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THE PROBLEM OF DEFICIENCY INDICES ADJACENCY MATRICES ON LOCALLY FINITE GRAPHS

SYLVAIN GOLÉNIA AND CHRISTOPH SCHUMACHER

ABSTRACT. In this note we answer negatively to our conjecture concerning the deficiency indices. More precisely, given any non-negative integer n , there is locally finite graph on which the adjacency matrix has deficiency indices (n, n) .

Let V be a countable set. Let $E := V \times V \rightarrow \{0, 1\}$ and assume that $E(x, y) = E(y, x)$, for all $x, y \in V$. Set that $E(x, x) = 0$, for all $x \in V$. We say that $x, y \in V$ are *neighbors* if $E(x, y) \neq 0$ and denote it by $x \sim y$. We say that $G := (E, V)$ is an unoriented simple graph with *vertices* V , *weights* E , and no loop. We associate to G the complex Hilbert space $\ell^2(V)$. We denote by $\langle \cdot, \cdot \rangle$ and by $\|\cdot\|$ the scalar product and the associated norm, respectively. The set of complex functions with compact support in V is denoted by $\mathcal{C}_c(G)$. The *adjacency matrix* is defined by:

$$(0.1) \quad (\mathcal{A}_{G, \circ} f)(x) := \sum_{y \sim x} f(y), \text{ with } f \in \mathcal{C}_c(G).$$

The operator is symmetric and thus closable. We denote the closures by \mathcal{A}_G . We denote the domains by $\mathcal{D}(\mathcal{A}_G)$, and its adjoint by $(\mathcal{A}_G)^*$. Even in the case of a locally finite tree G , \mathcal{A}_G may have many self-adjoint extensions. We investigate the number of possible self-adjoint extensions of the adjacency matrix by computing its deficiency indices. Given a closed and densely defined symmetric operator T acting on a complex Hilbert space, the deficiency indices of T are defined by $\eta_{\pm}(T) := \dim \ker(T^* \mp i) \in \mathbb{N} \cup \{+\infty\}$. The operator T possesses a self-adjoint extension if and only if $\eta_+(T) = \eta_-(T)$. If this is the case, we denote the common value by $\eta(T)$. T is self-adjoint if and only if $\eta(T) = 0$. Since the operator \mathcal{A}_G commutes with the complex conjugation, its deficiency indices are equal. This means that \mathcal{A}_G possesses a self-adjoint extension. Note that $\eta(\mathcal{A}_G) = 0$ if and only if \mathcal{A}_G is essentially self-adjoint on $\mathcal{C}_c(G)$.

In [MO, Mü], one constructs adjacency matrices for simple trees with positive deficiency indices. In fact, it follows from the proof that the deficiency indices are infinite in both references. As a general result, a special case of [GS, Theorem 1.1] gives that, given a locally finite simple tree G , one has the following alternative:

$$(0.2) \quad \eta(\mathcal{A}_G) \in \{0, +\infty\}.$$

The value of $\eta(\mathcal{A}_G)$ is discussed in [GS] and is linked with the growth of the tree.

In [MW, Section 3], one finds:

Theorem 0.1. *For all $n \in \mathbb{N} \cup \{\infty\}$, there is a simple graph G , such that $\eta(\mathcal{A}_G) = n$.*

Using standard perturbation theory, e.g., [GS, Proposition A.1], to prove this result it is enough to point out one graph G for which

$$(0.3) \quad \eta(\mathcal{A}_G) = 1.$$

In [MW], they unfortunately used a tree for their proof and one knows by (0.2) that trees cannot satisfy (0.3). On the other hand, in [GS] we conjectured that for any simple graph, one has (0.2). In this note we prove that Theorem 0.1 is true and we disprove our conjecture. We rely on the decomposition of anti-trees, see [BK, Theorem 4.1].

Let S_n , $n \in \mathbb{N}$, be nonempty, finite and pairwise disjoint sets. We set $s_n := |S_n|$, $V := \cup_{n \in \mathbb{N}} S_n$ and $|x| := n$ for $x \in S_n$. Set also $E(x, y) = 1$, if $||x| - |y|| = 1$, and $E(x, y) = 0$ otherwise. We define:

$$P := \bigoplus_{n \in \mathbb{N}} P_n, \text{ where } P_n f(x) := \frac{1}{s_n} \mathbf{1}_{S_n}(x) \sum_{y \in S_n} f(y),$$

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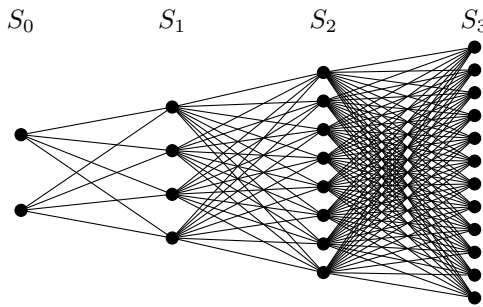


FIGURE 1. An antitree with $s_0 = 2$, $s_1 = 4$, $s_2 = 8$, and $s_4 = 12$

for all $f \in \ell^2(V)$. Note that $P = P^2 = P^*$ and $\text{rank } P\mathbf{1}_{S_n} = 1$ for all $n \in \mathbb{N}$. Given $f \in \ell^2(V)$ such that $f = Pf$, i.e., f is radially symmetric, we set $\tilde{f}(|x|) := f(x)$, for all $x \in V$. Note that

$$P\ell^2(V) = \{f \in \ell^2(V), \sum_{n \in \mathbb{N}} s_n |\tilde{f}(n)|^2 < \infty\} \simeq \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}),$$

where $(s_n)_{n \in \mathbb{N}}$ is now a sequence of weights. The key remark of [BK, Theorem 4.1] is that

$$\mathcal{A}_G = P\mathcal{A}_G P \text{ and } \widetilde{\mathcal{A}_G P f}(|x|) = s_{|x|-1} \widetilde{P f}(|x| - 1) + s_{|x|+1} \widetilde{P f}(|x| + 1),$$

for all $f \in \mathcal{C}_c(V)$ with the convention that $s_{-1} = 0$. Using the unitary transformation $U: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}, s_n)$ given by $U\tilde{f}(n) = \sqrt{s_n}\tilde{f}(n)$, we see that \mathcal{A}_G is unitarily equivalent to the direct sum of 0 and of a Jacobi matrix acting on $\ell^2(\mathbb{N})$, with 0 on the diagonal and the sequence $(\sqrt{s_n}\sqrt{s_{n+1}})_{n \in \mathbb{N}}$ on the off-diagonal. Finally using [Ber, Page 504 and 507], we derive that: Given $\alpha > 0$ and $s_n := \lfloor n^\alpha \rfloor$,

$$\eta(\mathcal{A}_G) = \begin{cases} 0, & \text{if } \alpha \leq 1, \\ 1, & \text{if } \alpha > 1. \end{cases}$$

Notation: The set of nonpositive integers is denoted by \mathbb{N} , note that $0 \in \mathbb{N}$. Given a set X and $Y \subseteq X$ let $\mathbf{1}_Y$ be the multiplication operator by the characteristic function of Y .

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