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Daniel Barlet

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# A finiteness theorem for $S$ -relative formal Brieskorn modules.

Daniel Barlet.\*

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## Abstract

We give a general result of finiteness for holomorphic families of Brieskorn modules constructed from a holomorphic family of one parameter degeneration of compact complex manifolds acquiring (general) singularities.

AMS CLASSIFICATION. 32 S 25, 32 S 40, 32 S 50.

KEY WORDS. Brieskorn module, (a,b)-module, asymptotic expansion, Gauss-Manin connection, filtered differential equation.

## Introduction

In this article we are interested in the following situation :

Let  $\varphi : X \rightarrow T$  be a holomorphic proper and surjective map between complex manifolds such that outside an hypersurface  $S \subset T$  the fibers of  $\varphi$  are smooth. Then around the generic point of  $S$  we may assume that  $T$  is locally isomorphic to  $S_0 \times D$  where  $D$  is a small open disc with center  $0$  in  $\mathbb{C}$  and  $S_0$  an open set in  $S$ . We can consider the restriction of  $\varphi$  over  $S_0 \times D$  as a holomorphic family parametrized by  $S_0$  of one parameter degeneration of compact complex manifolds acquiring singularities over  $S_0 \times \{0\}$ . Our aim is first to give in such a local situation (without properness but with suitable hypotheses, always satisfied in the absolute case) a  $S$ -relative version of the construction given in [B. II] theorem 2.1.1.

In the second part we show that the properness assumption allows to obtain a finiteness result for the family parametrized by  $S$  of Brieskorn modules (see [Br.70],

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[S.89], [B.93] and [B.95]) obtained via the cohomology of the direct image of the complex constructed in the local setting. This may be seen as a generalisation of a result on variation of the mixed Hodge structures associated to these degenerations. Note that the result in the absolute case is a rather simple consequence of the result of [B.II], but the generalization of the local result to the relative case is quite tricky, although it needs rather strong hypotheses.

The finiteness result, which is new already in the absolute case, is a key tool to produce holomorphic families of Brieskorn modules when we consider holomorphic families of one parameter degenerations of compact complex manifolds acquiring arbitrary singularities.

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# 1 The general construction of (a,b)-modules in the relative case.

## 1.1 Our situation.

NOTATIONS. Let  $S$  be a reduced complex space. We shall say that  $\pi : \mathcal{X} \rightarrow S$  is a  $S$ -**manifold** when  $\mathcal{X}$  is a reduced complex space and  $\pi$  a holomorphic map which is  $S$ -smooth. By definition this means that locally on  $\mathcal{X}$  we have a  $S$ -isomorphism of  $\mathcal{X}$  with a product  $S \times U$  where  $U$  is an open set in  $\mathbb{C}^{n+1}$ . Such an isomorphism will be called a  $S$ -relative chart or a  $S$ -relative system of coordinates on  $\mathcal{X}$ .

In such a situation we have on  $\mathcal{X}$  a locally free  $\mathcal{O}_{\mathcal{X}}$ -sheaf of  $S$ -**relative holomorphic differential forms**  $\Omega_{/S}^{\bullet}$  which corresponds, via the local  $S$ -isomorphisms above, to the sheaf of holomorphic differential forms on  $U$  with holomorphic coefficients on  $S \times U$ .

For a holomorphic function  $f$  on  $\mathcal{X}$  the  $S$ -**relative differential** is defined as a section of the sheaf  $\Omega_{/S}^1$  and denoted  $d_{/S}f$ . If  $d_{/S}f_x \neq 0$ , then  $f$  may be chosen as the first coordinate of a local  $S$ -relative coordinate system on  $\mathcal{X}$  near the point  $x$ . The relative differential  $d_{/S}$  is well defined and of degree 1 on the graded sheaf  $\Omega_{/S}^{\bullet}$ . The relative differential  $d_{/S}$  is  $\mathcal{O}_S$ -linear and the  $S$ -relative de Rham complex is a complex of  $\mathcal{O}_S$  modules. The de Rham lemma with holomorphic parameters gives that the complex  $(\Omega_{/S}^{\bullet}, d_{/S}^{\bullet})$  has zero cohomology sheaves in positive degrees ; so it is a resolution of  $\mathcal{O}_S$ .

THE SITUATION (@). We shall consider a holomorphic function  $f : \mathcal{X} \rightarrow D$  where  $D$  is a disc with center 0 in  $\mathbb{C}$  and where  $\pi : \mathcal{X} \rightarrow S$  is a  $S$ -manifold.

We shall define  $\mathcal{Y} := f^{-1}(0)$  and  $\mathcal{Z} := \{x \in \mathcal{X} / d_{/S}f_x = 0\}$ .

HYPOTHESIS (H0). We shall assume that  $\mathcal{Y}$  has codimension 1 in each fiber of  $\pi$  and that  $\mathcal{Z} \subset \mathcal{Y}$  has codimension  $\geq 2$  in each fiber of  $\pi$ .

HYPOTHESIS (H1). We shall assume that locally on  $\mathcal{X}$  there exists a proper  $S$ -modification  $\tau : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  such that  $\tilde{\mathcal{Y}} := \tau^{-1}(\mathcal{Y})$  is locally on  $\tilde{\mathcal{X}}$  a  $S$ -relative normal crossing divisor. That is to say that  $f$  admits a simultaneous desingularisation over  $S$ .

HYPOTHESIS (H2). We shall assume that locally on  $\mathcal{Z}$  we shall find local  $S$ -relative Milnor fibration.

REMARK. When we assume (H1) we may weaken the hypothesis (H0) in asking only the inclusion  $\mathcal{Z} \subset \mathcal{Y}$ .

The precise meaning of these hypotheses will be given in section 1.3. But remark already that in the absolute case ( i. e. for  $S = \{0\}$ ), they are always satisfied for any non zero germ of holomorphic function in  $\mathbb{C}^{n+1}$ .

## 1.2 The sheaf $\mathcal{A}_S$ on a reduced complex space.

In this paragraph we consider the situation (@) with the hypothesis (H0).

Let  $S$  be a reduced complex space. We define the following sheaves on  $S$ :

1. The sheaf  $\mathcal{O}_S[[a]]$  of commutative  $\mathcal{O}_S$ -algebras defined as the presheaf

$$U \mapsto \Gamma(U, \mathcal{O}_S)[[a]] := \prod_{j=0}^{\infty} \Gamma(U, \mathcal{O}_S).a^j.$$

We shall denote  $(\mathcal{O}_S[[a]])_{s_0}$  the germ at a point  $s_0 \in S$  to distinguish it from the algebra  $\mathcal{O}_{S,s_0}[[a]]$  which is strictly bigger in general. Note that both are noetherian rings but the proof of this fact for the first one uses Oka's theorem to have the coherence of  $\mathcal{O}_S$  and Cartan theorem B on Stein neighbourhoods of the point  $s_0$  in  $S$ .

2. The sheaf  $\mathcal{A}_S^0$  of non commutative  $\mathcal{O}_S$ -algebras defined as the presheaf

$$U \mapsto \Gamma(U, \mathcal{O}_S)[a] \langle b \rangle := \bigoplus_{j=0}^{\infty} b^j \cdot \Gamma(U, \mathcal{O}_S)[a]$$

with the commutation relation  $a.b - b.a = b^2$  and more generally with the relation  $T(a).b = b.T(a) + b.T'(a).b$  where  $T'(a)$  is the usual derivative in  $a$  of the polynomial  $T(a) \in \Gamma(U, \mathcal{O}_S)[a]$ .

3. The sheaf  $\mathcal{A}_S$  of non commutative  $\mathcal{O}_S$ -algebras defined as the presheaf

$$U \mapsto \Gamma(U, \mathcal{O}_S)[[a]] \langle\langle b \rangle\rangle := \prod_{j=0}^{\infty} b^j \cdot \Gamma(U, \mathcal{O}_S)[[a]]$$

with the commutation relation  $T(a).b = b.T(a) + b.T'(a).b$  and the fact that the right and left actions of  $\mathcal{O}_S[[a]]$  are continuous with the  $b$ -adic filtration.

REMARK. This sheaf is the formal completion (in  $a$ , that is to say in a local coordinate near 0 in  $\mathbb{C}$ ) of the sheaf of  $S$ -relative microdifferential operators of order  $\leq 0$  on  $S \times \mathbb{C}$ . It has a natural action on the sheaf  $\mathcal{O}_S[[a]]$  which is given by the action of  $b$  defined as  $[b.g](s, a) := \int_0^a g(s, x).dx$ . It is easy to see that it extends as an  $\mathcal{O}_S[[a]]$ -linear action (see the lemma 1.2.3).

NOTATIONS. Let  $E$  be a left  $\mathcal{A}_S^0$ -module. We define  $A(E)$  and  $B(E)$  respectively as the  $a$ -torsion and  $b$ -torsion of  $E$ . Remark that  $B(E)$  is a  $\mathcal{A}_S^0$ -submodule of  $E$  but that  $A(E)$  is not stable by  $b$  in general. So we shall define also  $\tilde{A}(E)$  as the maximal sub  $\mathcal{A}_S^0$ -submodule contained in  $A(E)$ .  $\square$

AN EXAMPLE OF  $\mathcal{A}_S$ -MODULE. Let  $\Lambda$  be a finite subset in  $]0, 1] \cap \mathbb{Q}$ ,  $k \geq 0$  an integer. Define  $\Xi_{\Lambda, S}^{(k)}$  as the free  $\mathcal{O}_S[[b]]$ -module generated by  $e_0(\lambda), \dots, e_k(\lambda)$ , where  $\lambda$  is in  $\Lambda$ , with the action of  $a$  given by

$$a.e_j(\lambda) = \lambda.b.e_j(\lambda) + b.e_{j-1}(\lambda)$$

with the convention that  $e_{-1}(\lambda) = 0$  for each  $\lambda \in \Lambda$ .  
 If we think to  $e_j(\lambda)$  as

$$x^{\lambda-1} \cdot \frac{(\text{Log } x)^j}{j!}$$

with  $b := \int_0^x$  and  $a := \times x$  it is easy to see that  $\Xi_{\Lambda, S}^{(k)}$  is the standard sheaf with holomorphic coefficients in  $\mathcal{O}_S$  for multivalued asymptotic expansions in  $x$  with logarithmic terms of degree at most  $k$  and monodromy with spectrum in  $\exp(2i\pi \cdot \lambda)$ ,  $\lambda \in \Lambda$ .

It is easy to see that  $\Xi_{\Lambda, S}^{(k)}$  is also a free  $\mathcal{O}_S[[a]]$ -sheaf with the same generators. Both structures of  $\mathcal{O}_S[[b]]$  and  $\mathcal{O}_S[[a]]$ -modules define a structure of  $\mathcal{A}_S$ -module on  $\Xi_{\Lambda, S}^{(k)}$ .

More generally, for  $V$  a finite dimensional complex vector space, we shall consider the  $\mathcal{A}_S$ -module  $\Xi_{\Lambda, S}^{(k)} \otimes_{\mathbb{C}} V$  where  $a$  and  $b$  act as the identity on  $V$ . This is again a free finite type module on  $\mathcal{O}_S[[a]]$  and  $\mathcal{O}_S[[b]]$ .  $\square$

**THE COMPLEX  $(K_{/S}^\bullet, d_{/S}^\bullet)$ .** In our situation  $(\textcircled{a})$ , under the hypothesis  $(H0)$  we define the complex  $(\hat{\Omega}_{/S}^\bullet, d_{/S}^\bullet)$  which is the formal completion in  $f$  of the holomorphic  $S$ -relative de Rham complex. Recall that, by definition, the sheaf  $\hat{\Omega}_{/S}^\bullet$  is defined by the presheaf on  $\mathcal{Y}$

$$U \longrightarrow \lim_{\leftarrow -N} [\hat{\Omega}_{/S}^\bullet(U)/f^N \cdot \hat{\Omega}_{/S}^\bullet(U)]$$

and that  $d_{/S}^\bullet$  is induced by the  $S$ -relative de Rham differential.

Our smoothness assumption for the map  $\pi : \mathcal{X} \rightarrow S$  implies that this complex is a resolution of  $\pi^{-1}(\mathcal{O}_S)_{|Y}$  with  $\mathcal{O}_S$ -linear differentials.

Then we define the sub-complex  $(K_{/S}^\bullet, d_{/S}^\bullet)$  where the subsheaf  $K_{/S}^p$  is the kernel of the map<sup>1</sup>

$$d_{/S} f \wedge : \hat{\Omega}_{/S}^p \rightarrow \hat{\Omega}_{/S}^{p+1}$$

given by the left exterior multiplication by  $d_{/S} f$ . We shall denote  $E^p$  the  $p$ -th cohomology sheaf of this complex. Now the  $\mathcal{O}_S[[a]]$ -linearity of the differential  $d_{/S}$  on this sub-complex shows that these cohomology sheaves are  $\mathcal{O}_S[[a]]$ -modules in an obvious way. We shall explain now that, via the inverse of the Gauss-Manin connection, there exists a natural action of  $b$  on these cohomology sheaves in such a way that they become  $\mathcal{A}_S^0$ -modules.

**ACTION OF  $b$  ON THE COHOMOLOGY SHEAVES  $E^\bullet$ .** Consider now the exact sequence of complexes of  $\mathcal{O}_S[[a]]$ -modules on  $\mathcal{Y}$

$$0 \rightarrow (K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow (\hat{\Omega}_{/S}^\bullet, d_{/S}^\bullet) \xrightarrow{d_{/S} f \wedge} (I_{/S}^\bullet, d_{/S}^\bullet)[+1] \rightarrow 0. \quad (1)$$

---

<sup>1</sup>we shall note  $K_{f/S}^p$  when we want to precise the function  $f$  we consider.

where  $I_{/S}^p := d_{/S}f \wedge \hat{\Omega}_{/S}^{p-1}$ . Note that  $K_{/S}^0$  and  $I_{/S}^0$  are zero and that  $I_{/S}^1 = \hat{\mathcal{O}}_{\mathcal{Y}} \cdot d_{/S}f$ . The natural inclusions  $I_{/S}^p \subset K_{/S}^p$  for all  $p \geq 0$  are compatible with the differential  $d_{/S}$ . This leads to an exact sequence of complexes of  $\mathcal{O}_S$ -modules

$$0 \rightarrow (I_{/S}^\bullet, d_{/S}^\bullet) \rightarrow (K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow ([K_{/S}/I_{/S}]^\bullet, d_{/S}^\bullet) \rightarrow 0. \quad (2)$$

where  $[K_{/S}/I_{/S}]^\bullet$  is a coherent  $\mathcal{O}_{\mathcal{X}}$ -module with support contained in  $\mathcal{Y}$ . So there exists, thanks to the Nullstellensatz, locally on  $\mathcal{Y}$ , an integer  $N$  such that  $f^N \cdot K_{/S}^p \subset I_{/S}^p$  for any  $p \geq 0$ ; so we have

$$[K_{/S}/I_{/S}]^p \simeq K_{/S}^p/I_{/S}^p \quad \forall p \geq 0 \quad (3)$$

In particular, when we shall assume that  $\mathcal{Y}$  is  $S$ -proper, the direct image theorem of H. Grauert will give the  $\mathcal{O}_S$ -coherence of the direct images  $R^q\pi_*([K_{/S}/I_{/S}]^p)$ . This will be a key point for our finiteness theorem (see section 2).

We have a natural inclusion  $f^*(\hat{\Omega}_{S \times D/S}^1) \subset K_{/S}^1 \cap \text{Ker } d_{/S}$ , and this gives a sub-complex concentrated in degree 1 with zero differential of  $(K_{/S}^\bullet, d_{/S}^\bullet)$ . As in [B.II], we shall consider also the quotient complex  $(\tilde{K}_{/S}^\bullet, d_{/S}^\bullet)$ . So we have the exact sequence

$$0 \rightarrow f^*(\hat{\Omega}_{S \times D/S}^1) \rightarrow (K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow (\tilde{K}_{/S}^\bullet, d_{/S}^\bullet) \rightarrow 0. \quad (4)$$

This exact sequence corresponds to the (a,b)-module version of the exact sequence involving the nearby cycles and the vanishing cycles sheaves in the usual context. Note that with our hypothesis (H0) the sheaf  $K^1 \cap \text{Ker } d = I_{/S}^1 \cap \text{Ker } d$  is the constant sheaf on  $\mathcal{Y}$  of fiber  $\mathbb{C}[[f]].df$  and the cohomology sheaves of the complex  $(\tilde{K}_{/S}^\bullet, d_{/S}^\bullet)$  are supported in  $\mathcal{Z}$ .

We want to describe now the natural actions of  $a$  and  $b$  on the cohomology sheaves of the complexes  $(K_{/S}^\bullet, d_{/S}^\bullet), (I_{/S}^\bullet, d_{/S}^\bullet), ([K_{/S}/I_{/S}]^\bullet, d_{/S}^\bullet), f^*(\hat{\Omega}_{S \times D/S}^1), (\tilde{K}_{/S}^\bullet, d_{/S}^\bullet)$ . As in all cases we have complexes of  $\mathcal{O}_S[[a]]$ -modules, with  $\mathcal{O}_S[[a]]$ -linear differentials, the action of  $a$  on the cohomology sheaves of these complexes is clear.

The next lemma will give the action of  $b$ .

**Lemma 1.2.1** *The exact sequence of complexes (1) induces for any  $p \geq 2$  a  $\mathcal{O}_S$ -linear isomorphism*

$$\partial^p : \mathcal{H}^p(I_{/S}^\bullet, d_{/S}^\bullet) \rightarrow \mathcal{H}^p(K_{/S}^\bullet, d_{/S}^\bullet)$$

and a short exact sequence of  $\mathcal{O}_S$ -modules on  $\mathcal{Y}$

$$0 \rightarrow \mathcal{O}_S \cdot d_{/S}f \rightarrow \mathcal{H}^1(I_{/S}^\bullet, d_{/S}^\bullet) \xrightarrow{\partial^1} \mathcal{H}^1(K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow 0 \quad (@@)$$

There is a canonical  $\mathcal{O}_S$ -linear splitting of  $\partial^1$

$$\tilde{b}^1 : \mathcal{H}^1(K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow \mathcal{H}^1(I_{/S}^\bullet, d_{/S}^\bullet)$$

For  $p \geq 2$  define  $\tilde{b}^p := (\partial^p)^{-1}$ . Then we have for each  $p$  the formula

$$a.\tilde{b} = \tilde{b} \circ \mathcal{H}^p(i) \circ \tilde{b} + \tilde{b}.a \quad (\text{F})$$

where  $i : (I_{/S}^\bullet, d_{/S}^\bullet) \rightarrow (K_{/S}^\bullet, d_{/S}^\bullet)$  is the obvious map of  $\mathcal{O}_S[[a]]$ -complexes.

PROOF. The fact that for  $p \geq 2$  the connector  $\partial^p$  is an isomorphism is clear, as the  $S$ -relative de Rham complex is a  $\mathcal{O}_S$ -linear resolution of  $\mathcal{O}_S$ . Now we shall construct the splitting  $\tilde{b}^1$  and verify the formula (F). Consider  $x \in K_{/S}^1$  such that  $d_{/S}x = 0$ . Thanks to the relative de Rham lemma we may write  $x = d_{/S}\xi$  with  $\xi \in \mathcal{O}_X$ . But the assumption that  $x$  is  $d_{/S}$ -closed implies that  $\xi$  is locally constant along the  $S$ -fibers of  $\mathcal{Y} \setminus \mathcal{Z}$  because near such a point, we may find  $S$ -relative coordinates  $(z_0, \dots, z_n)$  on  $\mathcal{X}$  such that  $f(s, z) = z_0$ . Then the relation  $d_{/S}\xi \wedge d_{/S}f = 0$  implies that  $\xi(s, z)$  is a function of  $(s, z_0)$  only. This proves our assertion, and the fact that  $\mathcal{Y}$  is locally connected allows to choose  $\xi$  vanishing (set theoretically) on  $\mathcal{Y}$ , because  $\mathcal{Y} \setminus \mathcal{Z}$  is dense in  $\mathcal{Y}$ . Moreover this choice is unique because in the previous local computation near a point in  $\mathcal{Y} \setminus \mathcal{Z}$ , if  $x = g(s, z_0).dz_0$  we have to choose  $\xi = G(s, z_0)$  with  $\frac{\partial G}{\partial z_0}(s, z_0) = g(s, z_0)$  and  $G(s, 0) = 0$ , which has a unique solution. So the continuity of  $\xi$  forces vanishing on all  $\mathcal{Y}$ .

Remark that an open dense set in  $\mathcal{Y}$  where we may write locally  $f(s, z) = z_0^k$  for some  $k \in \mathbb{N}^*$  is enough for the preceding discussion.

Now we define  $\tilde{b}^1[d_{/S}\xi] = [d_{/S}f \wedge \xi]$  with this unique choice of  $\xi$ . It is clearly  $\mathcal{O}_S$ -linear and

$$\partial^p \circ \tilde{b}^1[d_{/S}\xi] = \partial^1[d_{/S}f \wedge \xi] = [d_{/S}\xi]$$

as  $\partial^1 = d_{/S} \circ (d_{/S}f \wedge)^{-1}$ .

To finish the proof compute  $\tilde{b}(a[x]) + \tilde{b}(\mathcal{H}^p(i)[\tilde{b}[x]])$ . Writing again  $x = d_{/S}\xi$ , we get

$$a[x] + \mathcal{H}^p(i)[\tilde{b}[x]] = [f.d_{/S}\xi + d_{/S}f \wedge \xi] = [d_{/S}(f.\xi)]$$

and so

$$\tilde{b}(a[x] + \mathcal{H}^p(i)[\tilde{b}[x]]) = [d_{/S}f \wedge f.\xi] = a.\tilde{b}([x])$$

which concludes the proof. ■

**Definition 1.2.2** We shall define

1.  $b : \mathcal{H}^p(K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow \mathcal{H}^p(K_{/S}^\bullet, d_{/S}^\bullet)$  as  $b := \mathcal{H}^p(i) \circ \tilde{b}$ .
2.  $b : \mathcal{H}^p(I_{/S}^\bullet, d_{/S}^\bullet) \rightarrow \mathcal{H}^p(I_{/S}^\bullet, d_{/S}^\bullet)$  as  $b := \tilde{b} \circ \mathcal{H}^p(i)$ .
3.  $b = 0$  on the quotient complex (with trivial differential)  $([K_{/S}/I_{/S}]^\bullet, d_{/S}^\bullet)$ .
4.  $b := \int_0^z$  on  $f^*(\hat{\Omega}_{S \times D/S}^1) \simeq \mathcal{O}_S[[z]].dz$  (with also trivial differential).



5. and  $b$  is induced on  $\mathcal{H}^p(\tilde{K}_{f/S}, d_{/S})$  by  $b$  on  $\mathcal{H}^p(K_{/S}^\bullet, d_{/S}^\bullet)$ . Note that for  $p \neq 1$  these cohomology sheaves are canonically isomorphic and that for  $p = 1$  the injection in the exact sequence

$$0 \rightarrow f^*(\hat{\Omega}_{S \times D/S}^1) \rightarrow \mathcal{H}^1(K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow \mathcal{H}^1(\tilde{K}_{f/S}, d_{/S}) \rightarrow 0$$

is  $b$ -linear.

Note that under the hypothesis (H0) we have the vanishing of  $\mathcal{H}^1(\tilde{K}_{f/S}, d_{/S})$ , but this is not the case if under (H1) we weaken (H0) assuming only  $\mathcal{Z} \subset \mathcal{Y}$ .

An easy consequence of the previous lemma is that we have  $a.b - b.a = b^2$  for each cohomology sheaf of any of these complexes of  $\mathcal{O}_S[[a]]$ -modules.

THE COMPLEXES  $(\mathcal{K}_{/S}^{0,\bullet}, D_{/S}^\bullet)$  AND  $(\mathcal{K}_{/S}^\bullet, D_{/S}^\bullet)$ . Define first the subsheaves  $\mathcal{K}_{/S}^\bullet \subset \hat{\Omega}_{/S}^\bullet[[b]]$  by the condition

$$\Omega := \sum_{j \geq 0}^{+\infty} b^j \cdot \omega_j \in \mathcal{K}_{/S}^\bullet \quad \text{when} \quad d_{/S} f \wedge \omega_0 = 0.$$

Then define  $\mathcal{K}_{/S}^{0,\bullet} = \hat{\Omega}_{/S}^\bullet[b] \cap \mathcal{K}_{/S}^\bullet$ . The differential  $D_{/S}^\bullet$  is given by

$$D_{/S}^\bullet \left( \sum_{j=0}^{+\infty} b^j \cdot \omega_j \right) = \sum_{j=0}^{+\infty} b^j \cdot (d_{/S} \omega_j - d_{/S} f \wedge \omega_{j+1}).$$

It sends  $\mathcal{K}_{/S}^p$  to  $\mathcal{K}_{/S}^{p+1}$  and also  $\mathcal{K}_{/S}^{0,p}$  to  $\mathcal{K}_{/S}^{0,p+1}$ . Moreover it satisfies  $D_{/S}^{p+1} \circ D_{/S}^p = 0$  for each  $p \geq 0$ .

Now define the action of  $a$  and  $b$  as follows :

$$\begin{aligned} a. \sum_{j=0}^{+\infty} b^j \cdot \omega_j &= \sum_{j=0}^{+\infty} b^j \cdot (f \cdot \omega_j + (j-1) \cdot \omega_{j-1}) \quad \text{with the convention} \quad \omega_{-1} = 0 \\ b. \sum_{j=0}^{+\infty} b^j \cdot \omega_j &= \sum_{j=1}^{+\infty} b^j \cdot \omega_{j-1} \end{aligned}$$

It is clear that  $\mathcal{K}_{/S}^{0,\bullet}$  is a  $\mathcal{O}_{/S}[a]$ -module, but the fact that  $\mathcal{K}_{/S}^\bullet$  is a  $\mathcal{O}_S[[a]]$ -module is a consequence of the following lemma.

**Lemma 1.2.3** *Let  $\Omega \in \mathcal{K}_{/S}^\bullet$ . Then for each  $N \in \mathbb{N}$  and each  $(h, j) \in \mathbb{N}^2, h \leq j$  there exists a polynomial of degree  $\leq N$  and of valuation  $\geq N - h$  in  $\mathbb{C}[a]$  such that the  $j$ -th component of  $a^N \cdot \Omega$  is given by*

$$(a^N \cdot \Omega)_j = \sum_{h=0}^j T_{j,h}^N(a) \cdot \omega_{j-h}. \quad (*)$$

PROOF. We shall prove the relation (\*) and the estimations on the degree and valuation on  $T_{j,h}^N$  by induction on  $N$ . We have

$$\begin{aligned} (a^{N+1}.\Omega)_j &= f.(a^N.\Omega)_j + (j-1).(a^N.\Omega)_{j-1} \\ &= \sum_{h=0}^j (a.T_{j,h}^N(a) + (j-1).T_{j-1,h-1}^N(a)).\omega_{j-h} \end{aligned}$$

with the convention  $T_{j,h}^N = 0$  if  $j < 0$  or  $h < 0$ . This gives the induction relation

$$T_{j,h}^{N+1}(a) = a.T_{j,h}^N(a) + (j-1).T_{j-1,h-1}^N(a).$$

The assertions on the degree and on the valuation are immediate.  $\blacksquare$

Now the verification of the following facts are easy

- i)  $\mathcal{K}_{/S}^\bullet$  is a  $\mathcal{A}_S$ -module.
- ii) The differential  $D_{/S}^\bullet$  is  $\mathcal{A}_S$ -linear.

**Lemma 1.2.4** *Let  $X := \sum_{j=0}^{\infty} b^j.x_j$  be a  $D_{/S}$ -closed element in  $\mathcal{K}_{/S}^p$ . Then  $[X]$  is in  $b.\mathcal{E}^p$  if and only if  $x_0$  is in  $I_{/S}^p + d_{/S}K_{/S}^{p-1}$ .*

PROOF. A relation  $X = b.Y + D_{/S}U$  with  $Y \in \mathcal{K}_{/S}^p$  and  $U \in \mathcal{K}_{/S}^{p-1}$  implies  $x_0 = d_{/S}(u_0) - d_{/S}f \wedge u_1$ , so our assertion is clear.

Conversely, if  $x_0 = d_{/S}f \wedge w + d_{/S}(v)$  for some  $w \in \hat{\Omega}_{/S}^{p-1}$  and some  $v \in K_{/S}^{p-1}$  we have  $D_{/S}(v - b.w) = d_{/S}f \wedge w + d_{/S}(v) - b.d_{/S}(w)$  and then we have  $[X] = [X - D_{/S}(v - b.w)] = b.[Y]$  where  $Y := x_1 - d_{/S}(w) + \sum_{j=2}^{\infty} b^j.x_j$ . And  $D_{/S}(b.Y) = 0$  implies  $d_{/S}f \wedge (x_1 - d_{/S}(w)) = 0$ , so  $Y$  is in  $\mathcal{K}_{/S}^{p-1}$ .  $\blacksquare$

A corollary of this lemma is the fact that we have an injective  $\mathcal{O}_S$ -linear map  $\mathcal{E}^p/b.\mathcal{E}^p \rightarrow K_{/S}^p/I_{/S}^p$ .

An other easy consequence of this lemma is that, using the hypothesis (H0) and the Nullstellensatz, we may find locally on  $\mathcal{Y}$  an integer  $N$  such that  $a^N.\mathcal{E}^p \subset b.\mathcal{E}^p$ .

**Proposition 1.2.5 (quasi-isomorphism  $u^0$ .)** *The natural map given by the inclusion  $u^0(x) = x$  is a quasi-isomorphism  $u^0 : (K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow (K_{/S}^{0,\bullet}, D_{/S})$ . It preserves also the action of  $b$  on the cohomology sheaves, and so induces isomorphisms of  $\mathcal{A}_S^0$ -modules on the cohomology sheaves.*

PROOF. The identities  $D_{/S} \circ u^0 = u^0 \circ d_{/S}$  and  $a \circ u^0 = u^0 \circ a$  are easily verified. For the action of  $b$  on cohomology, take  $x \in K_{/S}^p$  such that  $d_{/S}(x) = 0$ . Then writing  $x = d_{/S}(\xi)$  for some  $\xi \in \hat{\Omega}_{/S}^{p-1}$ , we have  $b.[x] = [d_{/S}f \wedge \xi]$ . Now we have  $b.\xi \in \mathcal{K}_{/S}^{0,p-1}$  and  $D_{/S}(b.\xi) = -d_{/S}f \wedge \xi + b.d_{/S}(\xi)$ .

So we have  $D_{/S}(b.\xi) = -d_{/S}f \wedge \xi + x.b$  in  $\mathcal{K}_{/S}^{0,p}$ . This gives  $b \circ u^0 = u^0 \circ b$  in the cohomology sheaf  $\mathcal{H}^p(\mathcal{K}_{/S}^{0,\bullet}, D_{/S})$ .

We shall prove now the injectivity of the map  $\mathcal{H}^p(u^0)$ . Let  $x$  be in  $K_{/S}^p$  such that  $d_{/S}(x) = 0$  and assume that  $u^0(x) = D_{/S}Y$  where  $Y := \sum_{j=0}^N b^j.y_j$  with  $y_0 \wedge d_{/S}f = 0$ . Then we have

$$\begin{aligned} x &= d_{/S}(y_0) - d_{/S}f \wedge y_1 & \text{and} \\ d_{/S}(y_j) &= d_{/S}f \wedge y_{j+1} \quad \forall j \geq 1 \end{aligned} \quad (\textcircled{a})$$

For  $j = N$  we have  $d_{/S}(y_N) = 0$ , so we may write (locally)  $y_N = d_{/S}(\eta_N)$  for some  $\eta_N \in \hat{\Omega}_{/S}^{p-1}$ . Now we have  $d_{/S}(y_{N-1}) = d_{/S}f \wedge y_N = -d_{/S}(d_{/S}f \wedge \eta_N)$  so we obtain  $d_{/S}(y_{N-1}) + d_{/S}f \wedge \eta_N = 0$ . Then going on in this way, we construct  $\eta_j \in \hat{\Omega}_{/S}^{p-1}$  such that

$$y_j = d_{/S}(\eta_j) - d_{/S}f \wedge \eta_{j+1} \quad \forall j \in [1, N] \quad (\textcircled{a}\textcircled{a})$$

So we get  $x = d_{/S}(y_0) + d_{/S}(d_{/S}f \wedge \eta_1)$  and this shows that  $[x] = 0$  in the cohomology of  $(K_{/S}^\bullet, d_{/S}^\bullet)$ .

To prove the surjectivity on cohomology, take  $Y \in \mathcal{K}_{/S}^{0,p}$  such that  $D_{/S}Y = 0$ . Write  $Y = \sum_{j=0}^N b^j.y_j$ ; then the relation  $(\textcircled{a})$  is satisfied for each  $j \geq 0$  and arguing as above we may find  $\eta_j \in \hat{\Omega}_{/S}^{p-1}$  such that  $(\textcircled{a}\textcircled{a})$  holds for each  $j \geq 1$  (with  $\eta_j = 0$  for  $j \geq N+1$ ). Then let  $H := \sum_{j=1}^N b^j.\eta_j$ . We obtain

$$Y - D_{/S}H = y_0 + d_{/S}f \wedge \eta_1.$$

But  $y_0 + d_{/S}f \wedge \eta_1$  is in  $K_{/S}^p \cap \text{Ker } d_{/S}$  as  $d_{/S}(y_0) = d_{/S}f \wedge d_{/S}(\eta_1)$ . So we have  $\mathcal{H}^p(u^0)[y_0 + d_{/S}f \wedge \eta_1] = [Y]$ . ■

NOTATION. We shall identify via  $\mathcal{H}(u^0)$  the cohomology sheaf  $E^p$  of the complex  $(K_{/S}^\bullet, d_{/S}^\bullet)$  to the cohomology sheaf of the complex of  $\mathcal{A}_S^0$ -modules  $(\mathcal{K}_{/S}^{0,\bullet}, D_{/S})$ . We shall note  $\mathcal{E}^p$  the cohomology sheaf of the complex  $(\mathcal{K}_{/S}^\bullet, D_{/S}^\bullet)$ .

**Proposition 1.2.6 (the map  $u^1$ .)** *The obvious map  $u^1 : (\mathcal{K}_{/S}^{0,\bullet}, D_{/S}^\bullet) \rightarrow (\mathcal{K}_{/S}^\bullet, D_{/S}^\bullet)$  induces a map  $\mathcal{H}^p(u^1)$  with kernel contained in  $\cap_{m \geq 0} b^m.E^p$ .*

*If we have  $\cap_{m \geq 0} b^m(E^p) = \{0\}$  then the image of  $\mathcal{H}^p(u^1)$  is dense for the  $b$ -adic topology of  $\mathcal{E}^p$ .*

Note that we don't know that the  $b$ -adic topology of  $\mathcal{E}^p$  is separated under the hypothesis of the lemma (which is only  $(H0)$ ).

PROOF. We begin by the kernels in cohomology. Using the quasi-isomorphism  $u^0$  we may consider only some  $x_0 \in K_{/S}^p \cap \text{Ker } d_{/S}$  such that there exists an  $Y = \sum_{j=0}^\infty b^j.y_j$  in  $\mathcal{K}_{/S}^p$  such that  $D_{/S}Y = 0$ . Then we have

$$x_0 = d_{/S}(y_0) - d_{/S}f \wedge y_1 \quad \text{and} \quad d_{/S}(y_j) = d_{/S}f \wedge y_{j+1} \quad \forall j \geq 1.$$

As  $[d_{/S}(y_j)] = b.[d_{/S}(y_{j+1})]$  for each  $j \geq 1$  we obtain that  $[x_0] = [x_0 - d_{/S}(y_0)]$  is in  $b^m(E)$  for each  $m \geq 0$ .

To see that the image is dense, let  $X \in \mathcal{K}_{/S}^p$  such that  $D_{/S}X = 0$  and fix an integer  $N$ . Let  $X = \sum_{j=0}^{\infty} b^j.x_j$  and define  $X_N := \sum_{j=0}^N b^j.x_j$ . As we have  $d_{/S}(x_j) = d_{/S}f \wedge x_{j+1} \quad \forall j \geq 0$ , we have  $D_{/S}(X_N) = d_{/S}(x_N).b^N$ . As we have  $d_{/S}(x_j) = d_{/S}f \wedge x_{j+1}$  for all  $j \geq 0$  we have  $[d_{/S}(x_N)] \in \cap_{m \geq 0} b^m.E$ ; so we may find  $z_N$  such that  $d_{/S}(z_N) = d_{/S}(x_N)$  and  $d_{/S}f \wedge z_N = 0$ . We have  $D_{/S}(b^N.z_N) = b^N.d_{/S}(x_N)$  and so  $Y_N := X_N - b^N.z_N$  is  $D_{/S}$ -closed and satisfies  $X - Y_N \in b^N.\mathcal{K}_{/S}^p$ .  $\blacksquare$

Our first important result is the extension to the  $S$ -relative case of the construction given in [B.II] theorem 2.1.1.

**Theorem 1.2.7** *In our situation ( $\textcircled{a}$ ), so under the hypothesis (H0), (H1), (H2), the map  $u := u^1 \circ u^0$  induces for each  $p \geq 0$  an isomorphism of  $\mathcal{A}_S^0$ -modules which is also  $\mathcal{O}_S[[a]]$ -linear*

$$\mathcal{H}(u)^p : E^p \longrightarrow \mathcal{E}^p.$$

Moreover we have the following properties of the  $\mathcal{A}_S$ -modules  $\mathcal{E}^p, p \geq 0$  :

- i) *There exists locally on  $\mathcal{Y}$  an integer  $N$  such that  $a^N$  kills  $A(\mathcal{E}^p)$  for each  $p \geq 0$ .*
- ii) *We have  $B(\mathcal{E}^p) = A(\mathcal{E}^p)$  for each  $p \geq 0$ .*
- iii) *There exists locally on  $\mathcal{Y}$  an integer  $N$  such that  $b^N$  kills the  $b$ -torsion  $B(\mathcal{E}^p)$  of  $\mathcal{E}^p$  for each  $p \geq 0$ .*
- iv)  *$\cap_{m \geq 0} b^m.\mathcal{E}^p = \{0\}$  for each  $p \geq 0$ .*
- v) *For each  $y \in \mathcal{Y}$  the germ  $\mathcal{E}_y^{p+1}/A(\mathcal{E}_y^{p+1})$  may be embedded in some  $\mathcal{A}_{S,s_0}$ -module of asymptotic expansions  $\Xi_{\Lambda, S, s_0}^{(n)} \otimes V$  where  $s_0 := \pi(y)$ ,  $V := H^p(F_y, \mathbb{C})$  and  $\Lambda \subset ]0, 1[ \cap \mathbb{Q}$  is a finite subset. Here  $F_y$  is the Milnor fiber of  $f$  at  $y$ , which coincides with the Milnor fiber at  $y$  of  $f$  restricted to  $\pi^{-1}(s_0)$ , thanks to the hypothesis (H2). The set  $\{\exp(2i\pi.\lambda), \lambda \in \Lambda\}$  is the spectrum of the monodromy acting on  $H^p(F_y, \mathbb{C})$ .*

REMARK. This last fact v) implies that the  $(\mathcal{O}_S[[a]])_{s_0}$ -module  $E_y^p/A(E_y^p)$  (resp.  $(\mathcal{O}_S[[b]])_{s_0}$ -module  $E_y^p/A(E_y^p)$ ) is finitely generated by noetherianity (see the point 1. in the begining of section 1.2.)  $\square$

The proof of the theorem 1.2.7 will be given in the subsection 1.4.

### 1.3 The hypotheses (H1) and (H2).

Let me recall the following standard definitions which makes precise the hypothesis (H1).

**Definition 1.3.1** *We shall say that the divisor  $\mathcal{Y} := \{f = 0\}$  in the  $S$ -manifold  $\mathcal{X}$  is  $S$ -normal crossing if near each point  $y \in \mathcal{Y}$  there exists a system of local  $S$ -coordinate in  $\mathcal{X}$  such that we have  $f(s, z) = z^\alpha$  where  $\alpha$  is in  $\mathbb{N}^{n+1}$ .*

**Definition 1.3.2** *We shall say that the holomorphic function  $f : \mathcal{X} \rightarrow D$  on the  $S$ -manifold  $\mathcal{X}$  admits a **simultaneous desingularisation over  $S$**  if there exists a proper  $S$ -modification  $\tau : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  where  $\tilde{\mathcal{X}}$  is a  $S$ -manifold and such that  $\tilde{\mathcal{Y}} := \{f \circ \tau = 0\}$  is a  $S$ -normal crossing divisor in  $\tilde{\mathcal{X}}$ .*

Note that by a  $S$ -proper modification we mean that  $\tau$  is a proper holomorphic map such that there exists a closed analytic subset  $T \subset \mathcal{X}$  with the following properties :

- i) For each  $s \in S$  the fiber  $T_s$  has empty interior in  $\mathcal{X}_s := \pi^{-1}(s)$ .
- ii) The map  $\tau$  induces an isomorphism of  $\tilde{\mathcal{X}} \setminus \tau^{-1}(T)$  to  $\mathcal{X} \setminus T$ .

The above definition, which makes precise the hypothesis (H2) is may-be less standard.

**Definition 1.3.3** *In the situation (@) we say that we have locally on  $\mathcal{Y}$  a Milnor fibration for  $f$  is for each point  $y \in \mathcal{Y}$  there exists an open neighbourhood  $S'$  of  $\pi(y)$  is  $S$  and a  $S'$ -relative chart  $\varphi : X \simeq S' \times U$  where  $U$  is an open neighbourhood of the origin in  $\mathbb{C}^{n+1}$ , with the following property:*

- For any  $\varepsilon > 0$  sufficiently small, there exists  $\eta > 0$  such that the restriction of  $\pi \times f$  to  $X \cap f^{-1}(D_\eta) \cap \varphi^{-1}(S' \times B_\varepsilon)$  induces a  $S'$ -relative  $\mathcal{C}^N$ -fibration for  $N \gg n$  from the complement of  $\mathcal{Y}$  to  $S' \times D_\eta^*$  with fiber a manifold  $F_y$  with finite dimensional complex homology. Moreover we ask that this  $\mathcal{C}^N$ -fibration is independant of the choices of  $\varepsilon \ll 1$  and  $\eta \ll \varepsilon$ .

Of course, for  $S = \{0\}$  this is always satisfied thanks to Milnor [Mi.68]. In general this is a quite strong condition on the situation (@) which implies, for instance, the fact that the  $S$ -relative cohomology of the fibers of  $f$  define a local system on  $S \times D^*$  near  $\pi(y) \times f(y)$ . The corresponding relative Gauss-Manin meromorphic vector bundle has a meromorphic connection which has, along  $S \times \{0\}$ , a regular singularity thanks to [D. 70].

QUESTION. Is the condition (H1) implies the condition (H2) ?

### 1.3.1 Use of (H2).

We shall use this hypothesis (H2) via the following proposition :

**Proposition 1.3.4** *In the situation (ⓐ) with the hypotheses (H0), (H1) and (H2) the cohomology sheaves  $E^\bullet$  of the complex  $(K_{/S}^\bullet, d_{/S}^\bullet)$  are  $b$ -separated, that is to say satisfy  $\cap_{m \geq 0} b^m \cdot E^\bullet = \{0\}$ .*

Remark that, as  $\cap_{m \geq 0} b^m \cdot E^\bullet$  is a sub- $\mathcal{O}_S[[a]]$ -module stable by  $b$ , it is enough to prove the following key properties of  $E := E^\bullet$  :

- i) There exists locally on  $\mathcal{Y}$  and integer  $N$  such that  $a^N \cdot A(E) = \{0\}$ .
- ii)  $B(E) \subset A(E)$ .
- iii)  $\cap_{m \geq 0} b^m \cdot E \subset A(E)$ .

The proof of the property i) is given, under hypothesis (H0) and (H1) in subsection 1.4. The properties ii) and iii) are easy consequences of the next lemma and its corollary 1.3.6. Then the proof of the proposition is achieved as follows :

Let  $\tilde{A}(E)$  be the subset of  $A(E)$  consisting of  $x \in E$  such that  $\mathcal{A}_S^0 \cdot x \subset A(E)$ . Then from ii) we have  $B(E) \subset \tilde{A}(E)$  and using the lemma 1.4.1 we obtain, thanks to i) and ii), that  $b^{2N} \cdot \tilde{A}(E) = \{0\}$  and  $B(E) = \tilde{A}(E)$ , because on  $\tilde{A}(E)/B(E)$  the map  $b$  is injective and satisfies  $b^{2N} = 0$ . Now the remark above and iii) give that  $\cap_{m \geq 0} b^m \cdot E \subset \tilde{A}(E) = B(E)$ . The fact that  $b^{2N} \cdot B(E) = \{0\}$  implies  $\cap_{m \geq 0} b^m \cdot E = \{0\}$ . ■

So the proof of the proposition is a consequence of the following lemma and its corollary (compare with [M.74]).

**Lemma 1.3.5** *In the situation (ⓐ) with the hypothesis (H0) and (H2) let  $y \in \mathcal{Y}$  and denote  $F_y$  the Milnor fiber of  $f$  at  $y$ . Put  $\pi(y) := s_0$ . We have a natural  $(\mathcal{O}_S[[a]])_{s_0}$ -linear map*

$$\text{dev}_y : E_y^{p+1} \rightarrow (\Xi_{\Lambda, S}^{(n)})_{s_0} \otimes_{\mathbb{C}} H^p(F_y, \mathbb{C})$$

*which commutes with the actions of  $b$ . The kernel of  $\text{dev}_y$  is exactly the germ at  $y$  of the  $a$ -torsion  $A(E^{p+1})$  of  $E^{p+1}$ .*

PROOF. Let  $[\omega] \in E_y^{p+1}$  where  $\omega$  is a local section of  $K_{/S}^{p+1}$  which is  $d_{/S}$ -closed, and, for  $\gamma \in H_p(F_y, \mathbb{C})$  let  $(\gamma_{s,z}), (s, z) \in S \times D^*$  be the horizontal multivalued family of  $p$ -cycles in the fibers deduced from  $\gamma$ , and define

$$\text{dev}_y[\omega] := \left[ \int_{\gamma_{s,z}} \omega / d_{/S} f \right] \in (\Xi_{\Lambda, S}^{(n)})_{s_0}$$

where  $(\Xi_{\Lambda, S}^{(n)})_{s_0}$  is the germ at  $s_0$  of multivalued asymptotic expansions (in the local coordinate  $z$  in  $D$ , see the example in section 1.2. ) with  $\exp(2i\pi \cdot \Lambda)$  is the

set of eigenvalues of the local monodromy at  $y$  and where  $n = \dim_{\mathbb{C}} F_y$  bounds the degrees of the logarithms (in  $z$ ). This commutes with the action of  $\mathcal{A}_{S,s_0}$ . Then the fact that  $[\omega]$  is in the kernel of  $dev_y$  is clearly equivalent to the fact that  $[\omega]$  induces the zero cohomology class on the generic fibers of the  $S$ -relative Milnor fibration of  $f$  at  $y$ . So this exactly means that  $[\omega]$  is in  $A(E^{p+1})_y$ . ■

**Corollary 1.3.6** *In the situation  $(\textcircled{a})$  with the hypothesis (H0) and (H2) the properties ii) and iii) of the proposition 1.3.4 are satisfied.*

PROOF. It is enough to prove these properties for  $E_y^p$  for any  $p \geq 0$  and any  $y \in \mathcal{Y}$ . But they are immediate consequence of the fact that the  $\mathcal{A}_{S,s_0}$ -module  $(\Xi_{\Lambda,S}^{(n)})_{s_0} \otimes_{\mathbb{C}} H^p(F_y, \mathbb{C})$  has no  $a$ -torsion and is  $b$ -separated. ■

We have another important consequence of this lemma 1.3.5

**Corollary 1.3.7** *In the situation  $(\textcircled{a})$  with the hypothesis (H0) and (H2) the  $(\mathcal{O}_S[b])_{s_0}$ -module  $E_y^{p+1}/A(E_y^{p+1})$  is  $b$ -complete, for each  $y \in \pi^{-1}(s_0) \cap \mathcal{Y}$ .*

PROOF. Remark first that  $E_y^{p+1}/A(E_y^{p+1})$  is an  $(\mathcal{O}_S[[a]])_{s_0}$ -module of finite type using the noetherianity of this ring and the fact that the  $(\mathcal{O}_S[[a]])_{s_0}$ -module  $(\Xi_{\Lambda,S}^{(n)})_{s_0} \otimes_{\mathbb{C}} H^p(F_y, \mathbb{C})$  is of finite type. Remark then that the  $a$ -filtration and  $b$ -filtration on  $(\Xi_{\Lambda,S}^{(n)})_{s_0} \otimes_{\mathbb{C}} H^p(F_y, \mathbb{C})$  are equivalent. So a Cauchy sequence for the  $b$ -filtration in  $E_y^{p+1}/A(E_y^{p+1})$  gives a Cauchy sequence for the  $a$ -filtration in  $(\Xi_{\Lambda,S}^{(n)})_{s_0} \otimes_{\mathbb{C}} H^p(F_y, \mathbb{C})$  and then converges for the  $a$ -filtration to an element which is in  $E_y^{p+1}/A(E_y^{p+1})$ . Then it is enough to prove that the convergence for the  $a$ -filtration in  $E_y^{p+1}/A(E_y^{p+1})$  implies the convergence for the  $b$ -filtration. But this is a simple consequence of the hypothesis (H0) because we have, for each  $p \geq 0$ , locally on  $\mathcal{Y}$ , an integer  $N$  such that

$$a^N \cdot E^p \subset b \cdot E^p$$

as an easy consequence of the Nullstellensatz : as  $\{d_{/S}f = 0\} \subset \{f = 0\}$  the support of the coherent sheaf  $K_{/S}^p/I_{/S}^p$  is contained in  $\{f = 0\}$ . So there exists locally on  $\mathcal{Y}$  an integer  $N$  such that  $a^N \cdot E^p \subset b \cdot E^p$ . ■

### 1.3.2 Use of (H1).

The hypothesis (H1) will be usefull thanks to the following proposition.

**Proposition 1.3.8** *Let  $\mathcal{X} \xrightarrow{\pi} S$  be a  $S$ -relative complex manifold and let  $f : \mathcal{X} \rightarrow D$  be an holomorphic function satisfying the following assumptions :*

- i)  $\{d_{/S}f = 0\} \subset f^{-1}(0)$ .

ii) Locally on  $\mathcal{X}$  the function  $f$  admits a  $S$ -desingularization.

Then there exists locally on  $\mathcal{X}$  an integer  $N$  such that we have  $a^N.A(E^\bullet) = 0$  where  $E^\bullet$  are the cohomology sheaves of the complex  $(K_{f/S}^\bullet, d_{/S})$ .

First recall the following lemma (for a proof see for instance [B-S 04] proposition 2.2.)

**Lemma 1.3.9** *We consider the situation  $(\textcircled{a})$  where we assume that  $\mathcal{X} := S \times U$  where  $U \subset \mathbb{C}^{n+1}$  is a open neighbourhood of the origin and where  $\pi$  is the first projection. We assume that  $f(s, z) = z^\alpha$  where  $(z_0, \dots, z_n)$  are the coordinates on  $\mathbb{C}^{n+1}$  and where  $\alpha \in \mathbb{N}^{n+1}$ . Then the  $a$ -torsion  $A(E^\bullet)$  of the cohomology sheaves of the complex  $(K_{f/S}^\bullet, d_{/S})$  is 0 and  $b^{-1}.a$  acts bijectively on the cohomology sheaves  $E^\bullet$ .*

PROOF OF THE PROPOSITION. As the problem is local on  $\mathcal{X}$  we may assume the following facts :

- i)  $\mathcal{X} = S \times U$  where  $S$  is Stein and where  $U$  is a polydisc in  $\mathbb{C}^{n+1}$ .
- ii) We have a  $S$ -proper modification  $\tau : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  such that the zero set of  $\tilde{f} := f \circ \tau$  defines locally on  $\tilde{\mathcal{X}}$  a  $S$ -relative normal crossing divisor.

We have an "edge" map, for each open set  $\mathcal{X}'$  in  $\mathcal{X}$ , where  $\tilde{\mathcal{X}}' = \tau^{-1}(\mathcal{X}')$ ,

$$\varepsilon : H^q(\Gamma(\mathcal{X}', K_{f/S}^\bullet, d_{/S})) \rightarrow \mathbb{H}^q(\tilde{\mathcal{X}}', (K_{\tilde{f}/S}^\bullet, d_{/S})) \quad (\textcircled{a})$$

which is consequence of the equalities  $\tau_*(K_{\tilde{f}/S}^\bullet) = K_{f/S}^\bullet$  which are compatible with the  $S$ -relative de Rham differential. These maps define a morphism, denoted  $e$ , of the corresponding  $\mathcal{O}_S[[a]]$ -sheaves on  $\mathcal{X}$ .

CLAIM. There exists locally on  $\mathcal{X}$  an integer  $N_1$  such the sheaf map  $e$  induces an isomorphism between  $a^{N_1}.\mathcal{H}^q(K_{f/S}^\bullet, d_{/S})$  onto  $a^{N_1}.\mathcal{H}^q(R\tau_*(K_{\tilde{f}/S}^\bullet, d_{/S}))$  where  $R\tau_*(K_{\tilde{f}/S}^\bullet, d_{/S})$  is the direct image in the category of complexes of sheaves (compare with [G.65]).

PROOF OF THE CLAIM. First remark that for each  $i \geq 1$  the sheaves  $R^i\tau_*(K_{\tilde{f}/S}^\bullet)$  are  $a$ -completions of coherent sheaves on  $\mathcal{X}$  and supported in  $\mathcal{Y} := f^{-1}(0)$ . So on any relatively compact Stein open set in  $\mathcal{Y}$  we may find an integer  $N_1$  such that  $f^{N_1}.R^i\tau_*(K_{\tilde{f}/S}^\bullet) = 0$ . This means that, in the spectral sequence

$$E_2^{p,q} := \mathcal{H}^q(R^p\tau_*(K_{\tilde{f}/S}^\bullet, d_{/S}))$$

which converges to  $\mathcal{H}^m(\tau_*(K_{\tilde{f}/S}^\bullet, d_{/S}))$ , the sub-spectral sequence  $a^{N_1}.E_2^{p,q}$  which converges to  $a^{N_1}.\mathcal{H}^q(R\tau_*(K_{\tilde{f}/S}^\bullet, d_{/S}))$ , degenerates at  $E_2$  and gives that  $e$  induces the isomorphism of the claim. ■



END OF THE PROOF OF 1.3.8. Now it is enough to prove that there exists locally on  $\mathcal{X}$  an integer  $N_2$  such that if  $x \in \mathcal{H}^q(K_{\tilde{f}/S}^\bullet, d_{/S}^\bullet)$  is in the  $a$ -torsion of this sheaf, then its image in  $\mathcal{H}^q(R\tau_*(K_{\tilde{f}/S}^\bullet, d_{/S}^\bullet))$  is killed by  $a^{N_2}$  in this sheaf, because this allows to conclude by taking  $N := N_1 + N_2$ .

This is a easy exercise on hypercohomology because at each step we know that  $x$  is locally killed on  $\tilde{\mathcal{X}}$  by  $a$ , thanks to the proposition 1.3.8, and that the obstruction for the next step is in the sheaf  $R^i\tau_*(K_{\tilde{f}/S}^{q-i})$  so it is killed by  $a^{N_1}$ .

So we may take  $N_2 = (n + 1).(N_1 + 1)$ . ■

## 1.4 The proof of the theorem 1.2.7

### 1.4.1 Properties of $E^p$ .

We shall begin by the proof that the cohomology sheaves  $E^p$  satisfy the properties i) to v).

The property i) is proved in proposition 1.3.8, the properties iv) is obtained in the proposition 1.3.4 and the property v) in the lemma 1.3.5 and its corollary. The inclusion  $B(E^p) \subset A(E^p)$  is obtained in proposition 1.3.4 and the lemma 1.3.5 shows that  $\tilde{A}(E^p) = A(E^p)$ . Then the end of the proof of the proposition 1.3.4 gives  $B(E^p) = \tilde{A}(E^p)$ .

The only point left is the proof of property iii). But it is an easy consequence of the following lemma from [B.06] lemma 2.1.2. with the property i).

**Lemma 1.4.1** *The commutation relation  $a.b - b.a = b^2$  implies for each integer  $N$  the formula*

$$N!.b^{2N} = \sum_{j=0}^N (-1)^j \cdot \frac{N!}{(N-j)! \cdot j!} \cdot b^j \cdot a^N \cdot b^{N-j}.$$

It implies that an  $\mathcal{A}_S^0$ -module which is killed by  $a^N$  is killed by  $b^{2N}$ .

Now  $A(E^p)$  is killed (locally on  $\mathcal{Y}$ ) by some  $a^{N_0}$ . So the same is true for  $B(E^p)$  which is a  $\mathcal{A}_S^0$ -module. This proves property iii) and so the proof of our first step is complete. ■

### 1.4.2 Properties of $\mathcal{E}^p$ .

**Lemma 1.4.2** *Assume that we have  $b^N \cdot B(E^p) = 0$  then we have  $b^{N+1} \cdot B(\mathcal{E}^p) = 0$ .*

PROOF. Let  $X = \sum_{j=0}^{\infty} b^j \cdot x_j$  such that  $b^M \cdot X = D_{/S}W$  with  $M \geq N + 1$ .

Define  $U = \sum_{j=0}^{M-1} b^j \cdot w_j$  and  $V = \sum_{j=M}^{\infty} b^j \cdot w_j$ . Then we have

$$b^M \cdot X = D_{/S}(U) + b^{M-1} \cdot D_{/S}(b \cdot V) \tag{@}$$

and this implies  $D_{/S}(U) - b^{M-1} \cdot (d_{/S}f \wedge w_M) = 0$  and so

$$b^{M-1} \cdot [b \cdot X - D_{/S}(b \cdot V)] = b^{M-1} \cdot d_{/S}f \wedge w_M$$

and  $b.X = d_{/S}f \wedge w_M + D_{/S}(b.V)$ . Note that the relation

$$x_0 = -d_{/S}f \wedge w_{M-1} + d_{/S}(w_M),$$

consequence of  $(\textcircled{a})$ , implies  $d_{/S}f \wedge d_{/S}(w_M) = 0$ , so we have

$$\mathcal{H}^p(u)[b.d_{/S}(w_M)] = [b.X].$$

So  $b^{M-1}.[d_{/S}(w_M)]$  is in the kernel of  $\mathcal{H}^p(u)$  which is zero from our assumption. Then  $b^N.[d_{/S}(w_M)] = 0$  and so  $b^{N+1}.[X] = 0$  in  $\mathcal{E}^p$ .  $\blacksquare$

**Lemma 1.4.3** *Assume now that we have, locally on  $\mathcal{Y}$  an integer  $N$  such that  $b^N.B(\mathcal{E}^p) = 0$  then we have  $\cap_{m \geq 0} b^m(\mathcal{E}^p) = \{0\}$ .*

PROOF. Note first that the assertion is local on  $\mathcal{Y}$ . Let  $\Omega_0 \in \mathcal{K}_{/S}^p$  such that  $D_{/S}\Omega_0 = 0$  and such that for each  $m \geq 0$  there exists  $\Omega_m \in \mathcal{K}_{f/S}^p$  and  $U_m \in \mathcal{K}_{/S}^{p-1}$  such that  $D_{/S}\Omega_m = 0$  and  $b^m.\Omega_m = \Omega_0 + D_{/S}U_m$ . Then we have for each  $m \geq 0$

$$b^m.(\Omega_m - b.\Omega_{m+1}) = D_{/S}(U_m - U_{m+1})$$

and this means that  $[V_m] := [\Omega_m] - b.[\Omega_{m+1}]$  is in  $B(\mathcal{E}^p)$ . As we know, thanks to our assumption, that there exists locally on  $\mathcal{Y}$  an integer  $N$  such that that  $b^N.B(\mathcal{E}^p) = \{0\}$ , we obtain

$$\sum_{m=0}^{\infty} b^m.V_m = \sum_{m=0}^{N-1} b^m.V_m + D_{/S}W = \sum_{m=0}^{\infty} (b^m.\Omega_m - b^{m+1}.\Omega_{m+1}) = \Omega_0$$

where we have defined  $W := \sum_{m=2N}^{\infty} b^m.W_m$  using the fact that for each  $m \geq N$  we may write  $b^m.V_m = D_{/S}W_m$  for some  $W_m \in \mathcal{K}_{/S}^{p-1}$ , and where we use also that on a given open set  $U \subset \mathcal{Y}$  the  $\mathcal{O}_S(U)[[b]]$ -module  $\Gamma(U, \mathcal{K}_{/S}^\bullet)$  is complete for its  $b$ -adic topology. So we find that  $[\Omega_0]$  is in  $B(\mathcal{E}^p)$ . But then any  $[\Omega_m]$  is in  $B(\mathcal{E}^p)$  and for  $m \geq N$  this gives  $b^m.[\Omega_m] = [\Omega_0] = [0]$ .  $\blacksquare$

### 1.4.3 The $b$ -completion

**Lemma 1.4.4** *Let  $[X] \in \mathcal{E}^p$  such that  $b.[X] = 0$ . Then there exists  $[x] \in E^p$  such that  $b.[x] = 0$  and  $\mathcal{H}^p(u)[x] = [X]$ .*

PROOF. Write  $X = \sum_{j=0}^{\infty} b^j.x_j$ . Now if  $b.X = D_{/S}U$  where  $U := \sum_{j=0}^{\infty} b^j.u_j$  is a local section of  $\mathcal{K}_{/S}^{p-1}$  we shall have

$$0 = d_{/S}(u_0) - d_{/S}f \wedge u_1 \quad \text{and} \quad u_0 \in \mathcal{K}_{/S}^{p-1}.$$

So  $d_{/S}f \wedge d_{/S}(u_1) = 0$  and  $d_{/S}(u_1)$  is in  $K_{/S}^p \cap \text{Ker } d_{/S}$ . Then we have in  $E^p$  the relation  $b.[d_{/S}(u_1)] = [d_{/S}(u_0)] = 0$ .

Write  $U = u_0 + b.V$  and  $V = u_1 + b.W$ . Then  $b.W$  is in  $\mathcal{K}_{/S}^{p-1}$  and we have

$$D_{/S}(U - u_0) = b.X - d_{/S}u_0 \quad \text{and then} \quad X = D_{/S}V$$

But  $V \notin \mathcal{K}_{/S}^{p-1}$  in general. Nevertheless  $X - d_{/S}u_1 = D_{/S}(b.W)$  with  $[d_{/S}(u_1)]$  is the image by  $\mathcal{H}^p(u)$  of the class  $d_{/S}(u_1)$  in  $E^p$  which is in  $\text{Ker } b$ . This concludes the proof.  $\blacksquare$

**Corollary 1.4.5** *For any integer  $N \geq 1$  the map  $\mathcal{H}^p(u)$  sends  $\text{Ker } b^N$  in  $E^p$  onto  $\text{Ker } b^N$  in  $\mathcal{E}^p$ .*

PROOF. As the case  $N = 1$  is proved in the previous lemma it is enough to prove that if the statement is true for  $N \geq 1$  (and for  $N = 1$ ) then it is true for  $N + 1$ . Let  $X \in \text{Ker } b^{N+1}$  in  $\mathcal{E}^p$ . Then  $b.X$  may be written  $\mathcal{H}^p(u)(x)$  where  $x$  is in  $E^p$  and satisfies  $b^N.x = 0$ . Now, as  $\mathcal{H}^p(u)(x)$  is in  $b.\mathcal{E}^p$  we may write  $x = b.y$  for some  $y \in E^p$ , thanks to the lemma 1.4.6. Then  $Y := \mathcal{H}^p(u)(y)$  satisfies  $b.(X - Y) = 0$  and so we may find  $t \in E^p$  such that  $b.t = 0$  and  $\mathcal{H}^p(u)(t) = X - Y$ . Then  $y + t$  is in  $\text{Ker } b^{N+1}$  and is sent on  $X$  by  $\mathcal{H}^p(u)$ .  $\blacksquare$

**Lemma 1.4.6** *If  $x \in E^p$  is such that  $\mathcal{H}(u)[x]$  is in  $b^k.\mathcal{E}^p$  then  $x$  is in  $b^k.E^p$ .*

PROOF. We begin by the case  $k = 1$ . Then we may write

$$x = b.X + D_{/S}U$$

and this implies  $x = d_{/S}(u_0) - d_{/S}f \wedge u_1$ . So  $[d_{/S}(u_1)]$  is in  $E^p$  and  $[d_{/S}(u_0)] = 0$  in  $E^p$ . This gives  $[x] = -b[d_{/S}(u_1)]$ , concluding this case.

Assume now  $k \geq 1$  and that the case  $k$  is proved. Then consider  $[x] \in E^p$  such that

$$x = b^{k+1}.X + D_{/S}U.$$

Then the inductive assumption gives  $[y] \in E^p$  such that  $[x] = b^k.[y]$ . Then we have

$$b^k.y = b^{k+1}.X + b^k.D_{/S}(b.W) + D_{/S}V$$

where we put  $U = V + b^{k+1}.W$ . This implies

$$b^k.y = b^k.[d_{/S}(u_k)] + b^{k+1}.X + b^k.D_{/S}(b.W)$$

and so

$$y - d_{/S}u_k = b.X + D_{/S}(b.W).$$

But our formula above implies  $d_{/S}(u_j) = d_{/S}f \wedge u_{j+1} \quad \forall j \in [0, k-1]$  and  $d_{/S}(u_k) - d_{/S}f \wedge u_{k+1} = y$  showing that  $d_{/S}(u_j)$  is in  $E^p$  for all  $j \in [0, k]$  and that

$0 = [d_{/S}(u_0)] = b.[d_{/S}(u_1)] = \dots = b^k.[d_{/S}(u_k)]$  in  $E^p$ . The case  $k = 1$  applied to  $y - d_{/S}(u_k)$  gives  $[z] \in E^p$  such that  $y = d_{/S}(u_k) + b.z$  and then applying  $b^k$  we find that  $[x] = b^{k+1}.[z]$  as  $b^k.[d_{/S}(u_k)] = 0$ . ■

REMARK. This lemma implies that the  $b$ -filtration on  $E^p$  is induced by the  $b$ -filtration on  $\mathcal{E}^p$ . □

#### 1.4.4 The end of the proof.

As we have proved the properties i) to v) for the sheaves  $E^p$  the last point to prove is that the  $\mathcal{A}_S^0$ -linear map  $\mathcal{H}^p(u) : E^p \rightarrow \mathcal{E}^p$  is bijective for each  $p \geq 0$ . But we already prove its injectivity thanks to the proposition 1.2.6 and the proposition 1.3.4. Now the surjectivity comes from the  $b$ -density of its image proved in the proposition 1.2.6 and the following facts :

1. The map  $\mathcal{H}^p(u)$  induce a bijective map between  $B(E^p)$  and  $B(\mathcal{E}^p)$ .
2. The  $b$ -filtration on  $E^p$  is induced by the  $b$ -filtration of  $\mathcal{E}^p$ .
3. For each  $y \in \mathcal{Y}$  the quotient  $(E^p/B(E^p))_y$  is  $b$ -complete.
4. For each  $p \geq 0$   $\mathcal{E}^p$  is  $b$ -separated (and so is  $b$ -complete).

The point 1) is proved in the corollary 1.4.5.

The point 2) is proved in the lemma 1.4.6.

The point 3) is proved in the corollary 1.3.7.

The point 4) is proved in the lemmata 1.4.2 and 1.4.3 thanks to the property iii) for  $E^p$  already obtained. This conclude the proof of the theorem 1.2.7. ■

## 1.5 Functorial properties.

**Proposition 1.5.1** *In our situation ( $\textcircled{\text{A}}$ ) with the hypothesis (H0) the complex of  $\mathcal{A}_S$ -modules constructed in the theorem 1.2.7 has the following functorial properties:*

1. *For any holomorphic map  $\varphi : T \rightarrow S$  from a reduced complex space  $T$  there exists a natural pull-back map of complexes over the pull-back of sheaves of algebras  $\varphi^* : \mathcal{A}_S \rightarrow \mathcal{A}_T$*

$$\varphi^* : (\mathcal{K}_{f/S}^\bullet, D_{f/S}^\bullet) \rightarrow (\mathcal{K}_{g/T}^\bullet, D_{g/T}^\bullet)$$

*where  $g : \mathcal{X} \times_S T \rightarrow D$  is the composition of the first projection of the fiber product with  $f$ . If  $\varphi$  is a proper embedding, then  $\varphi^*$  is given by the natural map to the tensor product on  $\mathcal{O}_S$  by  $\mathcal{O}_T$ .*

2. For any  $S$ -holomorphic map  $\psi : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  between two  $S$ -manifolds and any holomorphic function  $f : \mathcal{X}_2 \rightarrow D$  satisfying (H0) and such that  $f \circ \psi$  satisfies (H0), we have a natural pull-back of  $\mathcal{A}_S$ -complexes

$$\psi^* : (\mathcal{K}_{f/S}^\bullet, D_{f/S}^\bullet) \rightarrow (\mathcal{K}_{g/S}^\bullet, D_{g/S}^\bullet)$$

where  $g := f \circ \psi$ .

These two "pull-back" maps are of course compatible with the the obvious pull-back of relative differential forms and so with the natural map

$$u : (K_{/S}^\bullet, d_{/S}^\bullet) \rightarrow (\mathcal{K}_{/S}^\bullet, D_{/S}^\bullet)$$

defined in the theorem 1.2.7.

PROOF. The verification is an easy exercise left to the reader. ■

## 2 The finiteness theorem.

We assume in this section that we are in the situation (@) with the hypotheses (H0), (H1), (H2).

**Definition 2.0.2** A left  $\mathcal{A}_S$ -module  $E$  on  $S$  is called  $S$ -**small** when the following conditions hold

- 1)  $\cap_{m \geq 0} b^m.E \subset A(E)$ .
- 2)  $B(E) \subset A(E)$ .
- 3) Locally on  $S$  there exists an integer  $N$  such that  $a^N.A(E) = 0$ .
- 4) The sheaves  $\text{Ker } b$  and  $\text{Coker } b$  are  $\mathcal{O}_S$ -coherent.

**Lemma 2.0.3** If a left  $\mathcal{A}_S$ -module  $E$  is  $S$ -small then  $E$  is a finite type  $\mathcal{O}_S[[b]]$ -module such that its  $b$ -torsion  $B(E)$  is  $\mathcal{O}_S$ -coherent.

PROOF. Using the lemma 1.4.1 the properties 2) and 3) imply that we have the relation  $b^{2N}.B(E) = 0$ .

We want to show now that the sheaves  $\text{Ker } b^\nu$  are  $\mathcal{O}_S$ -coherent, for all  $\nu \geq 1$ . As this is true by condition 4) for  $\nu = 1$ , assume that it is proved for  $\nu \geq 1$  and let us prove that  $\text{Ker } b^{\nu+1}$  is  $\mathcal{O}_S$ -coherent. Consider the obvious  $\mathcal{O}_S$ -linear map  $\text{Ker } b^\nu \rightarrow E/b.E$ , between two  $\mathcal{O}_S$ -coherent sheaves. Its kernel is coherent, and as we have  $\text{Ker } b^\nu \cap b.E = b(\text{Ker } b^{\nu+1})$  we obtain the coherence of  $b(\text{Ker } b^{\nu+1})$ . But now the exact sequence of  $\mathcal{O}_S$ -modules

$$0 \rightarrow \text{Ker } b \rightarrow \text{Ker } b^{\nu+1} \xrightarrow{b} b(\text{Ker } b^{\nu+1}) \rightarrow 0$$

gives the coherence of  $\text{Ker } b^{\nu+1}$ . This implies the coherence of  $B(E)$  thanks to our first remark.

Define now  $F := E/B(E)$ . Looking at the exact sequence of  $\mathcal{O}_S[[b]]$ -modules

$$0 \rightarrow B(E) \rightarrow E \rightarrow E/B(E) \rightarrow 0$$

we deduce the  $\mathcal{O}_S$ -coherence of  $F/b.F$  as follows :  
as  $F/b.F \simeq E/B(E) + b.E$  the exact sequence

$$0 \rightarrow B(E)/B(E) \cap b.E \rightarrow E/b.E \rightarrow F/b.F \rightarrow 0$$

and the coherence of  $B(E)/B(E) \cap b.E$  because  $B(E) \cap b.E = b.B(E) \simeq B(E)/\text{Ker } b$  is  $\mathcal{O}_S$ -coherent .

But now the condition 1) implies that  $F$  is  $b$ -complete and without  $b$ -torsion. So it is now standard to prove that  $F$  is a finite type  $\mathcal{O}_S[[b]]$ -module. We conclude that  $E$  is a finite type  $\mathcal{O}_S[[b]]$ -module.  $\blacksquare$

REMARK. We have proved that  $F/b.F$  is also  $\mathcal{O}_S$ -coherent. Then there exists a Zariski dense open set  $S' \subset S$  where  $F/b.F$  is locally free, and, as  $F$  has no  $b$ -torsion and is  $b$ -complete, it will be locally free over  $\mathcal{O}_S[[b]]$  on  $S'$ . On such an open set,  $E$  is locally a direct sum of its  $b$ -torsion with a free finite type  $\mathcal{O}_S[[b]]$ -module.  $\square$

**Definition 2.0.4** We shall say that a left  $\mathcal{A}_S$ -module  $E$  is **geometric** when  $E$  is  $S$ -small and when it associated family of  $(a,b)$ -module  $E/B(E)$  has geometric fibers at each point of  $S$ .

Our main result is the following finiteness theorem, which shows that the Gauss-Manin connection produces in our situation a  $\mathcal{A}_S$ -module which is geometric .

**Theorem 2.0.5** Let  $S$  be a reduced complex space and let  $\mathcal{X}$  be a  $S$ -relative manifold of pure relative dimension  $n + 1$ . Let  $f := \mathcal{X} \rightarrow D$  be a holomorphic function on an open disc  $D$  in  $\mathbb{C}$  with center 0. Assume that the properties (H0), (H1), (H2) are satisfied and that  $\mathcal{Y}$  is  $S$ -proper. Then the complex of  $\mathcal{A}_S$ -modules

$$R\pi_*(\mathcal{K}_{f/S}^\bullet, D_{f/S}^\bullet)$$

has cohomology sheaves which are geometric  $\mathcal{A}_S$ -modules for any degree.

PROOF. To show that  $E := \mathcal{H}^p[R\pi_*(\mathcal{K}_{f/S}^\bullet, d_{f/S}^\bullet)]$  is small, it is enough to prove that  $E$  satisfies the condition 4) of the definition 2.0.2, as the properties 1) 2) and 3) are given by the theorem 1.2.7.

Consider now the long exact sequence of hypercohomology of the exact sequence of complexes

$$0 \rightarrow (I_{f/S}^\bullet, d_{f/S}^\bullet) \rightarrow (K_{f/S}^\bullet, d_{f/S}^\bullet) \rightarrow ([K_{f/S}/I_{f/S}]^\bullet, d_{f/S}^\bullet) \rightarrow 0.$$

It contains the exact sequence

$$\begin{aligned} \cdots \rightarrow \mathcal{H}^{p-1}[R\pi_*([K_{/S}/I_{/S}]^\bullet, d_{/S}^\bullet)] &\rightarrow \mathcal{H}^p[R\pi_*(I_{/S}^\bullet, d_{/S}^\bullet)] \xrightarrow{R^p(i)} \\ &\rightarrow \mathcal{H}^p[R\pi_*(K_{/S}^\bullet, d_{/S}^\bullet)] \rightarrow \mathcal{H}^p[R\pi_*([K_{/S}/I_{/S}]^\bullet, d_{/S}^\bullet)] \rightarrow \cdots \end{aligned}$$

and we know that  $b$  is induced on the complex of  $\mathcal{A}_S$ -modules quasi-isomorphic to  $(K_{/S}^\bullet, d_{/S}^\bullet)$  by the composition  $i \circ \tilde{b}$  where  $\tilde{b}$  is a quasi-isomorphism of complexes of  $\mathcal{O}_S[[b]]$ -modules (see lemma 1.2.1 and the definition 1.2.2). This implies that the kernel and the cokernel of  $R^p(i)$  are isomorphic (as  $\mathcal{O}_S$ -modules) to  $\text{Ker } b$  and  $\text{Coker } b$  respectively. Now to prove that  $E$  satisfies condition 4) of the definition 2.0.2 it is enough to prove coherence of the  $\mathcal{O}_S$ -modules  $\mathcal{H}^j[R\pi_*([K_{/S}/I_{/S}]^\bullet, d_{/S}^\bullet)]$  for all  $j \geq 0$ .

But the sheaves  $[K_{/S}/I_{/S}]^j \simeq [\text{Ker } d_{/S}f / \text{Im } d_{/S}f]^j$  are coherent on  $\mathcal{X}$  and supported in  $\mathcal{Y}$ . The spectral sequence

$$E_2^{p,q} := H^q(R^p\pi_*([K_{/S}/I_{/S}]^\bullet, d_{/S}^\bullet))$$

which converges to  $\mathcal{H}^j[R\pi_*([K_{/S}/I_{/S}]^\bullet, d_{/S}^\bullet)]$ , is a bounded complex of coherent  $\mathcal{O}_S$ -modules by the direct image theorem of H. Grauert. This gives the desired finiteness.

To conclude the proof, we want to show that  $E/B(E)$  is geometric. But this is an easy consequence of the regularity of the Gauss-Manin connexion of  $f$  and of the monodromy theorem. ■

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