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On the longest path in a recursively partitionable graph

Julien Bensmail^{a,b}

^a*Univ. Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France*

^b*CNRS, LaBRI, UMR 5800, F-33400 Talence, France*

Abstract

A connected graph G with order n is said to be *recursively arbitrarily partitionable* (R-AP for short) if either it is isomorphic to K_1 , or for every sequence (τ_1, \dots, τ_k) of positive integers summing up to n there exists a partition (V_1, \dots, V_k) of $V(G)$ such that each V_i induces a connected R-AP subgraph of G on τ_i vertices. Since previous investigations, it is believed that a R-AP graph should be "almost traceable" somehow. We show that there does not exist a constant c such that every R-AP graph with order n must contain an elementary path on at least $n - c$ vertices for every n .

Keywords: recursively partitionable graph, longest elementary path

1. Introduction

Let n be a positive integer. A n -graph is a graph whose *order*, that is, its number of vertices, is n . We say that a connected n -graph G is *recursively arbitrarily partitionable* (R-AP for short) iff one of the following two conditions holds.

- The graph G is an isolated vertex.
- For every sequence (τ_1, \dots, τ_k) of positive integers performing a partition of n , there exists a partition (V_1, \dots, V_k) of $V(G)$ such that $G[V_i]$ is a connected R-AP subgraph of G on τ_i vertices for all $i \in \{1, \dots, k\}$.

The property of being R-AP was introduced in [5] as a strengthened version of the property of being *arbitrarily partitionable*. The latter property was itself introduced to deal with a problem of resource sharing between an arbitrary number of users (see [1, 2, 4, 7] for further details).

R-AP graphs have been mainly studied in the context of some simple classes of graphs. In particular, let us mention some works on trees [5], a class of unicyclic 1-connected graphs called *suns* [6], and a class of 2-connected graphs called *balloons* [3, 5]. Despite these efforts, we still do not know much about R-AP graphs though. However, all the previously mentioned investigations suggest

Email address: jbensmai@labri.fr (Julien Bensmail)

that the property of being R-AP is even closer to traceability¹ than is the one of being arbitrarily partitionable.

In particular, it is known that a R-AP tree or sun with order n has its longest elementary path passing through at least $n - 2$ of its vertices for every n . Additionally, for every n , all known R-AP balloons with order n have their longest elementary path of order at least $n - 4$. Regarding these observations, one could naively think that there should exist some small constant c such that the order of the longest elementary path of a R-AP n -graph is at least $n - c$ vertices for every n .

In this work, we show that such a constant does not exist by exhibiting a class \mathcal{C} of R-AP graphs such that for every $c \geq 1$, there exists, for some n , a n -graph in \mathcal{C} whose longest elementary path has order exactly $n - c$.

2. Definitions and preliminary results

One can easily observe that paths are all R-AP. Since the property of being R-AP is closed under edge-addition, the next result follows naturally.

Remark 1. Every traceable graph is R-AP.

Determining whether a n -graph is R-AP is laborious since, according to the original definition, one has to consider every partition of n . Thus, we usually prefer to check the following equivalent condition which derives from the fact that a R-AP graph is partitionable at will.

Remark 2 ([5]). A connected n -graph G is R-AP iff for every $\lambda \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ there exists a partition $(V_\lambda, V_{n-\lambda})$ of $V(G)$ such that $G[V_\lambda]$ and $G[V_{n-\lambda}]$ induce connected R-AP subgraphs of G on λ and $n - \lambda$ vertices, respectively.

We finally introduce the class of *caterpillar* graphs.

Definition 1. Let $a, b \geq 2$ be two positive integers. The *caterpillar* $Cat(a, b)$ is the tree obtained by linking one root vertex to one endvertex of each of two disjoint paths with order $a - 1$ and $b - 1$, respectively².

Two examples of caterpillars are given in Figure 1. R-AP caterpillars were fully characterized in [5].



Figure 1: The caterpillars $Cat(2, 3)$ and $Cat(3, 3)$

Theorem 1 ([5]). A caterpillar $Cat(a, b)$ is R-AP iff a and b take values in Table 1.

¹A *traceable* graph is a graph having a Hamiltonian path.

²Notice that a caterpillar $Cat(a, b)$ has order $a + b$.

a	b
2, 4	$\equiv 1 \pmod{2}$
3	$\equiv 1, 2 \pmod{3}$
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

Table 1: Values a and b ($a \leq b$) such that $Cat(a, b)$ is R-AP

3. Main result

In this article, we deal with the structure of *connected cycles* graphs.

Definition 2. Let $k \geq 1$ and $x, y \geq 0$ be three positive integers. The *connected cycles* graph $CC(k, x, y)$ is the graph with the following vertices.

- Let $u_1 \dots u_x$ and $v_1 \dots v_y$ be paths with order x and y , respectively;
- For every $i \in \{1, \dots, k\}$, let $a_i b_i e_i d_i c_i a_i$ be a cycle with length 5;
- For every $i \in \{1, \dots, k-1\}$, let $w_{i,i+1}$ be a vertex.

These vertices are linked in $CC(k, x, y)$ in the following way: $u_x a_1, v_y e_k \in E(CC(k, x, y))$ and we have $w_{i,i+1} e_i, w_{i,i+1} a_{i+1} \in E(CC(k, x, y))$ for every $i \in \{1, \dots, k-1\}$.

An example of connected cycles graph is depicted in Figure 2. Notice that $CC(k, 1, 1)$ has its longest elementary path of order $|V(CC(k, 1, 1))| - k$. Hence, our goal here is to prove that $CC(k, 1, 1)$ is R-AP for every $k \geq 1$. It will then follow that for some n , there exists a R-AP n -graph whose longest elementary path is arbitrarily smaller than n .

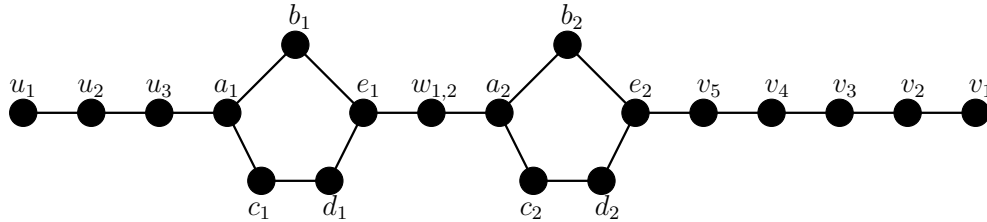


Figure 2: The connected cycles graph $CC(2, 3, 5)$

Before proving that $CC(k, 1, 1)$ is R-AP for every k , we introduce another graph structure we shall encounter while partitioning a connected cycles graph.

Definition 3. Let $k \geq 1$ and $x \geq 0$ be two positive integers. The *partial connected cycles* graph $PCC(k, x)$ is the graph obtained by removing the vertex e_k from $CC(k, x, 0)$.

We are now ready to prove our main result.

Theorem 2. *The graph $PCC(k, x)$ is R-AP for every $k \geq 1$ and $x \in \{1, 3, 4, 6\}$. The graph $CC(k, x, y)$ is R-AP for every $k \geq 1$ when one of the following two conditions is met: $x \in \{1, 3, 4, 6\}$ and $y \geq 1$, or $x = 2$ and $y \not\equiv 2 \pmod{3}$.*

Proof. The proof is by induction on k and uses the terminology introduced in Definition 2. For each value of k , we prove that the result is true for all possible values for both x and y . Recall that, according to Remark 2, a connected n -graph G is R-AP iff for every $\lambda \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ we can partition $V(G)$ into two parts V_λ and $V_{n-\lambda}$ inducing connected R-AP subgraphs of G with order λ and $n - \lambda$, respectively.

Case 1: $k = 1$.

First, the graphs $PCC(1, 1)$, $PCC(1, 3)$, $PCC(1, 4)$ and $PCC(1, 6)$ are R-AP since they are spanned by $Cat(2, 3)$, $Cat(3, 4)$, $Cat(3, 5)$ and $Cat(3, 7)$, respectively. We now prove the base case for connected cycles graphs by induction on y .

Case 1.a: $k = 1$ and $y = 1$.

Notice that $CC(1, 1, 1)$, $CC(1, 2, 1)$, $CC(1, 3, 1)$, $CC(1, 4, 1)$ and $CC(1, 6, 1)$ are R-AP since they are spanned by $Cat(2, 5)$, $Cat(3, 5)$, $Cat(2, 7)$, $Cat(3, 7)$ and $Cat(5, 7)$, respectively.

Case 1.b: $k = 1$ and arbitrary y .

Now, suppose that $CC(1, 1, i)$, $CC(1, 3, i)$, $CC(1, 4, i)$ and $CC(1, 6, i)$ are R-AP for every i from 1 up to $y - 1$, and similarly for $CC(1, 2, i)$ when $i \not\equiv 2 \pmod{3}$. Let us next consider a graph $C = CC(1, x, y)$ for some $x \in \{1, 3, 4, 6\}$ and $\lambda \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ where $n = 5 + x + y$. We show how to partition $V(C)$ into two parts V_λ and $V_{n-\lambda}$ satisfying the above conditions by giving a satisfying subset V_λ (each time, it is assumed that $V_{n-\lambda} = V(C) - V_\lambda$). First, since $n = 5 + x + y$, $x \leq 6$ and $y \geq 2$, then λ cannot exceed $y + 4$. If $\lambda \leq y - 1$, then we can partition C into a path of size λ and $CC(1, x, y - \lambda)$, which is R-AP by the induction hypothesis, by considering $V_\lambda = \{v_1, \dots, v_\lambda\}$. When $\lambda = y$, one can partition C into a path with order λ and $CC(1, x, 0)$, which is traceable, by choosing V_λ in a similar way. For $\lambda = y + 1$, consider $V_\lambda = \{v_1, \dots, v_y, e_1\}$ to obtain a partition of C into a path and $PCC(1, x)$. When $\lambda = y + 2$, consider $V_\lambda = \{v_1, \dots, v_y, e_1, b_1\}$ to induce two paths. For $\lambda = y + 3$, one can obtain a similar partition of C by considering $V_\lambda = \{v_1, \dots, v_y, e_1, d_1, c_1\}$. Finally, the case $\lambda = y + 4$ may only happen when C is $CC(1, 6, 2)$ or $CC(1, 6, 3)$, but in these cases C can be partitioned into a path and $CC(1, 0, y)$ (which is traceable) or a path and $Cat(3, 4)$ by considering $V_\lambda = \{u_1, \dots, u_6\}$ or $V_\lambda = \{u_1, \dots, u_6, a_1\}$, respectively.

When considering a graph $C = CC(1, 2, y)$, where $y \geq 3$ and $y \not\equiv 2 \pmod{3}$, then λ cannot exceed $y + 2$. For values $\lambda \leq y$ such that $y - \lambda \not\equiv 2 \pmod{3}$,

one can partition C as above into a path and a connected cycles graph which is R-AP by the induction hypothesis. When $\lambda \leq y$ and $y - \lambda \equiv 2 \pmod{3}$, consider as V_λ the first λ vertices of the ordering $(u_1, u_2, a_1, b_1, c_1, d_1, e_1, v_y, \dots, v_1)$ of $V(C)$ to obtain a partition of C into either a path and the traceable graph $CC(1, 0, y)$, a path and $Cat(2, 3)$, or a path and $CC(1, 2, y')$ with $y' \equiv 1 \pmod{2}$ which is R-AP by the induction hypothesis (when $y \equiv 1 \pmod{3}$), or either a path and $CC(1, 1, y)$ which is R-AP, two paths, a path and the traceable graph $CC(1, 2, 0)$, or a path and some graph $CC(1, 2, y'')$, with $y'' \equiv 0 \pmod{3}$, which is R-AP by the induction hypothesis (when $y \equiv 0 \pmod{3}$). When $\lambda = y + 1$, one can use the previous ordering of $V(C)$ once again to deduce V_λ since $y \geq 3$. Finally, for $\lambda = y + 2$, one may consider $V_\lambda = \{v_1, \dots, v_y, e_1, b_1\}$ to obtain a partition of C into two paths.

Case 2: arbitrary k .

Let us now suppose that the result is true for every i up to $k - 1$, and let us prove it for k . Consider first $C = PCC(k, x)$ for consecutive values of x in $(1, 3, 4, 6)$. As we did before, to prove that C is R-AP we show that there exists a partition of it satisfying our conditions for every possible value of λ . One may choose V_λ as follows.

- If $\lambda \equiv 1 \pmod{3}$, then one may consider, as V_λ , the first λ vertices of the ordering $(b_k, d_k, c_k, a_k, w_{k-1,k}, e_{k-1}, b_{k-1}, d_{k-1}, c_{k-1}, a_{k-1}, \dots, w_{1,2}, e_1, b_1, d_1, c_1, a_1, u_x, \dots, u_1)$ of $V(C)$. On the one hand, notice that $C[V_\lambda]$ is either a path or covered by a R-AP caterpillar or a partial connected cycles graph $PCC(k', x')$ with $k' \leq k - 1$ and $x' \in \{1, 3, 4, 6\}$, which is R-AP by the induction hypothesis. On the other hand, observe that $C[V_{n-\lambda}]$ is spanned by a connected cycles graph $CC(k'', x, y)$ for some $k'' \leq k - 1$ and y . The latter graph is R-AP according to the induction hypothesis.
- When $\lambda \equiv 2 \pmod{3}$, one can obtain the same partitions of C from the ordering $(d_k, c_k, b_k, a_k, w_{k-1,k}, e_{k-1}, d_{k-1}, c_{k-1}, b_{k-1}, a_{k-1}, \dots, w_{1,2}, e_1, d_1, c_1, b_1, a_1, u_x, \dots, u_1)$ of $V(C)$.
- Otherwise, one has to consider as V_λ the first λ vertices of the ordering $(u_1, \dots, u_x, a_1, b_1, c_1, d_1, e_1, w_{1,2}, \dots, a_{k-1}, b_{k-1}, c_{k-1}, d_{k-1}, e_{k-1}, w_{k-1,k}, a_k, b_k, c_k, d_k)$ of $V(C)$ when $x \in \{1, 4\}$, the ordering $(u_1, \dots, u_x, a_1, c_1, d_1, b_1, e_1, w_{1,2}, \dots, a_{k-1}, c_{k-1}, d_{k-1}, b_{k-1}, e_{k-1}, w_{k-1,k}, a_k, c_k, d_k, b_k)$ otherwise. The two induced subgraphs $C[V_\lambda]$ and $C[V_{n-\lambda}]$ are then R-AP. Indeed, on the one hand, $C[V_\lambda]$ is either isomorphic to a path or spanned by a connected cycles graph $CC(k', x, y)$ for $k' \leq k - 1$ and some y . On the other hand, the subgraph $C[V_{n-\lambda}]$ is spanned by some $PCC(k'', x')$ graph with $k'' \leq k$ and $x' \in \{1, 3, 4, 6\}$.

To end up proving the claim, one has to show that $CC(k, x, y)$ is R-AP for every $x \in \{1, 3, 4, 6\}$ and every $y \geq 1$, and for every $y \not\equiv 2 \pmod{3}$ when $x = 2$. This is done by induction on y once again.

First, let $C = CC(k, x, y)$ where x is taken consecutively in $(1, 3, 4, 6)$ and consider consecutive values of y . For some $\lambda \in \{1, \dots, y\}$, one can consider

$V_\lambda = \{v_1, \dots, v_\lambda\}$ so that C is partitioned into a path and $CC(k, x, y - \lambda)$ which is R-AP according to the induction hypothesis on y (when $\lambda \leq y - 1$) or k (otherwise). When $\lambda = y + 1$, one has to choose $V_\lambda = \{v_1, \dots, v_y, e_k\}$ so that C is partitioned into a path and a partial connected cycles graph which is R-AP by the induction hypothesis. For other values of λ , one may choose V_λ as done previously.

- If $\lambda \equiv 0 \pmod{3}$, one can consider, as V_λ , the λ first vertices from the ordering $(u_1, \dots, u_x, a_1, b_1, c_1, d_1, e_1, w_{1,2}, \dots, w_{k-1,k}, a_k, b_k, c_k, d_k, e_k, v_y, \dots, v_1)$ of $V(C)$ when $x \in \{1, 4\}$, from $(u_1, \dots, u_x, a_1, c_1, d_1, b_1, e_1, w_{1,2}, \dots, w_{k-1,k}, a_k, c_k, d_k, b_k, e_k, v_y, \dots, v_1)$ otherwise. The two induced subgraphs are then R-AP since they are traceable or isomorphic to connected cycles graphs which are R-AP according to the induction hypotheses on k and y .
- When $\lambda \not\equiv 0 \pmod{3}$ and $\lambda - (y + 1) \equiv 1 \pmod{3}$, then one may pick up, as V_λ , the λ first vertices from the ordering $(v_1, \dots, v_y, e_k, b_k, d_k, c_k, a_k, w_{k-1,k}, \dots, e_1, b_1, d_1, c_1, a_1, u_x, \dots, u_1)$ of $V(C)$. This choice of V_λ makes, on the one hand, $C[V_\lambda]$ being spanned by either a path or $CC(k', x', y)$ where $k' \leq k - 1$ and x' are such that the conditions of our claim are met and so that this graph is R-AP (in particular, we cannot simultaneously have $x' = 2$ and $y \equiv 2 \pmod{3}$). On the other hand, $C[V_{n-\lambda}]$ is spanned by some graph $CC(k'', x, y')$ for $k'' \leq k - 1$ and some y' and thus is R-AP according to the induction hypothesis.
- Otherwise, if $\lambda \not\equiv 0 \pmod{3}$ and $\lambda - (y + 1) \equiv 2 \pmod{3}$, then some similar partitions of C may be obtained from the ordering $(v_1, \dots, v_y, e_k, d_k, c_k, b_k, a_k, w_{k-1,k}, \dots, w_{1,2}, e_1, d_1, c_1, b_1, a_1, u_x, \dots, u_1)$ of $V(C)$.

Finally, partitions of $C = CC(k, 2, y)$, for consecutive values of y with $y \not\equiv 2 \pmod{3}$, may be obtained in a similar way using the following four orderings of $V(C)$.

1. $(v_1, \dots, v_y, e_k, d_k, c_k, b_k, a_k, w_{k-1,k}, \dots, w_{1,2}, e_1, d_1, c_1, b_1, a_1, u_2, u_1)$ when $\lambda, y \equiv 0 \pmod{3}$.
2. $(v_1, \dots, v_y, e_k, b_k, d_k, c_k, a_k, w_{k-1,k}, \dots, w_{1,2}, e_1, b_1, d_1, c_1, a_1, u_2, u_1)$ when $\lambda \equiv 0 \pmod{3}$ and $y \equiv 1 \pmod{3}$.
3. $(u_1, u_2, a_1, b_1, c_1, d_1, e_1, w_{1,2}, \dots, w_{k-1,k}, a_k, b_k, c_k, d_k, e_k, v_y, \dots, v_1)$ when $\lambda \equiv 1 \pmod{3}$.
4. $(u_1, u_2, a_1, c_1, d_1, b_1, e_1, w_{1,2}, \dots, w_{k-1,k}, a_k, c_k, d_k, b_k, e_k, v_y, \dots, v_1)$ when $\lambda \equiv 2 \pmod{3}$.

In each of these cases, the partition $(V_\lambda, V_{n-\lambda})$ of $V(C)$ induces either traceable subgraphs of C , or subgraphs of C which are R-AP according to our induction hypotheses. □

From Theorem 2, we deduce the following.

Corollary 1. *There does not exist a constant c such that, for every n , a R-AP n -graph has its longest elementary path of order at least $n - c$.*

Indeed, recall that, for every c , the connected cycles graph $CC(c + 1, 1, 1)$ is R-AP according to Theorem 2 and has its longest elementary path passing through $n - (c + 1)$ of its vertices.

4. Conclusions

Observe that any graph $CC(k, x, y)$ has order $n = x + y + 6k - 1$ while its longest elementary path passes through exactly $n - k$ of its vertices. Thus, such a graph remains almost traceable although its traceability degree cannot be characterized using some $n - c$ bound. Hence, even if the order of the longest elementary path of a R-AP n -graph is not constantly lower than n , it remains that there seems to be some strong relationship between these two parameters though. In particular, it would be next interesting to catch this relationship in the following way.

Question 1. What is the greatest c such that every R-AP n -graph has its longest elementary path of order at least $c \cdot n$ for every n ?

We considered the question for all classes of graphs which have been studied so far regarding the property of being R-AP (that is, trees [5], suns [6] and balloons [3]). Table 2 sums up the upper bounds on c which can be deduced from each of these classes (the reader is referred to the corresponding articles for further information).

Thanks to the balloon graph $B(1, 2, 2, 3, 4)$, we get that $11/14$ is the best upper bound on c known so far regarding Question 1.

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<i>Graphs class</i>	<i>Pattern</i>	<i>Best candidate</i>	<i>Upper bound</i>
Trees	$Cat(a, b)$	$Cat(2, 3)$	$c \leq 4/5$
	$T_3(a, b, c)$	$T_3(2, 4, 6)$	$c \leq 11/13$
Suns	$Sun(a, b)$	$Sun(1, 2)$	$c \leq 6/7$
	$Sun(a, b, c)$	$Sun(1, 2, 2)$	$c \leq 9/11$
Connected cycles graphs	$CC(k, x, y)$	$CC(1, 1, 1)$	$c \leq 6/7$
Partial connected cycles graphs	$PCC(k, x)$	$PCC(1, 1)$	$c \leq 4/5$
Balloons with four branches	$B(1, a, b, c)$	$B(1, 1, 1, 2)$	$c \leq 6/7$
	$B(2, a, b, c)$	$B(2, 2, 2, 3)$	$c \leq 9/11$
	$B(3, a, b, c)$	$B(3, 3, 4, 5)$	$c \leq 14/17$
	$B(4, a, b, c)$	$B(4, 4, 6, 10)$	$c \leq 11/13$
Balloons with five branches	$B(1, 1, a, b, c)$	$B(1, 1, 1, 2, 4)$	$c \leq 9/11$
	$B(1, 2, a, b, c)$	$B(1, 2, 2, 3, 4)$	$c \leq 11/14$
	$B(2, 2, a, b, c)$	$B(2, 2, 3, 4, 6)$	$c \leq 15/19$

Table 2: Question 1 studied regarding all known classes of R-AP graphs

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