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# Regularity of the extremal solutions for the Liouville system

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In this short note, we study the smoothness of the extremal solutions to the following system of equations:

$$(1) \quad \begin{cases} -\Delta u = \mu e^v & \text{in } \Omega, \\ -\Delta v = \lambda e^u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\lambda, \mu > 0$  are parameters and  $\Omega$  is a smoothly bounded domain of  $\mathbb{R}^N$ ,  $N \geq 1$ . As shown by M. Montenegro (see [6]), there exists a limiting curve  $\Upsilon$  in the first quadrant of the  $(\lambda, \mu)$ -plane serving as borderline for existence of classical solutions of (1). He also proved the existence of a weak solution  $u^*$  for every  $(\lambda^*, \mu^*)$  on the curve  $\Upsilon$  and left open the question of its regularity. Following standard terminology (see e.g. the books [3], [5] for an introduction to this vast subject),  $u^*$  is called an extremal solution. Our result is the following.

**Theorem 1** *Let  $1 \leq N \leq 9$ . Then, extremal solutions to (1) are smooth.*

**Remark 2** *C. Cowan ([1]) recently obtained the same result under the further assumption that  $(N-2)/8 < \lambda/\mu < 8/(N-2)$ .*

Any extremal solution  $u^*$  is obtained as the increasing pointwise limit of a sequence of regular solutions  $(u_n)$  associated to parameters  $(\lambda_n, \mu_n) = (1 - 1/n)(\lambda^*, \mu^*)$ . In addition, see [6],  $u_n$  is stable in the sense that the principal eigenvalue of the linearized operator associated to (1) is nonnegative. In other words, there exist  $\lambda_1 \geq 0$  and two positive functions  $\varphi_1, \psi_1 \in C^2(\overline{\Omega})$  such that

$$(2) \quad \begin{cases} -\Delta \varphi_1 - g'(v)\psi_1 = \lambda_1 \varphi_1 & \text{in } \Omega, \\ -\Delta \psi_1 - f'(u)\varphi_1 = \lambda_1 \psi_1 & \text{in } \Omega, \\ \varphi_1 = \psi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

where, in the context of (1),  $g(v) = e^v$  and  $f(u) = e^u$ . This motivates the following useful inequality.

Let  $f, g$  denote two nondecreasing  $C^1$  functions and consider the more general system

$$(3) \quad \begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

**Lemma 3** *Let  $N \geq 1$  and let  $(u, v) \in C^2(\overline{\Omega})^2$  denote a stable solution of (3). Then, for all  $\varphi \in C_c^1(\Omega)$ , there holds*

$$(4) \quad \int_{\Omega} \sqrt{f'(u)g'(v)} \varphi^2 \, dx \leq \int_{\Omega} |\nabla \varphi|^2 \, dx$$

**Remark 4** *As we just learnt, the same inequality has been obtained independently by C. Cowan and N. Ghoussoub. See [2].*

**Proof.** Since  $(u, v)$  is stable, there exist  $\lambda_1 \geq 0$  and two positive functions  $\varphi_1, \psi_1 \in C^2(\overline{\Omega})$  solving (2). Given  $\varphi \in C_c^1(\Omega)$ , multiply the first equation in (2) by  $\varphi^2/\varphi_1$  and integrate. Then,

$$(5) \quad \begin{aligned} \int_{\Omega} g'(v) \frac{\psi_1}{\varphi_1} \varphi^2 \, dx &\leq \int_{\Omega} \frac{\varphi^2}{\varphi_1} (-\Delta \varphi_1) \\ &= - \int_{\Omega} |\nabla \varphi_1|^2 \left( \frac{\varphi}{\varphi_1} \right)^2 + 2 \int_{\Omega} \frac{\varphi}{\psi_1} \nabla \varphi \nabla \varphi_1 \\ &= - \int_{\Omega} \left| \frac{\varphi}{\varphi_1} \nabla \varphi_1 - \nabla \varphi \right|^2 + \int_{\Omega} |\nabla \varphi|^2 \leq \int_{\Omega} |\nabla \varphi|^2. \end{aligned}$$

Working similarly with the second equation, we also have

$$(6) \quad \int_{\Omega} f'(u) \frac{\varphi_1}{\psi_1} \varphi^2 \, dx \leq \int_{\Omega} |\nabla \varphi|^2 \, dx$$

(4) then follows by combining the Cauchy-Schwarz inequality and (5)- (6).  $\square$

Thanks to the inequality (4), we obtain the following estimate.

**Lemma 5** *Let  $N \geq 1$ . There exists a universal constant  $C > 0$  such that any stable solution of (1) satisfies*

$$(7) \quad \int e^{u+v} \, dx \leq C |\Omega| \left( \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right).$$

**Proof.** Multiply the second equation in (1) by  $e^v - 1$  and integrate.

$$(8) \quad \begin{aligned} \lambda \int_{\Omega} e^{u+v} \, dx &\geq \lambda \int_{\Omega} e^u (e^v - 1) \, dx = \int_{\Omega} \nabla v \nabla (e^v - 1) \, dx \\ &= 4 \int_{\Omega} \left| \nabla (e^{v/2} - 1) \right|^2 \, dx. \end{aligned}$$

Using (4) with test function  $\varphi = e^{v/2} - 1$ , it follows that

$$(9) \quad \begin{aligned} \lambda \int_{\Omega} e^{u+v} dx &\geq 4\sqrt{\lambda\mu} \int_{\Omega} e^{\frac{u+v}{2}} (e^{v/2} - 1)^2 dx \\ &\geq 4\sqrt{\lambda\mu} \int_{\Omega} e^{\frac{u+v}{2}} e^v dx - 8\sqrt{\lambda\mu} \int_{\Omega} e^{\frac{u+v}{2}} e^{v/2} dx. \end{aligned}$$

By Young's inequality,  $e^{v/2} = \frac{1}{\sqrt{2}} e^{v/2} \cdot \sqrt{2} \leq \frac{1}{4} e^v + 1$ . So,

$$\int_{\Omega} e^{\frac{u+v}{2}} e^{v/2} dx \leq \frac{1}{4} \int_{\Omega} e^{\frac{u+v}{2}} e^v dx + \int_{\Omega} e^{\frac{u+v}{2}} dx.$$

Plugging this in (9), we obtain

$$(10) \quad \lambda \int_{\Omega} e^{u+v} dx + 8\sqrt{\lambda\mu} \int_{\Omega} e^{\frac{u+v}{2}} dx \geq 2\sqrt{\lambda\mu} \int_{\Omega} e^{\frac{u+v}{2}} e^v dx.$$

Similarly,

$$(11) \quad \mu \int_{\Omega} e^{u+v} dx + 8\sqrt{\lambda\mu} \int_{\Omega} e^{\frac{u+v}{2}} dx \geq 2\sqrt{\lambda\mu} \int_{\Omega} e^{\frac{u+v}{2}} e^u dx.$$

Multiply (10) and (11) to get

$$(12) \quad \begin{aligned} \lambda\mu \left( \int_{\Omega} e^{u+v} dx \right)^2 + 64\lambda\mu \left( \int_{\Omega} e^{\frac{u+v}{2}} dx \right)^2 + 8\sqrt{\lambda\mu}(\lambda+\mu) \int_{\Omega} e^{u+v} dx \int_{\Omega} e^{\frac{u+v}{2}} dx \geq \\ 4\lambda\mu \int_{\Omega} e^{\frac{u+v}{2}} e^u dx \int_{\Omega} e^{\frac{u+v}{2}} e^v dx. \end{aligned}$$

Using Young's inequality, the left-hand side in the above inequality is bounded above by

$$(13) \quad 2\lambda\mu \left( \int_{\Omega} e^{u+v} dx \right)^2 + C(\lambda + \mu)^2 \left( \int_{\Omega} e^{\frac{u+v}{2}} dx \right)^2,$$

where  $C$  is a universal constant. In addition, by the Cauchy-Schwarz inequality,

$$(14) \quad \int_{\Omega} e^{\frac{u+v}{2}} e^u dx \int_{\Omega} e^{\frac{u+v}{2}} e^v dx \geq \left( \int_{\Omega} e^{u+v} dx \right)^2.$$

Plugging (14) in (13) and remembering that (13) is an upper bound of the left-hand side in (12), we obtain

$$(15) \quad C(\lambda + \mu)^2 \left( \int_{\Omega} e^{\frac{u+v}{2}} dx \right)^2 \geq 2\lambda\mu \int_{\Omega} e^{\frac{u+v}{2}} e^u dx \int_{\Omega} e^{\frac{u+v}{2}} e^v dx.$$

By the Cauchy-Schwarz inequality and (14), we have

$$(16) \quad \left( \int_{\Omega} e^{\frac{u+v}{2}} dx \right)^2 \leq |\Omega| \int_{\Omega} e^{u+v} dx \\ \leq |\Omega| \left( \int_{\Omega} e^{\frac{u+v}{2}} e^u dx \int_{\Omega} e^{\frac{u+v}{2}} e^v dx \right)^{1/2}.$$

Using (16) in (15), we obtain

$$(17) \quad C \frac{(\lambda + \mu)^2}{\lambda \mu} |\Omega| \geq \left( \int_{\Omega} e^{\frac{u+v}{2}} e^u dx \int_{\Omega} e^{\frac{u+v}{2}} e^v dx \right)^{1/2}.$$

Applying once more (14), we obtain the desired estimate.  $\square$

We can now prove Theorem 1.

**Step 1. Case  $1 \leq N \leq 3$ .** It is enough to treat the case  $N = 3$ , the cases  $N = 1, 2$  being easier. By (8) and (7),  $e^{v/2} - 1$  is bounded in  $H_0^1(\Omega)$  (with a uniform bound with respect to  $\lambda$  and  $\mu$ ). By the Sobolev embedding, it follows that  $e^v$  is bounded in  $L^{\frac{N}{N-2}}(\Omega)$ . By (8) and elliptic regularity,  $u$  is bounded in  $W^{2, \frac{N}{N-2}}$ . For  $N = 3$ ,  $\frac{N}{N-2} > \frac{N}{2}$ . By Sobolev's embedding, we deduce that  $u$  is bounded, and so must be  $v$ . This implies the desired conclusion for the corresponding extremal solution.

**Step 2. General case.** We adapt a method introduced in [4]. Fix  $\alpha > 1/2$  and multiply the first equation in (1) by  $e^{\alpha u} - 1$ . Integrating over  $\Omega$ , we obtain

$$\mu \int_{\Omega} (e^{\alpha u} - 1) e^v dx = \alpha \int_{\Omega} e^{\alpha u} |\nabla u|^2 dx = \frac{4}{\alpha} \int_{\Omega} |\nabla (e^{\frac{\alpha u}{2}} - 1)|^2 dx$$

By (4),

$$\sqrt{\lambda \mu} \int_{\Omega} e^{\frac{u+v}{2}} (e^{\frac{\alpha u}{2}} - 1)^2 dx \leq \int_{\Omega} |\nabla (e^{\frac{\alpha u}{2}} - 1)|^2 dx.$$

Combining these two inequalities, we deduce that

$$(18) \quad \sqrt{\lambda \mu} \int_{\Omega} e^{\frac{u+v}{2}} (e^{\frac{\alpha u}{2}} - 1)^2 dx \leq \frac{\alpha}{4} \mu \int_{\Omega} (e^{\alpha u} - 1) e^v dx$$

Hence,

$$(19) \quad \sqrt{\lambda \mu} \int_{\Omega} e^{\frac{2\alpha+1}{2}u} e^{\frac{v}{2}} dx \leq \frac{\alpha}{4} \mu \int_{\Omega} e^{\alpha u} e^v dx + 2\sqrt{\lambda \mu} \int_{\Omega} e^{\frac{\alpha+1}{2}u} e^{\frac{v}{2}} dx$$

Let us estimate the terms on the right-hand side. By Hölder's inequality,

$$(20) \quad \int_{\Omega} e^{\alpha u} e^v dx \leq \left( \int_{\Omega} e^{\frac{2\alpha+1}{2}u} e^{\frac{v}{2}} dx \right)^{\frac{2\alpha-1}{2\alpha}} \left( \int_{\Omega} e^{\frac{u}{2}} e^{\frac{2\alpha+1}{2}v} dx \right)^{\frac{1}{2\alpha}}$$

Given  $\varepsilon > 0$ , it also follows from Young's inequality that

$$\int_{\Omega} e^{\frac{\alpha+1}{2}u} e^{\frac{v}{2}} dx \leq \frac{\varepsilon}{2} \sqrt{\frac{\mu}{\lambda}} \int_{\Omega} e^{\alpha u} e^v dx + \frac{1}{2\varepsilon} \sqrt{\frac{\lambda}{\mu}} \int_{\Omega} e^u dx.$$

Using (7), we deduce that

$$(21) \quad \int_{\Omega} e^{\frac{\alpha+1}{2}u} e^{\frac{v}{2}} dx \leq \frac{\varepsilon}{2} \sqrt{\frac{\mu}{\lambda}} \int_{\Omega} e^{\alpha u} e^v dx + \frac{1}{2\varepsilon} \sqrt{\frac{\lambda}{\mu}} C |\Omega| \left( \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right).$$

where  $C$  is the universal constant of Lemma 5.

So, gathering (19), (20), (21), and letting

$$X = \int_{\Omega} e^{\frac{2\alpha+1}{2}u} e^{\frac{v}{2}} dx \quad \text{and} \quad Y = \int_{\Omega} e^{\frac{2\alpha+1}{2}v} e^{\frac{u}{2}} dx,$$

we obtain

$$\sqrt{\lambda\mu} X \leq \left( \frac{\alpha}{4} + \varepsilon \right) \mu X^{\frac{2\alpha-1}{2\alpha}} Y^{\frac{1}{2\alpha}} + C \frac{\lambda}{\varepsilon} |\Omega| \left( \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right).$$

By symmetry, we also have

$$\sqrt{\lambda\mu} Y \leq \left( \frac{\alpha}{4} + \varepsilon \right) \lambda Y^{\frac{2\alpha-1}{2\alpha}} X^{\frac{1}{2\alpha}} + C \frac{\mu}{\varepsilon} |\Omega| \left( \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right).$$

Multiplying these inequalities, we deduce that

$$\left( 1 - \left( \frac{\alpha}{4} + \varepsilon \right)^2 \right) X Y \leq C_1 \left( \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right)^2 \left( 1 + X^{\frac{2\alpha-1}{2\alpha}} Y^{\frac{1}{2\alpha}} + Y^{\frac{2\alpha-1}{2\alpha}} X^{\frac{1}{2\alpha}} \right).$$

where  $C_1 = C \frac{|\Omega|}{\varepsilon} \left( \frac{\alpha}{4} + \varepsilon \right) > 0$ . Hence, for every  $\alpha < 4$ , either  $X$  or  $Y$  must be bounded (with a uniform bound with respect to  $\lambda$  and  $\mu$ ).

Without loss of generality,  $\lambda \geq \mu$  and by the maximum principle,  $v \geq u$ . It follows that  $e^u$  is bounded in  $L^p(\Omega)$  for every  $p = \alpha + 1 < 5$ . Using standard elliptic regularity, the result follows.  $\square$

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