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Sylvain Golenia, Christoph Schumacher

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**ADDENDUM: “THE PROBLEM OF DEFICIENCY INDICES FOR DISCRETE
SCHRÖDINGER OPERATORS ON LOCALLY FINITE GRAPHS” [J. MATH.
PHYS. (52), 063512 (2011)]**

SYLVAIN GOLÉRIA AND CHRISTOPH SCHUMACHER

ABSTRACT. In this note we answer negatively to our conjecture concerning the deficiency indices. More precisely, given any non-negative integer n , there is locally finite graph on which the adjacency matrix has deficiency indices (n, n) .

Given a closable and densely defined symmetric operator T acting on a complex Hilbert space, the deficiency indices of T are defined by $\eta_{\pm}(T) := \dim \ker(T^* \mp i) \in \mathbb{N} \cup \{+\infty\}$. The operator T possesses a self-adjoint extension if and only if $\eta_+(T) = \eta_-(T)$. If this is the case, we denote the common value by $\eta(T)$ and the self-adjoint extensions of T are parametrized by the unitary group $U(\eta(T))$, e.g., [RS78, Section X.1]. The operator T is essentially self-adjoint, i.e., its closure is self-adjoint, if and only if $\eta(T) = 0$. In this note we discuss the possible values of $\eta(T)$, when T is the adjacency matrix acting on a locally finite and simple graph.

We recall some standard definitions of graph theory to fix notation. A (simple, undirected) *graph* is a pair $G = (E, V)$, where V is a countable set and $E: V \times V \rightarrow \{0, 1\}$ is a symmetric function with $E(v, v) = 0$ for all $v \in V$. The elements of V are called *vertices*. Two vertices $v, w \in V$ with $E(v, w) = 1$ form an *edge* (v, w) , are *neighbours*, and we write $v \sim w$. The set of neighbours of $v \in V$ is $\mathcal{N}_G(v) := \{w \in V \mid v \sim w\}$. The number of neighbours of v is the *degree* $d_G(v) := |\mathcal{N}_G(v)|$ of v . The graph $G = (V, E)$ is *locally finite*, if $d_G(v) < \infty$ for all $v \in V$. In this note, all graphs are simple, undirected and locally finite.

A *path* of length $n \in \mathbb{N}$ in G is a tuple $(v_0, v_1, \dots, v_n) \in V^{n+1}$ such that $v_{j-1} \sim v_j$ for all $j \in \{1, \dots, n\}$. Such a path *connects* v_0 and v_n and is called *v_0 - v_n -path*. Being connected by a path is an equivalence relation on V , and the equivalence classes are called *connected components* of the graph. A graph is *connected*, if all its vertices belong to the same connected component. The vertex set V of a connected graph is equipped with the *graph metric* $\rho_G: V \times V \rightarrow \mathbb{R}$, $\rho_G(v, w) := \inf\{n \in \mathbb{N} \mid \text{there exists a } v\text{-}w\text{-path of length } n\}$. Note that we use the convention $0 \in \mathbb{N}$, so that each vertex is connected to itself with a path of length 0.

We now define trees. An edge $e \in V \times V$, $E(e) = 1$, in a connected graph $G = (E, V)$ is *pivotal*, if the graph G with the edge e removed, i.e. (\tilde{E}, V) with $\tilde{E}(e) = 0$ and $\tilde{E}(e') = E(e')$ for all $e' \in V \setminus \{e\}$, is disconnected. A *tree* is a connected graph, which has only pivotal edges.

We associate to a graph G the complex Hilbert space $\ell^2(V)$. We denote by $\langle \cdot, \cdot \rangle$ and by $\|\cdot\|$ the scalar product and the associated norm, respectively. The set of complex functions with finite support in V is denoted by $\mathcal{C}_c(G)$. One may define different discrete operators acting on $\ell^2(V)$. For instance, the (physical) *Laplacian* is defined by

$$(1) \quad (\Delta_{G, \circ} f)(x) := \sum_{y \in \mathcal{N}_G(x)} (f(x) - f(y)), \text{ with } f \in \mathcal{C}_c(G)$$

It is well known that it is symmetric and essentially self-adjoint on $\mathcal{C}_c(G)$, see [Woj07].

In this note we focus on the study of the *adjacency matrix* of G , which is defined by:

$$(2) \quad (\mathcal{A}_{G, \circ} f)(x) := \sum_{y \in \mathcal{N}_G(x)} f(y), \text{ with } f \in \mathcal{C}_c(G).$$

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This operator is symmetric and thus closable. We denote the closure by \mathcal{A}_G . We denote the domain by $\mathcal{D}(\mathcal{A}_G)$, and its adjoint by $(\mathcal{A}_G)^*$. Unlike the Laplacian, \mathcal{A} may have several self-adjoint extensions. We investigate its deficiency indices. Since the operator \mathcal{A}_G commutes with complex conjugation, its deficiency indices are equal, see [RS78, Theorem X.3]. This means that \mathcal{A}_G possesses a self-adjoint extension. Note that $\eta(\mathcal{A}_G) = 0$ if and only if \mathcal{A}_G is essentially self-adjoint on $\mathcal{C}_c(G)$.

In [MO85, Mü187], one constructs adjacency matrices for simple trees with positive deficiency indices. In fact, it follows from their proofs that the deficiency indices are infinite in both references. As a general result, a special case of [GS11, Theorem 1.1] gives that, given a locally finite simple tree G , one has the following alternative:

$$(3) \quad \eta(\mathcal{A}_G) \in \{0, +\infty\}.$$

The value of $\eta(\mathcal{A}_G)$ is discussed in [GS11] and linked with the growth of the tree.

In [MW89, Section 3], one finds:

Theorem 1. *For all $n \in \mathbb{N} \cup \{\infty\}$, there is a simple graph G , such that $\eta(\mathcal{A}_G) = n$.*

Their proof is unfortunately incomplete. However, the statement is correct, this is aim of this note. In [MW89], they provided simple and locally finite graph G such that $\eta(\mathcal{A}_G) \geq 1$ but did not check that $\eta(\mathcal{A}_G) = 1$. The problem comes from the fact that they considered a tree. More precisely, they referred to the works of [MO85, Mü187]. Therefore, (3) gives $\eta(\mathcal{A}_G) = \infty$ in their case. Keeping that in mind and strongly motivated by some other examples, we had proposed a drastically different scenario and had conjectured in [GS11] that that for any simple graph G , one has (3).

We now turn to the proof of Theorem 1 and therefore disprove our conjecture. First, we show that the validity of Theorem t:main is equivalent to the existence of a simple graph G with

$$(4) \quad \eta(\mathcal{A}_G) = 1.$$

Of course, Theorem t:main in particular states the existence of G . We focus on the other implication. We denote the positive integers with \mathbb{N}^* .

Lemma 2. *Let $n \in \mathbb{N}^*$ and G be a locally finite and connected graph. Then there exists a locally finite and connected graph \hat{G} such that*

$$\eta(\mathcal{A}_{\hat{G}}) = n \times \eta(\mathcal{A}_G).$$

Proof. Let $\hat{G} := (\hat{E}, \hat{V})$ be the disjoint union of n copies. We have: $\hat{G} := (\hat{E}, \hat{V})$ with $\hat{V} := \{1, \dots, n\} \times V$ and $\hat{E}((i, v), (j, w)) := \delta_{i,j} E(v, w)$. Note that $\eta(\mathcal{A}_{\hat{G}}) = n \times \eta(\mathcal{A}_G)$ since we have a direct sum. Take now $v_0 \in V$ and connect the copies of G by adding an edge between (i, v_0) and $(i+1, v_0)$, for all $i = 1, \dots, n-1$, and denote the resulting graph by \tilde{G} . Note that $\mathcal{A}_{\tilde{G}}$ is bounded perturbation of $\mathcal{A}_{\hat{G}}$. Therefore, by Proposition p:stab in Appendix A, we have $\eta(\mathcal{A}_{\tilde{G}}) = n \times \eta(\mathcal{A}_G)$. \square

Our example of a graph G with (4) is an antitree, a class of graphs which we define next. See also [BK]. The *sphere* of radius $n \in \mathbb{N}$ around a vertex $v \in V$ is the set $S_n(v) := \{w \in V \mid d_G(v, w) = n\}$. A graph is an *antitree*, if there exists a vertex $v \in V$ such that for all other vertices $w \in V \setminus \{v\}$

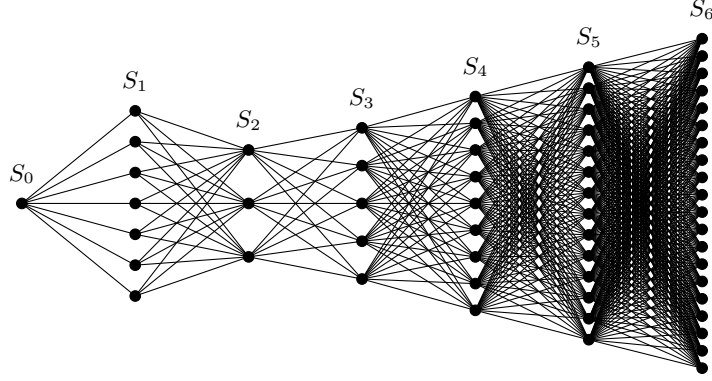
$$\mathcal{N}_G(w) = S_{n-1}(v) \cup S_{n+1}(v),$$

where $n = d_G(v, w) \geq 1$. See Figure 1 for an example. The distinguished vertex v is the *root* of the antitree. Antitrees are bipartite and enjoy *radial symmetry*, which means that each permutation of V , which fixes the spheres around the root, induces a graph isomorphism on G .

We denote the root by v , the spheres by $S_n := S_n(v)$, and their sizes by $s_n := |S_n|$. Further, $|x| := \rho_G(v, x)$ is the distance of $x \in V$ from the root. The operator $P: \ell^2(V) \rightarrow \ell^2(V)$, given by

$$Pf(x) := \frac{1}{s_{|x|}} \sum_{y \in S_{|x|}} f(y), \text{ for all } f \in \ell^2(V) \text{ and } x \in V,$$

averages a function over the spheres. Thereby, $P = P^2 = P^*$ is the orthogonal projection onto the space of radially symmetric functions in $\ell^2(V)$. A function $f: V \rightarrow \mathbb{C}$ is radially symmetric, if

FIGURE 1. An antitree with spheres S_0, \dots, S_6 .

it is constant on spheres, i.e., for all nodes $x, y \in V$ with $|x| = |y|$, we have $f(x) = f(y)$. For all radially symmetric f , we define $\tilde{f}: \mathbb{N} \rightarrow \mathbb{C}$, $\tilde{f}(|x|) := f(x)$, for all $x \in V$. Note that

$$P\ell^2(V) = \{f: V \rightarrow \mathbb{C}, f \text{ radially symmetric}, \sum_{n \in \mathbb{N}} s_n |\tilde{f}(n)|^2 < \infty\} \simeq \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}),$$

where $(s_n)_{n \in \mathbb{N}}$ is now a sequence of weights. The key observation of [BK, Theorem 4.1] is that

$$\mathcal{A}_G = P\mathcal{A}_G P \text{ and } \widetilde{\mathcal{A}_G P f}(|x|) = s_{|x|-1} \widetilde{P f}(|x|-1) + s_{|x|+1} \widetilde{P f}(|x|+1),$$

for all $f \in \mathcal{C}_c(V)$, with the convention $s_{-1} = 0$. Using the unitary transformation $U: \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}) \rightarrow \ell^2(\mathbb{N})$, $U\tilde{f}(n) = \sqrt{s_n}\tilde{f}(n)$, we see that \mathcal{A}_G is unitarily equivalent to the direct sum of 0 on $(P\ell^2(V))^\perp$ and a Jacobi matrix acting on $\ell^2(\mathbb{N})$ with 0 on the diagonal and the sequence $(\sqrt{s_n}\sqrt{s_{n+1}})_{n \in \mathbb{N}}$ on the off-diagonal.

Proposition 3. *Set $\alpha > 0$. Let G be the antitree with sphere sizes s_n , where $s_0 := 1$, $s_n := \lfloor n^\alpha \rfloor$, $n \geq 1$. Then,*

$$\eta(\mathcal{A}_G) = \begin{cases} 0, & \text{if } \alpha \in (0, 1], \\ 1, & \text{if } \alpha > 1. \end{cases}$$

Proof. Using Proposition 5 from Appendix A, we have $\eta(\mathcal{A}_G) = \eta(J)$, where J is the Jacobi matrix given by $a_n = \sqrt{s_n s_{n+1}}$ on the off-diagonal and $b_n = 0$ on the diagonal. Let \tilde{J} be the Jacobi matrix given by $\tilde{a}_n = \sqrt{n^\alpha (n+1)^\alpha}$ and $\tilde{b}_n = 0$. Now note that

$$\begin{aligned} 0 \leq \tilde{a}_n - a_n &\leq \sqrt{n^\alpha (n+1)^\alpha} - \sqrt{(n^\alpha - 1)((n+1)^\alpha - 1)} \\ &= \frac{(n+1)^\alpha + n^\alpha - 1}{\sqrt{n^\alpha (n+1)^\alpha} + \sqrt{(n^\alpha - 1)((n+1)^\alpha - 1)}} \xrightarrow{n \rightarrow \infty} 1, \end{aligned}$$

therefore $\tilde{a}_n - a_n$ is bounded. Hence, $\tilde{J} - J$ is a bounded operator, and by Proposition 5, cf. Appendix A, we have $\eta(J) = \eta(\tilde{J})$.

Now note $\sum_{n \in \mathbb{N}} \tilde{a}_n^{-1} = \infty$, iff $\alpha \leq 1$, and

$$\tilde{a}_{n-1} \tilde{a}_{n+1} = \sqrt{(n-1)^\alpha n^\alpha} \sqrt{(n+1)^\alpha (n+2)^\alpha} = \sqrt{(n^2-1)^\alpha} \sqrt{((n+1)^2-1)^\alpha} \leq n^\alpha (n+1)^\alpha = \tilde{a}_n^2.$$

By Theorem 4, see Appendix A, applied to \tilde{J} we get the result. \square

APPENDIX A. USEFUL FACTS

The theory of Jacobi matrices, as developed in [Ber68, Chapter VII], provides the following general theorem.

Theorem 4. *Let J be the Jacobi matrix with off-diagonal entries $a_n > 0$ and diagonal entries $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, acting on $\ell^2(\mathbb{N})$.*

- (1) If $\sum_{n \in \mathbb{N}} a_n^{-1} = \infty$, then J is essentially self-adjoint on $\mathcal{C}_c(\mathbb{N})$.
 (2) If $\sum_{n \in \mathbb{N}} a_n^{-1} < \infty$, $a_{n-1}a_{n+1} \leq a_n^2$ for all $n \geq n_0$ and $|b_n| \leq C$ for some constants $n_0, C > 0$, then J is not essentially self-adjoint on $\mathcal{C}_c(\mathbb{N})$ and has deficiency index 1.

We also recall that the deficiency indices are stable under the Kato-Rellich class of perturbation and refer to [GS11, Proposition A.1] for a proof.

Proposition 5. *Given two closed and densely defined symmetric operators S, T acting on a complex Hilbert space and such that $\mathcal{D}(S) \subset \mathcal{D}(T)$. Suppose there are $a \in [0, 1)$ and $b \geq 0$ such that*

$$(5) \quad \|Tf\| \leq a\|Sf\| + b\|f\|, \text{ for all } f \in \mathcal{D}(S).$$

Then, the closure of $(S+T)|_{\mathcal{D}(S)}$ is a symmetric operator that we denote by $S+T$. Moreover, one obtains that $\mathcal{D}(S) = \mathcal{D}(S+T)$ and that $\eta_{\pm}(S) = \eta_{\pm}(S+T)$. In particular, $S+T$ is self-adjoint if and only if S is self-adjoint.

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E-mail address: sylvain.golenia@u-bordeaux1.fr

INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UNIVERSITÉ BORDEAUX 1, 351, COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE

E-mail address: christoph.schumacher@mathematik.tu-chemnitz.de

FAKULTÄT FÜR MATHEMATIK DER TECHNISCHEN UNIVERSITÄT CHEMNITZ, REICHENHAINER STR. 41, 09126 CHEMNITZ, GERMANY