\omega\|_{\mathfrak{M}_{\rho}}. \quad (2.53)$$

Furthermore, we say that $u \in L^1(\Omega)$ is a subsolution of (2.52) in $\overline{\Omega}$, if

$$-\int_{\Omega} u L^* \zeta dx \leq \int_{\overline{\Omega}} \zeta d\omega := \int_{\Omega} \zeta d\mu - \int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda \quad \forall \zeta \in C_c^{1, L^*}(\overline{\Omega}), \zeta \geq 0. \quad (2.54)$$

Comparison principle applies, thus $u \leq \overline{\mathbb{G}}[\omega]$. A supersolution is defined similarly.

Remark. If $\omega = T[\mu, \lambda] \in \mathfrak{M}_{\alpha}^+(\overline{\Omega})$ its Lebesgue decomposition is $\omega_r + \omega_s = T[\mu_r, \lambda_r] + T[\mu_s, \lambda_s]$ where μ_r and λ_r are the absolutely continuous part with respect to the Hausdorff measures $d\mathcal{H}^N$ and $d\mathcal{H}^{N-1}$ and μ_s and λ_s the respective singular parts. Similarly if $\omega = T[\mu, \lambda]$, then $\omega = \omega^+ - \omega^-$ where $\omega^+ = T[\mu^+, \lambda^+]$ and $\omega^- = T[\mu^-, \lambda^-]$.

2.4 Regularity results

We define the class of measures $B_h^\Psi(\overline{\Omega})$ by

$$B_h^\Psi(\overline{\Omega}) := \{\omega \in \mathfrak{M}_\rho(\overline{\Omega}) : \overline{\mathbb{G}}[|\omega|] \in M_{\rho h}^\Psi(\Omega)\}. \quad (2.55)$$

By Proposition 2.4, this class remains unchanged if we replace $-\Delta$ by L and the Green operator for L by the one of $-\Delta$. If $\Psi(t) = t^{-p}$ and $h = 1$, the corresponding class of measures is larger than the usual

$$\tilde{B}^p(\overline{\Omega}) := \{\omega \in \mathfrak{M}_\rho(\overline{\Omega}) : \overline{\mathbb{G}}[|\omega|] \in L^p(\Omega)\} \quad (2.56)$$

which corresponds to negative Besov spaces: if $\omega = T[\mu, \lambda]$, then the regularity results for harmonic functions [9] and solution of Laplace equation [1] yields to

$$\tilde{B}^p(\overline{\Omega}) = B^{-\frac{2}{p}, p}(\Omega). \quad (2.57)$$

Example 1 If $h(x) = (\rho(x))^\beta$, with $\beta > -2$. Then $\omega = T[0, \lambda] \in B_{\rho^\beta}^\Psi(\overline{\Omega})$ if and only if $\overline{\mathbb{G}}[|\omega|] \in M_{\rho^{\beta+1}}(\Omega)$. This means that $\lambda \in B_\infty^{-s, p}(\partial\Omega)$ with $s = (\beta + 2)/p$ (see [11] for the definition of $B_q^{\alpha, p}$).

3 The main results

Definition 3.1 We say that a Caratheodory function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ belongs to the class $G_{h, \Psi}$ if there exist a nonnegative function $h \in L_\rho^1(\Omega)$, a continuous nondecreasing function \tilde{g} defined on \mathbb{R}_+ and vanishing at $r = 0$ such that $0 \leq g(x, r) \text{sign } r \leq h(x) \tilde{g}(|r|)$ in $\Omega \times \mathbb{R}$ and a continuous nonincreasing function $\Psi : [0, \infty) \rightarrow [0, \infty)$ with the property that

$$-\int_1^\infty \tilde{g}(s) d\Psi(s) < \infty. \quad (3.58)$$

Lemma 3.2 Let μ be a nonnegative measure in $\mathfrak{M}(\Omega)$ and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a Caratheodory function such that $0 \leq g(x, r) \text{sign } r \leq h(x) \tilde{g}(|r|)$ where $h \in L_\rho^1(\Omega)$ and \tilde{g} is a continuous and nondecreasing function \tilde{g} defined on \mathbb{R}_+ and vanishing at $r = 0$. Then

- (i) If $g \in G_{h, \Psi}$ and $\mu \in B_h^\Psi(\overline{\Omega})$, then $\tilde{g} \circ \overline{\mathbb{G}}[\mu] \in L^1(\Omega)$.
- (ii) if $\tilde{g} \circ \overline{\mathbb{G}}[\mu] \in L_{\rho h}^1(\Omega)$ and , then $\mu \in B_h^\Psi(\overline{\Omega})$ and $g \in G_{h, \Psi}$ with $\Psi(s) = \theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s)$, where $\lambda_{\overline{\mathbb{G}}[\mu]}(s)$ is defined by (2.42) with f replaced by $\overline{\mathbb{G}}[\mu]$ and $\theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s) = \int_{\lambda_{\overline{\mathbb{G}}[\mu]}(s)} d(\rho h)$.

Proof. This due to the fact that

$$\int_\Omega \tilde{g}(\overline{\mathbb{G}}[\mu]) \rho h dx = - \int_0^\infty \tilde{g}(s) d\theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s) \quad (3.59)$$

Therefore, if $\theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s) \leq \Psi(s)$, it proves (i). Conversely, if $\Psi(s) = \theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s)$, then $\mu \in B_h^\Psi(\overline{\Omega})$ and $g \in G_{h, \Psi}$. \square

The following existence result extends to one in [13]

Theorem 3.3 Assume g belongs to the class $G_{h,\Psi}$. Then for any $\omega \in B_h^\Psi(\overline{\Omega})$ there exists a function $u \in L^1(\Omega)$ such that $g \circ u \in L^1(\Omega)$ satisfying

$$\int_{\Omega} (-uL^*\zeta + g \circ u \zeta) dx = \int_{\overline{\Omega}} \zeta d\omega \quad \forall \zeta \in C_c^{1,L^*}(\overline{\Omega}). \quad (3.60)$$

Furthermore u is unique if $r \mapsto g(x, r)$ is nondecreasing for a.e. $x \in \Omega$.

Proof. It is essentially [13, Theorem 3.7]. Since $0 \leq g(x, r)\text{sign } r \leq h(x)\tilde{g}(|r|)$, we define the following truncation $g_k(\cdot, r)$ for any $k > 0$.

$$g_k(x, r) = g(x, r)\chi_{\Theta_k} \quad (3.61)$$

where $\Theta_k = \{x \in \Omega : h(x) \leq k\}$. Then $0 \leq g(x, r)\text{sign } r \leq k\tilde{g}(|r|)$ and there exists a solution u_k to

$$-Lu_k + g_k \circ u_k = \omega \quad \text{in } \overline{\Omega}. \quad (3.62)$$

Actually, in [13, Theorem 3.7] the proof is done with $\mu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ for any $\alpha \in [0, 1]$, but due to our definition of measures in $\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$, it is also valid in this case.

Step 2: Convergence when $k \rightarrow \infty$. By Brezis' estimates (see e.g. [13, Th 2.4]), for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, $\zeta \geq 0$, one has

$$\int_{\Omega} (-|u_k|L^*\zeta + \text{sign}(u_k)g_k(x, u_k)\zeta) dx \leq \int_{\overline{\Omega}} \zeta d|\omega|. \quad (3.63)$$

and

$$\|u_k\|_{L^1} + \|\rho g_k(\cdot, u_k)\|_{L^1_\rho} \leq C_1 \|\omega\|_{\mathfrak{M}_\rho}. \quad (3.64)$$

Furthermore, by estimates of Proposition 2.5 and since $|u_k| \leq \overline{\mathbb{G}}[|\omega|]$, there holds,

$$\|u_k\|_{M_\rho^{(N+1)/N}} + \|\nabla u_k\|_{M_\rho^{(N+1)/N}} \leq C \|\omega\|_{\mathfrak{M}_\rho}. \quad (3.65)$$

Since the right-hand side of (3.65) is bounded independently of k fixed, there exist a subsequence $\{u_{k_j}\}$ and a function $u \in W_{loc}^{1,q}(\Omega)$, for any $1 \leq q < (N+1)/N$, such that $u_{k_j} \rightarrow u$ a.e. in Ω - and thus $g_{k_j} \circ u_{k_j} \rightarrow g \circ u$ a.e. - and weakly in $W_{loc}^{1,q}(\Omega)$ when $k_j \rightarrow \infty$. Let $R > 0$ and $E \subset \Omega$ be a Borel subset, then

$$\begin{aligned} \int_E |g_{k_j} \circ u_{k_j}| \rho dx &\leq \int_{E \cap \{|u_{k_j}| \leq R\}} \tilde{g}(|u_{k_j}|) \rho h dx + \int_{E \cap \{|u_{k_j}| > R\}} \tilde{g}(|u_{k_j}|) \rho h dx \\ &\leq \tilde{g}(R) \int_E \rho h dx - \int_R^\infty \tilde{g}(s) d\theta_{u_{k_j}}(s), \end{aligned} \quad (3.66)$$

where, we recall it,

$$\theta_{u_{k_j}}(s) := \int_{\lambda_{u_{k_j}}(s)} d(\rho h).$$

Since $|u_{k_j}| \leq \overline{\mathbb{G}}[|\omega|]$, $\theta_{u_{k_j}}(s) \leq \theta_{\overline{\mathbb{G}}[|\omega|]}(s)$. By assumption,

$$\theta_{\overline{\mathbb{G}}[|\omega_n|]}(s) \leq C\Psi(s) \quad \forall s > 0,$$

with

$$C = C_{\rho h}^{\Psi}(\overline{\mathbb{G}}[|\omega|]).$$

Furthermore, by a standard integration by parts in Stieltjes integrals and for a.e. R ,

$$\begin{aligned} -\int_R^\infty \tilde{g}(s) d\theta_{u_{k_j}}(s) &= \tilde{g}(R)\theta_{u_{k_j}}(R) + \int_R^\infty \theta_{u_{k_j}}(s) d\tilde{g}(s) \\ &\leq \tilde{g}(R)\theta_{u_{k_j}}(R) + C \int_R^\infty \Psi(s) d\tilde{g}(s) \\ &\leq \tilde{g}(R)\theta_{u_{k_j}}(R) - C\tilde{g}(R)\Psi(R) - C \int_R^\infty \tilde{g}(s) d\Psi(s) \\ &\leq -C \int_R^\infty \tilde{g}(s) d\Psi(s). \end{aligned} \tag{3.67}$$

Since condition (3.58) holds

$$\lim_{R \rightarrow \infty} \int_R^\infty \tilde{g}(s) d\Psi(s) = 0. \tag{3.68}$$

Given $\epsilon > 0$, we first choose $R > 0$ such that

$$-C \int_R^\infty \tilde{g}(s) d\Psi(s) \leq \epsilon/2.$$

Then we put $\delta = \epsilon/(2(1 + \tilde{g}(R)))$ and derive

$$\int_E \rho dx \leq \delta \implies \int_E |g_{k_j}(u_{k_j})| \rho h dx \leq \epsilon.$$

Therefore $\{g_{k_j} \circ u_{k_j}\}$ is uniformly integrable in $L_\rho^1(\Omega)$. It follows by Vitali's convergence theorem

$$\lim_{k \rightarrow \infty} g_{k_j} \circ u_{k_j} = g \circ u \quad \text{in } L_\rho^1(\Omega). \tag{3.69}$$

Let $\zeta \in C_c^{1,L}(\overline{\Omega})$. If we let $k_j \rightarrow \infty$ in the equality

$$\int_\Omega (-u_{k_j} L^* \zeta + g_{k_j} \circ u_{k_j} \zeta) dx = \int_\Omega \zeta d\omega \tag{3.70}$$

we derive

$$\int_\Omega (-u L^* \zeta + g \circ u \zeta) dx = \int_\Omega \zeta d\omega. \tag{3.71}$$

Uniqueness follows classically if $g(x, \cdot)$ is nondecreasing. \square

The following extension of the previous result is an adaptation of [13, Th. 3.20]

Theorem 3.4 Assume g belongs to the class $G_{h,\Psi}$ and satisfies the following Δ_2 -condition

$$|g(x, r + r')| \leq \theta (|g(x, r)| + |g(x, r')|) + \ell(x) \quad \forall x \in \Omega, \forall (r, r') \in \mathbb{R} \times \mathbb{R}, \quad (3.72)$$

for some nonnegative $\ell \in L^1_\rho(\Omega)$. Suppose also that $r \mapsto g(x, r)$ is nondecreasing. If $\omega \in \mathfrak{M}_\rho(\overline{\Omega})$ has Lebesgue decomposition $\omega = \omega_r + \omega_s$ with regular part with respect to the Lebesgues measures ω_r and singular part ω_s , and if ω_s belongs to $B_h^\Psi(\overline{\Omega})$, then there exists a unique solution u to (3.60).

Proof. If g satisfies (3.72), g_k defined by (3.61) shares the same property with the same ℓ . Therefore, by [13, Th 3.12], there exists a solution u_k to (3.62). Actually, in this result it is only assume that ℓ in (3.72) is a constant, but the proof is valid if it is a nonnegative function in $L^1_\rho(\Omega)$. Let v_k and v'_k be weak solutions in $\overline{\Omega}$ of $-Lv_k + g_k \circ v_k = \omega_r^+$ and $-Lv'_k - g_k \circ (-v'_k) = \omega_r^-$ respectively. Set $w_k = v_k + \overline{\mathbb{G}}(\omega_s^+)$ and $w'_k = v'_k + \overline{\mathbb{G}}(\omega_s^-)$. Then $-Lw_k + g_k \circ w_k \geq \omega^+$ and $-Lw'_k - g_k \circ (-w'_k) \geq \omega^-$ in $\overline{\Omega}$. By monotonicity $-w'_k \leq u_k \leq w_k$, thus $g_k(-w'_k) \leq g_k(u_k) \leq g_k(w_k)$. The estimates (3.64) and (3.65) are satisfied, therefore there exist a function $u \in L^1(\Omega)$ and a subsequence u_{k_j} which converges to u a.e. in Ω . Furthermore

$$\begin{aligned} g_k(x, u_k) &\leq \theta (g_k(x, v_k) + g_k(x, \overline{\mathbb{G}}(\omega_s^+))) + \ell \\ &\leq \theta (g_k(x, v_k) + g(x, \overline{\mathbb{G}}(\omega_s^+))) + \ell \end{aligned} \quad (3.73)$$

Since the sequence $\{|g_k|\}$ increases, $\{v_k\}$ and $\{v'_k\}$ decrease. Therefore $v_k \downarrow v$ and $v'_k \downarrow v'$ which satisfy $-Lv + g \circ v = \omega_r^+$ and $-Lv' - g \circ (-v') = \omega_r^-$ respectively in $\overline{\Omega}$. Therefore $g_k \circ v_k \rightarrow g \circ v$ and $g_k \circ v'_k \rightarrow -g \circ (-v')$ in $L^1_\rho(\Omega)$ respectively. Since

$$g_k \circ \overline{\mathbb{G}}(\omega_s^+) \leq g \circ \overline{\mathbb{G}}(\omega_s^+)$$

and $\omega_s \in B_h^\Psi(\overline{\Omega})$, $g \circ \overline{\mathbb{G}}(\omega_s^+)$ by Lemma 3.2, the right-hand side term of inequality (3.73) is uniformly integrable in $L^1_\rho(\Omega)$. Similarly

$$g_k(x, u_k) \geq \theta (g_k(x, -v'_k) + g(x, -\overline{\mathbb{G}}(\omega_s^-))) - \ell \quad (3.74)$$

and the right-hand side of (3.74) is also uniformly integrable in $L^1_\rho(\Omega)$. We conclude as in Theorem 3.3. \square

4 Stability

Lemma 4.1 Let $\{\omega_n\} \subset B_h^\Psi(\overline{\Omega})$ be a sequence of measures such that $C_\rho^\Psi(\overline{\mathbb{G}}[|\omega_n|])$ is bounded independently of n . Then $\{\omega_n\}$ remains bounded in $\mathfrak{M}_\rho(\overline{\Omega})$. If $\omega_n \rightarrow \omega$ weakly in $\mathfrak{M}_\rho(\overline{\Omega})$, then $\omega \in B_h^\Psi(\overline{\Omega})$.

Proof. Since $C_\rho^\Psi(\overline{\mathbb{G}}[|\omega_n|])$ is uniformly bounded, the sequence $\{g \circ \overline{\mathbb{G}}[|\omega_n|]\}$ is bounded in $L^1_\rho(\Omega)$ by Lemma 3.2. Since $\omega_n \rightarrow \omega$ weakly in $\mathfrak{M}_\rho(\overline{\Omega})$, $\overline{\mathbb{G}}[\omega_n] \rightarrow \overline{\mathbb{G}}[\omega]$ in $L^1_\rho(\Omega)$ and, up to a subsequence, a.e. in Ω . Therefore, and up to sets of zero Lebesgue measure,

$$\lambda_{\overline{\mathbb{G}}[\omega]}(t) \subset \bigcap_{n \geq 0} \left(\bigcup_{p \geq n} \lambda_{\overline{\mathbb{G}}[\omega_p]}(t) \right) \subset \bigcap_{n \geq 0} \left(\bigcup_{p \geq n} \lambda_{\overline{\mathbb{G}}[\omega_p]}(t) \right) \subset \lambda_{\overline{\mathbb{G}}[\omega]}(t). \quad (4.75)$$

Therefore

$$\limsup_{n \rightarrow \infty} \theta_{\lambda_{\overline{\mathbb{G}}[\omega_n]}(t)} \leq \theta_{\lambda_{\overline{\mathbb{G}}[\omega]}(t)}. \quad (4.76)$$

Conversely, for any $x \in \lambda_{\overline{\mathbb{G}}[\omega]}(t)$, i.e. such that $\overline{\mathbb{G}}[\omega](x) > t$, there exists n_x such that $x \in \lambda_{\overline{\mathbb{G}}[\omega_n]}(t)$ if $n \geq n_x$. This implies

$$\lim_{n \rightarrow \infty} \chi_{\lambda_{\overline{\mathbb{G}}[\omega_n]}(t)} \chi_{\lambda_{\overline{\mathbb{G}}[\omega]}(t)} = \chi_{\lambda_{\overline{\mathbb{G}}[\omega]}(t)},$$

and

$$\liminf_{n \rightarrow \infty} \theta_{\lambda_{\overline{\mathbb{G}}[\omega_n]}(t)} \geq \theta_{\lambda_{\overline{\mathbb{G}}[\omega]}(t)}. \quad (4.77)$$

Since $\theta_{\lambda_{\overline{\mathbb{G}}[\omega_n]}(t)} \leq C_\rho^\Psi(\overline{\mathbb{G}}[|\omega_n|])\Psi(t)$ and the $C_\rho^\Psi(\overline{\mathbb{G}}[|\omega_n|])$ are bounded, it follows that ω belongs to $B_h^\Psi(\overline{\Omega})$. \square

Theorem 4.2 *Assume g belongs to the class $G_{h,\Psi}$ and $r \mapsto g(x, r)$ is nondecreasing for a.e. $x \in \Omega$. Let $\{\omega_n\} \subset B_h^\Psi(\overline{\Omega})$ be a sequence of measures such that $C_\rho^\Psi(\overline{\mathbb{G}}[|\omega_n|])$ is bounded independently of n which converges to ω weakly with respect to $C_1(\overline{\Omega})$. Then the solution u_n of*

$$-Lu_n + g \circ u_n = \omega_n \quad \text{in } \overline{\Omega} \quad (4.78)$$

converges to the solution u of

$$-Lu + g \circ u = \omega \quad \text{in } \overline{\Omega} \quad (4.79)$$

Proof. Since u_n satisfies the Brezis estimates (3.64) and (3.65), there exists a subsequence $\{u_{n_j}\}$ and $u \in L^1(\Omega)$ such that $u_{n_j} \rightarrow u$ a.e. in Ω and in $L^1(\Omega)$. As in the proof of Theorem 3.3, the problem is to prove the convergence of the $g \circ u_{n_j}$ in $L^1_\rho(\Omega)$. But this is clearly obtained by the uniform integrability, as in the proof of Theorem 3.3-Step 2, using the fact that, in (3.67), the $\theta_{u_{n_j}}$ are bounded by $\sup_n C_{\rho h}^\Psi(\overline{\mathbb{G}}[|\omega_n|])\Psi$. \square

Theorem 4.3 *Assume g belongs to the class $G_{h,\Psi}$, satisfies the Δ_2 -condition (3.72) and $r \mapsto g(x, r)$ is nondecreasing. Let $\{\omega_n\} \subset \mathfrak{M}_\rho(\overline{\Omega})$ has Lebesgue decomposition $\omega_n = \omega_{nr} + \omega_{ns}$ if $\{\omega_{ns}\} \subset B_h^\Psi(\overline{\Omega})$ are such that the $C_{\rho h}^\Psi(\overline{\mathbb{G}}[|\omega_{ns}|])$ are uniformly bounded, then the solutions u_n of (4.78) converges in $L^1(\Omega)$ to the solution u of (4.79).*

Proof. The argument follows the one of Theorem 3.4. Let v_n and v'_n be weak solutions in $\overline{\Omega}$ of $-Lv_n + g \circ v_n = \omega_{nr}^+$ and $-Lv'_n - g \circ (-v'_n) = \omega_{nr}^-$ respectively. Set $w_n = v_n + \overline{\mathbb{G}}(\omega_{ns}^+)$ and $w'_n = v'_n + \overline{\mathbb{G}}(\omega_{ns}^-)$. Then $-Lw_n + g \circ w_n \geq \omega_n^+$ and $-Lw'_n - g \circ (-w'_n) \geq \omega_n^-$. By monotonicity $-w'_n \leq u_n \leq w_n$, thus $g(-w'_n) \leq g(u_n) \leq g(w_n)$. The estimates (3.64) and (3.65) are satisfied therefore there exist a function $u \in L^1(\Omega)$ and a subsequence u_{n_j} which converges to u a.e. in Ω and in $L^1(\Omega)$. Furthermore

$$\begin{aligned} g(x, u_n) &\leq \theta \left(g(x, v_n) + g(x, \overline{\mathbb{G}}(\omega_{ns}^+)) \right) + \ell \\ &\leq \theta \left(g(x, v_n) + g(x, \overline{\mathbb{G}}(\omega_{ns}^+)) \right) + \ell \end{aligned} \quad (4.80)$$

Classically $v_n \rightarrow v$, $v'_n \rightarrow v'$ in $L^1(\Omega)$ which satisfy $-Lv + g \circ v = \omega_r^+$ and $-Lv' - g_k \circ (-v') = \omega_r^-$ respectively. Therefore $g \circ v_n \rightarrow g \circ v$ and $g \circ v'_n \rightarrow -g \circ (-v')$ in $L^1_\rho(\Omega)$ respectively. Since $C_{\rho h}^\Psi(\overline{\mathbb{G}}[\omega_{ns}])$ is uniformly bounded the $g \circ \overline{\mathbb{G}}[\omega_{ns}]$ are uniformly integrable in $L^1_\rho(\Omega)$ by Lemma 3.2. Therefore the $(g \circ u_n)^+$ are uniformly integrable in $L^1_\rho(\Omega)$. Similarly

$$g(x, u_n) \geq \theta (g(x, -v'_k) + g(x, -\overline{\mathbb{G}}(\omega_s^-)) - \ell) \quad (4.81)$$

and the $(g \circ u_n)^-$ are also uniformly integrable in $L^1_\rho(\Omega)$. The conclusion follows in the same way as in Theorem 3.4. \square

References

- [1] Adams D. R., Hedberg L. I.: *Function spaces and potential theory* Grundlehren Math. Wissen. **314**, Springer (1996).
- [2] Ancona A.: *Principe de Harnak à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine Lipschitzien*, Ann. Inst. Fourier (Grenoble) **28**, 169–213 (1978).
- [3] Bénilan Ph., Brezis H.: *Nonlinear problems related to the Thomas-Fermi equation*, J. Evolution Eq. **3**, 673-770 (2003).
- [4] Bénilan Ph., Brezis H., Crandall M.: *A semilinear elliptic equation in $L^1(\mathbb{R}^N)$* , Ann. Sc. Norm. Sup. Pisa Cl. Sci. 5 **Vol. 2**, 523555 (1975).
- [5] Bidaut-Véron M.F., Vivier L. : : *An elliptic semilinear equation with source term involving boundary measures: The subcritical case*, Rev. Mat. Iberoamericana **16**, 477513 (2000).
- [6] Brezis H.: *Some variational problems of the Thomas-Fermi type. Variational inequalities and complementarity problems*, Proc. Internat. School, Erice, 1978, pp. 53–73, Wiley, Chichester (1980).
- [7] Gmira A., Véron L.: *Boundary singularities of solutions of nonlinear elliptic equations*, Duke J. Math. **64**, 271-324 (1991).
- [8] Marcus M.: *Stability relative to weak convergence for a family of semilinear elliptic equations with measure data*, preprint (2012).
- [9] Marcus M., Véron L.: *A characterization of Besov spaces with negative exponents*, Around the Research of Vladimir Maz'ya I. Function Spaces, Springer Verlag International Mathematical Series , Vol. 11, 273-284 (2010).
- [10] Pinchover Y., *On positive solutions of second-order elliptic equations, stability results, and classification*, Duke Math. J. **57**, 955-980 (1988).
- [11] Triebel H., *Interpolation Theory, Function Spaces, Differential Operators*, North Holland Publ. Co. (1978).

- [12] Véron L.: *Weak and strong singularities of nonlinear elliptic equations*, Proc. Symposia in Pure Math. **45** Part 2, 477-495 (1986).
- [13] Véron L.: *Elliptic equations involving measures* in *Stationary partial differential equations* **Vol. I**, 593-712. Handb. Differ. Equ., North-Holland, Amsterdam (2004).
- [14] Véron L.: *Singularities of Solutions of Second Order Quasilinear Equations*, Pitman Research Notes in Math. **Vol. 353**, Longman, Harlow (1996).