



HAL
open science

The problem of deficiency indices adjacency matrices on locally finite graphs

Sylvain Golenia, Christoph Schumacher

► **To cite this version:**

Sylvain Golenia, Christoph Schumacher. The problem of deficiency indices adjacency matrices on locally finite graphs. 2012. hal-00717815v2

HAL Id: hal-00717815

<https://hal.science/hal-00717815v2>

Preprint submitted on 23 Jul 2012 (v2), last revised 25 Oct 2013 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE PROBLEM OF DEFICIENCY INDICES ADJACENCY MATRICES ON LOCALLY FINITE GRAPHS

SYLVAIN GOLÉNIA AND CHRISTOPH SCHUMACHER

ABSTRACT. In this note we answer negatively to our conjecture concerning the deficiency indices. More precisely, given any non-negative integer n , there is locally finite graph on which the adjacency matrix has deficiency indices (n, n) .

Let V be a countable set. Let $E := V \times V \rightarrow \{0, 1\}$ and assume that $E(x, y) = E(y, x)$, for all $x, y \in V$. Set that $E(x, x) = 0$, for all $x \in V$. We say that $x, y \in V$ are *neighbors* if $E(x, y) \neq 0$ and denote it by $x \sim y$. We say that $G := (E, V)$ is an unoriented simple graph with *vertices* V , *weights* E , and no loop. We associate to G the complex Hilbert space $\ell^2(V)$. We denote by $\langle \cdot, \cdot \rangle$ and by $\|\cdot\|$ the scalar product and the associated norm, respectively. The set of complex functions with compact support in V is denoted by $\mathcal{C}_c(G)$. The *adjacency matrix* is defined by:

$$(1) \quad (\mathcal{A}_{G, \circ} f)(x) := \sum_{y \sim x} f(y), \text{ with } f \in \mathcal{C}_c(G).$$

The operator is symmetric and thus closable. We denote the closures by \mathcal{A}_G . We denote the domains by $\mathcal{D}(\mathcal{A}_G)$, and its adjoint by $(\mathcal{A}_G)^*$. Even in the case of a locally finite tree G , \mathcal{A}_G may have many self-adjoint extensions. We investigate the number of possible self-adjoint extensions of the adjacency matrix by computing its deficiency indices. Given a closed and densely defined symmetric operator T acting on a complex Hilbert space, the deficiency indices of T are defined by $\eta_{\pm}(T) := \dim \ker(T^* \mp i) \in \mathbb{N} \cup \{+\infty\}$. The operator T possesses a self-adjoint extension if and only if $\eta_+(T) = \eta_-(T)$. If this is the case, we denote the common value by $\eta(T)$. T is self-adjoint if and only if $\eta(T) = 0$. Since the operator \mathcal{A}_G commutes with the complex conjugation, its deficiency indices are equal. This means that \mathcal{A}_G possesses a self-adjoint extension. Note that $\eta(\mathcal{A}_G) = 0$ if and only if \mathcal{A}_G is essentially self-adjoint on $\mathcal{C}_c(G)$.

In [MO, Mü], one constructs adjacency matrices for simple trees with positive deficiency indices. In fact, it follows from the proof that the deficiency indices are infinite in both references. As a general result, a special case of [GS, Theorem 1.1] gives that, given a locally finite simple tree G , one has the following alternative:

$$(2) \quad \eta(\mathcal{A}_G) \in \{0, +\infty\}.$$

The value of $\eta(\mathcal{A}_G)$ is discussed in [GS] and is linked with the growth of the tree.

In [MW, Section 3], one finds:

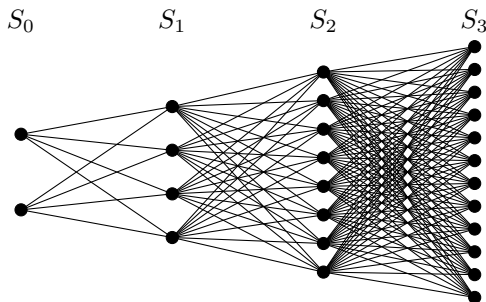
Theorem 1. *For all $n \in \mathbb{N} \cup \{\infty\}$, there is a simple graph G , such that $\eta(\mathcal{A}_G) = n$.*

Their proof is unfortunately incomplete. However, the statement is correct, this is aim of this note. Using standard perturbation theory, e.g., [GS, Proposition A.1], one sees that the validity of Theorem 1 is equivalent to the existence of a simple graph G for which

$$(3) \quad \eta(\mathcal{A}_G) = 1.$$

Date: Version of July 23, 2012.

Key words and phrases. adjacency matrix, deficiency indices, locally finite graphs.

FIGURE 1. An antitree with $s_0 = 2$, $s_1 = 4$, $s_2 = 8$, and $s_3 = 12$

In [MW], instead of pointing a simple graph such that (3) holds, they relied on a tree. More precisely, they refer to the works of [MO, Mü].

Keeping that in mind and strongly motivated by some other examples, we had proposed a drastically different scenario and had conjectured in [GS] that for any simple graph, one has (2).

We now turn to the proof of Theorem 1 and therefore disprove our conjecture. We rely on the decomposition of anti-trees, see [BK, Theorem 4.1].

Let S_n , $n \in \mathbb{N}$, be nonempty, finite and pairwise disjoint sets. We set $s_n := |S_n|$, $V := \cup_{n \in \mathbb{N}} S_n$ and $|x| := n$ for $x \in S_n$. Set also $E(x, y) = 1$, if $||x| - |y|| = 1$, and $E(x, y) = 0$ otherwise. We define:

$$P := \bigoplus_{n \in \mathbb{N}} P_n, \text{ where } P_n f(x) := \frac{1}{s_n} \mathbf{1}_{S_n}(x) \sum_{y \in S_n} f(y),$$

for all $f \in \ell^2(V)$. Note that $P = P^2 = P^*$ and $\text{rank } P \mathbf{1}_{S_n} = 1$ for all $n \in \mathbb{N}$. Given $f \in \ell^2(V)$ such that $f = Pf$, i.e., f is radially symmetric, we set $\tilde{f}(|x|) := f(x)$, for all $x \in V$. Note that

$$P\ell^2(V) = \{f \in \ell^2(V), \sum_{n \in \mathbb{N}} s_n |\tilde{f}(n)|^2 < \infty\} \simeq \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}),$$

where $(s_n)_{n \in \mathbb{N}}$ is now a sequence of weights. The key remark of [BK, Theorem 4.1] is that

$$\mathcal{A}_G = P \mathcal{A}_G P \text{ and } \widetilde{\mathcal{A}_G P f}(|x|) = s_{|x|-1} \widetilde{P f}(|x| - 1) + s_{|x|+1} \widetilde{P f}(|x| + 1),$$

for all $f \in \mathcal{C}_c(V)$ with the convention that $s_{-1} = 0$. Using the unitary transformation $U: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}, s_n)$ given by $U \tilde{f}(n) = \sqrt{s_n} \tilde{f}(n)$, we see that \mathcal{A}_G is unitarily equivalent to the direct sum of 0 and of a Jacobi matrix acting on $\ell^2(\mathbb{N})$, with 0 on the diagonal and the sequence $(\sqrt{s_n} \sqrt{s_{n+1}})_{n \in \mathbb{N}}$ on the off-diagonal. Finally using [Ber, Page 504 and 507], we derive that: Given $\alpha > 0$ and $s_n := \lfloor n^\alpha \rfloor$,

$$\eta(\mathcal{A}_G) = \begin{cases} 0, & \text{if } \alpha \leq 1, \\ 1, & \text{if } \alpha > 1. \end{cases}$$

Notation: The set of nonpositive integers is denoted by \mathbb{N} , note that $0 \in \mathbb{N}$. Given a set X and $Y \subseteq X$ let $\mathbf{1}_Y$ be the multiplication operator by the characteristic function of Y .

Acknowledgments: We would like to thank Matthias Keller for helpful discussions and Thierry Jecko for precious remarks.

REFERENCES

- [Ber] J.M. Berezanskii: *Expansions in eigenfunctions of selfadjoint operators*, Providence, R.I.: American Mathematical Society 1968. IX, 809 p. (1968).
[BK] J. Breuer and M. Keller: *Spectral Analysis of Certain Spherically Homogeneous Graphs*, preprint arXiv:1201.0118v1

- [GS] S. Golénia and C. Schumacher: *The problem of deficiency indices for discrete Schrödinger operators on locally finite graphs*, J. Math. Phys. 52 (2011), no. 6, 063512, 17 pp.
- [MO] B. Mohar and M. Omladic: *The spectrum of infinite graphs with bounded vertex degrees*, Graphs, hypergraphs and applications, Teubner Texte 73 (Teubner, Leipzig, 1985), pp. 122–125.
- [MW] B. Mohar and W. Woess: *A survey on spectra of infinite graphs*, J. Bull. Lond. Math. Soc. **21**, No.3, 209-234 (1989).
- [Mü] V. Müller: *On the spectrum of an infinite graph*, Linear Algebra Appl. 93 (1987) 187–189.
- [RS] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, Tome I–IV: Analysis of operators* Academic Press.

INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UNIVERSITÉ BORDEAUX 1, 351, COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE

E-mail address: `sylvain.golenia@u-bordeaux1.fr`

FAKULTÄT FÜR MATHEMATIK DER TECHNISCHEN UNIVERSITÄT CHEMNITZ, REICHENHAINER STR. 41, 09126 CHEMNITZ, GERMANY

E-mail address: `christoph.schumacher@mathematik.tu-chemnitz.de`