THE GEL'FAND PROBLEM FOR THE BIHARMONIC OPERATOR

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1. Introduction

A classical result attributed[1] to G.I. Barenblatt asserts that there exists infinitely many solutions to the equation

$$-\Delta u = 2e^u \qquad \text{in } \Omega, \tag{1.1}$$

whenever Ω is the unit ball of $\mathbb{R}^3$ and the equation is supplemented with a homogeneous Dirichlet boundary condition. The result appeared in a volume edited by I.M. Gel'fand [14], whose name the problem now bears. We refer the interested reader to the book [12] for some of the developments of this equation in the more than sixty years that separate us from Barenblatt's discovery. Let us simply mention that K. Nagasaki and T. Suzuki [19] completly classified the solutions found by Barenblatt according to their Morse index. Much of what can be said of the equation posed in a general domain Ω rests, through a blow-up analysis, upon Liouville-type theorems for finite Morse index solutions of the equation posed in entire space. This lead N. Dancer and A. Farina [9] to proving that in dimension $3 \le N \le 9$, any solution to (1.1) in $\Omega = \mathbb{R}^N$ is unstable outside every compact set and so, it has infinite Morse index.

In the present work, we consider the fourth-order analogue of the Gel'fand problem. In particular, we want to classify solutions of

$$\Delta^2 u = e^u \qquad \text{in } \mathbb{R}^N, \tag{1.2}$$

which are stable (resp. stable outside a compact set), that is, solutions such that

$$\int_\Omega e^u \varphi^2 \, dx \le \int_\Omega |\Delta \varphi|^2 \, dx \quad \text{for all } \varphi \in H^2(\Omega) \cap H_0^1(\Omega), \tag{1.3}$$

where Ω is $\mathbb{R}^N$(resp. the complement of some compact subset of $\mathbb{R}^N$). Consider first radial solutions. Noting that the equation is invariant under the scaling transformation

$$u_\lambda(x) = u(\lambda x) + 4 \ln \lambda, \quad x \in \mathbb{R}^N, \lambda > 0, \tag{1.4}$$

we may always assume that $u(0) = 0$.

Proposition 1.1. *Let $5 \le N \le 12$. Assume u is a radial solution of (1.2). Let $v = -\Delta u$, $0 = u(0)$, and $\beta = v(0)$. There exist $\beta_1 > \beta_0 > 0$ depending on N only such that*

- $\beta \ge \beta_0$.

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[1] see H. Brezis, [5].

- *If $\beta = \beta_0$, u is unstable outside every compact set.*
- *If $\beta \in (\beta_0, \beta_1)$, u is unstable, but u is stable outside a compact set.*
- *If $\beta \geq \beta_1$, u is stable.*

Remark 1.2. The fact that no radial solution exists for $\beta < \beta_0$ is due to G. Arioli, F. Gazzola and H.-C. Grunau [1]. In addition, E. Berchio, A. Farina, A. Ferrero, and F. Gazzola already observed in [3] that for $5 \leq N \leq 12$, u is stable outside a compact set[2] if and only if $\beta > \beta_0$. Our result characterizes stable radial solutions when $5 \leq N \leq 12$. See [3], [25] for the remaining cases $1 \leq N \leq 4$ and $N \geq 13$.

In particular, there is no hope of proving a result similar to that of Dancer and Farina in our context, without further restrictions on the solution. One might ask whether all stable solutions are radial, at least in dimension $5 \leq N \leq 12$. This is still not the case.

Theorem 1.3. *Assume $N \geq 5$. Take a point $x^0 = (x_1^0, \dots, x_N^0) \in \mathbb{R}^N$, parameters $\alpha_1, \dots, \alpha_N > 1 + N/2$, and let*

$$p(x) = \sum_{i=1}^{N} \alpha_i (x_i - x_i^0)^2. \tag{1.5}$$

Then, there exists a solution u of (1.2) such that

$$u(x) = -p(x) + \mathcal{O}(|x|^{4-N}) \quad \text{as } |x| \to \infty, \tag{1.6}$$

In particular, u is stable outside a compact set (resp. stable, if $\min_{i=1,\dots,N} \alpha_i$ is large enough) and u is not radial about any point if the coefficients α_i are not all equal.

Remark 1.4. Using the scaling (1.4), one immediately obtains a solution u of (1.2) such that $u(x) = -p(x) + C + \mathcal{O}(|x|^{4-N})$ as $|x| \to \infty$, under the sole assumption that $\alpha_i > 0$ for all $i = 1, \dots, N$.

Remark 1.5. In fact, any solution satisfying (1.6) has finite Morse index, thanks to the Cwikel-Lieb-Rosenblum formula. We thank A. Farina for this observation.

Remark 1.6. In dimension $N \geq 13$, the radial solution $u_{\beta 0}(x) = -4 \ln |x| + \mathcal{O}(1)$ is stable, see [3]. Observe that u_{β_0} does not satisfy (1.6). In dimension $5 \leq N \leq 12$, it would be interesting to determine whether, up to rescaling and rotation, all stable solutions do satisfy (1.6).

All stable solutions that we have encountered so far have quadratic behavior at infinity. In particular, letting

$$v = -\Delta u \quad \text{and} \quad \overline{v}(r) = \fint_{\partial B_r} v \, d\sigma,$$

these solutions satisfy $\overline{v}(\infty) > 0$, where [3]

$$\overline{v}(\infty) := \lim_{r \to +\infty} \overline{v}(r).$$

This motivates the following Liouville-type result.

[2] the notion of stability outside a compact set that we use here is stronger than the one given in [3]. One can easily check that the results of [3] remain true in our setting.

[3] For any solution of (1.2), v is superharmonic. In particular, its spherical average $\overline{v}$ is a decreasing function of r. We will prove that $v > 0$, so that $\overline{v}(\infty) \geq 0$ is always well-defined.

Theorem 1.7. *Assume $5 \le N \le 12$. Let u be a solution of (1.2) such that $\overline{v}(\infty) = 0$. Then, u is unstable outside every compact set.*

Remark 1.8. As observed in [3], in dimension $N = 4$, applying inequality (1.3) with a standard cut-off function, we easily see that if u is stable outside a compact set, then $e^u \in L^1(\mathbb{R}^4)$. Thanks to the work of C.-S. Lin [17], up to a rotation of space, u must satisfy

$$u(x) = -p(x) - 4\gamma \ln|x| + c_0 + \mathcal{O}(|x|^{-\tau}) \quad \text{as } |x| \to \infty, \tag{1.7}$$

where $p(x)$ is of the form (1.5) with $\alpha_i \ge 0$, $\gamma = \frac{1}{32\pi^2}\int_{\mathbb{R}^4} e^u\, dx \le 2$, c_0 and $\tau > 0$ are constants. Conversely, there exist solutions of the form (1.7), as proved by J. Wei and D. Ye in [28]. If in addition $\overline{v}(\infty) = 0$, then $\gamma = 2$ and up to translation and the scaling (1.4),

$$u(x) = -4\ln\left(1 + |x|^2\right) + \ln 384, \qquad \text{for all } x \in \mathbb{R}^4.$$

Now, let us turn to bounded domains. We begin by recalling a few known results on the Gel'fand problem for the biharmonic operator, when the domain is the unit ball and the equation is supplemented with a homogeneous Dirichlet boundary condition, i.e. for $\lambda > 0$, we consider the equation

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B, \\ u = |\nabla u| = 0 & \text{on } \partial B. \end{cases}$$

It is known that there exists an extremal parameter $\lambda^* = \lambda^*(N) > 0$ such that the problem has at least one solution (which is stable) for $\lambda < \lambda^*$, a weak stable solution u^* for $\lambda = \lambda^*$ and no solution for $\lambda > \lambda^*$ (see G. Arioli, F. Gazzola, F.-C. Grunau and E. Mitidieri [2]). The unique weak stable solution u^* associated to $\lambda = \lambda^*$ is classical if and only if $1 \le N \le 12$ (see the work by J. Dávila, I. Guerra, M. Montenegro and one of the authors [10], as well as a simplification due to A. Moradifam [18]). It is also known that if $N \ge 5$, the problem has a unique singular radial solution for some $\lambda = \lambda_S$ (as follows from the analysis in [2] and [10]) and, as in Barenblatt's result, that it has infinitely many regular radial solutions for the same value of the parameter (see the delicate analysis due to J. Dávila, I. Flores and I. Guerra [11]). The case of general domains with Dirichlet boundary conditions is essentially unexplored, due to the lack of a comparison principle. The case of homogeneous Navier boundary conditions, namely the equation

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \ge 1$, seems, for now, easier to deal with. Our results can be summarized as follows.

Theorem 1.9. *Let $N \ge 1$ and let Ω be a smoothly bounded domain of $\mathbb{R}^N$. Let u^* be the extremal solution of (1.8).*

- *If $1 \le N \le 12$, then $u^* \in C^\infty(\overline{\Omega})$.*
- *If $N \ge 13$, then $u \in C^\infty(\Omega \setminus \Sigma)$, where Σ is a closed set whose Hausdorff dimension is bounded above by*

$$\mathcal{H}_{dim}(\Sigma) \le N - 4p^*$$

and $p^* > 3$ is the largest root of the polynomial $(X - \frac{1}{2})^3 - 8(X - \frac{1}{2}) + 4$.

Remark 1.10. Theorem 1.9 was first proved for $1 \leq N \leq 8$ by C. Cowan, P. Esposito, and N. Ghoussoub, see [7]. As we were completing this work, we learnt that C. Cowan and N. Ghoussoub just improved their result to $1 \leq N \leq 10$, see [8]. The question of partial regularity in large dimension was studied by K. Wang for the classical Gel'fand problem (see [24]). Unfortunately, we could not understand a part of his proof[4], and our approach is somewhat different. We expect that the computed exponent p^* is not optimal. Similarly, the methods that we develop here will not[5] yield the expected critical curve (resp. dimension) for the Lane-Emden system (resp. for the MEMS problem), for which new ideas must be found.

Finally, our Liouville-type result, Theorem 1.7, will be used to prove the following.

Theorem 1.11. *Let $5 \leq N \leq 12$ and let Ω be a smoothly bounded domain of $\mathbb{R}^N$. Assume in addition that Ω is convex. Let $u \in C^4(\overline{\Omega})$ be any classical solution of (1.8) and let $v = -\Delta u$. There exists a compact subdomain $\omega \subset \Omega$ such that if*

$$\int_{B_r(x_0)} v \, dx \leq K r^{N-2}, \tag{1.9}$$

for every ball $B_r(x_0) \subset \omega$ and for some constant $K > 0$, then, there exists a number C depending only on λ, Ω, N, K and the Morse index of u, such that

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Definition 1.12. In the above, the Morse index of a solution u is the number of negative eigenvalues of the linearized operator $L = \Delta^2 - \lambda e^u$ with domain $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$. According to standard spectral theory, this number is finite.

Remark 1.13. If u is stable, then (1.9) automatically holds. See Lemma 5.2. We do not know whether this remains valid for solutions of bounded Morse index. In addition, it would be interesting to know how the number C depends on the Morse index of u. See the work of X.-F. Yang [30] for a result in this direction, in a subcritical setting.

Notation. For any given function f, $\overline{f}$ connotes the spherical average of f. We write $f \lesssim g$ (resp. $f \gtrsim g$), when there exists a numerical constant C such that $f \leq Cg$ (resp. $f \geq Cg$). $B_R(x)$ denotes the ball centered at x and of radius R. For shorthand, $B_R = B_R(0)$ and A_R is the annulus of radii R and $2R$.

2. Classification of stable radial solutions

We prove here Proposition 1.1. Take $\beta \geq \beta_0$ and let $u = u_\beta$ be the radial solution such that $u(0) = 0$ and $v(0) = \beta$. We claim that if β is large enough, then u_β is stable. We shall use the following inequality, found in [1]:

$$u_\beta \leq -\frac{\beta - \beta_0}{2N} r^2 \quad \text{for all } r > 0. \tag{2.1}$$

Simply choose $\overline{\beta} > \beta_0$ such that

$$e^{-\frac{\overline{\beta}-\beta_0}{2N} r^2} \leq \frac{N^2(N-4)^2}{16 r^4} \quad \text{for all } r > 0. \tag{2.2}$$

[4] the relation between the exponents in his interpolation inequality (3.18) seems incorrect.

[5] They should, however, improve somewhat the partial results given in [7], [8], [29], [27], [6].

Combining (2.1), (2.2), and the Hardy-Rellich inequality, we deduce that u_β is stable for $\beta \geq \overline{\beta}$. So, we may define $\Lambda = \{\beta > \beta_0 \mid u_\beta \text{ is stable}\}$ and $\beta_1 = \inf \Lambda$. By standard ODE theory, one easily proves that $\beta_1 = \min \Lambda$. According to [3], u_{β_0} is unstable. So, $\beta_1 > \beta_0$. Also, by a result in [1], solutions are ordered : if $\tilde{\beta} > \beta$, then $u_{\tilde{\beta}} \leq u_\beta$. So, Λ is the interval $[\beta_1, +\infty)$. □

3. Construction of nonradial solutions

We present here the proof of Theorem 1.3. Take a polynomial p of the form (1.5). Without loss if generality, we may assume that $x^0 = 0$. We look for a solution u of the form $u = -p(x) + z(x)$, so that z and $w = -\Delta z$ satisfy

$$\begin{cases} -\Delta z = w & \text{in } \mathbb{R}^N, \\ -\Delta w = e^{-p(x)} e^z & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

For $x \in \mathbb{R}^N$, let

$$Z(x) = (1 + |x|^2)^{2-N/2}, \quad W(x) = (1 + |x|^2)^{1-N/2}.$$

We claim that (Z, W) is a super-solution of (3.1). Indeed, straightforward calculations yield

$$\begin{aligned} -\Delta Z &\geq 2(N-4)W && \text{in } \mathbb{R}^N, \\ -\Delta W &= N(N-2)(1+|x|^2)^{-1-N/2} && \text{in } \mathbb{R}^N. \end{aligned}$$

Since $Z \leq 1$ and since we assumed that $p(x) \geq (1 + N/2)|x|^2$ in $\mathbb{R}^N$, we have

$$-\Delta W \geq (1+|x|^2)^{-1-N/2} e^Z \geq e^{-(1+N/2)|x|^2} e^Z \geq e^{-p(x)} e^Z \quad \text{in } \mathbb{R}^N,$$

which proves our claim.

Since the system is cooperative, and $(0, 0)$ and (Z, W) form a pair of ordered sub- and super-solutions, we obtain the existence of a solution of (3.1) which further satisfies $0 < z \leq Z$ and $0 < w \leq W$ in $\mathbb{R}^N$. Hence, $u(x) := -p(x)+z(x)$ is a solution of (1.2).

To prove that u is stable outside a compact set, let us observe again that $Z \leq 1$ in $\mathbb{R}^N$. So, we can find $\rho > 0$ large such that

$$e^{u(x)} \leq e^{Z(x)} e^{-p(x)} \leq e^{1-p(x)} \leq \frac{N^2(N-4)^2}{16|x|^4}, \quad \text{for all } |x| > \rho. \tag{3.2}$$

By the Hardy-Rellich inequality, u is stable outside $\overline{B_\rho}$. Remark now that if $\min_{i=1\dots N} \alpha_i > 0$ is large enough then (3.2) is valid for all $x \in \mathbb{R}^N$, so u is stable in $\mathbb{R}^N$. This ends the proof of Theorem 1.3. □

4. Regularity of the extremal solution in dimension $1 \leq N \leq 12$

In this section, we prove the first part of Theorem 1.9. Let u denote the minimal solution to (1.8) associated to a parameter $\lambda \in (\lambda^*/2, \lambda^*)$. Up to rescaling, we may assume that $\lambda = 1$. As observed by several authors (see [13], [8] or Lemma 6.1), the stability inequality (1.3) implies that

$$\int_\Omega e^{\frac{u}{2}} \varphi^2 \, dx \leq \int_\Omega |\nabla \varphi|^2 \, dx \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{4.1}$$

Let us write the problem (1.8) as a system in the following way:

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = e^u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Fix $\alpha > \frac{1}{2}$ and multiply the first equation in (4.2) by $e^{\alpha u} - 1$. Integrating over Ω, we obtain

$$\int_\Omega (e^{\alpha u} - 1)\, v \, dx = \alpha \int_\Omega e^{\alpha u} |\nabla u|^2 \, dx = \frac{4}{\alpha} \int_\Omega \left|\nabla \left(e^{\frac{\alpha u}{2}} - 1\right)\right|^2 \, dx.$$

By (4.1),

$$\int_\Omega e^{u/2} \left(e^{\frac{\alpha u}{2}} - 1\right)^2 \, dx \le \int_\Omega \left|\nabla \left(e^{\frac{\alpha u}{2}} - 1\right)\right|^2 \, dx.$$

Combining these two inequalities, we deduce that

$$\int_\Omega e^{u/2} \left(e^{\frac{\alpha u}{2}} - 1\right)^2 \, dx \le \frac{\alpha}{4} \int_\Omega (e^{\alpha u} - 1)\, v \, dx. \tag{4.3}$$

Similarly, multiply the second equation in (4.2) by $v^{2\alpha-1}$ and use (4.1) to deduce that

$$\int_\Omega e^{u/2} v^{2\alpha} \, dx \le \frac{\alpha^2}{2\alpha - 1} \int_\Omega e^u v^{2\alpha-1} \, dx. \tag{4.4}$$

By Hölder's inequality,

$$\int_\Omega e^u v^{2\alpha-1} \, dx \le \left(\int_\Omega e^{u/2} v^{2\alpha} \, dx\right)^{\frac{2\alpha-1}{2\alpha}} \left(\int_\Omega e^{u/2} e^{\alpha u} \, dx\right)^{\frac{1}{2\alpha}} \quad \text{and}$$

$$\int_\Omega e^{\alpha u} v \, dx \le \left(\int_\Omega e^{u/2} v^{2\alpha} \, dx\right)^{\frac{1}{2\alpha}} \left(\int_\Omega e^{u/2} e^{\alpha u} \, dx\right)^{\frac{2\alpha-1}{2\alpha}}.$$

Plugging these inequalities in (4.4) and (4.3) respectively, we deduce that

$$\left(\int_\Omega e^{u/2} v^{2\alpha} \, dx\right)^{\frac{1}{2\alpha}} \le \frac{\alpha^2}{2\alpha - 1} \left(\int_\Omega e^{u/2} e^{\alpha u} \, dx\right)^{\frac{1}{2\alpha}} \quad \text{and}$$

$$\int_\Omega e^{u/2} \left(e^{\frac{\alpha u}{2}} - 1\right)^2 \, dx \le \frac{\alpha}{4} \left(\int_\Omega e^{u/2} v^{2\alpha} \, dx\right)^{\frac{1}{2\alpha}} \left(\int_\Omega e^{u/2} e^{\alpha u} \, dx\right)^{\frac{2\alpha-1}{2\alpha}}.$$

Multiplying these inequalities, it follows that

$$\int_\Omega e^{u/2} \left(e^{\frac{\alpha u}{2}} - 1\right)^2 \, dx \le \frac{\alpha^3}{8\alpha - 4} \int_\Omega e^{(\alpha + \frac{1}{2})u} \, dx$$

so that

$$\left(1 - \frac{\alpha^3}{8\alpha - 4}\right) \int_\Omega e^{u/2} e^{\alpha u} \, dx \le 2 \int_\Omega e^{\frac{\alpha+1}{2} u} \, dx.$$

Apply Hölder's inequality to the right-hand side. Then,

$$\int_\Omega e^{\frac{\alpha+1}{2} u} \, dx \le \left(\int_\Omega e^{\frac{2\alpha+1}{2} u} \, dx\right)^{\frac{\alpha+1}{2\alpha+1}} |\Omega|^{\frac{\alpha}{2\alpha+1}}$$

and so

$$\left(1-\frac{\alpha^3}{8\alpha-4}\right)\left(\int_\Omega e^{\frac{2\alpha+1}{2}u}\,dx\right)^{\frac{\alpha}{2\alpha+1}} \le 2\,|\Omega|^{\frac{\alpha}{2\alpha+1}}\,. \tag{4.5}$$

Let $\alpha^* > 5/2$ denote the largest root of the polynomial $X^3 - 8X + 4$. We have just proved that e^u is uniformly bounded in $L^p(\Omega)$ for every $p < p^* = \alpha^* + \frac{1}{2}$. In particular, e^u is bounded in $L^p(\Omega)$ for some $p > 3$. Using elliptic regularity applied to (1.8), this implies that u^* is bounded, hence smooth whenever $N \le 12$. □

5. Growth of the L^1 norm

This section and the next provide preparatory results that will be used both for the proof of the Liouville-type theorem and the partial regularity result. We begin with the case of $\mathbb{R}^N$.

Lemma 5.1. *Assume $N \ge 5$ and let u be a solution of (1.2) which is stable (resp. stable outside the ball $\overline{B_{R_0}}$). Let $v = -\Delta u$, $\overline{v}$ its spherical average, and assume that $\overline{v}(\infty) = 0$. Let B_R denote the ball of radius R (resp. the annulus of radii R and $2R$). Then, there exists a constant C depending only on N (resp. on N, u, R_0) such that*

$$\int_{A_R} v\,dx \le CR^{N-2} \quad \textit{for every } R > 0 \textit{ (resp. } R > R_0). \tag{5.1}$$

Proof. We claim that $v > 0$. Since the equation is invariant under translation, it suffices to prove that $v(0) > 0$. Let $\overline{v}$ be the spherical average of v and assume by contradiction that $v(0) = \overline{v}(0) \le 0$. We have

$$\begin{cases} -\Delta\overline{v} = \overline{e^u} & \text{in } \mathbb{R}^N, \\ -\Delta\overline{u} = \overline{v} & \text{in } \mathbb{R}^N. \end{cases} \tag{5.2}$$

In particular, $-\Delta\overline{v} = -r^{1-N}(r^{N-1}v')' > 0$, and so $\overline{v}$ is a decreasing function of $r > 0$. Since we assumed that $\overline{v}(0) \le 0$, it follows that $\overline{v}(r) < 0$ for all $r > 0$. So, $\overline{u}$ is subharmonic. It follows that $\overline{u}$ is an increasing function of $r > 0$. In particular, $\overline{u}$ is bounded below and so, given $R > 2R_0$,

$$\int_{B_{2R}\setminus B_R} e^{\overline{u}}\,dx \ge e^{u(0)}\int_{B_{2R}\setminus B_R} dx \gtrsim R^N. \tag{5.3}$$

Take a cut-off function $\varphi \in C_c^2(B_4 \setminus B_{\frac{1}{2}})$, with $0 \le \varphi \le 1$ in $\mathbb{R}^N$ and $\varphi \equiv 1$ in $B_2 \setminus B_1$. Applying Jensen's inequality and the stability inequality (1.3) with test function $\varphi(x/R)$, we find

$$\int_{B_{2R}\setminus B_R} e^{\overline{u}}dx \le \int_{B_{2R}\setminus B_R} \overline{e^u}dx \lesssim R^{N-4}$$

which contradicts (5.3), if R is chosen large enough. Hence, $v(0) > 0$. Now, if u is stable, take $r > 0$, take a standard cut-off function $\varphi \in C_c^2(B_2)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ in B_1. Apply the stability inequality (1.3) with test function $\varphi(x/r)$ to get

$$\int_{B_r} e^u\,dx \lesssim r^{N-4}. \tag{5.4}$$

If u is only stable outside $\overline{B_{R_0}}$, fix $r > 2R_0$ and take a cut-off function $\varphi \in C_c^2(\mathbb{R}^N)$ such that $\varphi \equiv 0$ in B_{R_0}, $\varphi \equiv 1$ in $B_r \setminus B_{R_0+1}$, $\varphi \equiv 0$ outside B_{2r}, and $|\Delta\varphi| \lesssim r^{-2}$ in $B_{2r} \setminus B_r$. Using the stability inequality (1.3) with this test function we find that

$$\int_{B_r} e^u \, dx = \int_{B_{R_0+1}} e^u \, dx + \int_{B_r \setminus B_{R_0+1}} e^u \, dx \lesssim 1 + r^{N-4} \lesssim r^{N-4}. \tag{5.5}$$

Now, we rewrite the first equation in (5.2) as

$$-(r^{N-1}\overline{v}')' = r^{N-1}\overline{e^u}.$$

Integrate on $(0, r)$. By (5.4), which holds for every $r > 0$ if u is stable (resp. by (5.5), which holds for $r > 2R_0$ if u is stable outside $\overline{B_{R_0}}$),

$$-r^{N-1}\overline{v}'(r) = \int_0^r t^{N-1}\overline{e^u} dt \lesssim r^{N-4}.$$

We integrate once more between R and $+\infty$. Since $\overline{v}(\infty) = 0$, we obtain

$$\overline{v}(R) \lesssim R^{-2}.$$

Clearly, (5.1) follows. □

Now, we turn to an analogous result on bounded domains. Let us recall that the extremal solution is stable.

Lemma 5.2. *Let $N \geq 3$ and let u be the extremal solution of (1.8). Set $v = -\Delta u$. Fix $x_0 \in \Omega$ and let $R_0 = \operatorname{dist}(x_0, \partial\Omega)/2$. Then, there exists a constant $C > 0$ depending only on N, Ω, and R_0 such that for all $r < R_0$,*

$$\int_{B_r(x_0)} v \, dx \leq Cr^{N-2}. \tag{5.6}$$

Proof. Without loss of generality, we may assume that $x_0 = 0$ and $\lambda^* = 1$.

Step 1. There exists a constant $C = C(N, R_0)$ such that for all $r \in (0, R_0)$,

$$\int_{[r<|x|<R_0]} e^u |x|^{2-N} \, dx \leq Cr^{-2}. \tag{5.7}$$

To see this, consider the function $\psi : B_{2R_0} \to \mathbb{R}^N$ given by

$$\psi(x) = \begin{cases} a - b|x|^2 & \text{if } |x| < r, \\ |x|^{1-N/2} & \text{if } r \leq |x| < 2R_0, \end{cases} \tag{5.8}$$

where the constants $a = \frac{N+2}{4} r^{1-N/2}$ and $b = \frac{N-2}{4} r^{-1-N/2}$ are chosen so that ψ is C^1 and piecewise C^2. Take a standard cut-off function

$$\zeta \in C_c^2(B_2) \quad \text{such that } 0 \leq \zeta \leq 1 \text{ and } \zeta = 1 \text{ on } B_1, \tag{5.9}$$

so that $\varphi(x) = \psi(x)\zeta(x/R_0)$ belongs to $H_0^1(\Omega) \cap H^2(\Omega)$.

Since u is stable, we have

$$\int_{[r<|x|<R_0]} e^u |x|^{2-N} \, dx \leq \int_\Omega e^u \varphi^2 \, dx \leq \int_\Omega |\Delta\varphi|^2 \, dx. \tag{5.10}$$

By (5.8),

$$\int_{[|x|<r]} |\Delta\varphi|^2 \, dx \leq 4N^2 b^2 \int_{[|x|<r]} dx \leq Cb^2 r^N \leq C' r^{-2}. \tag{5.11}$$

Similarly,

$$\int_{[r<|x|<R_0]} |\Delta\varphi|^2 \, dx \le C \int_{[r<|x|]} |x|^{-2-N} \, dx \le C' r^{-2}. \tag{5.12}$$

Finally,

$$\int_{[R_0<|x|]\cap\Omega} |\Delta\varphi|^2 \, dx \le C(N, R_0) \le C' r^{-2}. \tag{5.13}$$

Collecting (5.10), (5.11), (5.12), and (5.13), the estimate (5.7) follows.

Step 2. Take as before a standard cut-off ζ satisfying (5.9) and let

$$\psi(x) = \int_{\mathbb{R}^N} |x-y|^{2-N} \zeta(y) \, dy, \qquad \text{for } x \in \mathbb{R}^N.$$

Then, there exists a constant C, depending on N only such that

$$\psi(x) \le C \min(1, |x|^{2-N}), \qquad \text{for } x \in \mathbb{R}^N. \tag{5.14}$$

Indeed, if $|x| > 4$ and $|y| < 2$,

$$|x-y| \ge |x| - |y| \ge \frac{1}{2}|x|$$

so that

$$\psi(x) \le 2^{N-2}|x|^{2-N} \int_{\mathbb{R}^N} \zeta(y) \, dy \le C|x|^{2-N},$$

while if $|x| \le 4$ and $|y| < 2$,

$$|x-y| \le |x| + |y| \le 6$$

and so

$$\psi(x) \le \|\zeta\|_{L^\infty(\mathbb{R}^N)} \int_{|x-y|<6} |x-y|^{2-N} \, dy \le C.$$

(5.14) follows.

Step 3. Take $r \in (0, R_0)$, ζ a standard cut-off function satisfying (5.9), and let φ_r be the solution to

$$\begin{cases} -\Delta\varphi_r = \zeta(x/r) & \text{in } \Omega, \\ \varphi_r = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.15}$$

Then, there exists a constant C depending on N only such that

$$\varphi_r(x) \le C r^2 \min\left(1, r^{N-2}|x|^{2-N}\right), \quad \text{for all } x \in \Omega. \tag{5.16}$$

This easily follows from the maximum principle, observing that a constant multiple of $r^2\psi(x/r)$ is a supersolution to the above equation.

Step 4. There exists a constant C depending on N and Ω only, such that

$$\int_\Omega e^u \, dx \le C. \tag{5.17}$$

This is an obvious consequence of Equation (4.5) (which holds for any $\alpha \in (\frac{1}{2}, 5/2]$) and Hölder's inequality.

Step 5. Multiply (5.15) by $v = -\Delta u$ and integrate by parts. Then,

$$\int_{[|x|<r]} v \, dx \le \int_\Omega v\zeta(x/r) \, dx = \int_\Omega e^u \varphi_r \, dx.$$

Using Step 3, we have

$$\int_{[|x|<r]} e^u \varphi_r \, dx \le Cr^2 \int_{[|x|<r]} e^u \, dx.$$

Using stability with test function $\zeta(x/r)$, we also have

$$\int_{[|x|<r]} e^u \, dx \le Cr^{N-4}$$

and so

$$\int_{[|x|<r]} e^u \varphi_r \, dx \le Cr^{N-2}.$$

By Step 1 and Step 3,

$$\int_{[r<|x|<R_0]} e^u \varphi_r \, dx \le Cr^N \int_{[r<|x|<R_0]} e^u |x|^{2-N} \, dx \le Cr^{N-2}.$$

Finally, using (5.17),

$$\int_{[R_0<|x|]\cap\Omega} e^u \varphi_r \, dx \le CR_0^{2-N} r^N \int_\Omega e^u \, dx \le C' r^{N-2}.$$

(5.6) follows. □

6. A bootstrap argument

Our next task consists in improving the L^1-estimates of the previous section to L^p-estimates for larger values of p. We do this through a bootstrap argument which is reminescent of the classical Moser iteration method, up to one major difference: we will take advantage of both the standard Sobolev inequality and the stability inequality (1.3). To be more precise, rather than (1.3), the following interpolated version of it will be used.

Lemma 6.1. *Let Ω be an open set of $\mathbb{R}^N$, $N \ge 1$. Assume that the stability inequality* (1.3) *holds. Then, for every $s \in [0,1]$,*

$$\int_\Omega e^{su} \varphi^2 \, dx \le \int_\Omega |(-\Delta)^{s/2} \varphi|^2 \, dx \quad \text{for all } \varphi \in H^2(\Omega) \cap H_0^1(\Omega).$$

In particular, for $s = \frac{1}{2}$,

$$\int_\Omega e^{u/2} \varphi^2 \, dx \le \int_\Omega |\nabla \varphi|^2 \, dx \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{6.1}$$

Proof Consider first the case where $\Omega = \mathbb{R}^N$. We apply complex interpolation between the family of spaces X_s, Y_s given for $0 \le s \le 1$ by

$$X_s = L^2((2\pi)^{-N} |\xi|^{4s} d\xi), \quad Y_s = L^2(e^{su} dx).$$

Recall that the inverse Fourier transform $\mathcal{F}^{-1} : X_0 \to Y_0$ satisfies $\|\mathcal{F}\|_{\mathcal{L}(X_0,Y_0)} = 1$. Furthermore, by the stability inequality (1.3) and Plancherel's theorem, we have

$$\int_{\mathbb{R}^N} e^u \varphi^2 \, dx \le \int_{\mathbb{R}^N} |\Delta \varphi|^2 \, dx = (2\pi)^{-n} \int_{\mathbb{R}^N} |\xi|^4 |\mathcal{F}(\phi)|^2 \, d\xi.$$

Thus, $\mathcal{F}^{-1} : X_1 \to Y_1$ satisfies $\|\mathcal{F}\|_{\mathcal{L}(X_1,Y_1)} \le 1$. By [22, Theorem 2], we deduce that $\|\mathcal{F}\|_{\mathcal{L}(X_s,Y_s)} \le 1$ for all $0 \le s \le 1$.

In the case where Ω is a bounded open set, simply repeat the above proof, using the spectral decomposition of the Laplace operator in place of the Fourier transform. More precisely, let (λ_k) denote the eigenvalues of the Laplace operator (with domain $D = H^2(\Omega) \cap H_0^1(\Omega)$), let $\hat{\varphi}_k$ be the k-th component of φ in the Hilbert basis of eigenfunctions associated to (λ_k) and interpolate between the family of weighted L^2-spaces X_s, Y_s corresponding to the norms

$$\|\varphi\|_{X_s}^2 = \sum_k \lambda_k^{2s} |\hat{\varphi}_k|^2, \quad \|\varphi\|_{Y_s}^2 = \int_\Omega e^{su} \varphi^2 \, dx.$$

In the case where Ω is an unbounded proper open set, take $k > 0$, let $\Omega_k = \Omega \cap B_k$ and

$$-\mu_k = \inf \left\{ \int_{\Omega_k} \left(|\Delta \varphi|^2 - e^u \varphi^2 \right) \, dx \ : \ \varphi \in H^2(\Omega_k) \cap H_0^1(\Omega_k), \|\varphi\|_{L^2(\Omega_k)} = 1 \right\}.$$

Then, the previous analysis leads to

$$\int_{\Omega_k} \left[(e^u - \mu_k)_+ \right]^s \varphi^2 \, dx \leq \int_{\Omega_k} |(-\Delta)^{s/2} \varphi|^2 \, dx \quad \text{ for all } \varphi \in H^2(\Omega_k) \cap H_0^1(\Omega_k).$$

By (1.3), $\lim_{k \to +\infty} -\mu_k \geq 0$ and the result follows. □

Our next lemma is simply the first step in the Moser iteration method: we multiply the equation by a power of its right-hand side, localize, and integrate.

Lemma 6.2. *Let Ω be an open set of $\mathbb{R}^N$, $N \geq 1$. Assume $(u, v) \in C^2(\Omega)^2$ solves*

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = e^u & \text{in } \Omega. \end{cases}$$

Take $\alpha > \frac{1}{2}$, $\varphi \in C_c^1(\Omega)$. Then, there exists a constant C depending on α only, such that

$$\frac{\sqrt{2\alpha - 1}}{\alpha} ||\nabla(v^\alpha \varphi)||_{L^2(\Omega)} \leq \|e^{\frac{u}{2}} v^{\alpha - \frac{1}{2}} \varphi\|_{L^2(\Omega)} + C||v^\alpha \nabla \varphi||_{L^2(\Omega)}. \tag{6.2}$$

and

$$\frac{2}{\sqrt{\alpha}} ||\nabla(e^{\frac{\alpha}{2} u} \varphi)||_{L^2(\Omega)} \leq \|e^{\frac{\alpha}{2} u} v^{\frac{1}{2}} \varphi\|_{L^2(\Omega)} + C||e^{\frac{\alpha}{2} u} \nabla \varphi||_{L^2(\Omega)}. \tag{6.3}$$

Proof. Since the computations are very similar, we prove only (6.2). We multiply $-\Delta v = e^u$ by $v^{2\alpha - 1} \varphi^2$ and we integrate. We obtain

$$\begin{aligned} \int_\Omega e^u v^{2\alpha - 1} \varphi^2 \, dx &= \int_\Omega \nabla v \cdot \nabla (v^{2\alpha - 1} \varphi^2) \, dx \\ &= \frac{2\alpha - 1}{\alpha^2} \int_\Omega |\nabla v^\alpha|^2 \varphi^2 \, dx + 2 \int_\Omega v^{2\alpha - 1} \varphi \nabla v \cdot \nabla \varphi \, dx \\ &= \frac{2\alpha - 1}{\alpha^2} \left(\int_\Omega |\nabla (v^\alpha \varphi)|^2 \, dx - \int_\Omega v^{2\alpha} |\nabla \varphi|^2 \, dx \right) \\ &\quad - \frac{2(\alpha - 1)}{\alpha^2} \int_\Omega v^\alpha \varphi \nabla \varphi \cdot \nabla v^\alpha \, dx. \end{aligned}$$

In the last term of the right hand side, replace $\varphi\nabla v^\alpha$ by $\nabla(v^\alpha\varphi) - v^\alpha\nabla\varphi$. Then,

$$\int_\Omega e^u v^{2\alpha-1}\varphi^2\,dx = \frac{2\alpha-1}{\alpha^2}\int_\Omega |\nabla(v^\alpha\varphi)|^2\,dx - \frac{1}{\alpha^2}\int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx - \frac{2(\alpha-1)}{\alpha^2}\int_\Omega v^\alpha\nabla\varphi\nabla(v^\alpha\varphi)\,dx,$$

which we rewrite as

$$(2\alpha-1)\int_\Omega |\nabla(v^\alpha\varphi)|^2\,dx = 2(\alpha-1)\int_\Omega v^\alpha\nabla\varphi\nabla(v^\alpha\varphi)\,dx + \int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx + \alpha^2\int_\Omega e^u v^{2\alpha-1}\varphi^2\,dx. \tag{6.4}$$

By the Cauchy-Schwarz inequality,

$$\left|\int_\Omega v^\alpha\nabla\varphi\nabla(v^\alpha\varphi)\,dx\right| \le \left(\int_{\mathbb{R}^n}|\nabla(v^\alpha\varphi)|^2\,dx\right)^{\frac12}\left(\int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx\right)^{\frac12},$$

Plugging this in (6.4), we obtain a quadratic inequality of the form

$$(2\alpha-1)X^2 \le 2|\alpha-1|AX + A^2 + B^2$$

where

$$X = ||\nabla(v^\alpha\varphi)||_2, \quad A = ||v^\alpha\nabla\varphi||_2 \quad \text{and} \quad B = \alpha\|e^{\frac12 u}v^{\alpha-\frac12}\varphi\|_2.$$

Solving the quadratic inequality, we deduce that

$$X \le \frac{|\alpha-1|A + \sqrt{|\alpha-1|^2A^2 + (2\alpha-1)(A^2+B^2)}}{2\alpha-1} \le \frac{B}{\sqrt{2\alpha-1}} + C_\alpha A.$$

(6.2) follows by replacing A, B and X with their values. □

We have just used the equation. Now, we use the stability assumption.

Lemma 6.3. *Make the same assumptions as in Lemma 6.2. Assume in addition that* (6.1) *holds for every $\varphi \in C_c^1(\Omega)$. Let $\alpha^\sharp, \alpha^*$ denote the largest two roots of the polynomial $X^3 - 8X + 4$. Then, for every $\alpha \in (\alpha^\sharp, \alpha^*)$, there exists a constant C depending on α only such that*

$$\int_\Omega |\nabla(v^\alpha\varphi)|^2\,dx \le C\int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx, \tag{6.5}$$

or

$$\int_\Omega |\nabla(e^{\frac{\alpha}{2}u}\varphi)|^2\,dx \le C\int_\Omega e^{\alpha u}|\nabla\varphi|^2\,dx, \tag{6.6}$$

for any $\varphi \in C_c^1(\Omega)$.

Proof. By Hölder's inequality,

$$\int_\Omega e^u v^{2\alpha-1}\varphi^2\,dx \le \left(\int_\Omega e^{\frac{2\alpha+1}{2}u}\varphi^2\,dx\right)^{\frac{1}{2\alpha}}\left(\int_\Omega e^{\frac{u}{2}}v^{2\alpha}\varphi^2\,dx\right)^{\frac{2\alpha-1}{2\alpha}}.$$

Using the stability inequality (6.1), we deduce that

$$\int_\Omega e^u v^{2\alpha-1}\varphi^2\,dx \le H^{\frac1\alpha}K^{2-\frac1\alpha},$$

where we set

$$H = ||\nabla(e^{\frac{\alpha}{2}u}\varphi)||_2 \ \text{ and } \ K = ||\nabla(v^\alpha\varphi)||_2.$$

Similarly,

$$\int_\Omega e^{\alpha u} v\varphi^2 \, dx \le K^{\frac{1}{\alpha}} H^{2-\frac{1}{\alpha}}.$$

Combining with (6.2)-(6.3), this gives

$$\frac{\sqrt{2\alpha-1}}{\alpha} H \le K^{\frac{1}{2\alpha}} H^{1-\frac{1}{2\alpha}} + C||e^{\frac{\alpha}{2}u}\nabla\varphi||_2, \tag{6.7}$$

$$\frac{2}{\sqrt{\alpha}} K \le H^{\frac{1}{2\alpha}} K^{1-\frac{1}{2\alpha}} + C||v^\alpha \nabla\varphi||_2. \tag{6.8}$$

Multiply (6.7) by (6.8). Then,

$$\left(\frac{2\sqrt{2\alpha-1}}{\alpha\sqrt{\alpha}} - 1\right) HK \le aH^{\frac{1}{2\alpha}} K^{1-\frac{1}{2\alpha}} + bK^{\frac{1}{2\alpha}} H^{1-\frac{1}{2\alpha}} + ab, \tag{6.9}$$

where

$$a = C||e^{\frac{\alpha}{2}u}\nabla\varphi||_2 \text{ and } b = C||v^\alpha\nabla\varphi||_2.$$

Note that for $\alpha \in (\alpha^\sharp, \alpha^*)$,

$$\delta := \frac{2\sqrt{2\alpha-1}}{\alpha\sqrt{\alpha}} - 1 > 0.$$

Introduce

$$X = K^{\frac{1}{2\alpha}} H^{1-\frac{1}{2\alpha}} \quad \text{and} \quad Y = H^{\frac{1}{2\alpha}} K^{1-\frac{1}{2\alpha}}.$$

Then, (6.9) can be rewritten as

$$\delta XY \le aY + bX + ab,$$

and so, either X is bounded by a multiple of a or Y by a multiple of b. In the former case, recalling (6.7), we obtain (6.6). In the latter case, (6.8) implies (6.5). □

In the two previous lemmata, we have used successively the equation and the stability assumption. Now, we apply the Sobolev inequality to set up a bootstrap procedure.

Lemma 6.4. *Make the same assumptions as in Lemma 6.3. Take $\alpha \in (\alpha^\sharp, \alpha^*)$ and for $R > 0$, let B_R denote a ball of radius R (resp. an annulus of radii R and $R/2$) contained in Ω. Assume that there exists a constant C depending on N and α only, such that for all $R > 0$ (resp. for all R large enough)*

$$\int_{B_R} (e^{\alpha u} + v^{2\alpha}) \, dx \le CR^{N-4\alpha}. \tag{H_α}$$

Then, $\left(H_{\frac{N}{N-2}\alpha}\right)$ also holds.

Bootstrapping the above lemma, we find

Corollary 6.5. *Make the same assumptions as in Lemma 6.3. Assume that (H_α) holds for some $\alpha \in (\alpha^\sharp, \alpha^*)$. Then,*

$$\begin{aligned} &\int_{B_R} e^{pu} \, dx \le CR^{N-4p}, \qquad \text{for all } p < p^* := \alpha^* + \frac{1}{2}, \\ &\int_{B_R} v^q \, dx \le CR^{N-2q}, \qquad \text{for all } q < q^* := \frac{2N}{N-2}\alpha^*. \end{aligned} \tag{6.10}$$

Remark 6.6. The inequality (H_α) can be further simplified if the boundary values/limiting behavior at infinity of the solution is known. See in particular Proposition A.1.

Proof of the lemma 6.4. Assume (H_α) is valid. By Lemma 6.3, either (6.5) or (6.6) holds.
Assume that (6.5) is valid (the other case is similar). Using the Sobolev embedding, we obtain

$$\left(\int_{\mathbb{R}^N} v^{2^*\alpha}\varphi^{2^*}dx\right)^{\frac{2}{2^*}} \leq \int_{\mathbb{R}^N} |\nabla(v^\alpha\varphi)|^2dx \lesssim \int_{\mathbb{R}^N} v^{2\alpha}|\nabla\varphi|^2dx.$$

Take a standard cut-off function $\psi \in C_c^1(B_2)$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in B_1, and $\psi = 0$ outside B_2. Apply the above inequality with $\varphi(x) = \psi(x/R)$ and use (H_α). Then,

$$\int_{B_R} v^{2^*\alpha}dx \lesssim R^{N-2.2^*\alpha}. \tag{6.11}$$

Going back to (6.7), we deduce similarly that

$$\int_{B_R} e^{\frac{2^*\alpha}{2}u}dx \lesssim R^{N-2^*.2\alpha}. \tag{6.12}$$

□

Proof of the corollary 6.5. By Hölder's inequality, if (H_α) holds for some α, then (H_β) holds for all $\beta \leq \alpha$. So, bootstrapping (H_α), we easily deduce that it holds for all $\alpha < \frac{N}{N-2}\alpha^*$. Fix at last $\alpha < \alpha^*$ and apply now stability (6.1), with test function $e^{\frac{\alpha}{2}u}\psi(x/R)$, to deduce that (6.10) holds for all $p = \alpha + \frac{1}{2} < p^* = \alpha^* + \frac{1}{2}$.
□

7. The Liouville theorem

We prove here Theorem 1.7. Assume by contradiction that there exists a solution u of (1.2) which is stable outside a compact set $K \subset B_{R_0}$ and such that $\overline{v}(\infty) = 0$.
Step 1. $e^u \in L^p(\mathbb{R}^N)$, for every $p < p^*$.

By Lemma 5.1, we have

$$\int_{A_R} v\ dx \leq CR^{N-2} \quad \text{for every } R > R_0, \tag{7.1}$$

where A_R is the annulus of radii R and $2R$. In addition, stability (1.3) implies that

$$\int_{A_R} e^u\ dx \leq CR^{N-4} \quad \text{for every } R > R_0, \tag{7.2}$$

Recall now the following standard elliptic estimate : for $p \in [1, \frac{N}{N-2})$,

$$||v||_{L^p(B_2\setminus B_1)} \lesssim ||\Delta v||_{L^1(B_2\setminus B_1)} + ||v||_{L^1(B_4\setminus B_{1/2})},$$

and its rescaled version

$$R^{-\frac{N}{p}}||v||_{L^p(A_R)} \lesssim R^{-N}\left(R^2||\Delta v||_{L^1(A_R)} + ||v||_{L^1(B_{4R}\setminus B_{R/2})}\right).$$

Applying this estimate respectively to u and v, we deduce from (7.1), (7.2) that (H_α) holds for any $\alpha \in [1, \frac{N}{N-2})$ and all large R. Hence, (6.10) holds. By a straightforward covering argument, Step 1 follows.

The rest of the proof is very similar to the one given in [9];

Step 2.

$$\lim_{|x|\to+\infty} |x|^4 e^u = 0.$$

By Step 1., given any $\delta > 0$, we can choose $\tilde{R}$ large enough such that

$$\int_{|y|\geq\tilde{R}} e^{pu} dx \leq \delta^p. \tag{7.3}$$

Let $x \in \mathbb{R}^N \backslash \overline{B_{R_0}}$, $|x| \geq 4\tilde{R}$. Set $R = \frac{2}{3}|x|$. This yields

$$B(x, R/4) \subset A_R = B_{2R} \backslash B_R \subset \{y \in \mathbb{R}^N \mid |y| \geq \tilde{R}\}.$$

Thus, we have

$$\begin{aligned} &\int_{B(x,R/4)} e^{pu}\, dx \leq CR^{N-4p}, \qquad \text{for all } p < p^* := \alpha^* + \frac{1}{2}, \\ &\int_{B(x,R/4)} v^q\, dx \leq CR^{N-2q}, \qquad \text{for all } q < q^* := \frac{2N}{N-2}\alpha^*. \end{aligned} \tag{7.4}$$

Next fix $\delta > 0$ and consider $w = e^u$. By Kato's inequality, w satisfies

$$-\Delta w - vw \leq 0 \quad \text{in } \mathbb{R}^N. \tag{7.5}$$

Take ε small enough such that $\frac{N}{2-\varepsilon} < \frac{2N}{N-2}\alpha^*$. Here, we have used the assumption $5 \leq N \leq 12$. The Serrin-Trudinger inequality [20, 23] for subsolutions to (7.5) ensures that for any $p < p^*$

$$||w||_{L^\infty(B(x,\frac{R}{8}))} \leq CR^{-\frac{N}{p}} ||w||_{L^p(B(x,\frac{R}{4}))}$$

where C depends to N, p and

$$R^\varepsilon ||v||_{L^{\frac{N}{2-\varepsilon}}(B(x,\frac{R}{4}))}.$$

In particular, for $p = \frac{N}{4}$ and using (7.4)

$$e^{u(x)} \leq CR^{-4} ||e^u||_{L^{N/4}(B(x,\frac{R}{4}))} \tag{7.6}$$

where C depends only to N. Combining (7.3) and (7.6) gives

$$e^{u(x)} \leq C\delta R^{-4} \lesssim \delta |x|^{-4},$$

which proves Step 2.

Step 3. By Step 2., there exists $R_1 > R_0$ such that

$$-\Delta \overline{v} \leq \frac{1}{2r^4} \quad \text{for all } r > R_1.$$

Hence,

$$\overline{v}'(r) \geq \frac{C(N)}{r^{N-1}} - \frac{1}{2(N-4)r^3} \quad \text{for all } r > R_1.$$

Integrating between r and $+\infty$, this yields

$$-\Delta \overline{u}(r) = \overline{v}(r) = \overline{v}(r) - \overline{v}(\infty) \leq \frac{1}{2r^2}\left(\frac{1}{2(N-4)} - \frac{C'(N)}{r^{N-4}}\right) \quad \text{for all } r > R_1.$$

Since $N \geq 5$ and choosing $R_2 > R_1$ large enough to have

$$\frac{1}{2(N-4)} - \frac{C'(N)}{r^{N-4}} \leq 1 \text{ for all } r > R_3,$$

we get

$$-\Delta \overline{u} \leq \frac{1}{2r^2} \text{ for all } r > R_3.$$

In the same way, we have for some $R_3 > R_2$

$$\overline{u}'(r) \geq -\frac{1}{r} \quad \text{for all } r > R_3.$$

Integrating the latter and taking the exponential, we get

$$r^4 e^{\overline{u}(r)} \geq cr^3 \quad \text{for all } r > R_3,$$

where the constant $c > 0$ does not depend on r. By Jensen's inequality, we have for any $r > R_3$,

$$cr^3 \leq r^4 e^{\overline{u}(r)} \leq r^4 \overline{e^u}(r) \leq \max_{|x|=r} |x|^4 e^{u(x)}.$$

This contradicts Step 2. □

8. Partial regularity of the extremal solution in dimension $N \geq 13$.

In this section, we prove the second part of Theorem 1.9. As in the previous sections, we interpret our equation as the system (4.2). Here, our task consists in showing that if a rescaled L^p norm of e^u is small on some ball B, then it remains small on any ball of smaller radius, which is included in B. This provides an estimate in Morrey spaces, for which an ϵ-regularity theorem is available, thanks to the Moser-Trudinger inequality.

Let u be the extremal solution of (1.8). By scaling, we may assume that $B_1(0) \subset\subset \Omega$. For any $x \in B_{\frac{1}{2}}(0)$, $1 \leq p < p^*$ and $0 < r \leq 1 - |x|$ we define

$$E(x,r) = r^{4p-N} \int_{B_r(x)} e^{pu} dy.$$

Lemma 8.1. *Assume $E(0,1) < 1$ and fix $p < q < p^*$. Then, there exists a positive constant $C = C(N,p,q) > 0$ such that*

$$E(x,r) \leq C\, E(0,1)^{1-\frac{p}{q}} \quad \text{for all } x \in B_{\frac{1}{2}}(0),\ 0 < r \leq \frac{1}{2}. \tag{8.1}$$

Proof. Let $u = u_1 + u_2$, where

$$\begin{cases} -\Delta u_1 = 0 & \text{in } B_1(0), \\ u_1 = u & \text{on } \partial B_1(0), \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = v & \text{in } B_1(0), \\ u_2 = 0 & \text{on } \partial B_1(0). \end{cases}$$

By the maximum principle, $u_1, u_2 > 0$ in $B_1(0)$. Write $E(x,r) = E_1 + E_2$, where

$$E_1 = r^{4p-N} \int_{B_r(x) \cap [u_2 \leq \theta]} e^{pu}\, dy \ , \quad E_2 = r^{4p-N} \int_{B_r(x) \cap [u_2 > \theta]} e^{pu}\, dy,$$

and where $\theta > 0$ will be fixed later on. By Kato's inequality, e^{pu_1} is subharmonic. By the mean value inequality, it follows that

$$⨍_{B_r(x)} e^{pu_1}\, dy \leq ⨍_{B_{\frac{1}{2}}(x)} e^{pu_1}\, dy. \tag{8.2}$$

Hence,

$$E_1 \le r^{4p-N} e^{p\theta} \int_{B_r(x)} e^{pu_1}\, dy \lesssim r^{4p} e^{p\theta} \int_{B_{\frac{1}{2}}(x)} e^{pu}\, dy \le r^{4p} e^{p\theta} \int_{B_1(0)} e^{pu}\, dy \le r^{4p} e^{p\theta} E(0,1). \tag{8.3}$$

To estimate E_2, we first note that

$$\big|B_r(x) \cap [u_2 > \theta]\big| \le e^{-p\theta} \int_{B_r(x)\cap[u_2>\theta]} e^{pu}\, dy \le \frac{r^N E(x,r)}{r^{4p} e^{p\theta}}. \tag{8.4}$$

By Hölder's inequality, we also have

$$\int_{B_r(x)\cap[u_2>\theta]} e^{pu}\, dy \le \left(\int_{B_r(x)} e^{qu}\, dy\right)^{\frac{p}{q}} \Big|\big\{B_r(x)\cap[u_2>\theta]\big\}\Big|^{1-\frac{p}{q}}. \tag{8.5}$$

Recall from Section 6 that

$$\int_{B_r(x)} e^{qu}\, dy \lesssim r^{N-4q}.$$

Using this together with (8.4), (8.5), we find

$$E_2 \le r^{4p-N} \left(\int_{B_r(x)} e^{qu}\, dy\right)^{\frac{p}{q}} \Big|\big\{B_r(x)\cap[u_2>\theta]\big\}\Big|^{1-\frac{p}{q}} \lesssim \left[\frac{E(x,r)}{r^{4p}e^{p\theta}}\right]^{1-\frac{p}{q}}. \tag{8.6}$$

Combining (8.3) and (8.6), for all $x \in B_{\frac{1}{2}}(0)$ and $0 < r \le \frac{1}{2}$ we find

$$E(x,r) \lesssim r^{4p} e^{\theta p} E(0,1) + \left[\frac{E(x,r)}{r^{4p}e^{p\theta}}\right]^{1-\frac{p}{q}}. \tag{8.7}$$

If for some $x \in B_{\frac{1}{2}}(0)$ and $0 < r \le \frac{1}{2}$ we have

$$r^{4p} E(0,1) \ge \left[\frac{E(x,r)}{r^{4p}}\right]^{1-\frac{p}{q}}, \tag{8.8}$$

then, using the assumption $E(0,1) < 1$, we find

$$E(x,r) \le r^{4p\frac{2q-p}{q-p}} E(0,1)^{\frac{q}{q-p}} \le E(0,1)^{1-\frac{p}{q}},$$

which is the desired inequality. So, assume that (8.8) does not hold. Then, we can find $\theta > 0$ such that

$$r^{4p}(e^{p\theta})^{2-\frac{p}{q}} E(0,1) = \left[\frac{E(x,r)}{r^{4p}}\right]^{1-\frac{p}{q}},$$

that is,

$$r^{4p} e^{p\theta} E(0,1) = \left[\frac{E(x,r)}{r^{4p}e^{p\theta}}\right]^{1-\frac{p}{q}}.$$

Using this fact in (8.7) we derive

$$E(x,r) \lesssim r^{4p} e^{\theta p} E(0,1) \lesssim \Big[E(0,1)E(x,r)\Big]^{\frac{q-p}{2q-p}},$$

and finally $E(x,r) \lesssim E(0,1)^{1-\frac{p}{q}}$. □

Proof of Theorem 1.9 completed. Let Σ be the singular set of u. Let $1 < p < q < p^*$ be fixed. We claim that

$$\Sigma \subset \left\{ x \in \Omega : \limsup_{r\to 0} r^{4p-N} \int_{B_r(x)} e^{pu} > 0 \right\}. \tag{8.9}$$

Assume to the contrary that there exists $x_0 \in \Sigma$ such that $\lim_{r\to 0} E(x_0, r) = 0$. Fix $\varepsilon_0 > 0$. Then, there exists $\rho = \rho(\varepsilon_0) > 0$ such that $E(x_0, r) \le \varepsilon_0$ for all $0 < r \le \rho$. For the sake of clarity let us assume that $x_0 = 0$ and $\rho = 1$. Thus, we can apply Lemma 8.1 and obtain

$$E(x, r) \le C(N, p, q)\varepsilon_0^{1-\frac{p}{q}} \quad \text{for all } x \in B_{\frac{1}{2}}(0)\,, 0 < r \le \frac{1}{2}.$$

Using Hölder's inequality this implies that $e^u \in M^{N/4}(B_{\frac{1}{2}}(0))$ and

$$\|e^u\|_{M^{N/4}(B_{\frac{1}{2}}(0))} \le C(N, p, q)\varepsilon_0^{\frac{1}{p}-\frac{1}{q}}.$$

Let

$$w(x) = \int_{\mathbb{R}^N} |x - y|^{4-N} e^{u(y)} dy\,, \quad x \in B_{\frac{1}{2}}(0)$$

and $\tilde{w} = u - w$. By Lemma 7.20 in [15] it follows that $e^{\beta w} \in L^1(B_{\frac{1}{2}}(0))$ for all

$$0 < \beta < \frac{C(N)}{\|e^u\|_{M^{\frac{N}{4}}(B_{\frac{1}{2}}(0))}}.$$

Since $\tilde{w}$ is biharmonic in $B_{\frac{1}{2}}(0)$, it follows that $e^u \in L^\beta(B_{1/4}(0))$ for all

$$0 < \beta < C(N, p, q)\varepsilon_0^{\frac{1}{q}-\frac{1}{p}}.$$

Letting $\varepsilon_0 << 1$ small, we have $e^u \in L^\beta(B_{1/4}(0))$ for some $\beta > N/4$, which, by standard regularity theory, yields $u \in L^\infty(B_{1/8}(0))$ and contradicts $0 \in \Sigma$.

Hence, (8.9) holds for all $1 < p < p^*$. By [12, Lemma 5.3.4], it follows that $\mathcal{H}^{N-4p}(\Sigma) = 0$ for all $1 < p < p^*$. Thus $\mathcal{H}_{dim}(\Sigma) \le N - 4p^*$. □

9. A scaling argument

This last section is devoted to the proof of Theorem 1.11. By rescaling, we may always assume that $\lambda = 1$. By standard elliptic regularity, it suffices to show that $u \le C$ in Ω. We assume to the contrary that there exists a sequence of solutions u_n of fixed index k, such that $M_n := \max_{\overline{\Omega}} u_n \to +\infty$, as $n \to +\infty$. Let x_n denote a corresponding point of maximum of u_n. Passing to a subsequence if necessary, we may assume that there exists $x_0 \in \overline{\Omega}$, such that $x_n \to x_0$, as $n \to +\infty$. Since Ω is convex, we can also assert that u_n is uniformly bounded in a fixed neighborhood of the boundary i.e. $x_n \in \omega \subset\subset \Omega$ for large n. See e.g. [26] for this standard boundary estimate.

We use a scaling argument. Let $r_n = e^{-M_n/4}$ and $U_n(x) = u_n(x_n + r_n x) - M_n$, for $x \in \Omega_n := \frac{1}{r_n}(\Omega - x_n)$. Then, U_n solves

$$\begin{cases} \Delta^2 U_n = e^{U_n} & \text{in } \Omega_n, \\ U_n + M_n = \Delta U_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{9.1}$$

Since $x_n \in \omega \subset\subset \Omega$, $\Omega_n \to \mathbb{R}^N$, as $n \to +\infty$. We claim that (U_n) is uniformly bounded on compact sets of $\mathbb{R}^N$. To see this, fix a ball B_R and n so large that $B_R \subset \Omega_n$. Write $-\Delta U_n = V_n = V_n^1 - V_n^2$, where V_n^1 solves

$$\begin{cases} -\Delta V_n^1 = e^{U_n} & \text{in } B_R, \\ V_n^1 = 0 & \text{on } \partial B_R, \end{cases}$$

and V_n^2 is harmonic in B_R. Since $U_n \le 0$, V_n^1 is uniformly bounded in B_R. Using the assumption (1.9), we also have

$$\int_{B(x_n, r_n R)} v_n \, dx \le C(r_n R)^{N-2}. \tag{9.2}$$

In other words, $V_n = -\Delta U_n$ is bounded in $L^1(B_R)$. Since V_n^1 is uniformly bounded in B_R, it follows that V_n^2 is bounded in $L^1(B_R)$. Since V_n^2 is harmonic, V_n^2 is uniformly bounded in $B_{R/2}$. Similarly, write $U_n = U_n^1 - U_n^2$, where

$$\begin{cases} -\Delta U_n^1 = V_n & \text{in } B_{R/2}, \\ U_n^1 = 0 & \text{on } \partial B_{R/2}, \end{cases}$$

and $U_n^2 \ge 0$ is harmonic in $B_{R/2}$. Since V_n is bounded in $B_{R/2}$, so is U_n^1. Since $U_n^2(0) = U_n^1(0)$, we may now apply Harnack's inequality to conclude that U_n^2 is uniformly bounded in $B_{R/4}$. Hence, U_n is uniformly bounded on compact subsets of $\mathbb{R}^N$. So, we may pass to the limit in the first line of (9.1) and find a solution U of finite Morse index to (1.2). Thanks to (9.2), if $V = -\Delta U$ and $R > 0$, then

$$\int_{B_R} V \, dx \le C R^{N-2},$$

which is possible only if $\overline{V}(\infty) = 0$. This contradicts Theorem 1.7.

□

Appendix A. Appendix

The following is a consequence of Kato's inequality. It is inspired by a similar result of P. Souplet [21] (see also [7] for similar inequalities on bounded domains).

Proposition A.1. *Let u be a stable solution of (1.2) and $v = -\Delta u$. Then,*

$$v \ge \sqrt{2} e^{u/2} \quad \text{in } \mathbb{R}^N.$$

Proof. Let $w := \sqrt{2} e^{u/2} - v$. A straightforward calculation yields

$$\Delta w \ge \frac{1}{\sqrt{2}} e^{u/2} w \quad \text{in } \mathbb{R}^N. \tag{A.1}$$

Since $v > 0$ (see the proof of Lemma 5.1), we have $w^+ \le \sqrt{2} e^{u/2}$ in $\mathbb{R}^N$. We multiply (A.1) by w^+ and integrate over $B_R(0)$, for $R > 0$. We obtain

$$\int_{B_R(0)} |\nabla w^+|^2 + \frac{1}{\sqrt{2}} \int_{B_R(0)} e^{u/2} |w^+|^2 \le \int_{\partial B_R(0)} w^+ \frac{\partial w^+}{\partial R}. \tag{A.2}$$

For all $R > 0$ define

$$f(R) := \frac{1}{2} \int_{\partial B_1(0)} |w^+(Rx)|^2 d\sigma(x).$$

Then (A.2) reads

$$\int\limits_{B_R(0)} |\nabla w^+|^2 + \frac{1}{\sqrt{2}} \int\limits_{B_R(0)} e^{u/2}|w^+|^2 \leq R^{N-1} f'(R). \tag{A.3}$$

Since u is stable and $e^u \geq 2|w^+|^2$ in $\mathbb{R}^N$, we have

$$R^{N-4} \gtrsim \int\limits_{B_R(0)} e^u ds \geq \int_0^R r^{N-1} f(r) dr.$$

In particular, f cannot be strictly increasing in a given interval $[S, +\infty)$. Hence, there exists an increasing sequence $\{R_j\}$ such that $R_j \to \infty$ and $f'(R_j) \leq 0$. Now, letting $R = R_j$ in (A.3), we find $w^+ \equiv 0$, that is, $v \geq \sqrt{2} e^{u/2}$ in $\mathbb{R}^N$.

□

References

[1] Gianni Arioli, Filippo Gazzola, and Hans-Christoph Grunau, *Entire solutions for a semilinear fourth order elliptic problem with exponential nonlinearity*, J. Differential Equations **230** (2006), no. 2, 743–770, DOI 10.1016/j.jde.2006.05.015. MR2269942 (2007i:35064)

[2] Gianni Arioli, Filippo Gazzola, Hans-Christoph Grunau, and Enzo Mitidieri, *A semilinear fourth order elliptic problem with exponential nonlinearity*, SIAM J. Math. Anal. **36** (2005), no. 4, 1226–1258 (electronic), DOI 10.1137/S0036141002418534. MR2139208 (2006c:35070)

[3] Elvise Berchio, Alberto Farina, Alberto Ferrero, and Filippo Gazzola, *Existence and stability of entire solutions to a semilinear fourth order elliptic problem*, J. Differential Equations **252** (2012), no. 3, 2596–2616. MR2860632

[4] Elvise Berchio and Filippo Gazzola, *Some remarks on biharmonic elliptic problems with positive, increasing and convex nonlinearities*, Electron. J. Differential Equations (2005), No. 34, 20 pp. (electronic). MR2135245 (2006e:35067)

[5] Haim Brezis, *Is there failure of the inverse function theorem?*, Morse theory, minimax theory and their applications to nonlinear differential equations, New Stud. Adv. Math., vol. 1, Int. Press, Somerville, MA, 2003, pp. 23–33. MR2056500 (2005h:35083)

[6] Craig Cowan, *Liouville theorems for stable Lane-Emden systems and biharmonic problems*, http://arxiv.org/abs/1207.1081 (4 july 2012).

[7] Craig Cowan, Pierpaolo Esposito, and Nassif Ghoussoub, *Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains*, Discrete Contin. Dyn. Syst. **28** (2010), no. 3, 1033–1050, DOI 10.3934/dcds.2010.28.1033. MR2644777 (2011e:35085)

[8] Craig Cowan and Nassif Ghoussoub, *Regularity of semi-stable solutions to fourth order nonlinear eigenvalue problems on general domains*, http://fr.arxiv.org/abs/1206.3471 (15 june 2012).

[9] E. N. Dancer and Alberto Farina, *On the classification of solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$: stability outside a compact set and applications*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1333–1338, DOI 10.1090/S0002-9939-08-09772-4. MR2465656 (2009h:35123)

[10] Juan Dávila, Louis Dupaigne, Ignacio Guerra, and Marcelo Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity*, SIAM J. Math. Anal. **39** (2007), no. 2, 565–592, DOI 10.1137/060665579. MR2338421 (2008h:35053)

[11] Juan Dávila, Isabel Flores, and Ignacio Guerra, *Multiplicity of solutions for a fourth order problem with exponential nonlinearity*, J. Differential Equations **247** (2009), no. 11, 3136–3162, DOI 10.1016/j.jde.2009.07.023. MR2569861 (2010j:35115)

[12] Louis Dupaigne, *Stable solutions of elliptic partial differential equations*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 143, Chapman & Hall/CRC, Boca Raton, FL, 2011. MR2779463

[13] Louis Dupaigne, Alberto Farina, and Boyan Sirakov, *Regularity of the extremal solution for the Liouville system*, to appear in Proceedings of the ERC Workshop on Geometric Partial Differential Equations. Ed. Scuola Normale Superiore di Pisa.

[14] I. M. Gel′fand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. (2) **29** (1963), 295–381. MR0153960 (27 #3921)

[15] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364 (2001k:35004)

[16] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1972/73), 241–269. MR0340701 (49 #5452)

[17] Chang-Shou Lin, *A classification of solutions of a conformally invariant fourth order equation in* $\mathbf{R}^n$, Comment. Math. Helv. **73** (1998), no. 2, 206–231, DOI 10.1007/s000140050052. MR1611691 (99c:35062)

[18] Amir Moradifam, *The singular extremal solutions of the bi-Laplacian with exponential nonlinearity*, Proc. Amer. Math. Soc. **138** (2010), no. 4, 1287–1293, DOI 10.1090/S0002-9939-09-10257-5. MR2578522 (2011c:35126)

[19] Ken'ichi Nagasaki and Takashi Suzuki, *Spectral and related properties about the Emden-Fowler equation* $-\Delta u = \lambda e^u$ *on circular domains*, Math. Ann. **299** (1994), no. 1, 1–15, DOI 10.1007/BF01459770. MR1273074 (95f:35190)

[20] James Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302.

[21] Philippe Souplet, *The proof of the Lane-Emden conjecture in four space dimensions*, Adv. Math. **221** (2009), no. 5, 1409–1427, DOI 10.1016/j.aim.2009.02.014. MR2522424 (2010h:35088)

[22] Elias M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. **83** (1956), 482–492. MR0082586 (18,575d)

[23] Neil S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

[24] Kelei Wang, *Partial regularity of stable solutions to the Emden equation*, Calculus of Variations and Partial Differential Equations **44** (2012), 601-610. 10.1007/s00526-011-0446-3.

[25] Guillaume Warnault, *Liouville theorems for stable radial solutions for the biharmonic operator*, Asymptotic Analysis **69** (2010), 87-98.

[26] Juncheng Wei, *Asymptotic behavior of a nonlinear fourth order eigenvalue problem*, Comm. Partial Differential Equations **21** (1996), no. 9-10, 1451–1467, DOI 10.1080/03605309608821234. MR1410837 (97h:35066)

[27] Juncheng Wei, Xingwang Xu, and Wen Yang, *Classification of stable solutions to biharmonic problems in large dimensions*.http://www.math.cuhk.edu.hk/∼wei/publicationpreprint.html

[28] Juncheng Wei and Dong Ye, *Nonradial solutions for a conformally invariant fourth order equation in* $\mathbb{R}^4$, Calc. Var. Partial Differential Equations **32** (2008), no. 3, 373–386, DOI 10.1007/s00526-007-0145-2. MR2393073 (2009h:35158)

[29] ———, *Liouville Theorems for finite Morse index solutions of Biharmonic problem*. http://www.math.cuhk.edu.hk/∼wei/publicationpreprint.html

[30] Xue-Feng Yang, *Nodal sets and Morse indices of solutions of super-linear elliptic PDEs*, J. Funct. Anal. **160** (1998), no. 1, 223–253, DOI 10.1006/jfan.1998.3301. MR1658692 (99j:35058)

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