Dynamic Boundary Stabilization of Linear and Quasi-Linear Hyperbolic Systems

Felipe Castillo Buenaventura, Emmanuel Witrant, Christophe Prieur, Luc Dugard

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***Abstract*— Systems governed by hyperbolic partial differential equations with dynamics associated with their boundary conditions are considered in this paper. These infinite dimensional systems can be described by linear or quasi-linear hyperbolic equations. By means of Lyapunov based techniques, some sufficient conditions are derived for the exponential stability of such systems. A polytopic approach is developed for quasi-linear hyperbolic systems in order to guarantee stability in a given region around an equilibrium point. An isentropic inviscid flow model is used to illustrate some of the main results.**

I. Introduction

Lyapunov function based techniques are commonly used for the stability analysis of dynamical systems, such as those described by partial differential equations (PDE). Many distributed physical systems are described by strict hyperbolic PDE. One of the main properties of this class of PDE is the existence of the so-called Riemann transformation which is a powerful tool for the proof of classical solutions, the analysis and the control, among other properties [1]. This kind of systems with infinite dimensional dynamics is relevant for a wide range of physical systems having an engineering interest. Among the potential applications, hydraulic networks [2], road traffic networks [3], gas flow in pipelines [4] or flow regulation in deep pits [5] are of significant importance.

The stability problem of the boundary control in hyperbolic systems has been considered for a long time in the literature, as reported in [6] [7] [8], among other references. Most results consider that the boundary control can react fast enough when compared to the travel time of the waves. More precisely, no time response limitation is considered at the boundary conditions. In applications such as the ones addressed in [9] [10], the wave travel time can be considered much larger than the actuator time response, allowing to establish a static relationship between the control input and the boundary condition. Nevertheless, there are applications where the dynamics associated with the boundary control cannot be neglected. Discrete approximations of this kind of systems have been used to address this problem (see [11]).

The present paper focuses on the stability problem of linear and quasi-linear hyperbolic systems in presence of dynamic behavior at the boundary conditions. To demonstrate asymptotic stability of this kind of hyperbolic systems, Riemann coordinates are used along with a Lyapunov function formulation. Sufficient conditions are derived on the system in terms of boundary conditions to prove Lyapunov stability. The sufficient conditions and the stability property are presented in a linear matrix inequality (LMI) framework.

In this paper, the general results obtained are applied on an homogeneous system of conservation laws to the design of a stabilizing boundary control of an isentropic and inviscid flow in a pipe with constant cross section. More prescisely, the problem of the regulation of the air pressure, density and speed inside a pipe is addressed. The physical model is a strict quasi-linear hyperbolic system since the presence of friction or thermal sources is not considered. This model is written in terms of the Euler equations introduced in [12], which are commonly used in compressible flow dynamics to describe the flow in ducts.

The paper is organized as follows. First in Section II, the class of hyperbolic systems under consideration is given. In Section III, the main stability results for linear hyperbolic systems with dynamic behavior at the boundary conditions are presented. In Section IV, an extention for quasi-linear hyperbolic systems is developed using a polytopic approach that allows to ensure stability in a defined region around an equilibrium point. Finally, the main result is applied to a boundary regulation of pressure, density and speed in a pipe with isentropic and inviscid air flow (see Section V).

II. Problem Formulation

In this section, a brief introduction on linear and quasi-linear hyperbolic systems in one dimension space is given and the specific hyperbolic systems considered in this work are presented. Let n be a positive integer and Ω be an open non-empty convex set of $\mathbb{R}^n$. Consider the general class of quasi-linear hyperbolic systems of order n defined as follows:

$$\partial_t s(x,t) + F(s(x,t))\partial_x s(x,t) = 0 \tag{1}$$

where ∂_t and ∂_x denote the partial derivative with respect to t and x respectively, $s(x,t) \in \Omega$, and $F : \Omega \to \mathbb{R}^{n\times n}$ is a continuously differentiable function called the characteristic matrix of (1). Consider the special case where the system (1) is strictly hyperbolic, then (1) accepts a bijection $\xi(s) \in \Theta \subset \mathbb{R}^n$, at least locally, such that the sytem can be transformed into a system of coupled transport equations [13]:

$$\begin{aligned}&\partial_t \xi_i(x,t) + \lambda_i(\xi(x,t))\partial_x \xi_i(x,t) = 0 \\ &i \in [1, ..., n]\end{aligned} \tag{2}$$

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where $\xi_i(x,t)$ are called the Riemann coordinates of (1), which are constant along the characteristic curves described by

$$\frac{dx}{dt} = \lambda_i(\xi(x,t)) \tag{3}$$

and where $\xi = [\xi_1, \xi_2, ..., \xi_n]^T$.

A. Linear Hyperbolic Systems with Dynamic Boundary Conditions

Consider the following linear hyperbolic equation in Riemann coordinates:

$$\partial_t \xi(x,t) + \Lambda \partial_x \xi(x,t) = 0 \quad \forall\, x \in [0,1], t \geq 0 \tag{4}$$

where Λ is a diagonal and invertible matrix in $\mathbb{R}^{n\times n}$ such that $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ with

$$\lambda_1 < ... < \lambda_m < 0 < \lambda_{m+1} < ... < \lambda_n \tag{5}$$

The state description can be partitioned as: $\xi = \begin{bmatrix} \xi_- \\ \xi_+ \end{bmatrix}$ where ξ_- is in $\mathbb{R}^m$ and ξ_+ is in $\mathbb{R}^{n-m}$. Let define:

$$\Lambda^+ = diag(|\lambda_1|, |\lambda_2|, ..., |\lambda_n|) \tag{6}$$

The problem of stability of linear hyperbolic systems has been considered by [6], [10], [8], among others, using static boundary conditions defined as:

$$\begin{pmatrix} \xi_-(1,t) \\ \xi_+(0,t) \end{pmatrix} = G \begin{pmatrix} \xi_-(0,t) \\ \xi_+(1,t) \end{pmatrix} \tag{7}$$

for a given matrix $G \in \mathbb{R}^{n\times n}$. The linear hyperbolic systems with dynamics associated with their boundary conditions are less explored in literature, although there are approaches using finite-dimensional approximations such as in [11] that have succesfully stabilized this kind of systems. Consider the following dynamics for the boundary conditions:

$$\begin{aligned} \dot{X}_c &= AX_c + Bu \\ Y_c &= X_c \end{aligned} \tag{8}$$

with

$$X_c = \begin{pmatrix} \xi_-(1,t) \\ \xi_+(0,t) \end{pmatrix}, \quad u = KY_\xi, \quad Y_\xi = \begin{pmatrix} \xi_-(0,t) \\ \xi_+(1,t) \end{pmatrix} \tag{9}$$

where $K \in \mathbb{R}^{n\times n}$ and $u \in \mathbb{R}^n$. Given a continuously differentiable function $\xi^0 : [0,1] \to \Theta$ that satisfies the zero-order and one-order compatibility conditions [14], then the initial condition can be defined as:

$$\xi(x,0) = \xi^0(x), \quad \forall\, x \in [0,1] \tag{10}$$

B. Quasi-Linear Hyperbolic Systems with Dynamic Boundary Conditions

Consider the following quasi-linear hyperbolic equation in Riemann coordinates:

$$\partial_t \xi(x,t) + \Lambda(\xi) \partial_x \xi(x,t) = 0 \quad \forall\, x \in [0,1], t \geq 0 \tag{11}$$

where Λ is a diagonal matrix function $\Lambda : \Theta \to \mathbb{R}^{n\times n}$ such that $\Lambda(\xi) = diag(\lambda_1(\xi), \lambda_2(\xi), ..., \lambda_n(\xi))$ with

$$\begin{aligned} &\lambda_1(\xi) < ... < \lambda_m(\xi) < 0 < \lambda_{m+1}(\xi) < ... < \lambda_n(\xi) \\ &\forall\, \xi \in \Theta \end{aligned} \tag{12}$$

Using an equivalent notation to the one used in the linear hyperbolic case, the same boundary conditions (8) along with the initial condition (10) can be considered for (11). Notice that the main difference between the linear and quasi-linear hyperbolic system is that for the quasi-linear system, the propogation speed depends on the state ξ.

III. Stability of Linear Hyperbolic Systems with Dynamic Boundary Conditions

The first problem under consideration is the stability analysis of (4), (8) and (10). The aim of this section is to use Lyapunov functions to state a sufficient condition for the exponential stability of (4), (8) and (10). The main results obtained for linear hyperbolic systems can be consolidated in the following theorem:

Theorem 1. *Consider the system* (4), (8) *and* (10). *Assume that there exists a diagonal positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that the following LMI is satisfied*

$$\begin{bmatrix} QA^T + AQ + \Lambda^+ Q & BY \\ Y^T B^T & -\Lambda^+ Q \end{bmatrix} \prec 0 \tag{13}$$

where $K = YQ^{-1}$, then there exist two constants $\alpha > 0$ and $M > 0$ such that, for all continuously differentiable functions $\xi^0 : [0,1] \to \Theta$ satisfying the zero-order and one-order compatibility conditions, the solution of (4), (8) *and* (10) *satisfies, for all $t \geq 0$,*

$$\begin{aligned} &||X_c(t)||^2 + ||\xi(x,t)||_{L^2(0,1)} \leq \\ &Me^{-\alpha t}\left(||X_c(0)||^2 + ||\xi^0(x)||_{L^2(0,1)}\right) \end{aligned} \tag{14}$$

Proof. Considering the system (4), it is possible to replace $\xi(x,t)$ by $\begin{pmatrix} \xi_-(1-x,t) \\ \xi_+(x,t) \end{pmatrix}$ and obtain a PDE whose corresponding diagonal characteristic matrix function is Λ^+. Therefore, it can be may assumed without loss of generality that $m = 0$ and $\Lambda^+ = \Lambda$ and that the boundary conditions (8) have the following form:

$$\begin{aligned} &\dot{X}_c = AX_c + Bu \\ &X_c = \xi(0,t), \quad u = KY_\xi, \quad Y_\xi = \xi(1,t) \end{aligned} \tag{15}$$

Given a diagonal positive definite matrix P, consider, as an extension of the Lyapunov function proposed in [7], the quadratic Lyapunov function candidate defined for all continuously differentiable functions $\xi : [0,1] \to \Theta$ as:

$$V(\xi, X_c) = X_c^T P X_c + \int_0^1 \left(\xi^T P \xi\right) e^{-\mu x} dx \tag{16}$$

where μ is a positive scalar that will be precised below. Computing the time derivative of V along the solutions of (4), (8) and (10) yields the following:

$$\begin{aligned}\dot{V} &= X_c^T \left(A^T P + PA\right) X_c \\ &+ Y_\xi^T K^T B^T P X_c + X_c^T PBK Y_\xi \\ &- \left[e^{-\mu x} \xi^T \Lambda P \xi\right] \big|_0^1 - \mu \int_0^1 \left(\xi^T \Lambda P \xi\right) e^{-\mu x} dx\end{aligned} \tag{17}$$

After some rearrangements, (17) can be written in terms of the boundary conditions (15) as follows:

$$\begin{aligned}\dot{V} &= -\mu X_c^T \Lambda P X_c - \mu \int_0^1 \left(\xi^T \Lambda P \xi\right) e^{-\mu x} dx \\ &+ \begin{bmatrix} X_c \\ Y_\xi \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PBK \\ +\Lambda P + \mu \Lambda P & \\ K^T B^T P & -e^{-\mu} \Lambda P \end{bmatrix} \\ &\times \begin{bmatrix} X_c \\ Y_\xi \end{bmatrix}\end{aligned} \tag{18}$$

Notice that (13) is equivalent to consider that

$$\begin{bmatrix} A^T P + PA + \Lambda P & PBK \\ K^T B^T P & -\Lambda P \end{bmatrix} \prec 0 \tag{19}$$

which is obtained by multiplying both sides of (19) by $diag\left(P^{-1}, P^{-1}\right)$ and performing the variable transformations $Q = P^{-1}$ and $Y = KQ$. Thus, for a small enough and positive μ, the third term of (18) is always negative. Consider the following inequality:

$$\begin{aligned}&\lambda_{min}(P) ||\xi(x,t)||_{L^2(0,1)} \leq \xi(x,t) P \xi(x,t) \\ &\leq \lambda_{max}(P) ||\xi(x,t)||_{L^2(0,1)}\end{aligned} \tag{20}$$

where $\lambda_{min}(P)$ and $\lambda_{max}(P)$ denote respectively the minimum and maximum eigenvalue of P. This can also be applied with Λ, implying that:

$$\dot{V} \leq -\mu \frac{\lambda_1 \lambda_{min}(P)}{\lambda_{max}(P)} V(\xi) \tag{21}$$

Therefore for a sufficiently small $\mu > 0$, the function (16) is a Lyapunov function for the hyperbolic system (4), (8), and (10). This concludes the proof.

IV. Stability of Quasi-Linear Hyperbolic Systems with Dynamic Boundary Conditions

The stability of quasi-linear hyperbolic systems has been exhautively studied in literature. A proof of the Lyapunov stability of (11) under the static boundary conditions (7) has been investigated in details in [6], assuming that $\rho_1(K) < 1$ (where $\rho_1(K) = Inf\{||\Delta K \Delta^{-1}||; \Delta \in D_{n,+}\}$ and $D_{n,+}$ denotes the set of $n \times n$ real diagonal positive definite matrices) and using as a Lyapunov function candidate:

$$V(\xi) = V_1(\xi) + V_2(\xi, \xi_x) + V_3(\xi, \xi_x, \xi_{xx}) \tag{22}$$

with

$$\begin{aligned} V_1(\xi) &= \int_0^1 \left(\xi^T Q(\xi) \xi\right) e^{-\mu x} dx \\ V_2(\xi, \xi_x) &= \int_0^1 \left(\xi_x^T R(\xi) \xi_x\right) e^{-\mu x} dx \\ V_3(\xi, \xi_x, \xi_{xx}) &= \int_0^1 \left(\xi_{xx}^T S(\xi) \xi_{xx}\right) e^{-\mu x} dx \end{aligned} \tag{23}$$

where $Q(\xi)$, $R(\xi)$ and $S(\xi)$ are symmetric positive definite matrices. In this paper, the stability of system (11), (8) and (10) is studied in a different way by introducing a polytopic approach in the characteristic matrix $\Lambda(\xi)$.

Let define a non empty convex set $\Xi \subset \Theta$ and a map $T : \Xi \to Z_\varphi$. Consider the following polytopic linear representation of the nonlinear characteristic matrix:

$$\Lambda(\xi) = \sum_{i=1}^{2^l} \alpha_i(\varphi) \Lambda(w_i) \tag{24}$$

$\forall \xi \in \Xi$ and therefore $\forall \varphi \in Z_\varphi$, where φ is a varying parameter vector that takes values in the parameter space Z_φ (a convex set) such that [15]:

$$Z_\varphi := \{[\varphi_1, ..., \varphi_l]^T \in \mathbb{R}^l, \varphi_i \in [\overline{\varphi}_i, \underline{\varphi}_i] \, \forall \, i = 1...l\} \tag{25}$$

where l is the number of varying parameters, $\alpha_i(\varphi)$ is a scheduling function $\alpha_i : Z_\varphi \to [0,1]$, w_i are the vertices of the polytope formed by all extremities of each varying parameter $\varphi \in Z_\varphi$ and $\sum_{i=1}^{2^l} \alpha_i(\varphi) \Lambda(w_i) : Z_\varphi \to \mathbb{R}^{n \times n}$. In general, all the admissible values of the vector φ are constrained in an hyperrectangle in the parameter space Z_φ. The scheduling functions $\alpha_i(\varphi)$ are defined as (see [15]):

$$\alpha_i(\varphi) = \frac{\prod_{k=1}^l |\varphi_k - C(w_i)_k|}{\prod_{k=1}^l |\overline{\varphi}_k - \underline{\varphi}_k]|} \tag{26}$$

where:

$$\begin{aligned} C(w_i)_k = &\{\varphi_k | \varphi_k = \overline{\varphi}_k \text{ if } (w_i)_k = \underline{\varphi}_k \\ &or\, \varphi_k = \underline{\varphi}_k \text{ otherwise}\} \end{aligned} \tag{27}$$

which exhibits the following properties:

$$\alpha_i(\varphi) \geq 0, \quad \sum_{i=1}^{2^l} \alpha_i(\varphi) = 1 \tag{28}$$

Consider (11) as an equivalent parameter varying hyperbolic system defined by:

$$\begin{aligned} &\partial_t \xi(x,t) + \sum_{i=1}^{2^l} \alpha_i(\varphi)\Lambda(w_i)\partial_x \xi(x,t) = 0 \\ &\forall\, \varphi \in Z_\varphi, \quad \forall\, x \in [0,1], \quad t \geq 0 \end{aligned} \tag{29}$$

It is clear that φ depends on ξ. However, as long as ξ remains in the set Ξ, the varying parameters φ_i can be considered as independent varying parameters (LPV framework [16]) that change the characteristic matrix, giving as a result, a conservative tool for stability analysis. Using (29), the following theorem states some sufficient conditions to ensure exponential stability for system (11), (8) and (10) in a defined region Z_φ.

Theorem 2. *Consider the system* (11), (8) *and* (10). *Assume that there exists a diagonal positive definite matrix* $Q \in \mathbb{R}^{n\times n}$ *such that the following LMI is satisfied* $\forall i \in [1, ..., l]$

$$\begin{bmatrix} QA^T + AQ + \Lambda^+(w_i)Q & BY \\ Y^T B^T & -\Lambda^+(w_i)Q \end{bmatrix} \prec 0 \tag{30}$$

where $K = YQ^{-1}$, *then there exist two constants* $\alpha > 0$ *and* $M > 0$ *such that, for all continuously differentiable functions* $\xi^0 : [0,1] \to \Xi$ *satisfying the zero-order and one-order compatibility conditions, the solution of* (11), (8) *and* (10) *satisfies* (14), *for all* $t \geq 0$

Proof. Consider once again the Lyapunov function candidate (16). Computing the time derivative of V along the solutions of (11), (8) and (10) yields the following:

$$\begin{aligned} \dot{V} &= X_c^T \left(A^T P + PA\right) X_c + Y_\xi^T K^T B^T P X_c \\ &+ X_c^T PBKY_\xi - \sum_{i=1}^{2^l} \alpha_i(\varphi) \left[e^{-\mu x}\xi^T \Lambda(w_i) P \xi\right] \Big|_0^1 \\ &- \sum_{i=1}^{2^l} \alpha_i(\varphi)\mu \int_0^1 \left(\xi^T \Lambda(w_i) P \xi\right) e^{-\mu x} dx \end{aligned} \tag{31}$$

Using the same procedure performed in the proof of Theorem 1, assuming once again that $\mu > 0$ is small enough and the fact that by definition, $\sum_{i=1}^{2^l} \alpha_i(\varphi) = 1$ and $\alpha_i \geq 0$, gives that (16) is a Lyapunov function for the system (11), (8) and (10) as long as (30) is satisfied.

This polytopic approach guarantees the stability and robustness of the quasi-linear hyperbolic system in a determined region Z_φ, which cannot be achieved with the approach presented in Theorem 1 as it would only guarantee the stability of (11), (8) and (10) in a small enough neighborhood around the equilibrium.

V. Illustrating Example: Isentropic Inviscid Flow in a Pipe with Constant Cross Section

In this section, the air flow inside a constant cross section pipe is modeled using the Euler equations. The stabilization problem is solved by using boundary control computed using Riemann coordinates as presented in Sections III and IV. Two boundary controllers are designed: one to stabilize the system in a neighborhood around a steady-state equilibrium by using Theorem 1 and a second one to stabilize the system in a defined region around the system's equilibrium by using Theorem 2.

Consider the Euler equations expressed in terms of the primitive variables: density (ρ), speed (u) and pressure (p),

$$\partial_t W + A(W)\partial_x W + C(W) = 0 \tag{32}$$

where

$$W = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}; \quad A(W) = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & a^2\rho & u \end{bmatrix}$$

$a = \sqrt{\frac{\gamma p}{\rho}}$ is the speed of sound, γ is the specific heat ratio and $C(W)$ is a function that describes the friction losses and heat exchanges. Since in this example, the isentropic case is analyzed, then $C(W) = 0$. The eigenvalues of the characteristic matrix $A(W)$, called the characteristic velocities, are:

$$\lambda_1(W) = u + a,\ \lambda_2(W) = u,\ \lambda_3(W) = u - a \tag{33}$$

and their respective Riemann invariants (see [17]):

$$a + \frac{\gamma - 1}{2}u, \quad \sqrt{\frac{p}{\rho^\gamma}}, \quad a - \frac{\gamma-1}{2}u \tag{34}$$

Let assume that the velocities (33) verify:

$$\lambda_3(W) < 0 < \lambda_2(W) < \lambda_1(W) \tag{35}$$

which enforces (32) to be a strict hyperbolic system and ensures the existence of a transformation to the Riemann coordinates. Consider the following change of coordinates:

$$\begin{aligned} \xi_1 &= \sqrt{\frac{\gamma p}{\rho}} + \frac{\gamma-1}{2}u - \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}} - \frac{\gamma-1}{2}\tilde{u} \\ \xi_2 &= \sqrt{\frac{p}{\rho^\gamma}} - \sqrt{\frac{\tilde{p}}{\tilde{\rho}^\gamma}} \\ \xi_3 &= \sqrt{\frac{\gamma p}{\rho}} - \frac{\gamma-1}{2}u - \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}} + \frac{\gamma-1}{2}\tilde{u} \end{aligned} \tag{36}$$

where ($\tilde{W} = [\tilde{\rho}, \tilde{u}, \tilde{p}]^T$) is an arbitrary steady-state. With these new coordinates (ξ_1, ξ_2, ξ_3), system (32) can be rewritten in the quasi-linear hyperbolic form (11) as follows:

$$\partial_t \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{bmatrix} \lambda_1(\xi) & 0 & 0 \\ 0 & \lambda_2(\xi) & 0 \\ 0 & 0 & \lambda_3(\xi) \end{bmatrix} \partial_x \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{37}$$

Since the change of coordinates (36) is a mapping, ρ, u and p can be expressed in terms of the Riemann invariants as:

$$\begin{aligned} u &= \frac{\xi_1 - \xi_3 + (\gamma - 1)\tilde{u}}{\gamma - 1} \\ \rho &= \left(\frac{\xi_1 - \frac{\gamma-1}{2}(u - \tilde{u}) + \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}}}{\sqrt{\gamma}\left(\xi_2 + \sqrt{\frac{\tilde{p}}{\tilde{\rho}^\gamma}}\right)} \right)^{\frac{2}{\gamma-1}} \\ p &= \rho^\gamma \left(\xi_2 + \sqrt{\frac{\tilde{p}}{\tilde{\rho}^\gamma}} \right)^2 \end{aligned} \tag{38}$$

Note that the equilibrium $[\tilde{\rho}, \tilde{u}, \tilde{p}]^T$ expressed in terms of Riemann coordinates is $[0, 0, 0]^T$.

A. Boundary Control Design using a Linear Hyperbolic Model

In this subsection, a boundary control for system (32) is designed in an equilibrium point using the results obtained in Theorem 1. It is clear that (32), evaluated near the steady-state equilibrium $\tilde{\xi} = 0$, can be considered as a linear hyperbolic system of the form (4). Considering $\tilde{W} = [1.16, 20, 100000]^T$ and the dynamic boundary conditions (8) defined with:

$$A = \begin{bmatrix} -300 & 10 & 13 \\ 15 & -40 & 5 \\ 4 & 9 & -300 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{39}$$

the following K is obtained applying Theorem 1:

$$K = \begin{bmatrix} -31.91 & -0.28 & -2.58 \\ -4.12 & -0.56 & 2.53 \\ -2.96 & 0.21 & -32.4 \end{bmatrix} \tag{40}$$

with the respective diagonal positive definite matrix associated with the Lyapunov function (16):

$$P^{-1} = Q = \begin{bmatrix} 0.013 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.013 \end{bmatrix} \tag{41}$$

To illustrate this result, numerical simulations of (32) with (8) and (40) are performed with an initial condition $W_0 = W(x, 0)$ close enough to the equilibrium in order to consider Λ constant. Let define $W_0 = [1.168, 21, 101000]^T$. Figures 1, 2 and 3 show the results obtained when using the boundary control (40).

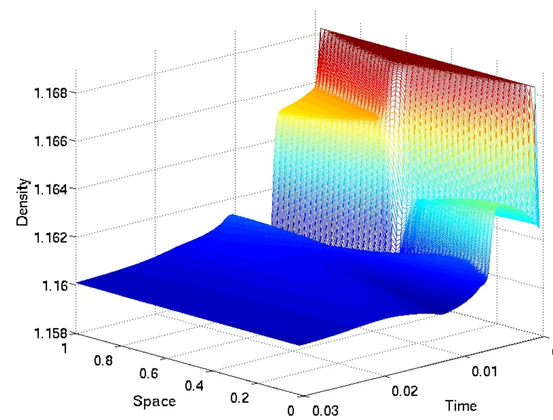

Fig. 1: Time evolution of the density profile ρ using the controller for linear hyperbolic systems

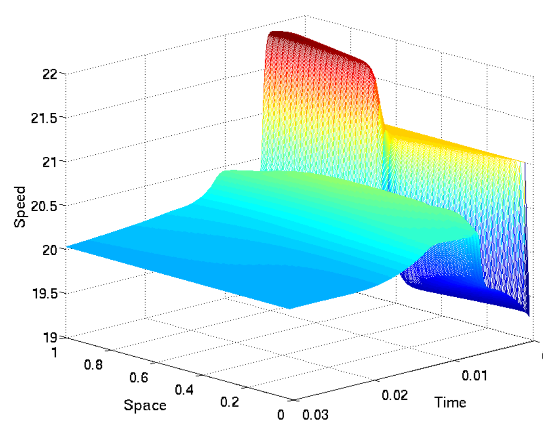

Fig. 2: Time evolution of the speed profile u using the controller for linear hyperbolic systems

B. Boundary Control for the Quasi-Linear Model

In this subsection, a boundary control for system (32) is designed with proved stability in a region described by a polytope around an equilibrium point. The characteristic matrix defined in terms of physical quantities can be expressed as follows:

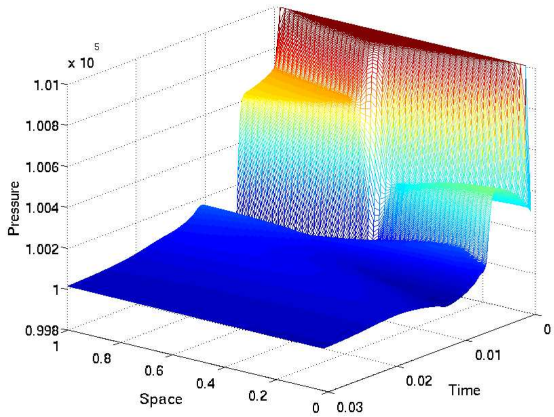

Fig. 3: Time evolution of the pressure profile p using the controller for linear hyperbolic systems

$$\Lambda(W) = \begin{bmatrix} a+u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & a-u \end{bmatrix} \tag{42}$$

Define the varying parameter $\varphi = [a, u]^T$, which is enough to describe the propagation speed of the Riemann invariants. Define the limits on each parameter as $[\overline{a}, \underline{a}]$ and $[\overline{u}, \underline{u}]$ to describe the region Z_φ where the stability of system (32) is ensured by using Theorem 2. Consider the equilibrium $\tilde{W} = [1.16, 20, 100000]^T$ once again. $\tilde{W}$ imposes at the equilibrium point $\tilde{u} = 20$ and $\tilde{a} = 347$. Taking this into account, the region Z_φ is defined by setting the following limits of each parameter:

$$\overline{a} = 355,\ \underline{a} = 340,\ \overline{u} = 40,\ \underline{u} = 5 \tag{43}$$

Applying Theorem 2, the following controller is obtained:

$$K = \begin{bmatrix} -19.95 & -0.037 & -0.84 \\ -4.11 & -0.063 & 1.83 \\ 0.5 & 0.027 & -15.7 \end{bmatrix} \tag{44}$$

with the respective diagonal positive definite matrix associated with the Lyapunov function (16):

$$P_{-1} = Q = \begin{bmatrix} 0.022 & 0 & 0 \\ 0 & 0.16 & 0 \\ 0 & 0 & 0.022 \end{bmatrix} \tag{45}$$

Figures 4, 5 and 6 present the results obtained in the numerical simulations using (44). Notice that the condition $\varphi \in Z_\varphi\, \forall\, t > 0$ is satisfied (Figures 5 and 7).

To illustrate the differences between the approaches presented in this work, the controller (40) derived using Theorem 1, valid in a neighborhood close enough to the equilibrium $\tilde{W}$ where Λ may be considered constant, is simulated under the same conditions where (44) was tested.

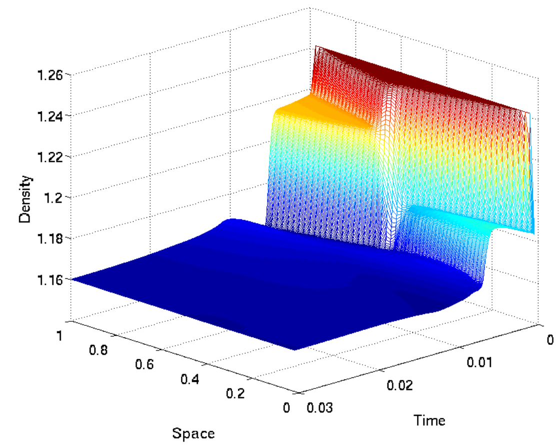

Fig. 4: Time evolution of the density profile ρ using the polytopic approach

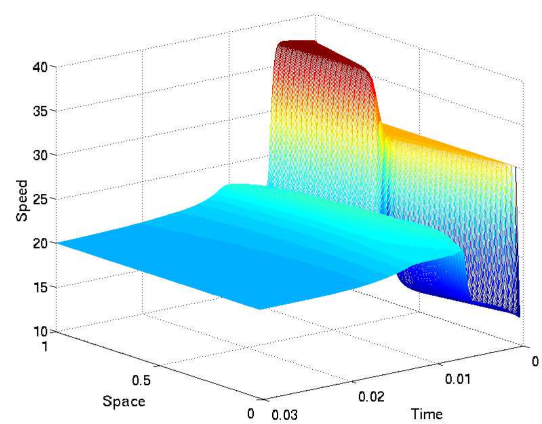

Fig. 5: Time evolution of the speed profile u using the polytopic approach

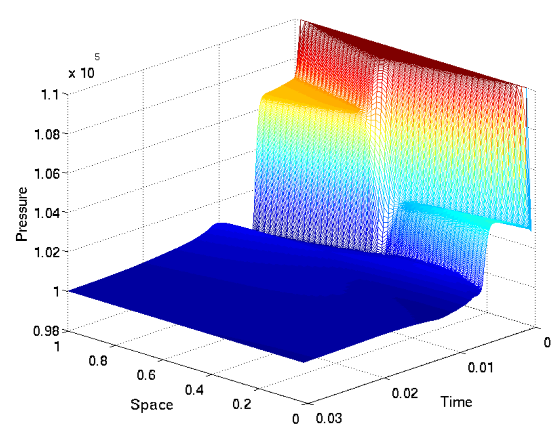

Fig. 6: Time evolution of the pressure profile p using the polytopic approach

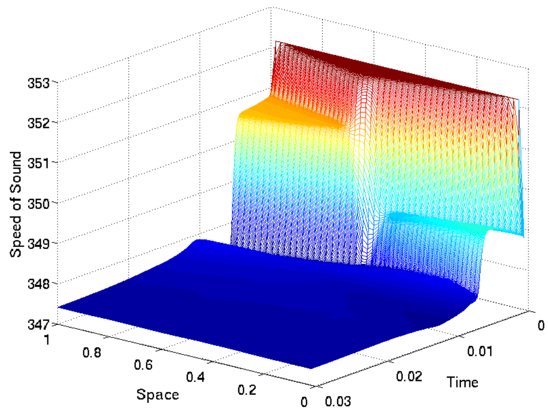

Fig. 7: Time evolution of the speed of sound a profile using the polytopic approach

The results are presented in Figures 8 and 9. Notice that the polytopic controller presents a better response together with the guarantee of stability.

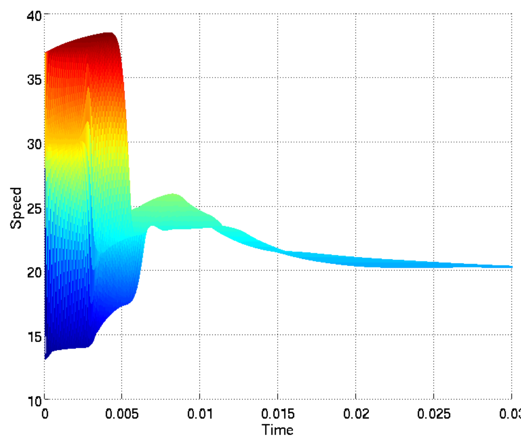

Fig. 8: Time evolution of the particle speed profile using the linear approach of Theorem 1

VI. Conclusion

In this paper, some sufficient conditions for the exponential stability of a linear and a quasi-linear hyperbolic PDE system with dynamics associated with the boundary conditions were derived. The stability analysis has been done using a Lyapunov function which allows to express the stability conditions in an LMI framework. A polytopic approach was implemented to guarantee the stability of quasi-linear hyperbolic system inside a defined polytope. A simulation example has shown the effectiveness of the contributions presented in this work and the advantages in terms of stability guarantee and robustness of the polytopic approach. This work could be implemented in different kind of physical systems governed by hyperbolic PDE's. This application may be considered in future work.

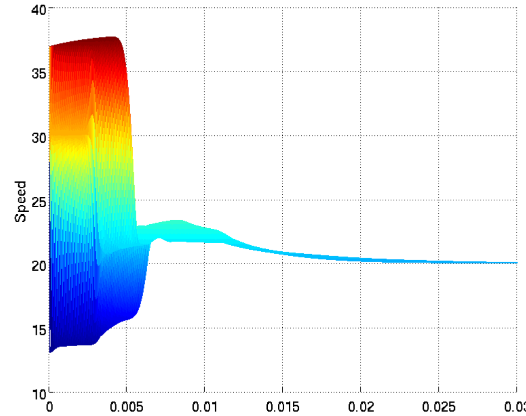

Fig. 9: Time evolution of the density profile using the polytopic approach of Theorem 2

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