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Abstract

Along a transit line, vehicle traffic and passenger traffic are jointly subject to variability in travel time and vehicle load hence crowding. The paper provides a stochastic model of passenger physical time and generalized time, including waiting on platform and in-vehicle run time from access to egress station. Five sources of variability are addressed: (i) vehicle headway which can vary between the stations provided that each service run maintains its rank throughout the local distributions of headways; (ii) vehicle order in the schedule of operations; (iii) vehicle capacity; (iv) passenger arrival time; (v) passenger sensitivity to quality of service. The perspective of the operator, which pertains to vehicle runs, is distinguished from the user's one at the disaggregate level of the individual trip, as in survival theory. Analytical properties are established that link the distributions of vehicle headways, vehicle run times, passenger wait times, passenger travel times, and their counterparts in generalized time, in terms of distribution functions, mean, variance and covariance. Many of them stem from Gaussian and log-normal approximations.

Keywords

Vehicle load. Headway rank. Passenger exposure. Wait time. Run time. Journey time.

1. Introduction

Background. The operations of a transit line, and even more of a network of lines, are submitted to variability in a number of ways. On the operator side, vehicle type may not be homogeneous, the passenger load depends on the service schedule and varies along the route, traffic disruptions arise due to causes either internal (such as human error, material incident, passenger incident or accident…) or external (such as adverse weather, malevolent intrusion, conflict with another flow…). On the demand side, the passenger experiences travel conditions along his trip, from service waiting and platform occupancy at the access station up to station egress passing by vehicle occupancy and its journey time, which vary according to the occurrence of the trip in a series of reiterations and also between passengers on a given occurrence. A major issue pertains to service reliability: any disruption causing a large delay induces a significant loss in quality of service, and the frequent reiteration of such events will make the passenger reconsider his travel decision of network route and even of transportation mode. Stated Preferences surveys have shown that frequent significant delays amount to additional travel time in a more than proportional way: for instance, the factor of proportionality was estimated to 1.5 for delays of more than 10 minutes occurring three out of 20 times in Paris suburban railways (Kroes *et al*, 2005). Such behavioral patterns must be

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taken into account in network planning, both within network traffic assignment models and the cost-benefit analysis of transportation projects.

Objective. The paper's objective is to provide a stochastic model of traffic variability and passenger exposure along a transit route. The model is designed as a sophisticated time-flow relationship at the level of the service route, in a matrix form between the stations of access and egress; thus it can be used as a component in a passenger traffic assignment model to a transit network. On the supply side, the model assumptions involve the statistical distribution of the local vehicle headways at station nodes and of the local run times along inter-station links, together with the distribution of vehicle capacity in terms of seated and standing passengers. On the demand side, a spatial pattern is assumed for the access-egress matrix of passenger flows, together with a statistical distribution (temporal pattern say on a day-to-day basis) of a volume index.

The model yields the following outcomes: (i) the distribution of vehicle journey times by pair of access-egress stations, together with the distribution of passenger loading ; (ii) the distribution of passenger physical time by access-egress pair ; (iii) the distribution of passenger generalized time by access-egress pair, assuming that crowding density adds discomfort cost to travel times. Thus the interplay of operations variability with the spatial pattern and temporal distribution of passenger flows is captured in an explicit and consistent framework.

Approach. The paper deals with the physics of traffic operations and passenger exposure to travel conditions both of service operations and vehicle load. The main variables of vehicle traffic, passenger traffic and passenger travel are cast into a probabilistic framework in the form of random variables. Variability sources are identified, among which the major one is the heterogeneity of vehicle headways. Analytical properties are established between the main model variables, in the form of functional relationships linking the CDF, PDF, mean and variance of them. This is achieved by making convenient specific assumptions: noteworthy assumptions include the conservation of headway rank by service run, normal approximations for headways and vehicle loads, or alternatively log-normal approximations when the interest lies in a product rather than in a sum of variables. Overall, the paper blends up probabilistic analysis taken mostly from the theory of renewal and survival, with traffic analysis at the two levels of transit vehicles and passengers, respectively. Previous analytical work along that line has addressed vehicle traffic only (e.g. Carey and Kwiencinski, 1994; Meester and Muns, 2007) or passenger traffic restricted to the issue of passenger waiting at a platform station, as in Bowman and Turnquist (1981). Our specific assumptions extend the scope to a "transit leg" that includes the in-vehicle journey from access to egress station as well as waiting on the access platform. Recent work has shown the distribution of travel conditions and the distinction between the operator and user perspectives: Islam and Vandebona (2011) on the basis of micro-simulation and Yuan *et al* (2011) on the basis of traffic observations.

Structure. The rest of the paper is organized in five sections. Vehicle traffic is considered first, by focusing on headways and deriving some consequences on journey times by pair of entry-exit stations (Section 2). Then, passenger load by vehicle is characterized with respect to headway rank and the index of demand volume (Section 3). Next, we turn our attention to passenger exposure to in-vehicle crowding, wait time and travel time (Section 4). The consequences of service irregularity and other variations affect not only the physical times but also the "generalized time" which takes into account the discomfort of specific travel states (Section 5). Lastly, the conclusion points to the model scope, limitations and potential developments (Section 6).

2. On vehicle headways and journey times

In this paper, a transit line operated along a single service route in a single direction is considered. The stations are indexed by $m \in M$ and the sections or links between adjacent stations by $a \in A$. Each vehicle run is characterized by a trajectory in space and time. The journey time is made up of the run times on the sections plus the dwell times at the stations.

The objective of this section is to model the statistical distribution of vehicle run times between station pairs along the line. The statistical population of interest is the set of runs during a reference period, for instance the morning peak hour of working days.

First, we shall model the distribution of vehicle headways (§ 2.1). Second, their propagation between stations is addressed in § 2.2. Then, a postulate is made about the "conservation of headway rank" (§ 2.3), which entails specific properties for the distribution of vehicle headways (§ 2.4) and that of journey times (§ 2.5).

2.1 On vehicle headways

Denote by $\eta_m(i)$ the time between the departure of vehicle i from station m and that of the previous vehicle, $i-1$. In the population of vehicle runs, the Cumulated Distribution Function (CDF) of η_m is denoted as H_m with inverse function $H_m^{(-1)}$. Let us recall classical properties:

i) The service frequency at station m during the reference period, f_m, is the reciprocal of the average headway: $f_m = 1/E[\eta_m]$.

ii) Service irregularity is related to the deviation of η_m from its average value. It can be assessed by the variance of this distribution, $V[\eta_m]$, or equivalently by its standard deviation $\sigma[\eta_m]$ or the relative dispersion $\gamma[\eta_m] = \sigma[\eta_m]/E[\eta_m]$.

Assuming that the incoming passengers at station m arrive independently from one another and from service schedule, their arrivals can be modeled as a Poisson process and, if the process intensity is medium or high, then it can be safely assumed that the number of passengers waiting for a given vehicle is proportional to the headway (neglecting any capacity constraint). Furthermore, the distribution of passenger waiting times at m stems from that of vehicle headway in a specific way (see Section 4).

2.2 Spatial propagation

The instant of departure of vehicle i from station m, $h_m(i)$, is separated from that of the next station, $h_{m+1}(i)$, by the run time along section $a \approx (m, m+1)$ plus the stop time at $m+1$, altogether denoted as $t_a(i)$:

$$h_{m+1}(i) = h_m(i) + t_a(i)\,. \tag{2.1}$$

Note that we also have:

$$\eta_m(i) = h_m(i) - h_{m-1}(i)\,.$$

So that from vehicle $i-1$ to vehicle i, the headways at service stations satisfy:

$$\eta_m(i) = \eta_{m-1}(i) + \tau_a(i)\,, \tag{2.2}$$

wherein $\tau_a(i) = t_a(i) - t_a(i-1)$ is the difference in travel time along a and m.

Service operations and exogenous influences may affect the distribution of τ_a and, in turn, that of η_m. The influences on the mean and variance are of crucial interest. By the linearity of expectation:

$$\mathrm{E}[\eta_m] = \mathrm{E}[\eta_{m-1}] + \mathrm{E}[\tau_a], \tag{2.3}$$

whereas, by the bi-linearity of covariance,

$$\mathrm{V}[\eta_m] = \mathrm{V}[\eta_{m-1}] + \mathrm{V}[\tau_a] + 2\,\mathrm{cov}(\eta_{m-1}, \tau_a). \tag{2.4}$$

Formula (2.2) and its consequences (2.3-4) state the propagation of vehicle headways from station to station.

2.3 On the conservation of headway rank

Of course, the conservation of schedule order is assumed along the line, under a First In – First Out discipline. Let us focus on the rank of each run in the "local" distribution of headway, characterized by the fractile $\alpha_m = \mathrm{H}_m(\eta_m)$. In this study, the postulate of conservation of headway rank is made:

$$\forall i, \forall m \neq n, \alpha_m(i) = \alpha_n(i) = \alpha(i). \tag{2.5}$$

This states that if a vehicle run is associated to a relatively low (resp. large) headway at a given station, it is associated to relatively low (resp. large) headways at all the stations of the line. However, local magnitudes may differ, only the rank remains stable.

The postulate is realistic enough in various instances:

- when the operations are regular along the line, the headway at the initial station is maintained from station to station.
- If most of traffic disruptions occur on a given section a, then the main source of variation pertains to τ_a and the rank in its distribution may be assumed to apply on the rest of the line as well.

The most noteworthy consequence is the functional dependency between the headways along the line:

$$\alpha_m = \mathrm{H}_m(\eta_m) = \alpha = \alpha_{m-1} = \mathrm{H}_{m-1}(\eta_{m-1}), \text{ hence} \tag{2.6}$$

$$\eta_m = \mathrm{H}_m^{-1} \circ \mathrm{H}_{m-1}(\eta_{m-1}). \tag{2.7}$$

Thus $\tau_a = \eta_m - \eta_{m-1}$ also is a function of η_{m-1}.

Assuming further that the dependency is linear, i.e. $\tau_a = \lambda\eta_{m-1} + \mu$ for some parameters $\lambda \geq 0$ and μ, then it would hold that

$$\mathrm{cov}(\eta_{m-1}, \tau_a) = \sigma[\eta_{m-1}].\sigma[\tau_a]. \tag{2.8}$$

This relationship notably holds for random variables η_{m-1} and τ_a that are distributed along a similar pattern, i.e. when

$$\frac{\tau_a - \mathrm{E}[\tau_a]}{\sigma[\tau_a]} \approx \frac{\eta_{m-1} - \mathrm{E}[\eta_{m-1}]}{\sigma[\eta_{m-1}]}.$$

This holds notably for perfectly correlated normal variables: in this case a valuable complementary property is that η_m is normal, too, yielding normal variables for headway and section time variation along the line.

2.4 Vehicle journey time with respect to schedule order

Let us turn to the journey time of each vehicle run with respect to its order in the schedule of operations, denoted by i. Let r denote a reference station and $m \geq r$ a subsequent station in the selected direction of traffic, $t_{rm}(i)$ be the journey time of vehicle run i between the instants of departure from r and m, $h_m(i)$ and $h_r(i)$ respectively. It holds that

$$t_{rm}(i) = h_m(i) - h_r(i), \text{ so}$$

$$t_{rm}(i) = t_{rm}(i-1) + \eta_m(i) - \eta_r(i) \text{ and}$$

$$t_{rm}(i) = t_{rm}(0) + \sum_{j=1}^{i} \left(\eta_m(j) - \eta_r(j)\right), \tag{2.9}$$

Wherein vehicle run #0 is an ideal vehicle run of nominal performance which immediately precedes the reference period. By the linearity of expectation, it then follows that

$$\mathrm{E}[T_{rm}(i)] = T_{rm}(0) + i(\mathrm{E}[\eta_m] - \mathrm{E}[\eta_r]). \tag{2.10}$$

Given the fact that the α(i) are assumed i.i.d., the runs are mutually independent, which implies that:

$$\mathrm{V}[T_{rm}(i)] = i.\mathrm{V}[\eta_m - \eta_r], \tag{2.11}$$

Under the conservation of headway rank and the assumption of normality, $\eta_m = \eta_r + \sum_{a\in[r,m]} \tau_a$ satisfies that $\sigma[\eta_m] = \sigma[\eta_r] + \sum_{a\in[r,m]} \sigma[\tau_a]$, which entails that

$$\sigma[\eta_m] - \sigma[\eta_r] = \sum_{a\in[r,m]} \sigma[\tau_a] = \sigma[\eta_m - \eta_r]. \tag{2.12}$$

Combining (2.12) and (2.11), we get that

$$\sigma[T_{rm}(i)] = \sqrt{i}\ \sigma[\eta_m - \eta_r]. \tag{2.13}$$

Of course the assumption of conservation of headway rank and the run independence are likely to interfere in practice. However, eqns (2.10) and (2.13) give some insight into the progressive deterioration of the vehicle journey time with respect to the order of the run in the schedule of operations, when submitted to irregularity and random disruptions.

3. Vehicle loading

So far, two sources of variability have been made explicit: headway rank, denoted as α, and the order in the schedule, denoted as i. In this section, two other sources are introduced, namely the level of passenger transport demand, denoted as β, and the train capacity, denoted as κ. Sources α and β jointly influence the vehicle load in passengers. Sources α, β and κ jointly influence the ratio of load to capacity by vehicle run.

This section establishes some analytical properties of the passenger load and load ratio along a transit line, by taking into account the demand (passenger flow) between stations of entry and exit.

3.1 Assumptions about passenger demand

A reference period of given duration is considered for line operations. In fact it refers in some average way to a population of periods, for instance the morning peak hour throughout a series of working days. To depict the variability of periods, let us associate to each period its level β of passenger demand, with CDF B in the population of periods.

Within a given period, passenger flow is modeled as a stationary random process, with macroscopic properties as follows: between any pair $r < s$ of stations along the line, the passenger flow arriving at r and destined to s during time interval $[h, h']$ amounts to $\beta q_{rs}(h'-h)$. Thus the set of $[q_{rs} : r < s]$ describes the spatial structure of passenger demand per unit of time.

Across the population of periods, we could define β so as to satisfy that $\mathrm{E}[\beta] = 1$; however we shall keep $\mathrm{E}[\beta]$ in the formulae for the sake of traceability.

3.2 Vehicle loading conditional on β

Assuming that passenger demand is not restrained by vehicle capacity, at each station r of entry a given vehicle run will attract incoming passengers in proportion to its local headway, η_r. On section a, the vehicle load denoted by y_a consists in those passengers having entered at station $r \le a$ (with obvious notation for $\le$ and $\ge$ for position along the line):

$$y_{a,\beta} = \beta \sum_{r \le a, s \ge a} q_{rs} \eta_r \ . \tag{3.1}$$

Then, on average:

$$\mathrm{E}[y_{a,\beta}] = \beta \sum_{r \le a, s \ge a} q_{rs} \, \mathrm{E}[\eta_r] . \tag{3.2}$$

Keeping to the postulate of conservation of headway rank, the vehicle run is characterized by its fractile α so that $\eta_r = \mathrm{H}_r^{(-1)}(\alpha)$. Then

$$y_{a,\beta}(\alpha) = \beta \sum_{r \le a, s \ge a} q_{rs} \, \mathrm{H}_r^{(-1)}(\alpha) . \tag{3.3}$$

Denote by $\mathrm{Y}_{a,\beta}$ the CDF of y_a conditional on β. Then:

$$\mathrm{Y}_{a,\beta}^{-1} = \beta \sum_{r \le a, s \ge a} q_{rs} \, \mathrm{H}_r^{-1} . \tag{3.4}$$

Furthermore, as in the previous section the sum of totally dependent random variables sharing a Gaussian pattern satisfies that

$$\sigma[y_{a,\beta}] = \beta \sum_{r \le a, s \ge a} q_{rs} \sigma[\eta_r] . \tag{3.5}$$

3.3 Vehicle loading, overall distribution

Let us now aggregate the analysis with respect to β. Denoting $\xi_a = \sum_{r \le a, s \ge a} q_{rs} \eta_r$ the random variable of reference link flow and by X_a its CDF, it holds generally that:

$$\mathrm{Y}_a(z) = \Pr\{\beta \xi_a \le z\} = \int \mathrm{X}_a(z/\beta) \, \mathrm{dB}(\beta) . \tag{3.6}$$

In reality, demand level β may influence vehicle operations – for instance because the number of boarding and alighting passengers may determine the dwelling time. However, for simplicity, independence is assumed in this model, yielding that:

$$\mathrm{E}[y_a] = \mathrm{E}[\beta] \, . \sum_{r \le a, s \ge a} q_{rs} \, \mathrm{E}[\eta_r] . \tag{3.7}$$

$\mathrm{V}[y_a] = \mathrm{V}[\beta] \, . \mathrm{E}[\xi_a]^2 + \mathrm{E}[\beta^2] . \mathrm{V}[\xi_a]$ due to ([2]), hence

[2] $\mathrm{var}[XY] = \mathrm{E}[X^2 Y^2] - \mathrm{E}[XY]^2 \ = \mathrm{E}[X^2]\mathrm{E}[Y^2] - \mathrm{E}[X]^2 \mathrm{E}[Y]^2 \ = \ \mathrm{E}[X^2]\mathrm{V}[Y] + \mathrm{E}[Y]^2 \mathrm{V}[X]$

$$V[y_a] = V[\beta] . \left(\sum_{r \le a, s \ge a} q_{rs} E[\eta_r]\right)^2 + E[\beta^2] . \left(\sum_{r \le a, s \ge a} q_{rs} \sigma[\eta_r]\right)^2 . \quad (3.8)$$

To gain further insight into the structure of influences, let us add to the assumption of Gaussian headways the approximation of the resulting flow, ξ_a, by a log-normal variable with same mean and standard deviation, $E[\xi_a]$ and $\sigma[\xi_a]$. Denote by m_a and s_a, respectively, the mean and standard deviation of $\ln \xi_a$. From the classical properties of log-normal distributions, these are related to the moments of ξ_a by:

$$E[\xi_a] = \exp(m_a + \tfrac{1}{2} s_a^2)$$

$$\sigma^2[\xi_a] = (\exp(s_a^2) - 1) . E[\xi_a]^2$$

Assuming lastly that $\beta \approx \mathrm{LN}(m_\beta, s_\beta)$, then the link load $\beta \xi_a \approx \mathrm{LN}(m_\beta + m_a, \sqrt{s_\beta^2 + s_a^2})$.

3.4 Vehicle loading ratio

Vehicle capacity, denoted as κ, pertains to the number of seats plus a reference number of positions for passenger standing with sufficient comfort (e.g. 4 persons per square meter). Heterogeneous vehicles may be used to operate the transit line, leading to the variability of capacity hence of the ratio of passenger load to capacity.

Let us denote that ratio as

$$z_a = y_a / \kappa = \beta \xi_a / \kappa . \quad (3.9)$$

While it is quite natural to assume the independence of β and κ, it would be a wise policy of line operations to assign vehicle types according to the planned headways, by associated larger capacity to larger headways so as to balance the load ratio across the runs. Under such a balancing policy, the load ratio could be analyzed in the same way as vehicle load by replacing $\eta_r(\alpha)$ with $\eta_r(\alpha) / \kappa_\alpha$. On the contrary, a negligent policy may be modeled based on the assumption of independence between κ and α as well as β. Then the load ratio would have mean and variance as follows:

$$E[z_a] = E[\beta] . E[\kappa^{-1}] . \sum_{r \le a, s \ge a} q_{rs} E[\eta_r] . \quad (3.10)$$

$$V[z_a] = E[\kappa^{-2}] V[y_a] + V[\kappa^{-1}] E[y_a]^2 . \quad (3.11)$$

4. Passenger exposition to physical time

Let us come to the perspective of the user at the level of the individual trip, as opposed to the operator's one at the level of the vehicle run.

4.1 User's exposure

Let us recall some basic properties of renewal theory (e.g. Kleinrock, 1975, pp. 169 *sq*). Denote by H_r^o the CDF of headway duration η_r and by $\dot{H}_r^o$ its PDF, with superscript o to mark the operator's perspective. A user willing to board at r arrives on platform at a random instant, which will belong to a headway interval of duration η with a probability proportional to η: in the user's perspective, marked by superscript u,

$$\dot{H}_r^u(\eta) \propto \eta \dot{H}_r^o(\eta) . \quad (4.1)$$

By integration, the factor of proportionality amounts to $1/\mathrm{E}[\eta_r^{\mathrm{o}}]$. The moments of η_r^{u} stem from those of η_r^{o} at the next order:

$$\mathrm{E}[(\eta_r^{\mathrm{u}})^k] = \mathrm{E}[(\eta_r^{\mathrm{o}})^{k+1}]/\mathrm{E}[\eta_r^{\mathrm{o}}]. \tag{4.2}$$

Consider now the size of the passenger group that includes the individual user, to board in a vehicle run at station r, n_r^{u}. Its probability density stems from the density $\mathrm{f}^{\mathrm{u}}(\beta,\eta)$ of pair (β,η), which is related to the PDF $\mathrm{f}^{\mathrm{o}}(\beta,\eta)$ in the following way:

$$\mathrm{f}^{\mathrm{u}}(\beta,\eta) \propto \beta\eta\ \mathrm{f}^{\mathrm{o}}(\beta,\eta), \tag{4.3}$$

where f^{o} is the PDF of passenger group sizes from the perspective of the operator. Assuming independence between β and η, then $\mathrm{f}^{\mathrm{o}}(\beta,\eta) = \dot{\mathrm{B}}^{\mathrm{o}}(\beta).\dot{\mathrm{H}}_r^{\mathrm{o}}(\eta)$: thus independence is maintained in the user's perspective, since

$$\begin{aligned}\mathrm{f}^{\mathrm{u}}(\beta,\eta) &\propto \beta\eta\ \dot{\mathrm{B}}^{\mathrm{o}}(\beta).\dot{\mathrm{H}}_r^{\mathrm{o}}(\eta)\\ &= \dot{\mathrm{B}}^{\mathrm{u}}(\beta).\dot{\mathrm{H}}_r^{\mathrm{u}}(\eta)\end{aligned} \tag{4.4}$$

In which $\dot{\mathrm{B}}^{\mathrm{u}}(\beta) = \beta\dot{\mathrm{B}}^{\mathrm{o}}(\beta)/\mathrm{E}[\beta^{\mathrm{o}}]$ and $\dot{\mathrm{H}}_r^{\mathrm{u}}(\eta) = \eta\dot{\mathrm{H}}_r^{\mathrm{o}}(\eta)/\mathrm{E}[\eta_r^{\mathrm{o}}]$.

As $n_r^{\mathrm{u}} = \beta.\eta$, its CDF is

$$\mathrm{N}_r^{\mathrm{u}}(x) = \Pr\{\beta\eta \le x\} = \int \mathrm{N}_{r,\beta}^{\mathrm{u}}(x)\,\mathrm{dB}^{\mathrm{u}}(\beta) = \int \mathrm{H}_r^{\mathrm{u}}(x/\beta)\,\mathrm{dB}^{\mathrm{u}}(\beta). \tag{4.5}$$

The independence property enables us to establish the mean and variance of group size as follows:

$$\mathrm{E}[n_r^{\mathrm{u}}] = \mathrm{E}[\beta^{\mathrm{u}}]\mathrm{E}[\eta_r^{\mathrm{u}}] = \frac{\mathrm{E}[(\beta^{\mathrm{o}})^2]}{\mathrm{E}[\beta^{\mathrm{o}}]}\frac{\mathrm{E}[(\eta_r^{\mathrm{o}})^2]}{\mathrm{E}[\eta_r^{\mathrm{o}}]}. \tag{4.6}$$

$$\begin{aligned}\mathrm{V}[n_r^{\mathrm{u}}] &= \mathrm{E}[\beta^{\mathrm{u}2}]\mathrm{V}[\eta_r^{\mathrm{u}}] + \mathrm{V}[\beta^{\mathrm{u}}](\mathrm{E}[\eta_r^{\mathrm{u}}])^2\\ &= \frac{\mathrm{E}[(\beta^{\mathrm{o}})^3]}{\mathrm{E}[\beta^{\mathrm{o}}]}\left(\frac{\mathrm{E}[(\eta_r^{\mathrm{o}})^3]}{\mathrm{E}[\eta_r^{\mathrm{o}}]} - \frac{\mathrm{E}[(\eta_r^{\mathrm{o}})^2]^2}{\mathrm{E}[\eta_r^{\mathrm{o}}]^2}\right) + \left(\frac{\mathrm{E}[(\eta_r^{\mathrm{o}})^2]}{\mathrm{E}[\eta_r^{\mathrm{o}}]}\right)2\left(\frac{\mathrm{E}[(\beta^{\mathrm{o}})^3]}{\mathrm{E}[\beta^{\mathrm{o}}]} - \frac{\mathrm{E}[(\beta^{\mathrm{o}})^2]^2}{\mathrm{E}[\beta^{\mathrm{o}}]^2}\right).\end{aligned} \tag{4.7}$$

4.2 Vehicle load by link as experienced by the user

Depending on his entry station e, the user travelling along link $a \ge e$ experiences there a vehicle passenger load as follows, wherein $\eta_{r,e}^{\mathrm{u}}$ depends on the entry station:

$$y_{a,e}^{\mathrm{u}} = \beta\textstyle\sum_{r\le a, s\ge a} q_{rs}\eta_{r,e}^{\mathrm{u}}, \tag{4.8}$$

Given the value η of η_e^{u}, headway rank is $\alpha = \mathrm{H}_e^{\mathrm{o}}(\eta)$: the conservation postulate in the operator's perspective is maintained in the user's perspective. Then, conditionally to η:

$$y_{a,e,\eta,\beta}^{\mathrm{u}} = \beta\textstyle\sum_{r\le a, s\ge a} q_{rs}\ \mathrm{H}_r^{\mathrm{o}(-1)} \circ \mathrm{H}_e^{\mathrm{o}}(\eta). \tag{4.9}$$

From the equation above stems the unconditional variable $y_{a,e}^{\mathrm{u}}$. Its CDF is given by:

$$\mathrm{Y}_{a,e}^{\mathrm{u}}(z) = \Pr\{y_{a,e}^{\mathrm{u}} \le z\} = \int \Pr\{y_{a,e,\eta,\beta}^{\mathrm{u}} \le z\}\ \mathrm{f}^{\mathrm{u}}(\beta,\eta)\,\mathrm{d}\eta\,\mathrm{d}\beta. \tag{4.10}$$

By successive transformations:

$$\begin{aligned} y^{\mathrm{u}}_{a,e,\eta,\beta} \le z \;&\Leftrightarrow \textstyle\sum_{r\le a, s\ge a} q_{rs}\, \mathrm{H}_r^{\mathrm{o}(-1)}(\alpha_\eta) \le z/\beta \\ &\Leftrightarrow \mathrm{X}_a^{\mathrm{o}(-1)}(\alpha_\eta) \le z/\beta \\ &\Leftrightarrow \alpha_\eta \le \mathrm{X}_a^{\mathrm{o}}(z/\beta) \\ &\Leftrightarrow \eta \le \mathrm{H}_e^{\mathrm{o}(-1)} \circ \mathrm{X}_a^{\mathrm{o}}(z/\beta) \end{aligned}$$

Thus $\Pr\{y^{\mathrm{u}}_{a,e,\beta} \le z\} = \mathrm{H}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)} \circ \mathrm{X}_a^{\mathrm{o}}(z/\beta)$, and:

$$\mathrm{Y}^{\mathrm{u}}_{a,e}(z) = \int \Pr\{y^{\mathrm{u}}_{a,e,\beta} \le z\}\, \mathrm{f}^{\mathrm{u}}(\beta)\, \mathrm{d}\beta = \int \mathrm{H}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)} \circ \mathrm{X}_a^{\mathrm{o}}(z/\beta)\ \mathrm{dB}^{\mathrm{u}}(\beta). \tag{4.11}$$

To gain insight into the consequences, let us approximate the distribution of headways in the operator's perspective by a log-normal distribution with parameters m_e^{o} and s_e^{o}. Then, by standard properties of the log-normal distribution, $\eta_e^{\mathrm{u}} \approx \mathrm{LN}(m_e^{\mathrm{o}} + s_e^{\mathrm{o}2}, s_e^{\mathrm{o}})$. Denoting by Φ the reduced Gaussian CDF, then $\mathrm{H}_e^{\mathrm{o}(-1)}(t) = \exp(m_e^{\mathrm{o}} + s_e^{\mathrm{o}}\Phi^{(-1)}(t))$ and $\mathrm{H}_e^{\mathrm{u}}(x) = \Phi(\frac{\ln x - m_e^{\mathrm{o}}}{s_e^{\mathrm{o}}} - s_e^{\mathrm{o}})$, so that $\mathrm{H}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)}(t) = \Phi(\frac{m_e^{\mathrm{o}} + s_e^{\mathrm{o}}\Phi^{(-1)}(t) - m_e^{\mathrm{o}}}{s_e^{\mathrm{o}}} - s_e^{\mathrm{o}}) = \Phi(\Phi^{(-1)}(t) - s_e^{\mathrm{o}})$.

Further on, let us approximate $\xi_a^{\mathrm{o}} \approx \mathrm{LN}(m_a, s_a)$: then $\mathrm{X}_a^{\mathrm{o}}(t) = \Phi(\frac{\ln t - m_a}{s_a})$. Combining, we get that $\mathrm{H}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)} \circ \mathrm{X}_a^{\mathrm{o}}(t) = \Phi(\frac{\ln t - m_a}{s_a} - s_e^{\mathrm{o}})$, which shows that $\xi_a^{\mathrm{u}} \approx \mathrm{LN}(m_a + s_a s_e^{\mathrm{o}}, s_a)$. So, in this case,

$$\mathrm{Y}^{\mathrm{u}}_{a,e}(z) = \int \Phi(\frac{\ln(z/\beta) - m_a}{s_a} - s_e^{\mathrm{o}})\ \mathrm{dB}^{\mathrm{u}}(\beta) = \int \Pr\{\xi^{\mathrm{u}}_{a,e}.\beta \le z\}\, \mathrm{dB}^{\mathrm{u}}(\beta) = \Pr\{\ln \xi^{\mathrm{u}}_{a,e} + \ln \beta^{\mathrm{u}} \le \ln z\}.$$

Under the last assumption that $\beta^{\mathrm{o}} \approx \mathrm{LN}(m_\beta, s_\beta)$, it comes out that

$$\mathrm{Y}^{\mathrm{u}}_{a,e}(z) = \Phi(\frac{\ln(z) - m_a - s_a s_e^{\mathrm{o}} - m_\beta - s_\beta^2}{\sqrt{s_a^2 + s_\beta^2}}), \tag{4.12}$$

Which shows that $y^{\mathrm{u}}_{a,e} \approx \mathrm{LN}(m_a + s_a s_e^{\mathrm{o}} + m_\beta + s_\beta^2, \sqrt{s_a^2 + s_\beta^2})$.

From this stems the average volume experienced by an individual user,

$$\mathrm{E}[y^{\mathrm{u}}_{a,e}] = \exp(m_a + s_a s_e^{\mathrm{o}} + m_\beta + s_\beta^2 + \tfrac{1}{2}(s_a^2 + s_\beta^2)) = \mathrm{E}[\beta^{\mathrm{u}}]\mathrm{E}[\mathrm{X}_a^{\mathrm{o}}]\exp(s_a s_e^{\mathrm{o}}).$$

The ratio to the average vehicle load in the operator's perspective amounts to

$$\begin{aligned} \frac{\mathrm{E}[y^{\mathrm{u}}_{a,e}]}{\mathrm{E}[y_a^{o}]} &= \frac{\mathrm{E}[(\beta^{\mathrm{o}})^2]}{\mathrm{E}[\beta^{\mathrm{o}}]^2}\exp[s_a s_e^{\mathrm{o}}] = \exp[s_\beta^2 + s_a s_e^{\mathrm{o}}] \\ &= (1+\gamma_\beta^2)\exp[\sqrt{\ln(1+\gamma_a^2)}\sqrt{\ln(1+\gamma_e^2)}] \\ &\approx (1+\gamma_\beta^2)(1+\gamma_a\gamma_e) \ \text{ for not too large relative dispersions} \end{aligned} \tag{4.13}$$

4.3 Run time

In section 2.3 some statistical properties of run time have been established for vehicles: schedule order i determines the mean and variance of run time $T_{rs}(i)$. Any user that arrives at station r at a given instant h will board a vehicle of order i which is random due to irregularity, so he will get a random run time. The precise definition of $i(h)$ as a random

variable is difficult except for Markovian vehicle runs which would yield a Poisson distribution but at the price of assuming a large amount of variability. For simplicity, let us assume here that $i(h)$ has a uniform discrete distribution derived from $i \approx 1+\text{int}[(h-h_0)/\text{E}[\eta_r]]$ on the reference period $[h_0, h_1]$. Let $I = i(h_1)$ and $1/I$ be the elemental probability of $i \in \{1,..I\}$.

The average run time is

$$\begin{aligned} \text{E}[t_{rs}^{\text{u}}] &= \tfrac{1}{I}\sum_{i=1}^{I}\text{E}[t_{rs}(i)] = t_{rm}(0) + (\text{E}[\eta_s] - \text{E}[\eta_r])\tfrac{1}{I}\sum_{i=1}^{I} i \\ &= t_{rm}(0) + \tfrac{I+1}{2}\Delta\text{E} \text{ by setting } \Delta\text{E} = \text{E}[\eta_s] - \text{E}[\eta_r] \end{aligned} \tag{4.14}$$

By the law of total variance, the variance of the run time is made of an interclass part plus an intra-class part in the following way:

$$\begin{aligned} \text{V}[t_{rs}^{\text{u}}] &= \tfrac{1}{I}\sum_{i=1}^{I}(\text{E}[t_{rs}(i)] - \text{E}[t_{rs}^{\text{u}}])^2 + \tfrac{1}{I}\sum_{i=1}^{I}\sigma^2[t_{rs}(i)] \\ &= \tfrac{1}{I}\Delta\text{E}^2\sum_{i=1}^{I}(i - \tfrac{I+1}{2})^2 \qquad + \tfrac{1}{I}\sum_{i=1}^{I} i(\Delta\sigma)^2 \text{ by setting } \Delta\sigma = \sigma[\eta_s - \eta_r] \\ &= \Delta\text{E}^2\,\tfrac{I^2-1}{24} + (\Delta\sigma)^2\,\tfrac{I+1}{2} = \tfrac{I+1}{2}\left(\Delta\text{E}^2\,\tfrac{I-1}{12} + (\Delta\sigma)^2\right) \end{aligned} \tag{4.15}$$

4.4 Wait time

The user wait time on the station platform, w_e, amounts to the residual span (or lifetime) of the on-going headway interval. From survival theory, its PDF is

$$\dot{\text{W}}_e(x) = \frac{1-\text{H}_e^{\text{o}}(x)}{\text{E}[\eta_e^{\text{o}}]}. \tag{4.16}$$

This leads to the following relationships between the moments of the two variables:

$$\text{E}[{w_e}^k] = \frac{\text{E}[(\eta_e^{\text{o}})^{k+1}]}{(k+1)\,\text{E}[\eta_e^{\text{o}}]} = \text{E}[(\eta_e^{\text{u}})^k]/(k+1) \tag{4.17}$$

So it holds that

$$\text{E}[w_e] = \frac{\text{E}[(\eta_e^{\text{o}})^2]}{2\,\text{E}[\eta_e^{\text{o}}]} = \tfrac{1}{2}\text{E}[\eta_e^{\text{u}}], \tag{4.18}$$

$$\text{V}[w_e] = \tfrac{1}{3}\text{E}[(\eta_e^{\text{u}})^2] - \tfrac{1}{4}\text{E}[\eta_e^{\text{u}}]^2 = \tfrac{1}{4}\text{V}[\eta_e^{\text{u}}] + \tfrac{1}{12}\text{E}[(\eta_e^{\text{u}})^2]. \tag{4.19}$$

Furthermore, η_e^{u} is correlated to w_e and so are the headway rank and all derived variables such as $y_{a,e}^{\text{o/u}}$. For instance, $\text{E}[w_e.\eta_e^{\text{u}}] = \tfrac{1}{2}\text{E}[(\eta_e^{\text{u}})^2]$ so $\text{cov}[w_e, \eta_e^{\text{u}}] = \tfrac{1}{2}\text{V}[\eta_e^{\text{u}}]$.

4.5 Travel time

The travel time of a user between stations r and s is composed by the wait time at r, w_r, plus the run time between the two stations, t_{rs}^{u}:

$$\tilde{t}_{rs}^{\text{u}} = w_r + t_{rs}^{\text{u}}. \tag{4.20}$$

By the linearity of expectation,

$$\text{E}[\tilde{t}_{rs}^{\text{u}}] = \text{E}[w_r] + \text{E}[t_{rs}^{\text{u}}]. \tag{4.21}$$

There may be some correlation between the two components. However independence may be assumed as a crude approximation, yielding:

$$V[\tilde{t}_{rs}^{u}] = V[w_r] + V[t_{rs}^{u}]. \quad (4.22)$$

4.6 Platform crowding

A related issue pertains to the number of passengers waiting on platform at a given station r. At any instant, this number is proportional to the level of the incoming flow, $\beta.\sum_{s>r} q_{rs}$, times the time elapsed since the departure of the last vehicle. From survival theory (e.g. Kleinrock, *op cit*), the latter is the random variable $\eta_r^u - w_r$. Thus the passenger stock amounts to

$$S_r = \beta.(\sum_{s>r} q_{rs})(\eta_r^u - w_r). \quad (4.23)$$

Independence of β and η_r implies that $\eta_r^u - w_r$ is independent of β, yielding

$$E[S_r] = \tfrac{1}{2} E[\eta_r^u] E[\beta](\sum_{s>r} q_{rs}) \quad (4.25)$$

$$V[S_r] = (\sum_{s>r} q_{rs})^2 \left[E[\beta^2] V[\eta_r^u] + V[\beta].E[\eta_r^u]^2\right]. \quad (4.26)$$

The perspective of either the operator or the user must be specified by setting the adequate distribution of β.

5. On passenger generalized time

To a trip-maker, the "generalized time" of travel is a comprehensive disutility to capture both the physical travel time and the quality of service during the trip. Each physical state (e.g. sitting in-vehicle) or transition (e.g. vehicle egress) within the trip sequence, is associated with a specific penalty factor: from 1 for sitting in-vehicle to 2 for standing in-vehicle under dense crowding or more for waiting in crowd with no traffic information. The physical time spent in a given state is multiplied by its penalty factor to yield the generalized time of that state. This is aggregated along the trip sequence to yield the generalized time of the trip. It is used in discrete choice models of network route or transportation mode. It is also the basis to evaluate the benefits and costs of a transport plan to the community.

5.1 The formation of generalized time

The notion of generalized time involves penalty factors that vary across the individual trip-makers. Small persons resent standing in a crowd more than tall ones do. In general, old persons move and walk more slowly than younger ones. People are more or less sensitive to fatigue. Let ε denote the particular sensitivity of a given individual.

Wait time w_r and link time t_a are transformed into generalized times, denoted as $\omega_{r\varepsilon}$ and $\theta_{a\varepsilon}$, respectively. The generalized travel time amounts to

$$\lambda_{rs,\varepsilon} = \omega_{r\varepsilon} + \sum_{a\in]r,s[} \theta_{a\varepsilon}. \quad (5.1)$$

To model the dependency of ω and θ on the crowding density, assume that

$$\omega_{r\varepsilon} = w_r \, \psi_{r\varepsilon}(S_r), \quad (5.2)$$

$$\theta_{a\varepsilon} = t_a^u \, \varphi_{a\varepsilon}(y_a^u, \kappa). \quad (5.3)$$

Formulae (5.1-3) provide a basis to analyze the influence of passenger flow on travel disutility. Taking wait time and link time as random variables, then so are $\omega_{r\varepsilon}$, $\theta_{a\varepsilon}$ and $\lambda_{rs,\varepsilon}$ conditionally to ε. From the previous section, w_r and S_r are correlated. Link loads y_a^{u} along successive links are correlated, too. Furthermore, platform variables and link loads are correlated due to headway rank. As all the correlations are positive, the generalized travel time conditionally to ε is subject to large relative dispersion.

5.2 In-vehicle discomfort

Let us focus on in-vehicle time and the influence of crowding density on its specific penalty factor. A well-known model is the so-called BPR function (e.g. Spiess and Florian, 1989):

$$\varphi_{a1}(y_a^{\mathrm{u}}, \kappa) = 1 + c_a \left(\frac{y_a}{\kappa}\right)^{b_a}, \tag{5.4}$$

in which exponent b_a takes positive values such as 1 or 4, whereas factor c_a takes positive values between 0 and 3 typically. Formulae (5.4) and (5.3) state that crowding discomfort inflicts a specific additional cost of $t_a^{\mathrm{u}} c_a (y_a / \kappa)^{b_a}$ to the physical link time. In the operator's perspective (resp. the user's one), the average additional cost is evaluated as

$$\mathrm{SC}^{\mathrm{o/u}} = \mathrm{E}[t_a^{\mathrm{o/u}} c_a (y_a^{\mathrm{o/u}} / \kappa)^{b_a}] = c_a \, \mathrm{E}[t_a^{\mathrm{o/u}}] \mathrm{E}[(y_a^{\mathrm{o/u}} / \kappa)^{b_a}]. \tag{5.5}$$

Assuming that capacity is homogeneous, the two notions differ by a ratio of

$$\frac{\mathrm{SC}^{\mathrm{u}}}{\mathrm{SC}^{\mathrm{o}}} = \frac{\mathrm{E}[y_a^{\mathrm{u}\ b_a}]}{\mathrm{E}[y_a^{\mathrm{o}\ b_a}]}. \tag{5.6}$$

Using the log-normal approximation, $y^b \approx \mathrm{LN}(bm_y, bs_y)$ so

$$\frac{\mathrm{SC}^{\mathrm{u}}}{\mathrm{SC}^{\mathrm{o}}} = \exp[bs_\beta^2 + bs_a s_e^o + \tfrac{1}{2} b(b-1)(s_a^2 + s_\beta^2)]. \tag{5.7}$$

Assuming further that $s_a = s_e^o$, it comes out that

$$\frac{\mathrm{SC}^{\mathrm{u}}}{\mathrm{SC}^{\mathrm{o}}} = \rho^{b(b+1)} \text{ wherein } \rho = \exp[\tfrac{1}{2}(s_a^2 + s_\beta^2)]. \tag{5.8}$$

5.3 Numerical instance

To fix ideas, let us assume that $\gamma_a = 0.3$ and $\gamma_\beta = 0.2$, yielding $s_a \approx 0.3$ and $s_\beta \approx 0.2$. Then $\rho = 1.13$ and the ratio varies from 1.13 to 3.5 as b is changed from 1 to 4. Fig. 1 depicts the variation of the disutility factor φ_{a1} with respect to the apparent occupancy ratio, $\mathrm{E}[y_a^o / \kappa]$. For a given apparent ratio, the experienced crowding density is equal to the disutility factor at $b = 1$ and $c = 1$, minus one: it differs from the apparent ratio in a significant yet not major amount.

Irregularity also affects the base travel time, $\mathrm{E}[t_a]$. Between stations r and s, from (4.14) the related additional cost amounts to $\mathrm{ST} = \tfrac{1}{2}(I+1)(\mathrm{E}[\eta_s] - \mathrm{E}[\eta_r])$. Denoting by f_r the service frequency delivered at station r during a reference period of length H, $I = f_r$ and $\mathrm{E}[\eta_r] = H / f_r$ while $\mathrm{E}[\eta_s] = H / f_s$. Then, $\mathrm{ST} \approx \tfrac{1}{2} H.(f_r / f_s - 1)$. For instance, along the line A of the regional railways in the Paris area, at the morning peak hour westwards, the

service frequency is reduced from $f_r = 30$/hour upstream of the centre, to $f_s = 27$/hour downstream. The resulting additional time is about 3' per trip. The train capacity is about 2,000 passengers and the apparent occupancy ratio of 83% upstream. The additional cost per trip, from nominal quality of service of $T_0 = 15'$ to personal experience, amounts to

$$(T_0 + \mathrm{ST}).(1 + c(\mathrm{E}[y_a^{\mathrm{u}}]/\kappa)^b) - T_0 \approx 21.7\text{'} \text{ if } b = 2 \text{ and } c = 1,$$

Whereas a naive evaluation by the operator would yield

$$T_0(1 + c(\mathrm{E}[y_a^{\mathrm{o}}]/\kappa)^b) - T_0 \approx 10.3\text{'} \text{ only.}$$

The discrepancy between the two evaluations would be much larger for larger values of exponent b. This demonstrates the need for accurate estimations of penalty functions and a consistent, user-oriented evaluation of vehicle crowding in the cost-benefit assessment of transport plans.

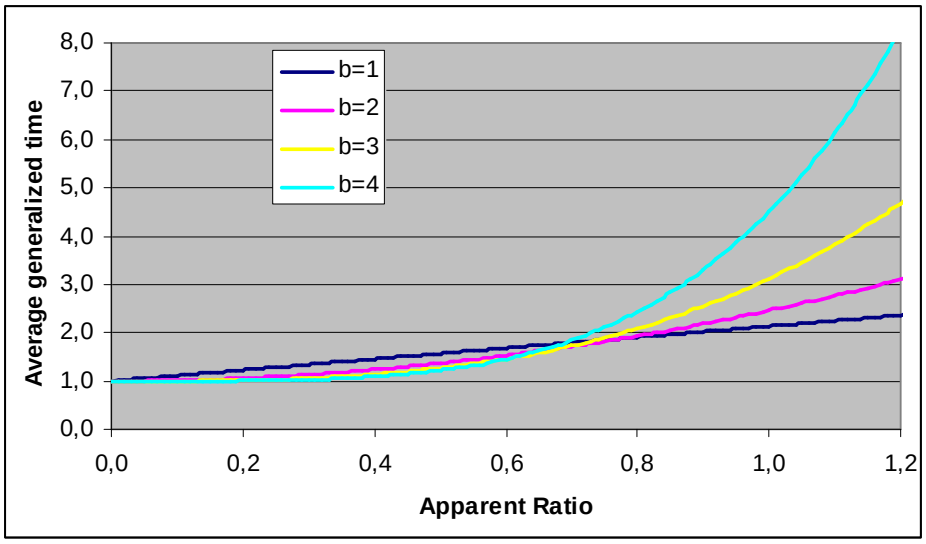

Figure 1. Generalized time versus Occupancy ratio, according to variability parameter.

6. Conclusion

A model of traffic along a transit line has been provided at both levels of traffic unit, the vehicle versus the passenger. The perspectives of the operator and the user have been identified. Based on a powerful postulate, the conservation of headway rank, it has been shown that service irregularity and demand variations, as well as other factors such as vehicle order in schedule, vehicle size and passenger sensitivity to quality of service, affect the passenger conditions of travel significantly. Crowding density above a ratio of say 80% exerts major influence on generalized travel time. The operator perspective is plagued with bias that must be corrected to represent passenger conditions objectively.

The model captures a set of variability sources. Analytical formulae have been established to assess their respective effects. The main postulate is the conservation of headway rank. Gaussian or log-normal approximations have been made to yield convenient approximations; in the authors' opinion their effect is innocuous.

The established properties will be useful in models of traffic assignment to a transit network, as they pertain to travel conditions hence to the leg quality of service, which determines the passenger travel choice of a network route.

Further work is required to analyze transit lines serviced by a set of routes: vehicle type and load will depend on the route and the joint operations. On the passenger side, between some station pairs a subset of routes will be used, yielding reduced waiting time but more diverse in-vehicle conditions. Another research topic pertains to the feedback of vehicle load on the operating conditions, as in the assignment model of Leurent *et al* (2011).

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