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On the entropy of the mean field spin glass model

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Abstract: From the study of a functional equation relating the Gibbs measures at two different tempratures we prove that the specific entropy of the Gibbs measure of the Sherrington-Kirkpatrick Spin Glass Model vanishes at the inverse temperature $\beta = 4\log 2$.

1 Introduction and main results

Over the last decade, mean field models of spin glasses have motivated increasingly many studies by physicists and mathematicians [1, 4, 5, 6, 7, 8, 10]. The existence of infinite volume limit of thermodynamic quantities is now rigorously established thanks to the development of numerous remarkable analytical techniques. For the Sherrington-Kirkpatrick model, the first major results of Guerra and Toninelli [5] on existence and uniqueness of the free energy, are generalized by Aizenman, Sims and Starr [2] in a scheme giving variational upper bounds on the free energy. Talagrand [10], under some conditions on the overlap function, contributed to the entirely rigorous account of the original formulae proposed by Parisi [7].

An interesting question, related to the behaviour of Gibbs measures, is the study of their specific entropy. Despite the numerous developments achieved lately on this model, the study of the properties of the entropy is still missing in the literature. The specific entropy decreases with the temperature and the high temperature entropy can easily be estimated. By lowering the temperature the entropy should eventually vanish and an early result, given in [1], corroborates the idea that the entropy does not vanish very fast. In this note we estimate the value of the (low) temperature at which the mean entropy of the Gibbs measure vanishes.

The approach we use here is totally self-contained. From the low-temperature results, we need solely the existence of the thermodynamic limit of the quenched specific free energy and its self-averaging property.

We first recall some basic definitions. Suppose that a finite set of n sites is given. With each site we associate the one-spin space $\Sigma := \{1, -1\}$. The natural configuration space is then the product space $\Sigma_n = \{-1, 1\}^n = \Sigma^n$, with $\mathrm{card}\Sigma_n = 2^n$ equipped with the uniform probability measure ν_n. For each $\sigma \in \Sigma_n$, the finite volume Hamiltonian of the

model is given by the following real-valued function on Σ_n

$$H_n(\sigma) = -\frac{1}{\sqrt{n}} \sum_{1\le i<j\le n} J_{ij}\sigma_i\sigma_j,$$

where the family of couplings $J = (J_{ij})_{1\le i<j\le n}$ are independent centred Gaussian random variables of variance 1.

At the inverse temperature $\beta = \frac{1}{T} > 0$, the disorder dependent partition function $Z_n(\beta, J)$, is given by the sum of the Boltzmann factors

$$Z_n(\beta, J) = \sum_{\sigma} e^{-\beta H_n(\sigma,J)}.$$

Moreover, if E_J denotes the expectation with respect to the randomness J, it is very simple to show that $E_J Z_n(\beta, J) = 2^n e^{\frac{\beta^2}{4}(n-1)}$.

When the randomness J is fixed, the corresponding conditional Gibbs probability measure is denoted by $\mu_{n,\beta}(\sigma|J)$ and given by:

$$\mu_{n,\beta}(\sigma|J) = \frac{e^{-\beta H_n(\sigma,J)}}{Z_n(\beta, J)}.$$

The entropy of $\mu_{n,\beta}$, is defined as usual by $S(\mu_{n,\beta,J}) = -\sum_\sigma \mu_{n,\beta}(\sigma|J)\log\mu_{n,\beta}(\sigma|J)$.

The real functions

$$f_n(\beta) = \frac{1}{n} E_J \log Z_n(\beta, J)$$

and

$$\overline{f}_n(\beta) = \frac{1}{n}\log E_J Z_n(\beta, J),$$

define the quenched average of the specific free energy and the annealed specific free energy respectively. The ground state energy density $-\epsilon_n(J)$ is given by

$$-\epsilon_n(J) = \frac{1}{n}\inf_{\sigma\in\Sigma_n} H_n(\sigma, J).$$

At the low temperature region ($\beta > 1$), the following two infinite volume limits

$$\lim_{n\to\infty} f_n(\beta, J) = f_\infty(\beta),$$

and,

$$-\lim_{n\to\infty}\epsilon_n(J) = \lim_{\beta\to\infty}\frac{f_\infty(\beta)}{\beta} = -\epsilon_0$$

exist for almost all J and are non random; this result has been rigorously proved by Guerra and Toninelli [5].

The main results of this note are stated in the following and proved in the next section.

Proposition: *Almost surely, at the inverse temperature $\beta_* = 4\log 2 = 2.77258\cdots$, the thermodynamic limit of the quenched free energy is given by*

$$f_\infty(\beta_*) = \lim_{n\to\infty}\frac{1}{n}E_J\log Z_n(\beta_*, J) = f_\infty(1) + (\beta_* - 1)\log 2 = \frac{\beta_*^2}{4} + \frac{\beta_1^2}{4} = \beta_*\log 2 + \frac{1}{4}.$$

The Parisi formula provides with the expression of the free energy for the entire low temperature region in terms of a functional equation; the pertinence of the precise calculation of the limit at a particular value of the temperature stems from its usefulness in determining the point where the entropy vanishes. This gives new insight to the behaviour of the model and is summarised in the following

Theorem: *At the inverse temperature $\beta_* = 4\log 2 = 2.77258\cdots$, the specific entropy $s(\mu_{\beta_*})$ of the Gibbs measure vanishes almost surely:*

$$s(\mu_{\beta_*}) := \lim_{n\to\infty} \frac{1}{n} S(\mu_{n,\beta_*,J}) = -\lim_{n\to\infty} \frac{1}{n} \sum_{\sigma} \mu_{n,\beta_*}(\sigma|J) \log \mu_{n,\beta_*}(\sigma|J) = 0.$$

Remark: The formulation of the above statement assumes that the limit $\lim_{n\to\infty} \frac{1}{n} S(\mu_{n,\beta_*,J})$ exists and is independent of J. This follows from general principles and can immediately be obtained from the existence and self-averaging of the low temperature specific free energy.

2 Proof of the main results

Notice first, that for all $\beta > 0$, the quenched limit $f_\infty(\beta)$ exists and is a convex function of β [5]. Let $\beta_1 \equiv 1$. From the high temperature results [1], we have, almost surely, that

$$\begin{aligned} f_\infty(\beta_1) &= \lim_{n\to\infty} \frac{1}{n} E_J \log Z_n(\beta_1, J) = \lim_{n\to\infty} \frac{1}{n} \log E_J Z_n(\beta_1, J) \\ &= \overline{f}_\infty(\beta_1) = \log 2 + \frac{\beta_1^2}{4} \\ &= \log 2 + \frac{1}{4}. \end{aligned}$$

The following figure 1 illustrates the definition of the inverse temperature β_*; the annealed free energy $\overline{f}_\infty(\beta) = \log 2 + \frac{\beta^2}{4}$ is plotted as a function of β and the straight line is defined by $\frac{\beta}{\beta_1} f_\infty(\beta_1) \equiv \beta f_\infty(\beta_1)$. The two graphs intersect at $\beta_1 = 1$ and $\beta_* = 4\log 2 = 2,77258\cdots$. One can now easily check that, at $\beta = \beta_*$, the annealed free energy $\overline{f}_\infty(\beta_*)$ is simply related to $f_\infty(\beta_1)$ by the following relationship

$$\overline{f}_\infty(\beta_*) = \frac{\beta_*^2}{4} + \log 2 = \frac{\beta_*}{\beta_1}\left(\frac{\beta_* \beta_1}{4} + \frac{\beta_1}{\beta_*} \log 2\right) = \frac{\beta_*}{\beta_1}\left(\log 2 + \frac{1}{4}\right) = \frac{\beta_*}{\beta_1} f_\infty(\beta_1).$$

We denote by T the mapping $T : \mu_{n,\beta_1}(\sigma|J) \mapsto \mu_{n,\beta}(\sigma|J)$ defining, for all $\beta > \beta_1$, the Gibbs probability measure $\mu_{n,\beta}(\sigma|J)$ via the functional equation

$$\mu_{n,\beta}(\sigma|J) := \frac{\exp(-\beta H_n(\sigma, J))}{Z_n(\beta, J)} = \mu_{n,\beta_1}^{\beta/\beta_1}(\sigma|J) \frac{Z_n^{\beta/\beta_1}(\beta_1, J)}{Z_n(\beta, J)}.$$

Notice that β/β_1 is a non dimensional quantity. Moreover the value β_1 fixes the temperature scale *i.e.* the temperature $\beta > \beta_1$ is expressed in units where $\beta_1 \equiv 1$.

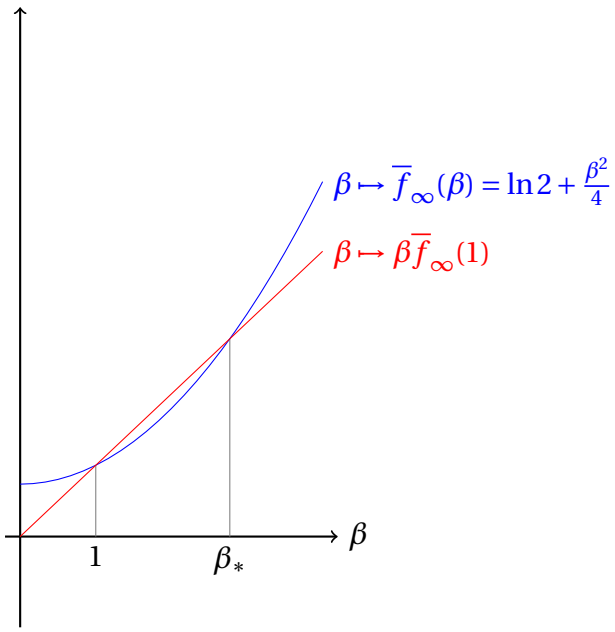

Figure 1: The value $\beta_* = 4\log 2$, is given by the intersection of the graph of the annealed free energy $\overline{f}_\infty(\beta)$ with the straight line $\beta\overline{f}_\infty(1)$.

Since $\mu_{n,\beta}$ is a probability on the configuration space, summing up over the configurations σ and taking the thermodynamic limit, we have indeed

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_\sigma \mu_{n,\beta}(\sigma|J) = \lim_{n\to\infty} \frac{1}{n} \log \sum_\sigma \mu_{n,\beta_1}^{\beta/\beta_1}(\sigma|J) + \alpha_\infty(\beta_1,\beta) = 0,$$

where the limit $\alpha_\infty(\beta_1,\beta)$ is given by

$$\begin{aligned}
\alpha_\infty(\beta_1,\beta) &= \lim_{n\to\infty} \frac{\beta}{\beta_1}\frac{1}{n}\log Z_n(\beta_1,J) - \lim_{n\to\infty}\frac{1}{n}\log Z_n(\beta,J) \\
&= \lim_{n\to\infty} \frac{\beta}{\beta_1}\frac{1}{n} E_J \log Z_n(\beta_1,J) - \lim_{n\to\infty}\frac{1}{n} E_J \log Z_n(\beta,J) \text{ (due to the self-averaging)} \\
&= \frac{\beta}{\beta_1} f_\infty(\beta_1) - f_\infty(\beta).
\end{aligned}$$

The existence, for all $\beta > \beta_1$, of the limit $\alpha_\infty(\beta_1,\beta)$ follows immediately from the existence of the two limits $f_\infty(\beta_1)$ and $f_\infty(\beta)$. Now, by making use of the relation between the limits $\overline{f}_\infty(\beta_*)$ and $f_\infty(\beta_1)$, one can check, that for $\beta = \beta_*$, the limit $\alpha_\infty(\beta_1,\beta_*)$ gives the deviation of the free energy $f_\infty(\beta_*)$ from its mean value :

$$a_\infty := \alpha_\infty(\beta_1,\beta_*) = \frac{\beta_*}{\beta_1} f_\infty(\beta_1) - f_\infty(\beta_*) = \overline{f}_\infty(\beta_*) - f_\infty(\beta_*).$$

The proof of the proposition reduces thus in determining the value a_∞.

Proof of the Proposition: At $\beta = \beta_1$, the quenched limit $f_\infty(\beta_1)$ equals the annealed one $\overline{f}_\infty(\beta_1) = \beta_1^2/4 + \log 2$, where the term $\beta_1^2/4$ comes from the mean value of the Boltzmann factor (*i.e.* the typical behaviour and the mean behaviour coincide at this temperature). Since for $\beta > \beta_1$, the typical and the average behaviour are no longer the same, we use the standard large deviations argument in order to make the deviant behaviour at β_* look like the typical behaviour at β_1.

The affine mapping T on measures induces a transformation on the free energies reading $\overline{f}_\infty(\beta_*) = \frac{\beta_*^2}{4} + \log 2 = \frac{\beta_*}{\beta_1} f_\infty(\beta_1)$. It follows that the pre-image of the term $\beta_*^2/4$ — coming from the average of the Boltzmann factor — (point C of the figure 2), is $\beta_1\beta_*/4 = \beta_1 \log 2 = \log 2$ (point C'); one gets the value of the free energy $f_\infty(\beta_1)$ if the term $\beta_1^2/4 = 1/4$ is added to this pre-image. We remark that the sheer particularity of the two temperatures β_1 and β_* is that the pre-image of $\log 2$ is $1/4$! Therefore, to obtain the quenched limit at β_* is enough to add to the image of $\log 2$ (*i.e.* to the segment OC) the value $1/4$ (segment CB).

One can now easily check that the difference of the two limits $f_\infty(\beta_*)$ and $f_\infty(\beta_1)$, is simply given by the segment OA:

$$f_\infty(\beta_*) - f_\infty(\beta_1) = (\beta_* - \beta_1)\log 2.$$

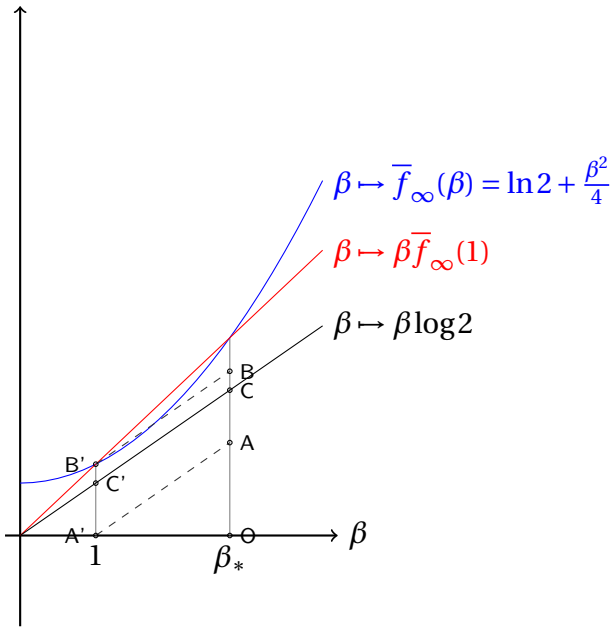

Figure 2: The affine map T maps C' to C. The length of the segment $A'B'$ corresponds to the value $f_\infty(\beta_1)$ that is parallel transported to the segment AB. The segment OB equals $f_\infty(\beta_*)$. The dashed lines $A'A$ and $B'B$ are parallel to $C'C$.

Hence,

$$f_\infty(\beta_*) = (\beta_* - \beta_1)\log 2 + f_\infty(\beta_1) = \beta_* \log 2 + \frac{\beta_1^2}{4} = \frac{\beta_*^2}{4} + \frac{1}{4} = 2.1718\cdots,$$

and, moreover

$$\begin{aligned} a_\infty &= \overline{f}_\infty(\beta_*) - f_\infty(\beta_*) \\ &= \frac{\beta_*}{\beta_1} f_\infty(\beta_1) - f_\infty(\beta_*) \\ &= \frac{\beta_*\beta_1}{4} - \frac{\beta_1^2}{4}. \end{aligned}$$

One can check that the value of $f_\infty(\beta_*)$ is slightly lower than the bound one can obtain by making use of the spherical model (2.2058$\cdots$). □

Proof of the Theorem: For β_*, we have

$$\begin{aligned} s(\mu_{\beta_*}) &= \lim_{n\to\infty} \frac{1}{n} S(\mu_{n,\beta_*,J}) = -\lim_{n\to\infty} \frac{1}{n} \sum_\sigma \mu_{n,\beta_*}(\sigma|J) \log \mu_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J) \frac{Z_n^{\beta_*/\beta_1}(\beta_1, J)}{Z_n(\beta_*, J)} \\ &= -\lim_{n\to\infty} \frac{1}{n} \sum_\sigma \mu_{n,\beta_*}(\sigma|J) \log \mu_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J) - \alpha_\infty, \end{aligned}$$

and, by the positivity of the entropy one checks readily that

$$\lim_{n\to\infty} \frac{1}{n} \sum_\sigma \mu_{n,\beta_*}(\sigma|J) \log \mu_{n,\beta_1}(\sigma|J) \le -\frac{\beta_1}{\beta_*}\alpha_\infty = \frac{1}{4\beta_*} - \frac{1}{4}.$$

In the following, we shall show that this inequality is saturated. For this we introduce a slightly different notation.

Let $W_{n,\beta_1}(\sigma|J) = e^{-\beta_1 H_n(\sigma,J)}/2^{n\beta_1}$ be the random weight associated with each configuration $\sigma \in \Sigma_n$. The Gibbs measures $\mu_{n,\beta_1}(\sigma|J)$ and $\mu_{n,\beta_*}(\sigma|J)$ are now given by

$$\mu_{n,\beta_1}(\sigma|J) = \frac{W_{n,\beta_1}(\sigma|J)}{\sum_\sigma W_{n,\beta_1}(\sigma|J)},$$

and,

$$\mu_{n,\beta_*}(\sigma|J) = \frac{W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J)}{\sum_\sigma W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J)}.$$

We have indeed, from the high temperature results,

$$\lim_{n\to\infty} \frac{1}{n} E_J \log \sum_\sigma W_{n,\beta_1}(\sigma|J) = \lim_{n\to\infty} \frac{1}{n} \log E_J \sum_\sigma \frac{e^{-\beta_1 H_n(\sigma,J)}}{2^{n\beta_1}} = \frac{\beta_1^2}{4} + \log 2 - \beta_1 \log 2 = \frac{1}{4},$$

and, from the previous proposition,

$$\lim_{n\to\infty} \frac{1}{n} E_J \log \sum_\sigma W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J) = \lim_{n\to\infty} \frac{1}{n} E_J \log \sum_\sigma \left(\frac{e^{-\beta_1 H_n(\sigma,J)}}{2^{n\beta_1}} \right)^{\frac{\beta_*}{\beta_1}} = \frac{\beta_*^2}{4} - \beta_* \log 2 + \frac{\beta_1^2}{4} = \frac{1}{4},$$

i.e. the behaviour of the sums $\sum_\sigma W_{n,\beta_1}(\sigma|J)$ and $\sum_\sigma W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J)$ is the same. Thus, for the comparison of the two measures, namely for distinguishing between the behaviour of the summands $W_{n,\beta_1}(\sigma|J)$ and $W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J)$ we need additional information.

We introduce the relative entropy density $s(\mu_{\beta_*}|\mu_{\beta_1})$ of the measure μ_{β_*} w.r.t. the measure μ_{β_1} which gives the extend to which the measure μ_{β_*} "differs" from the measure μ_{β_1}:

$$s(\mu_{\beta_*}|\mu_{\beta_1}) := \lim_{n\to\infty}\frac{1}{n}S(\mu_{n,\beta_*}|\mu_{n,\beta_1}) = \lim_{n\to\infty}\frac{1}{n}\sum_\sigma \mu_{n,\beta_*}(\sigma|J)\log\frac{\mu_{n,\beta_*}(\sigma|J)}{\mu_{n,\beta_1}(\sigma|J)}.$$

This limit exists and it is a non-negative function vanishing in the case the two measures are equal. We notice moreover that

$$\begin{aligned} s(\mu_{\beta_*}|\mu_{\beta_1}) &= \lim_{n\to\infty}\frac{1}{n}\sum_\sigma \mu_{n,\beta_*}(\sigma|J)\log\frac{\mu_{n,\beta_*}(\sigma|J)}{\mu_{n,\beta_1}(\sigma|J)} \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_\sigma \mu_{n,\beta_*}(\sigma|J)\log\frac{W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J)}{W_{n,\beta_1}(\sigma|J)} \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_\sigma \mu_{n,\beta_*}(\sigma|J)\log W_{n,\beta_1}^{\frac{\beta_*}{\beta_1}-1}(\sigma|J). \end{aligned}$$

Obviously, $W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J) \le \sum_\sigma W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J)$. Hence,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_\sigma \mu_{n,\beta_*}(\sigma|J)\log W_{n,\beta_1}^{\frac{\beta_*}{\beta_1}-1}(\sigma|J) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_\sigma W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J)\right)^{1-\frac{\beta_1}{\beta_*}}.$$

and, consequently,

$$s(\mu_{\beta_*}|\mu_{\beta_1}) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_\sigma W_{n,\beta_1}^{\beta_*/\beta_1}(\sigma|J)\right)^{1-\frac{\beta_1}{\beta_*}} = \frac{1}{4\beta_*}(\beta_*-\beta_1)$$

where the equality of the limsup and the limit is a consequence of the positivity of $s(\mu_{\beta_*}|\mu_{\beta_1})$.

Using now the functional definition of the measure one gets

$$\begin{aligned} s(\mu_{\beta_*}|\mu_{\beta_1}) &= \lim_{n\to\infty}\frac{1}{n}\sum_\sigma \mu_{n,\beta_*}(\sigma|J)\log\frac{\mu_{n,\beta_*}(\sigma|J)}{\mu_{n,\beta_1}(\sigma|J)} \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_\sigma \mu_{n,\beta_*}(\sigma|J)\log \mu_{n,\beta_1}^{\frac{\beta_*}{\beta_1}-1}(\sigma|J) + \alpha_\infty \\ &= \frac{1}{4\beta_*}(\beta_*-\beta_1). \end{aligned}$$

Recalling that $\alpha_\infty = \frac{\beta_1}{4}(\beta_*-\beta_1)$, it follows immediately that

$$\lim_{n\to\infty}\frac{1}{n}\sum_\sigma \mu_{n,\beta_*}(\sigma|J)\log\mu_{n,\beta_1}(\sigma|J) = \frac{1}{4\beta_*}-\frac{1}{4} = -\frac{\beta_1}{\beta_*}\alpha_\infty$$

which proves the theorem. □

Remarks: Another interesting quantity is the relative entropy density $s(\mu_{\beta_*}|\nu)$ of the measure μ_{n,β_*} w.r.t. the uniform measure $\nu_n(\sigma)$:

$$\begin{aligned} s(\mu_{\beta_*}|\nu) &= \lim_{n\to\infty}\frac{1}{n}S(\mu_{n,\beta_*}|\nu_n) = \lim_{n\to\infty}\frac{1}{n}\sum_{\sigma}\mu_{n,\beta_*}(\sigma|J)\log\frac{\mu_{n,\beta_*}(\sigma|J)}{\nu_n(\sigma)} \\ &= -s(\mu_{\beta_*})+\log 2 \\ &= \log 2. \end{aligned}$$

(We recall that $s(\mu_{\beta_1}|\nu) = -s(\mu_{\beta_1})+\log 2 = \frac{\beta_1^2}{4} = \frac{1}{4}$).

One can also easily check that the value of the limit α_∞ corresponds to the entropy difference $\alpha_\infty = s(\mu_{\beta_1}) - s(\mu_{\beta_*})$.

3 Concluding remarks

In this note we showed that the mean entropy of the Gibbs measure vanishes at the inverse temperature $\beta_* = 4\log 2$. A related question concerns the Hausdorff dimension of the support of the Gibbs measure. From our result on the entropy one can easily show that this dimension vanishes at β_*.

A last observation concerns the value of the temperature β_*: it is obtained from the relationship between the free energies $\bar{f}_\infty(\beta_*)$ and $f_\infty(1)$; moreover, one can readily check that $\beta_* = \beta_c^2$, where $\beta_c = 2\sqrt{\log 2}$ is the critical temperature of the Random Energy Model (REM). The REM is defined by 2^n energy levels $E_i (i = 1,\cdots,n)$, a family of random, independent, identically distributed random variables; many results are qualitatively the same as those of the SK model. It would be interesting to clarify this relationship in order to obtain some information on the behaviour and properties of the Gibbs measure at low temperatures. Both β_c and β_* are to be compared with the value at $\beta_1 \equiv 1$, *i.e.* the maximum value of β where the free energies of the two models coincide. What we learn by the comparison of the two models is that the Gibbs measure of the SK has seemingly a richer structure than for the REM. As a matter of fact, the entropy of the REM vanishes at β_c while the entropy of the SK model is still strictly positive at this point.

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