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## OPTIMIZATION OF LAMINATED COMPOSITES BY USING GENETIC ALGORITHM AND THE POLAR DESCRIPTION OF PLANE ANISOTROPY

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**Abstract.** *The object of the present work is the development and application of a quite general approach to optimal design of composite laminates where elastic symmetries can also be explicitly expressed as criteria of the optimisation process. Our formulation is in the form of a highly non linear and non convex single- or multi-objective optimisation problem subject to equality and inequality constraints. We show here applications to the design of maximum stiffness, maximum buckling load, maximum eigenfrequencies, maximum strength for laminated plates, as well as combinations of the afore mentioned criteria; all types of elastic symmetries can be taken into account. We show here a number of numerical solutions found using the genetic algorithm BIANCA.*

**Keywords:** laminate; elastic symmetries; stiffness; strength; vibration; anisotropy; buckling; optimization; polar method; genetic algorithm.

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## 1 Introduction

The use of composite laminates finds many applications in mechanical structures, particularly inspired by the need of replacing metals in order to obtain lightweight structures. However, composite materials show very special features in comparison to metals because of their heterogeneity, and the architecture of their reinforcement structure can be tailored in order to obtain different types of anisotropy and couplings among different behaviours, which can find such applications as in smart and adaptive structures.

Therefore, the concept of design and optimisation of the constitutive material is established as a fundamental step in the process of design and optimisation of composite structures [1].

Number of research works in this area showed how design of composite laminates raises very hard difficulties, and that is based on several different reasons. First of all, the manifold of variables in the design process, which are all the constitutive parameters of a laminate (number of layers; material, thickness and orientation of each layer) and which might take continuous real values as well as discrete values. On the other hand, the cumbersome dependence of the laminate properties on the constitutive parameters, particularly the orientation angles: from the mathematical side, relations are highly non linear, and on the physical side laminates show complex properties of anisotropies and couplings.

The afore mentioned difficulties, which affect at the same time the theoretical formulation of the design problems as well as the numerical methods for resolution, imposed some limits in the treatment of design and optimisation of composite laminates.

On one hand, the use is to restrict the number and nature of variables in the design process, especially in terms of orientation angles which are classically chosen within the discrete set  $0, \pm 45, 90$  [2-5]. On the other hand, authors use to introduce some simplifying hypothesis in order to ensure the respect of fundamental properties of composites such as some kinds of elas-

tic symmetries. For instance, it is common use to choose symmetric stacking sequences in order to satisfy elastic uncoupling, and balanced or cross-ply laminates for in-plane orthotropy [2-8]. On the other hand, this choice induces some approximations when the design of laminate flexural behaviour is concerned: in a general way, symmetric stacking sequences are assumed to be orthotropic in bending, even if this is not true [1-8].

All the burden of restrictions and simplifying hypothesis strongly reduce the search space in the process of optimal design of composite laminates. For instance, it is known that symmetric stacking sequences represent only one “trivial” solution among the great number of quasi-trivial non-symmetric exact solutions for elastic uncoupling [9]. Moreover, the development of modern manufacture technologies, together with the more and more stringent need for improving performances and satisfying simultaneously a larger panel of design criteria for new applications, push the research beyond the classically accepted limitations in the design and optimisation of composite materials.

A major difficulty is the very cumbersome expression of conditions for elastic symmetries in terms of the classical Cartesian components and, as a consequence, authors are not able to include such conditions in a synthetic and general formulation of design problems for composite laminates. Only few empirical solutions are known to a limited number of elastic symmetries, and they are universally applied: symmetric stacking sequences for uncoupling, balanced or cross-ply laminates for orthotropy, Werren-Norris solutions for isotropy. Concerning orthotropy and isotropy, these solutions apply only to in-plane behaviour, whilst for symmetries of bending behaviour general empirical rules are not available, and authors accept some approximation [1-8]. Only few authors dealt with elastic symmetries applying the lamination parameters of Tsai and Pagano [10] but a general methodology for the design of composite laminates is not established.

The object of the present work is the development and application of an approach to the optimal design of composite laminates based on the use of the polar method [11], which allows the explicit expression of conditions on elastic symmetries such as uncoupling and bending orthotropy. The generality of our approach relies on the absence of any simplifying hypothesis introduced a priori in order to automatically satisfy some given properties.

The framework of our approach is the Classical Laminated Plate Theory (CLPT) which is a simple model for laminates [1]. It is well known that when dealing with fine evaluation of the stress state within a laminate, the CLPT is no longer valid, however it is still the reference for the evaluation of elastic symmetries, which are important properties for anisotropic and coupled structured materials such as laminated composites.

In our approach, the design of composite laminates is formulated as a highly non linear and non convex optimisation problem, where all combinations of elastic symmetries as well as different required criteria can be taken into account simultaneously in the form of objective functions and/or constraints (equality as well as inequality constraints).

As far as the design variables are concerned, our formulation includes the dependence upon all the constitutive parameters of a composite laminate (number of layers; material properties, thickness and orientation of each layer). Of course, the high number of variables and the different classes (continuous, discrete, grouped) they belong to, dramatically increase the dimension of the design space. As a consequence, we developed a robust numerical tool for solving our optimisation problems, which is the genetic algorithm BIANCA.

In the following, we introduce the polar method, we describe our approach to the general formulation of optimal design of composite laminates and we detail the applications to maximization of stiffness, buckling load, first eigenfrequency and strength. Further, we give several examples of solutions to optimal design problems of laminates found by the use of BIANCA.

## 2 The polar method for plane elasticity and the classical laminated plate theory

One of the major difficulties in tailoring properties of composite laminates is dealing with anisotropy and coupling among different behaviours. Even if we limit the scope to the linear elastic behaviour of composite laminated plates according to Love-Kirchhoff model, the designer must take into account at least the different behaviours of the laminate with respect to in-plane and bending loads, as well as the elastic coupling between in-plane and bending. The stiffness behaviours of a laminate are represented in the CLPT (Classical Laminated Plate Theory) by tensors  $\mathbf{A}$ ,  $\mathbf{D}$  and  $\mathbf{B}$  respectively:

$$\begin{aligned}\mathbf{N} &= \mathbf{A}\boldsymbol{\varepsilon} + \mathbf{B}\boldsymbol{\kappa} \\ \mathbf{M} &= \mathbf{B}\boldsymbol{\varepsilon} + \mathbf{D}\boldsymbol{\kappa}\end{aligned}\quad (1)$$

where  $\mathbf{N}$  and  $\mathbf{M}$  are the in-plane loads and bending moments,  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\kappa}$  are the plate mid-plane strains and curvatures, respectively. Tensors  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{D}$  depend on the constitutive parameters of the laminate according to CLPT equations:

$$\mathbf{A}, \mathbf{B}, \mathbf{D} = \frac{1}{m} \sum_{k=1}^N \mathbf{Q}_k(\delta_k) (z_k^m - z_{k-1}^m) \quad (2)$$

where  $N$  is the number of constitutive layers of the laminate,  $\mathbf{Q}(\delta_k)$  is the stiffness tensor of the  $k^{\text{th}}$  constitutive layer oriented by angle  $\delta_k$  with respect to the reference axes of the laminated plate, and exponent  $m = 1, 2, 3$  for in-plane, coupling and bending behaviour, respectively. It is well known that the three elastic behaviours can have distinct shapes of anisotropy and, in particular, elastic coupling  $\mathbf{B}$  is non zero for a general laminate, whilst generally applications require the respect of some kind of elastic symmetry (at least orthotropy) and elastic uncoupling ( $\mathbf{B} = \mathbf{O}$ ). We remind here the definition of the homogenised stiffness tensors  $\mathbf{A}^*$ ,

$$\mathbf{B}^* \text{ and } \mathbf{D}^*: \quad \mathbf{A}^* = \frac{1}{h} \mathbf{A}, \quad \mathbf{B}^* = \frac{2}{h^2} \mathbf{B}, \quad \mathbf{D}^* = \frac{12}{h^3} \mathbf{D}.$$

The lamina and laminate stiffness tensors  $\mathbf{Q}_k$ ,  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{D}$  can be represented within the framework of the polar method [11]. Generally, for a fourth order tensor  $\mathbf{L}$  of the class of plane elasticity, we give here the relations between its Cartesian components  $L_{ijkl}$  and its polar components  $T_0, T_1, R_0, R_1, \Phi_0$  and  $\Phi_1$ :

$$\begin{aligned}
L_{1111} &= T_0 + 2T_1 + R_0 \cos 4\Phi_0 + 4R_1 \cos 2\Phi_1, \\
L_{1112} &= R_0 \sin 4\Phi_0 + 2R_1 \sin 2\Phi_1, \\
L_{1122} &= -T_0 + 2T_1 - R_0 \cos 4\Phi_0, \\
L_{1212} &= T_0 - R_0 \cos 4\Phi_0, \\
L_{2212} &= -R_0 \sin 4\Phi_0 + 2R_1 \sin 2\Phi_1, \\
L_{2222} &= T_0 + 2T_1 + R_0 \cos 4\Phi_0 - 4R_1 \cos 2\Phi_1.
\end{aligned} \tag{3}$$

The important feature of the polar method is that the conditions for elastic symmetries assume a very simple form [12] in terms of polar components and, moreover, polar components are tensor invariants, as shown in Table 1. We remind that the polar method sheds a new light on the different shapes of elastic symmetries; for a more detailed discussion on these topics we refer to previous works of the authors [12-13].

### 3 Elastic symmetries as active criteria in laminate design: problems of “type 0”

The introduction of the polar method for plane elasticity [11-12] allows a decisive step forward in the analysis and design of elastic symmetries thanks to the very simple form of the conditions shown in Table 1. Each class of elastic symmetry corresponds to one polar invariant equal to a given value (often equal to zero). Therefore, the formulation of the design problem of a laminate with respect to elastic symmetries can be seen as a system of equations, which are highly non linear in terms of the laminate constitutive parameters, particularly orientation angles. Authors developed a direct approach of resolution of the equations issued from Table 1 in order to obtain laminates with given elastic symmetries, particularly uncoupling and quasi-homogeneity [9, 14]. Nevertheless, the direct approach is limited in terms of

the number of equations and variables to deal with, that is to say the number of symmetries imposed and the number of constitutive layers of the laminate.

In order to overcome these limitations, we propose here a more synthetic and general approach in the form of a single-objective non-constrained optimization problem, where the objective function  $I(\mathbf{P}(\mathbf{x}))$  to minimise is a quadratic form of the normalised polar invariants  $P_i$  of the laminate:

$$I(\mathbf{P}(\mathbf{x})) = \mathbf{P} \cdot \mathbf{H} \mathbf{P} = H_{ij} P_i P_j, \quad i, j = 1, \dots, 18 \quad (4)$$

where matrix  $\mathbf{H}$  is dependent on the elastic symmetries that the designer chooses as active design criteria, and  $\mathbf{x}$  is the vector of constitutive parameters of the laminate, which are the active variables of the optimisation problem. The objective function  $I(P_i)$  is positive semi-definite and its minima are zeros, which correspond to solutions of the design problem. Expressions for parameters  $P_i$  are:

$$\begin{aligned} P_1 &= \frac{T_0^A}{h M}, \quad P_2 = \frac{T_1^A}{h M}, \quad P_3 = \frac{R_0^A}{h M}, \quad P_4 = \frac{R_1^A}{h M}, \quad P_5 = \Phi_0^A, \quad P_6 = \Phi_1^A, \\ P_7 &= \frac{2 T_0^B}{h^2 M}, \quad P_8 = \frac{2 T_1^B}{h^2 M}, \quad P_9 = \frac{2 R_0^B}{h^2 M}, \quad P_{10} = \frac{2 R_1^B}{h^2 M}, \quad P_{11} = \Phi_0^B, \quad P_{12} = \Phi_1^B, \\ P_{13} &= \frac{12 T_0^D}{h^3 M}, \quad P_{14} = \frac{12 T_1^D}{h^3 M}, \quad P_{15} = \frac{12 R_0^D}{h^3 M}, \quad P_{16} = \frac{12 R_1^D}{h^3 M}, \quad P_{17} = \Phi_0^D, \quad P_{18} = \Phi_1^D. \end{aligned} \quad (5)$$

where  $M$  is a normalisation factor, which can be chosen in various ways and it is generally a tensor norm. Of course, parameters  $P_i$  are functions of the laminate constitutive parameters (number of layers, material, thickness and orientation of each layer) according to the equations of CLPT, which are therefore the variables of the optimisation problem.

Through the non-constrained minimisation of the objective function  $I(\mathbf{P}(\mathbf{x}))$ , we can take into account all admissible combinations of elastic symmetries for both in-plane and bending behaviours, as well as uncoupling. It is also possible to account for relative shapes and posi-

tions of in-plane and bending symmetries; for instance, we can express the condition for quasi-homogeneity, which is the coincidence of the homogenised in-plane and bending behaviours, or we can impose the same principal directions in in-plane and bending orthotropies.

We give here an example of the expression of the objective function  $I(\mathbf{P}(\mathbf{x}))$  in the case of design of composite laminates showing in-plane and bending orthotropy with coincident orthotropic axes, as well as uncoupling; this is the combination of elastic symmetries which we require in our calculations shown in section 5. In the following expression,  $\bar{K}$  and  $\tilde{K}$  are the in-plane and bending factor of orthotropic shape:

$$I(\mathbf{P}) = (P_5 - P_6)^2 - \left(\bar{K} \frac{\pi}{4}\right)^2 + (P_{17} - P_{18})^2 - \left(\tilde{K} \frac{\pi}{4}\right)^2 + (P_6 - P_{18})^2 + \sum_{i=7}^{10} P_i^2 = 0. \quad (6)$$

As a consequence, non-zero components of matrix  $\mathbf{H}$  are defined as:

$$H_{ii} = 1, i = 5, 7, \dots, 10, 17; H_{66} = H_{1818} = 2; H_{56} = H_{618} = H_{1718} = -1. \quad (7)$$

For a more detailed description of the objective function  $I(\mathbf{P})$  for various combination of design criteria, see [13].

The set of design problems for elastic symmetries of laminates in the form of a single-objective unconstrained minimisation of the form  $I(\mathbf{P})$  are called design problems of “type 0”, and the corresponding formulation is:

$$\min I(\mathbf{P}(\mathbf{x})) \quad (8)$$

#### 4 A general formulation of optimal design problems for laminates: optimisation problems of “type 1” and “type 2”

For the first time, through the application of the polar method, a general approach is proposed for the design of all various classes of elastic symmetries for composite laminates (formulation of design problems of “type 0”, section 3). In this approach, elastic symmetries are

considered as proper design criteria, and they do not have to be enforced through the choice of a special class of stacking sequence: all possible solutions are theoretically considered by the proposed formulation. Therefore, this approach allows a complete and exhaustive exploration of the whole design space, which proved to be very important since most problems of elastic symmetries for laminates showed to admit infinity of solutions that could not be accessible otherwise [9, 14].

#### 4.1 *Laminate optimisation problems of “type 1”: symmetries and constraints*

Since our formulation of the design of elastic symmetries is a single-objective non-constrained optimisation problem, further design criteria can be introduced in the form of constraints (equality and/or inequality) to the optimisation problem. Our formulation takes the form of a constrained mathematical optimisation problem, that we call design problems of “type 1” (in opposition to the unconstrained problem of “type 0” that we introduced in the previous section):

$$\begin{aligned} & \min I(\mathbf{P}(\mathbf{x})) \\ & \text{such that : } \begin{cases} h_j(\mathbf{x}) = 0, j = 1, \dots, n_h \\ g_k(\mathbf{x}) \leq 0, k = 1, \dots, n_g \\ \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U \end{cases} \end{aligned} \quad (9)$$

where  $\mathbf{x}$  is the vector of constitutive parameters of the laminate, which are the active variables of the optimisation problem. Box constraints can be imposed on  $\mathbf{x}$  in terms of lower and upper bounds,  $\mathbf{x}_L$  and  $\mathbf{x}_U$  respectively. Constraints functions  $h_j(\mathbf{x})$  and  $g_k(\mathbf{x})$  can represent any properties of the laminate, which are relevant for the design process (stiffness moduli, buckling load, strength criteria, etc.).

Typically, formulation (9) can be applied to tailoring of the stiffness response of laminates. For instance, the problem of design of orthotropy for in-plane and/or bending behaviour to-

gether with a minimum requirement on the stiffness response in the principal orthotropy directions takes the form:

$$\begin{aligned} & \min I(\mathbf{P}(\mathbf{x})) \\ \text{such that: } & \begin{cases} E_A^{\max} \geq E_A^*, \text{ and/ or: } E_A^{\min} \geq e_A^* \\ E_D^{\max} \geq E_D^*, \text{ and/ or: } E_D^{\min} \geq e_D^* \\ \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U \end{cases} \end{aligned} \quad (10)$$

where  $E_A^{\max}$  and  $E_A^{\min}$  are Young moduli in the principal and secondary directions for in-plane orthotropy respectively, whilst  $E_A^*$  and  $e_A^*$  are minimum required values, which are fixed according to design needs ( $E_D^{\max}$ ,  $E_D^{\min}$ ,  $E_D^*$  and  $e_D^*$  are corresponding quantities for bending). Optimisation variables  $\mathbf{x}$  can be composed by any constitutive parameters of the laminate (material, thickness and orientation of each layer).

The same form as formulation (10) could express the design of composite laminates with respect to some strength criteria and/or to a condition of stability against buckling under a given load for the laminated plate.

#### 4.2 Laminate optimisation problems of “type 2”: symmetries and new objective functions

When building an optimisation problem of “type 1”, it can be difficult to estimate a priori an acceptable level of constraint (strength, buckling resistance, etc.), and more generally it is not always possible to transform some design criteria into constraints of the optimisation problem. On the contrary, in real world problems the optimal design of composites is more naturally expressed in terms of the search of laminates optimising some properties, such as minimising weight and/or maximising stiffness and/or strength, and at the same time respecting conditions on elastic symmetries, such as uncoupling, isotropy and/or orthotropy for in-plane and bending. Authors who dealt with these subjects [1-8] expressed the optimisation problem in terms of appropriate objective functions (weight, strength, buckling resistance,

etc.), and they did not explicitly write conditions for the requirements on elastic symmetries but they applied well-known simplifying hypothesis on the stacking sequence in order to ensure the respect of symmetries.

As shown in the previous section, solutions responding to given conditions on elastic symmetries are the minima as well as the zeros of the quadratic form  $I(\mathbf{P})$ . As a consequence, we can put a general optimisation problem for a composite laminate in the following form, which we denote as design problem of “type 2”:

$$\begin{aligned} & \min f(\mathbf{x}) \\ & \text{such that: } \begin{cases} I(\mathbf{P}(\mathbf{x})) = 0 \\ \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U \end{cases} \end{aligned} \quad (11)$$

where the objective function  $f(\mathbf{x})$  can be any design criterion for the laminate (stiffness, strength, buckling resistance, etc.). The variables  $\mathbf{x}$  can be all the constitutive parameters of the laminate.

It is worth note that since function  $I(\mathbf{P})$  is positive semi-definite, the equality constraint in expression (11) can be translated into an inequality condition:  $I(\mathbf{P}(\mathbf{x})) < \varepsilon$  ( $\varepsilon$  is a small positive real number). Therefore, problem (11) can either be treated as a constrained optimisation problem with an inequality constraint.

For instance, following the formulation given in expression (11), we can state the problem of maximisation of stiffness for a composite laminate:

$$\begin{aligned} & \max E_1^A \\ & \text{such that: } I(\mathbf{P}(\mathbf{x})) < \varepsilon \end{aligned} \quad (12)$$

where  $E_1^A$  is the in-plane Young modulus in principal orthotropy direction.

It is worth note here that a combination of several design criteria can be taken into account through the introduction of additional objective functions in the optimisation problem (11).

Moreover, we can consider further equality and/or inequality constraints applied to problem (11), as it is shown for instance in expressions (9) and (10).

We can give a more detailed description of expressions (11) when applied to a particular optimisation problem, and in the following sub-sections we will describe in detail the cases of maximisation of buckling load, eigenfrequencies and strength for a composite laminated plate, under constraints on the respect of given elastic symmetries.

#### 4.2.1. Maximisation of buckling load and eigenfrequencies for rectangular laminated plates

Instability of laminated plates is a very important issue when these very thin structures are loaded in compression. Also an important topic is the optimal design of laminates with respect to free vibration response, since resonance can be induced by an excitation frequency which is close to the natural frequencies of the plate [1-8, 15]. Nevertheless, authors do not take directly into account conditions on elastic symmetries in the search for maximum buckling load or maximum natural frequencies.

We consider here the case of a rectangular simply supported uncoupled orthotropic laminated plate loaded in compression by in-plane loads of stress resultants  $N_x$  and  $N_y$ , whilst shear load  $N_{xy}$  is zero (Figure 1), whilst  $a$  and  $b$  are the lengths of the plate sides.

In this case, buckling shape is sinusoidal, and being  $m$  and  $n$  the number of half waves in the  $x$  and  $y$  direction respectively, the critical value of load multiplier  $\lambda$  inducing buckling is:

$$\lambda_{crit}(m, n) = \frac{\pi^2 [D_{11}(m/a)^4 + 2(D_{12} + 2D_{66})(m/a)^2(n/b)^2 + D_{22}(n/b)^4]}{(m/a)^2 N_x + (n/b)^2 N_y} \quad (13)$$

In equation (13),  $D_{ij}$  ( $i, j = 1, 2, 6$ ) are the Cartesian components of the bending stiffness tensor expressed in the reference system  $Oxy$ . We notice that expression (13) applies to orthotropic uncoupled plates, and therefore bending orthotropy is a necessary condition as

well as elastic uncoupling. Moreover, the principal axes of orthotropy have to be coincident with the plate axes.

Optimisation of buckling load of a simply supported laminated plate can be expressed as:

$$\begin{aligned} & \max (\min_{m,n} \lambda_{\text{crit}}) \\ & \text{such that : } I(\mathbf{P}(\mathbf{x})) < \varepsilon \end{aligned} \quad (14)$$

where function  $I(\mathbf{P})$  corresponds to the necessary conditions of elastic uncoupling and bending orthotropy with the principal axes oriented along the plate axes.

In the same way, the free vibration modes of a simply supported rectangular plate are sinusoidal in both  $x$  and  $y$  directions, and the natural vibration frequency  $\omega_{mn}$  for an  $mn$ -mode is:

$$\frac{\omega_{mn}}{\pi^2} = \sqrt{\frac{D_{11}(m/a)^4 + 2(D_{12} + 2D_{66})(m/a)^2(n/b)^2 + D_{22}(n/b)^4}{\rho h}} \quad (15)$$

where  $\rho$  denotes the mass density of the plate and  $h$  its thickness.

Therefore, maximisation of the eigenfrequency of a given free vibration mode (given values of  $m$  and  $n$ ) for a simply supported laminated plate can be expressed as:

$$\begin{aligned} & \max \omega_{mn} \\ & \text{such that : } I(\mathbf{P}(\mathbf{x})) < \varepsilon \end{aligned} \quad (16)$$

where the constraint function  $I(\mathbf{P})$  corresponds to the necessary conditions of elastic uncoupling and bending orthotropy with the principal axes oriented along the plate axes.

#### 4.2.2. Maximisation of strength for laminated plates

Maximisation of strength of composite laminates is a very delicate issue in order to prevent failure of composite plates and many authors dealt with this subject [1, 4, 6-8].

Mechanisms of failure for composite laminates are complex and related to different phenomena because of the heterogeneous nature of such materials. However, the resistance to

failure of a unidirectional composite can be measured by comparing homogeneous functions of the stresses and strains to material strength limits. Several failure criteria are available for orthotropic materials [1], each one characterised by a different limit function, and they apply to each single lamina within a stacking sequence. Therefore, laminate strength prediction is carried out by evaluating the state of stress within each ply based on the laminate theory and, as it is well known, stresses vary from one ply to another because of their different orientation, different properties and different position through the thickness. Normally, first failure of any layer is not acceptable because it degrades the stiffness and the strength of the whole laminate. For this reason, a verification of the failure limit for each layer is necessary.

These reasons motivated our choice for an objective function in strength optimisation of composite laminates based upon a global quantity which is a measure of the laminate load state. According to Park [6], we used a quadratic form  $f(\boldsymbol{\epsilon})$  of the strain components to be minimised, which is a measure of the norm of the strain vector:

$$f(\boldsymbol{\epsilon}) = \epsilon_{xx}^2 + \epsilon_{yy}^2 + \frac{1}{2} \gamma_{xy}^2. \quad (17)$$

Function  $f(\boldsymbol{\epsilon})$  represents the global response of the laminate to the applied state of load and it does not depend on a ply level strain calculation. We can also express the optimisation problem in terms of maximisation of a strength function  $R_{index}$ :

$$R_{index} = \frac{1}{f(\boldsymbol{\epsilon})}. \quad (18)$$

Moreover, we chose here to apply the Hoffman criterium on each layer, which is an extension of Hill criterium by the introduction of linear terms of the stress components [1]. For a plane state of stress expressed in the principal orthotropy directions  $x_1$  and  $x_2$ , and for a transversely isotropic material in the plane  $x_2x_3$ , the failure envelop for the Hoffman criterium is expressed as:

$$-\frac{\sigma_{11}^2}{X_c X_t} + \frac{\sigma_{11} \sigma_{22}}{X_c X_t} - \frac{\sigma_{22}^2}{Y_c Y_t} + \frac{X_c + X_t}{X_c X_t} \sigma_{11} + \frac{Y_c + Y_t}{Y_c Y_t} \sigma_{22} + \frac{\tau_{12}^2}{S_{12}^2} = 1, \quad (19)$$

where  $X_c$ ,  $X_t$ ,  $Y_c$ ,  $Y_t$  and  $S_{12}$  are the layer strength limits in tension and compression along the two material axes and for shear, respectively.

Finally, we can formulate the design problem of composite laminates with respect to strength as it follows:

$$\begin{aligned} & \max R_{index} \\ & \text{such that : } f_{Hoffman} < 1 \\ & I(\mathbf{P}(\mathbf{x})) < \varepsilon \end{aligned} \quad (20)$$

where function  $I(\mathbf{P}(\mathbf{x}))$  expresses conditions over elastic symmetries (uncoupling, orthotropy, and so on).

## 5 Numerical resolution of optimal design problems for laminates by the use of the genetic algorithm BIANCA

The mechanical behaviour of composite laminates depends on harmonic functions of the layer orientation angles. Therefore, in our formulation of design problems for composite laminates as shown in previous sections, the objective functions as well as functions of constraints are generally highly non linear and non convex.

Moreover, optimisation variables are all the constitutive parameters of the laminate, which can be of various types (continuous and/or discrete). Nevertheless, in real world problems for design of composite laminates as well as in most engineering problems, variables are essentially discrete. They might also belong to the so-called grouped type, as it is the case for the constitutive materials of the elementary layer: the choice of one particular material corresponds to a fixed set of mechanical properties.

Therefore, we can state that design of composite laminates leads to hard combinatorial optimisation problems defined over a highly multidimensional search space, which are very difficult to solve.

Starting from these considerations, we put an important effort in the development of a robust and effective numeric strategy for solving our optimisation problems, and our choice fell on genetic algorithms and, more generally, on evolutionary strategies [16, 17]. We developed our own research program: the genetic algorithm BIANCA, but for lack of space, we are not going to give here details about the structure and the numerical strategies developed within BIANCA. A detailed and exhaustive description can be found in [13, 18-19]. All the results shown in the following subsections are obtained by the use of BIANCA, and the quality of these results proves the efficacy of our theoretical approach as well as the effectiveness of the numerical method that we chose.

For sake of generality, in the following examples we chose a single constitutive material for the elementary layer: all calculations are run using the T300/5208 carbone/epoxy unidirectional composite. Its mechanical properties are given in Tables 2. The thickness of the elementary layer is:  $t = 0.127$  mm.

The number of plies is fixed for each calculation, and it is generally not very large (between 10 and 16 in the following examples), even if this is not a limit for our theoretical and numerical approach. The choice is motivated by the very nature of such combinatorial optimisation problems, where the smaller is the number of plies, the more difficult is to find a solution, since the number of successful combinations is dramatically reduced. For instance, it is empirically known that the minimum number of layers is 12 in order to achieve isotropic laminates [10, 14]. In the following examples, we show the possibility of finding solutions to optimal design problems of laminates with small numbers of layers through our approach, and that could be useful in many applications where reduction of weight is an issue.

In the following subsections, we will give examples of solutions for various types of problems shown in sections 3 and 4. In order to verify the quality of the solutions found, one can enter the stacking sequences shown as results in the behaviour equations.

### 5.1 A design problem of type 0: in-plane and bending orthotropy and uncoupling

We show here the case of design of given elastic symmetries, leading to a single-objective unconstrained non linear and non convex programming problem as we showed in section 2. In the present case, the objective function  $I(\mathbf{P}(\mathbf{x}))$  expresses such elastic symmetries as in-plane orthotropy ( $K_A = 0$ ), bending orthotropy ( $K_D = 0$ ), coincidence of orthotropy axes between in-plane and bending behaviours, and uncoupling.

We fixed the number of layers  $n = 10$ , whilst the optimisation variables are the stacking sequence (which is completely free) and the corresponding values of the orientation angles. The angles can take all values between  $-90^\circ$  and  $90^\circ$ , with a precision of  $p = 0.01^\circ$ .

By running BIANCA, we can find several solutions to this problem; an example of these solutions is the following laminate:

$$[-12.63/5.97/5.82/17.46/-15.50/7.13/-16.64/-12.10/13.68/0]. \quad (21)$$

In order to appreciate the precision of solution (21) and to verify the respect of elastic symmetries, we give the corresponding values for the global objective function  $I(\mathbf{P})$  and for each sub-objective in Table 3 (we remind that exact solutions correspond to  $I(\mathbf{P}) = 0$ , see section 2), as well as its elastic polar parameters for in-plane, bending and coupling stiffness in Table 4.

From Table 4, we can remark that the laminate (21) responds to orthotropy for in-plane and bending ( $\Phi_0^A - \Phi_1^A \approx 0^\circ$ , and:  $\Phi_0^D - \Phi_1^D \approx 0^\circ$ ) and the coincidence of orthotropy axes is also satisfied:  $\Phi_1^A - \Phi_1^D \approx 0^\circ$ . In the same way, the laminate is almost uncoupled, since polar pa-

rameters for tensor  $\mathbf{B}$  corresponds to 0.5% of the maximum admissible coupling ( $B_{\max}^* = 42866$  MPa for a carbone-epoxy laminate).

Elastic symmetries are also evident looking at the polar curves of the elastic components for laminate (21), as illustrated in Figure 2 (the curve for the coupling coefficient  $B_{11}$  is concentrated around the origin, since  $\mathbf{B} = \mathbf{O}$ ). Units are in MPa and the horizontal and vertical axes represent the  $x$ - and  $y$ - directions, respectively.

In order to achieve the results shown here, we ran BIANCA with a population of 50 individuals, and quasi-optimal individuals were obtained within 21 iterations (generations), as we can see in Figure 3.

We remind here that this kind of optimisation problem for elastic symmetries can be enriched by adding constraints over different properties of the laminate (stiffness, strength, etc.), thus resulting in a formulation as in expression (9), i.e. a design problem of type 1. We ran a number of calculations over constrained problems belonging to this type using BIANCA and we were able to find lots of solutions [21]. Nevertheless, for sake of conciseness, we do not show them here, since in the next sections we will show more solutions found by running BIANCA on complex constrained problems of type 2.

## 5.2 *A problem of type 2: maximisation of buckling resistance with constraints over elastic symmetries and in-plane stiffness*

The formulation of the optimisation problem is here as in expression (14), where the objective function is the smallest buckling multiplier  $\lambda_{\text{crit}}$  to be maximised and constraint functions are over elastic symmetries. In addition, we search for a highly stiff orthotropic plate; thus, constraints are imposed over the extension Young moduli along the orthotropic axes:

$$\begin{aligned}
& \max (\min_{m,n} \lambda_{\text{crit}}) \\
& \text{such that : } I(\mathbf{P}(\mathbf{x})) < \varepsilon \quad (\varepsilon = 10^{-4}) \\
& \quad E_x^A \geq 60 \text{ GPa} \\
& \quad E_y^A \geq 30 \text{ GPa}
\end{aligned} \tag{22}$$

The required elastic symmetries here, expressed by the function  $I(\mathbf{P}(\mathbf{x}))$ , are in-plane and bending orthotropy ( $K_A = K_D = 1$ ), coincidence of the orthotropic axes in extension and bending, uncoupling. In addition, the orthotropic axes of the laminate must coincide with the axes of the plate. The precision of the solution in terms of elastic symmetries is fixed:  $\varepsilon = 10^{-4}$ .

The aspect ratio of the rectangular plate is fixed:  $a/b = 1.5$ , and compressive loads along the sides of the plate are:  $N_x = N_y = 1 \text{ N/mm}$ .

We fixed the number of layers  $n = 16$ , whilst the optimisation variables are the stacking sequence (which is completely free) and the corresponding values of the orientation angles. The angles can take all values between  $-90^\circ$  and  $90^\circ$ , with a discrete precision of  $p = 1^\circ$ .

We show here an example of solution to problem (22) found by running BIANCA:

$$[-24/39/-47/37/32/-47/-6/-47/55/59/18/-38/-38/19/-40/42]. \tag{23}$$

The maximum value achieved for the buckling critical multiplier is  $\lambda_{\text{opt}} = 6.86 \cdot 10^6$ , whilst the achieved values for the constraint functions are:  $E_x^A = 60607 \text{ MPa}$ ,  $E_y^A = 31157 \text{ MPa}$  and  $I(\mathbf{P}(\mathbf{x})) = 8.80 \cdot 10^{-5}$ . In Table 5, we give the detail for the contribution of each required elastic symmetry to the global function of constraint  $I(\mathbf{P}(\mathbf{x}))$ .

From Table 6, we can remark that the laminate (23) responds to the required shape of orthotropy for extension and bending ( $\Phi_0^A - \Phi_1^A \approx 45^\circ$ , and:  $\Phi_0^D - \Phi_1^D \approx 45^\circ$ ) and the coincidence of orthotropy axes is also satisfied:  $\Phi_1^A - \Phi_1^D \approx 0^\circ$ , being the principal orthotropic axes aligned with the axes of the plate ( $\Phi_1^A \approx \Phi_1^D \approx 0^\circ$ ). In the same way, the laminate is almost uncoupled, since polar parameters for tensor  $\mathbf{B}$  corresponds to 0.6% of the maximum admissible coupling ( $B_{\text{max}}^* = 42866 \text{ MPa}$  for a carbone-epoxy laminate).

We also give here the Cartesian components of the homogenised stiffness tensors  $\mathbf{A}^* \mathbf{B}^* \mathbf{D}^*$  for laminate (23) (units are GPa):

$$A_{11}^* = 86.22 ; A_{12}^* = 33.69 ; A_{16}^* = 0.11 ; A_{22}^* = 44.35 ; A_{26}^* = -0.17 ; A_{66}^* = 37.96 ;$$

$$B_{11}^* = 0.88 ; B_{12}^* = 0.03 ; B_{16}^* = 0.57 ; B_{22}^* = 0.96 ; B_{26}^* = 0.06 ; B_{66}^* = 0.03 ;$$

$$D_{11}^* = 90.34 ; D_{12}^* = 35.31 ; D_{16}^* = 0.09 ; D_{22}^* = 37.00 ; D_{26}^* = -0.04 ; D_{66}^* = 39.58.$$

The respect of elastic symmetries is confirmed by the graphics of polar variation of stiffness properties for laminate (23), shown in Figure 4. Units are in MPa and the horizontal and vertical axes represent the  $x$ - and  $y$ - directions, respectively.

The effectiveness of the genetic algorithm BIANCA is proved by the graphics of average and best values of the objective function over the generations, shown in Figures 5. Calculations were run over a population composed of  $n = 400$  individuals, and quasi-optimal individuals were obtained after 150 generations.

### 5.3 *A problem of type 2: maximisation of strength with constraints over elastic symmetries and stiffness*

The formulation adopted here for the optimisation of strength for a laminated plate is the one given in expression (20), where the objective function is the strength parameter  $R_{index}$  to be maximised, and constraint functions are over elastic symmetries and the respect of a strength criterion for each layer in the laminate. In addition, we search for a highly stiff orthotropic plate; thus, constraints are imposed over the in-plane Young modulus along the transverse orthotropic axes:

$$\begin{aligned} & \max R_{index} \\ & \text{such that : } f_{Hoffman} < 1 \\ & I(\mathbf{P}(\mathbf{x})) < \varepsilon \quad (\varepsilon = 10^{-4}) \\ & E_y^A \geq 45 \text{ GPa} \end{aligned} \tag{24}$$

The required elastic symmetries here, expressed by the function  $I(\mathbf{P}(\mathbf{x}))$ , are in-plane and bending orthotropy ( $K_A = K_D = 0$ ), coincidence of the orthotropic axes in extension and bending, uncoupling. In addition, the orthotropic axes of the laminate must coincide with the axes of the plate. The precision of the solution in terms of elastic symmetries is fixed:  $\varepsilon = 10^{-4}$ .

Compressive loads along the sides of the plate are given:  $N_x = 10^5$  N/mm,  $N_y = 0$  N/mm.

We fixed the number of layers  $n = 16$ , whilst the optimisation variables are the stacking sequence (which is completely free) and the corresponding values of the orientation angles. The angles can take all values between  $-90^\circ$  and  $90^\circ$ , with a discrete precision of  $p = 1^\circ$ .

We show here an example of solution to problem (24) found by running BIANCA:

$$[0/-6/-84/-5/42/4/-1/5/-72/-22/65/-84/5/-14/5/4]. \quad (25)$$

The maximum value achieved for the strength objective function is  $R_{\text{index}} = 2.0 \cdot 10^7$ , whilst the achieved values for the constraint functions over transverse stiffness and elastic symmetries are:  $E_y^A = 49468$  MPa and  $I(\mathbf{P}(\mathbf{x})) = 7.74 \cdot 10^{-5}$ . In Table 7, we give the detail for the contribution of each required elastic symmetry to the global function of constraint  $I(\mathbf{P}(\mathbf{x}))$ .

From Table 8, we can remark that the laminate (25) responds to the required shape of orthotropy for extension and bending ( $\Phi_0^A - \Phi_1^A \approx 0^\circ$ , and:  $\Phi_0^D - \Phi_1^D \approx 0^\circ$ ) and the coincidence of orthotropy axes is also satisfied:  $\Phi_1^A - \Phi_1^D \approx 0^\circ$ , being the principal orthotropic axes aligned with the axes of the plate ( $\Phi_1^A \approx \Phi_1^D \approx 0^\circ$ ). In the same way, the laminate is almost uncoupled, since polar parameters for tensor  $\mathbf{B}$  corresponds to 0.8% of the maximum admissible coupling ( $B_{\text{max}}^* = 42866$  MPa for a carbone-epoxy laminate).

We also give here the Cartesian components of the homogenised stiffness tensors  $\mathbf{A}^* \mathbf{B}^* \mathbf{D}^*$  for laminate (25) (units are GPa):

$$\begin{aligned} A_{11}^* &= 127.45 ; A_{12}^* = 10.08 ; A_{16}^* = -0.17 ; A_{22}^* = 50.34 ; A_{26}^* = 0.03 ; A_{66}^* = 14.35 ; \\ B_{11}^* &= -1.35 ; B_{12}^* = -0.01 ; B_{16}^* = 0.34 ; B_{22}^* = 1.37 ; B_{26}^* = 0.14 ; B_{66}^* = -0.01 ; \end{aligned}$$

$$D_{11}^* = 149.70 ; D_{12}^* = 6.50 ; D_{16}^* = -0.01 ; D_{22}^* = 35.23 ; D_{26}^* = -0.13 ; D_{66}^* = 10.78.$$

The respect of elastic symmetries is confirmed by the graphics of polar variation of stiffness properties for laminate (25), shown in Figure 6.

The effectiveness of the genetic algorithm BIANCA is proved by the graphics of average and best values of the objective function over the generations, shown in Figures 7. Calculations were run over a population composed of  $n = 400$  individuals, and quasi-optimal individuals were obtained after less than 150 generations.

#### 5.4 *A multi-objective optimization problem: maximisation of natural frequency and of in-plane principal stiffness with constraints over elastic symmetries*

We show here an example of calculation run using BIANCA and dealing with a multi-objective optimisation problem for composite laminates: maximisation of the first natural frequency and maximisation of the principal in-plane Young modulus  $E_x^A$ . At the same time, as usual, we impose the necessary constraints over required elastic symmetries expressed by function  $I(\mathbf{P}(\mathbf{x}))$ . The corresponding formulation of the resulting multi-objective optimisation problem is given as:

$$\begin{aligned} & \max (\min_{m,n} \omega_{mn}) \\ & \text{and} \\ & \max E_x^A \\ & \text{such that : } I(\mathbf{P}(\mathbf{x})) < \varepsilon \quad (\varepsilon = 10^{-4}) \end{aligned} \tag{26}$$

The required elastic symmetries here, expressed by the function  $I(\mathbf{P}(\mathbf{x}))$ , are extension and bending orthotropy ( $K_A = K_D = 0$ ) with coincident axes and uncoupling. The precision of the solution in terms of elastic symmetries is fixed:  $\varepsilon = 10^{-4}$ .

We ran the calculations using the strategy we developed within BIANCA in order to deal with multi-objective optimisation problems, which is a combination of *fitness sharing* and *niche*

methods [18, 19]. We used a population composed of  $n = 200$  individuals, and we obtained the Pareto front within 50 generations.

We fixed the number of layers  $n = 10$ , whilst the optimisation variables are the stacking sequence (which is completely free) and the corresponding values of the orientation angles. The angles can take all values between  $-90^\circ$  and  $90^\circ$ , with a discrete precision of  $p = 1^\circ$ .

In this case, the result of a run of BIANCA is the group of non-dominated individuals (Pareto front) belonging to the final generation, as shown in Figure 8.

The designer can choose among the design solutions belonging to the Pareto front according to some additional criterion (feasibility, mechanical properties, ...). We give here an example corresponding to the point called “solution 1” in Figure 8, which is situated at one end of the Pareto front: it shows the highest value of Young modulus  $E_x^A$  and the smallest value of the fundamental frequency. Its stacking sequence is the following:

$$[14/-2/-25/-7/-1/-3/17/1/15/-15]. \quad (27)$$

For laminate (27) the smallest fundamental frequency is  $\omega_{11} = 33.21$  Hz and the principal in-plane Young modulus is  $E_x^A = 159000$  MPa. Solution (27) satisfies as well the conditions over elastic symmetries, as it is shown in details in Table 9.

From Table 10, we can remark that the laminate (27) responds to the required shape of orthotropy for extension and bending ( $\Phi_0^A - \Phi_1^A \approx 0^\circ$ , and:  $\Phi_0^D - \Phi_1^D \approx 0^\circ$ ) and the coincidence of orthotropy axes is also satisfied:  $\Phi_1^A - \Phi_1^D \approx 0^\circ$ , being the principal orthotropic axes aligned with the axes of the plate ( $\Phi_1^A \approx \Phi_1^D \approx 0^\circ$ ). In the same way, the laminate is almost uncoupled, since polar parameters for tensor  $\mathbf{B}$  corresponds to 3.4% of the maximum admissible coupling ( $B_{\max}^* = 42866$  MPa for a carbone-epoxy laminate).

We also give here the Cartesian components of the homogenised stiffness tensors  $\mathbf{A}^* \mathbf{B}^* \mathbf{D}^*$  for laminate (27) (units are GPa):

$$\begin{aligned}
A^*_{11} &= 166.96 ; A^*_{12} = 9.58 ; A^*_{16} = -0.75 ; A^*_{22} = 11.82 ; A^*_{26} = -0.57 ; A^*_{66} = 13.86 ; \\
B^*_{11} &= -0.62 ; B^*_{12} = 0.14 ; B^*_{16} = -1.81 ; B^*_{22} = 0.35 ; B^*_{26} = -0.94 ; B^*_{66} = -0.14 ; \\
D^*_{11} &= 163.83 ; D^*_{12} = 11.08 ; D^*_{16} = 1.18 ; D^*_{22} = 11.96 ; D^*_{26} = -0.42 ; D^*_{66} = 15.36.
\end{aligned}$$

The respect of elastic symmetries is confirmed by the graphics of polar variation of stiffness properties for laminate (27), shown in Figure 9.

### 5.5 Comparison with solutions existing in the literature

We can notice that the stacking sequences given as examples in sub-sections 5.1 to 5.4 (laminates (21), (23), (25) and (27)), are all required to be orthotropic in bending. Of course, the respect of bending orthotropy is satisfied within a certain numerical precision and, in the previous sub-sections, we give the value of the partial objective function for bending orthotropy (Tables 3, 5, 7 and 9), as well as the corresponding polar angles  $\Phi_0$  and  $\Phi_1$  for bending (Tables 4, 6, 8 and 10): the objective function can be compared to the optimum required value, which is zero, and the angular difference  $\Phi_0 - \Phi_1$  can be compared to the polar condition for orthotropy (see Table 1). Nevertheless, we can give a different measure of the respect of bending orthotropy in terms of the more usual Cartesian components  $D_{16}$  and  $D_{26}$ . More precisely, we define the deviation  $\Delta_{\text{ortho}}$  from bending orthotropy in terms of the ratio between the off-axes components  $D_{16}$  and  $D_{26}$  and a tensor norm  $\|D\|$ :

$$\Delta_{\text{ortho}} = \max \left\{ \frac{|D_{16}|}{\|D\|}, \frac{|D_{26}|}{\|D\|} \right\} \quad (28)$$

where the tensor norm  $\|D\|$  is defined as (but a different definition of norm is also possible):

$$\|D\|^2 = T_0^2 + 2T_1^2 + R_0^2 + 4R_1^2 \quad (29)$$

Values of  $\Delta_{\text{ortho}}$  for solutions (21) to (27) are given in Table 11, and they show the very good agreement with the requirement of bending orthotropy.

In a similar way, we define a measure of the deviation from uncoupling  $\Delta_{\text{uncoupling}}$  as:

$$\Delta_{\text{uncoupling}} = \max \left\{ \frac{\|B\|}{\|A\|}, \frac{\|B\|}{\|D\|} \right\} \quad (30)$$

where  $\|A\|$ ,  $\|B\|$ ,  $\|D\|$  are the tensor norms of in-plane, coupling and bending, respectively.

Values given in Table 12 show the very good agreement with the requirement of uncoupling for solutions from (21) to (27).

In addition, both Tables 11 and 12 show the good repeatability of the precision achieved for the solutions obtained by running BIANCA on very different problems of design of composite laminates (different numbers of layers, objective functions and constraints).

We apply now the same measure of deviation from bending orthotropy  $\Delta_{\text{ortho}}$  to some solutions found in the literature [5, 20-23]. The references that we consider range over a long time (1992 to 2009), and a common feature is an usual assumption in the optimal design of composite laminates with respect to buckling, strength, and so on: the required property of bending orthotropy is considered to be satisfied by any symmetric stacking sequence, and therefore the design of bending behaviour is simply by-passed. From the values shown below, we can see that symmetric stacks, which are exactly uncoupled, might be quite far from matching the condition of bending orthotropy even when they are made of a large number of layers.

We start in the chronological order with the paper by R. Haftka and J. L. Walsh [20], dealing with stacking sequence optimisation with respect to buckling resistance: the problem is the maximisation of the buckling factor for simply supported rectangular orthotropic laminated plates loaded in compression (in this case, the objective function can be expressed analytically and it is the same that we use in our paper) with or without constraints on stiffness and strength, and the choice is the use of symmetric stacking sequences with orientation angles belonging to the set  $\{0^\circ; \pm 45^\circ; 90^\circ\}$ . Being the solutions shown in this paper symmetric, all the laminates are exactly uncoupled, but they are sometimes far from being orthotropic: the corresponding values of  $\Delta_{\text{ortho}}$  range from 1% to 28%. Here some examples:

in [20], Figure 2, stacking sequence  $[45/90/-45/90_3/-45/45]_s$ ,  $\lambda_{cr} = 94.29$ :  $\Delta_{ortho} = 12.3\%$

in [20], Figure 2, stacking sequence  $[90/45/90_4/-45/90]_s$ ,  $\lambda_{cr} = 61.99$ :  $\Delta_{ortho} = 17.9\%$

in [20], Figure 5, stacking sequence  $[-45/45/90_3]_s$ ,  $\lambda_{cr} = 1.03$ :  $\Delta_{ortho} = 17.4\%$

in [20], Figure 5, stacking sequence  $[-45/90_2/45/90/0]_s$ ,  $\lambda_{cr} = 1.32$ :  $\Delta_{ortho} = 28.1\%$

Another example of solutions, which are far from being orthotropic, is given in the paper of M. Walker et al. [21]: the problem is the maximisation of the buckling factor for rectangular laminated plates loaded in compression under different boundary conditions (the objective function is calculated by the FE method), and the choice is the use of the symmetric and balanced angle-ply stack  $[\theta/-\theta/-\theta/\theta]$  (angle  $\theta$  varies continuously between 0 and 90). The corresponding values of  $\Delta_{ortho}$  vary continuously between 0% (for the trivial case of  $\theta = 0$ ) up to 69% for  $\theta = 30$ .

Later on, the problem of maximisation of buckling loads of composite panels was treated in a paper by Liu et al. [5]: plates are rectangular and simply supported, loaded in compression and shear; the stacking sequences are symmetric and orientation angles belong to the set  $\{0^\circ; \pm 45^\circ; 90^\circ\}$ . We measure here the deviation from orthotropy  $\Delta_{ortho}$  of some solutions shown in this paper, when they are considered strictly symmetric:

in [5], page 32, sequence  $[\pm 45/90_4/\pm 45/90_4/0_4/\pm 45/90_2/\pm 45/0_4/\pm 45/0_2]_s$ :  $\Delta_{ortho} = 1.5\%$

in [5], page 33, sequence  $[\pm 45/0_2/\pm 45/0_2]_s$ :  $\Delta_{ortho} = 10.7\%$

More recently, Erdal and Sonmez [22] dealt with maximisation of the buckling load for simply supported rectangular laminated plates loaded in compression. The search for the optimal laminates is performed among the set of symmetric stacks in order to satisfy the uncoupling. Firstly, they considered orientation angles belonging to the set  $\{0^\circ; \pm 45^\circ; 90^\circ\}$  and they show the existence of multiple optima with reference to previous works in the literature (in this paper, see Table 1, page 50): even if they consider the case of 64-ply laminates (rather

high number of layers), the measure of the deviation from orthotropy  $\Delta_{\text{ortho}}$  is around 1% for these solutions. Further on, they suggest that these results can be improved by the introduction of non standard orientation angles, and they give examples of solutions in their paper [22], Tables 2-5, page 51: the measure  $\Delta_{\text{ortho}}$  ranges from 38% up to 80% for these solutions!

Very recently, a paper by F.X. Irisarri et al. [23] appeared about multi-objective optimisation of laminated composites. The problem is again the maximisation of the buckling load of a simply supported rectangular plate loaded in compression, taking into account several load cases simultaneously. As it is classically done, the stacking sequences are supposed to be symmetric in order to achieve uncoupling, and the search is performed among balanced angle-ply laminates in order to obtain orthotropy (in fact, this ensures only in-plane orthotropy). Therefore, the expression for the buckling critical factor  $\lambda_{\text{crit}}$  is given analytically in terms of the bending components  $D_{11}$ ,  $D_{22}$ ,  $D_{12}$  and  $D_{66}$  by neglecting components  $D_{16}$  and  $D_{26}$ , which are considered to be zero. A synthesis of results is given in the paper ([23], Tables 3-6) together with a measure of the deviation from bending orthotropy, and we observe that the measure used in [23] rises up to 20% for the proposed solutions. In order to compare with our previous estimations, we also apply our measure  $\Delta_{\text{ortho}}$  to some of the solutions shown in [23]:

reference laminate,  $[45_2/90/-45_2/90/45/90/-45/0_2/45/0_2/-45/0]_s$ :  $\Delta_{\text{ortho}} = 14.6\%$

Table 3, 30-ply laminate  $[45/90/45/-45/45/90/-45_2/45/0_5]_s$ :  $\Delta_{\text{ortho}} = 16.7\%$

Table 3, 31-ply laminate  $[90/-45/45/-45_2/90/0_2/45_2/0_{(5 \text{ et } 1/2)}]_s$ :  $\Delta_{\text{ortho}} = 15.9\%$

Table 5, 31-ply laminate  $[(-45/90)_2/(-45/0)_2/45_2/0_2/45/0/45/0_{(1/2)}]_s$ :  $\Delta_{\text{ortho}} = 36.3\%$

We first notice that the measure of the deviation from bending orthotropy proposed by in [23] is underestimated (in their ratio, they should have chosen the denominator as  $\min\{D_{11}, D_{22}\}$ ). However, the deviation from bending orthotropy seems to be too large to be considered as negligible.

The important deviation from bending orthotropy of such solutions, which are supposed to be orthotropic in bending, sheds a doubt on the validity of the optimisation process: results for maximum buckling load are based on an analytical formula which neglects the effect of components  $D_{16}$  and  $D_{26}$ , when in fact these components are far from being negligible. At least, it would be important to investigate the effects of such anisotropic behaviour on the actual stability or strength of a laminated plate. Therefore, we can state that bending orthotropy is an important property for laminates, and classically authors by-pass this problem considering that components  $D_{16}$  and  $D_{26}$  are close to zero in the case of laminates made of a large number of layers (especially in the case of symmetric balanced angle-ply laminates). The measure  $\Delta_{\text{ortho}}$  shows that this is not always true, and design with respect to bending orthotropy should be taken into account in the optimisation of composite laminates. Our approach allows to do so, and the examples of solutions (21), (23), (25) and (27) show that good precision for bending orthotropy and for elastic uncoupling can be achieved even for laminates with smaller numbers of layers (in this case, these properties are generally more difficult to obtain).

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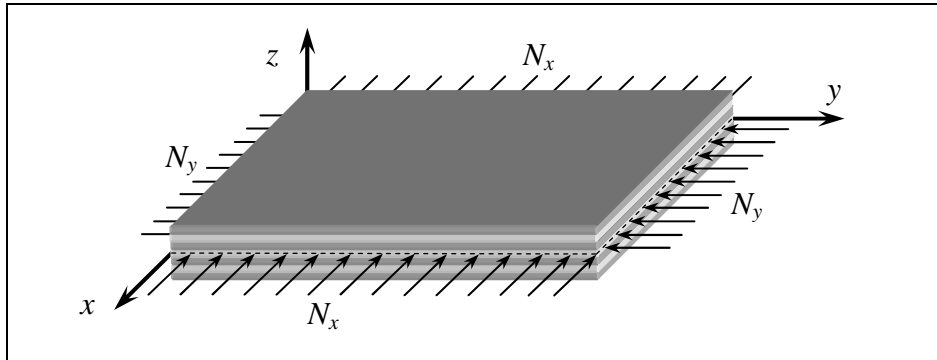


Figure 1 – A rectangular laminated plate under in-plane compressive loads  $N_x$  and  $N_y$

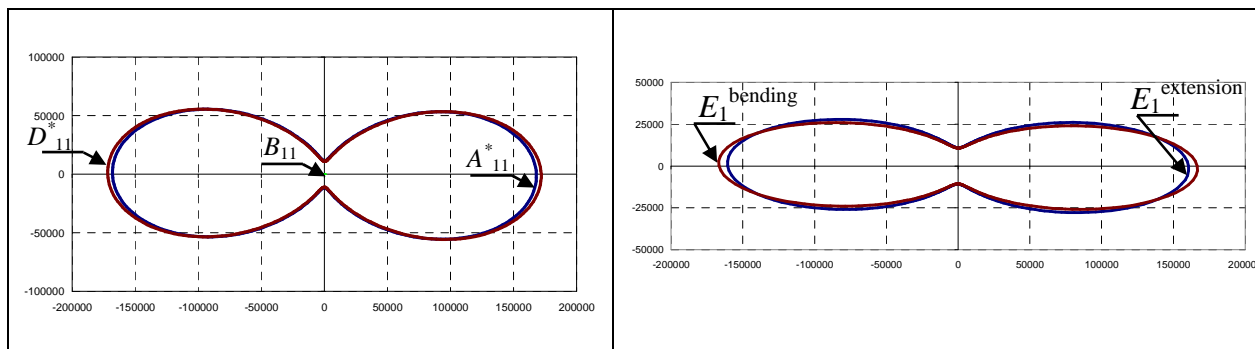


Figure 2 – Polar representation of the elastic properties for laminate (21).

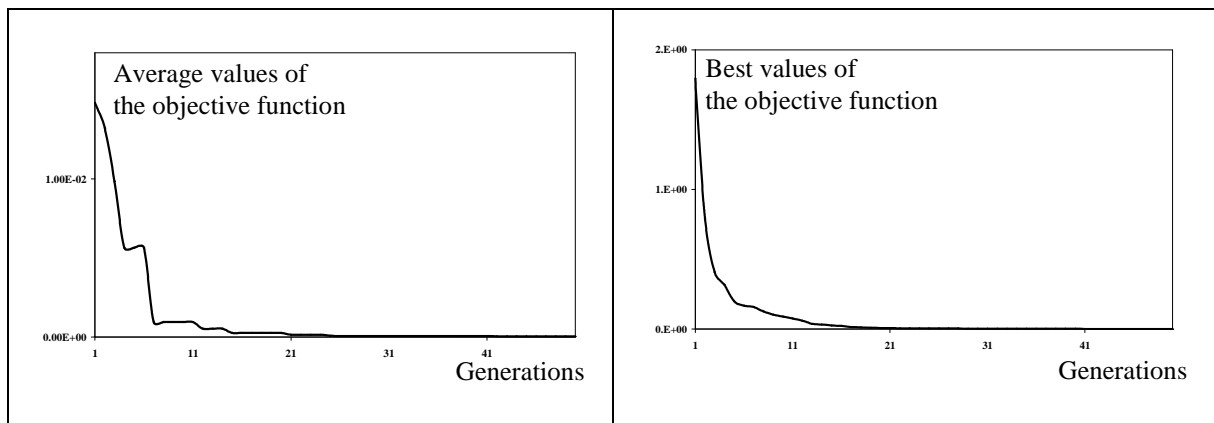


Figure 3 – Average and best values of the objective function vs. generations.

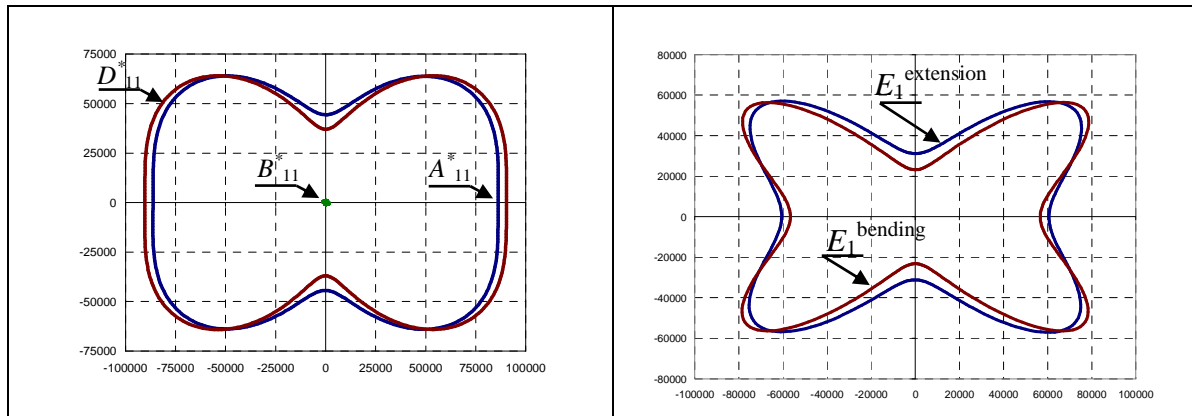


Figure 4 – Polar representation of the elastic properties for laminate (23).

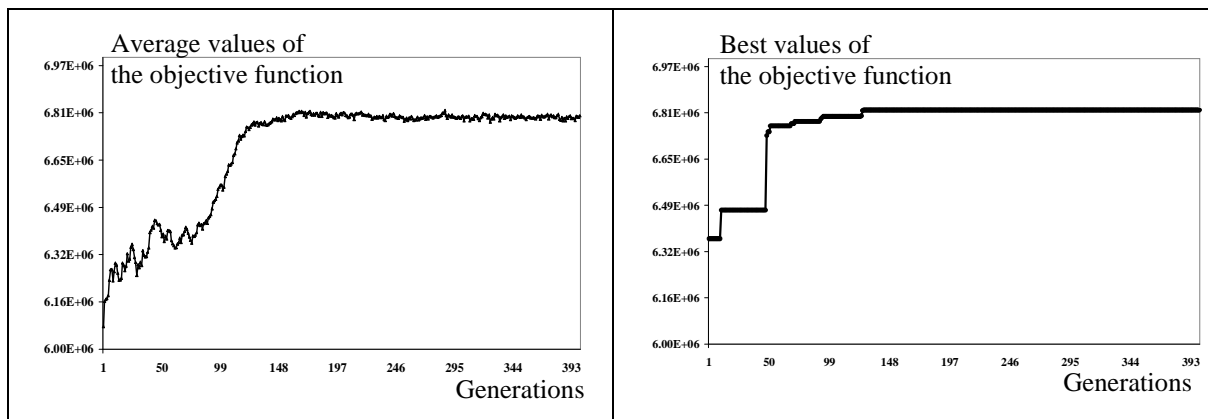


Figure 5 – Average and best values of the objective function vs. generations.

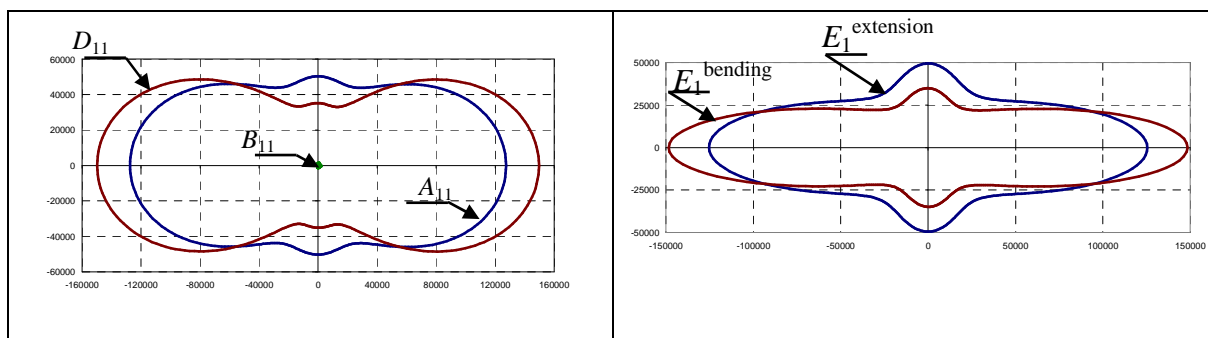


Figure 6 – Polar representation of the elastic properties for laminate (25).

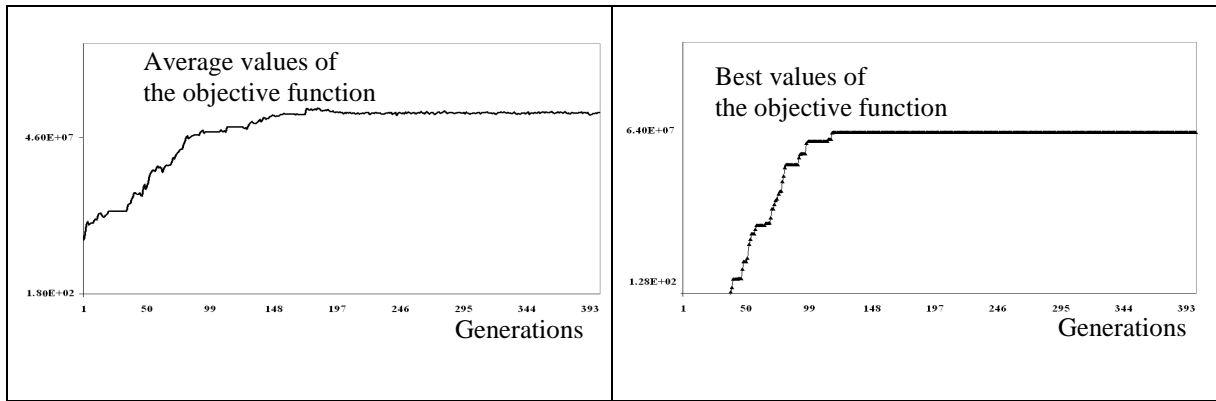


Figure 7 – Average and best values of the objective function vs. generations.

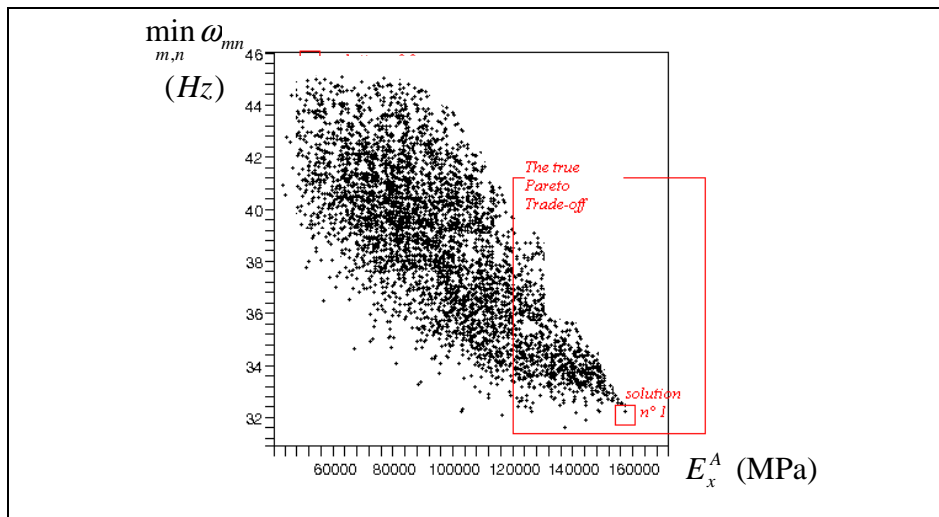


Figure 8 – Final population and Pareto front for problem (26)

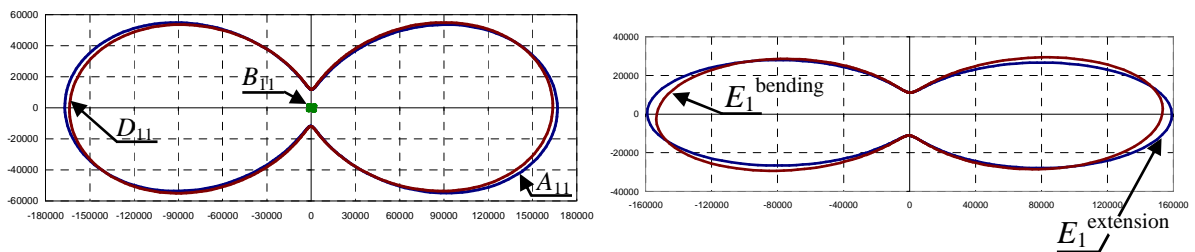


Figure 9 – Polar representation of the elastic properties for laminate (27).

CLASS OF SYMMÉTRY	POLAR CONDITION
Orthotropy	$\Phi_0 - \Phi_1 = K \frac{\pi}{4}, K = 0 \text{ or } 1$
	$R_0 = 0$
Square symmetry	$R_1 = 0$
Isotropy	$R_0 = 0, R_1 = 0$

Table 1. Polar conditions for elastic symmetries.

$E_1$ (MPa)	$E_2$ (MPa)	$G_{12}$ (MPa)	$\nu_{12}$	$\rho$ (kg/m <sup>3</sup> )
181000	10300	7170	0.28	1578
$X_t$ (MPa)	$X_c$ (MPa)	$Y_t$ (MPa)	$Y_c$ (MPa)	$S_{12}$ (MPa)
1500	1500	40	246	68

Table 2. Mechanical properties for T300/5208.

Global objective function $I(\mathbf{P})$	In-plane orthotropy	Bending orthotropy	Coupling	Coincidence of orthotropy axes
$6.32 \cdot 10^{-6}$	$1.24 \cdot 10^{-6}$	$7.40 \cdot 10^{-10}$	$5.06 \cdot 10^{-6}$	$9.39 \cdot 10^{-9}$

Table 3. Values of the objective function for solution (21).

Polar parameters	$T_0$ (MPa)	$T_1$ (MPa)	$R_0$ (MPa)	$R_1$ (MPa)	$\Phi_0$ (°)	$\Phi_1$ (°)
In-plane, A	26880	24744	13417	19605	-0.61	-0.67
Bending, D	26880	24744	15122	20117	-0.66	-0.66
Coupling, B	0	0	93	31	39	68

Table 4. Elastic polar parameters for laminate (21).

Global objective function $I(\mathbf{P})$	In-plane orthotropy	Bending orthotropy	Coupling	Coincidence of orthotropy axes
$8.80 \cdot 10^{-5}$	$3.76 \cdot 10^{-6}$	$4.55 \cdot 10^{-10}$	$7.48 \cdot 10^{-5}$	$4.90 \cdot 10^{-6}$

Table 5. Values of the objective function for solution (23).

Polar parameters	$T_0$ (MPa)	$T_1$ (MPa)	$R_0$ (MPa)	$R_1$ (MPa)	$\Phi_0$ ( $^\circ$ )	$\Phi_1$ ( $^\circ$ )
In-plane, A	26880	24744	11084	5234	44.81	-0.08
Bending, D	26880	24744	12698	6667	44.93	0.05
Coupling, B	0	0	129	141	-24.45	-17.44

Table 6. Elastic polar parameters for laminate (23).

Global objective function $I(\mathbf{P})$	In-plane orthotropy	Bending orthotropy	Coupling	Coincidence of orthotropy axes
$7.74 \cdot 10^{-5}$	$1.04 \cdot 10^{-7}$	$4.56 \cdot 10^{-6}$	$7.26 \cdot 10^{-5}$	$2.06 \cdot 10^{-6}$

Table 7. Values of the objective function for solution (25).

Polar parameters	$T_0$ (MPa)	$T_1$ (MPa)	$R_0$ (MPa)	$R_1$ (MPa)	$\Phi_0$ ( $^\circ$ )	$\Phi_1$ ( $^\circ$ )
In-plane, A	26880	24744	12530	9640	-0.12	-0.10
Bending, D	26880	24744	16102	14309	0.05	-0.07
Coupling, B	0	0	102	362	20.55	80.17

Table 8. Elastic polar parameters for laminate (25).

Global objective function $I(\mathbf{P})$	In-plane orthotropy	Bending orthotropy	Coupling	Coincidence of orthotropy axes
$7.3 \cdot 10^{-4}$	$4.5 \cdot 10^{-5}$	$1.5 \cdot 10^{-4}$	$3.5 \cdot 10^{-4}$	$1.8 \cdot 10^{-4}$

Table 9. Values of the objective function for solution (27).

Polar parameters	$T_0$ (MPa)	$T_1$ (MPa)	$R_0$ (MPa)	$R_1$ (MPa)	$\Phi_0$ ( $^\circ$ )	$\Phi_1$ ( $^\circ$ )
In-plane, A	26880	24744	13022	19395	-0.10	-0.49
Bending, D	26880	24744	11553	18895	1.00	0.29
Coupling, B	0	0	228	349	-26.86	-49.98

Table 10. Elastic polar parameters for laminate (27).

Solution	(20)	(22)	(24)	(26)
$\Delta_{\text{ortho}}$	0.009%	0.2%	0.2%	0.9%

Table 11. Deviation from bending orthotropy  $\Delta_{\text{ortho}}$  for solutions (21) to (27)

Solution	(20)	(22)	(24)	(26)
$\Delta_{\text{uncoupling}}$	0.2%	1.3%	1.4%	2.4%

Table 12. Deviation from unncoupling  $\Delta_{\text{uncoupling}}$  for solutions (21) to (27)