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# STEIN'S METHOD FOR BROWNIAN APPROXIMATIONS

L. COUTIN AND L. DECREUSEFOND

ABSTRACT. Motivated by a theorem of Barbour, we revisit some of the classical limit theorems in probability from the viewpoint of the Stein method. We setup the framework to bound Wasserstein distances between some distributions on infinite dimensional spaces. We show that the convergence rate for the Poisson approximation of the Brownian motion is as expected proportional to  $\lambda^{-1/2}$  where  $\lambda$  is the intensity of the Poisson process. We also exhibit the speed of convergence for the Donsker Theorem and for the linear interpolation of the Brownian motion. By iterating the procedure, we give Edgeworth expansions with precise error bounds.

## 1. INTRODUCTION

Among the classics in probability theory, one can cite the approximation in distribution of a Brownian motion by a normalized compensated Poisson process of intensity going to infinity or the celebrated Donsker theorem which says that a symmetric random walk conveniently normalized also approaches a Brownian motion in distribution. Though the topology of the convergence in distribution is known to derive from a distance on the space of probability measures, to the best of our knowledge, we are aware of only one result precising the speed of convergence in one of these two theorems. In [2], Barbour estimated the distance between the distribution of a normalized compensated Poisson process of intensity  $\lambda$  and the distribution of a Brownian motion. The common space on which these two processes are compared is taken as the space of rcl functions, denoted by  $\mathfrak{D}([0, 1], \mathbf{R})$  equipped with the distance:

$$d_0(\omega, \eta) = \inf_{\Phi \in \text{Hom}([0,1])} (\|\omega \circ \Phi - \eta\|_\infty + \|\Phi - \text{Id}\|_\infty),$$

where  $\text{Hom}([0, 1])$  is the set of increasing homeomorphisms of  $[0, 1]$ . It is proved in [2] that the speed of convergence is not  $\lambda^{-1/2}$  as expected but that there exists a non negligible corrective term. This additional term exists because the sample-paths of the two processes do not really belong to the same space: Continuous functions are a rather special class of rcl functions and sample-paths of Poisson process even normalized are never continuous whatever the value of the intensity. Thus there is an unavoidable gap between the two kind of trajectories in the considered approximation. Actually, the additional term is related to the modulus of continuity

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of the Brownian motion, i.e. in some sense, it measures the cost to approximate a continuous function by a purely discontinuous one.

We circumvent this problem by considering Poisson and Brownian sample-paths as elements of the same space. In fact, the Poisson sample-paths, like the trajectories of the other processes we are considering in this paper, belong to a much smaller space than  $\mathfrak{D}([0, 1], \mathbf{R})$ . They all are piecewise differentiable, i.e. of the form  $\sum_{n \in \mathbf{N}} r_n(t - t_n) \mathbf{1}_{[t_n, t_{n+1})}$  where  $(t_n, n \geq 1)$  is an increasing sequence of real and  $r_n$  are differentiable functions. An indicator function is not continuous but it has more property than being rcll. In particular, it belongs to  $\mathcal{I}_{\beta, p}$  for any  $p \geq 1$  and any  $\beta < 1/p$  (see Appendix A for definition). On the other hand, Brownian trajectories are  $(1/2 - \epsilon)$ -Hölder continuous so that they belong to  $\mathcal{I}_{\beta, p}$  any  $p \geq 1$  and any  $\beta < 1/2$ . Therefore, the natural candidates to support both the distribution of piecewise differentiable processes and that of the Brownian motion are the spaces  $\mathcal{I}_{\beta, 2}$  for any  $\beta < 1/2$ . The original problem is then reduced to the computation of the distance between between a given distribution and a Gaussian law on some Hilbert space.

The Stein method is known for a long time to give the speed of convergence of many Gaussian approximations (see for instance [5] and references therein). The usual approach requires some sort of coupling to derive the pertinent estimates. It is only recently that the mixing of Stein approach and Malliavin calculus proved its efficiency (see [18] for a thorough analysis of this line of thought): The search of ad-hoc couplings in the Stein method is there bypassed by using integration by parts formula in the sense of Malliavin calculus. In particular, it has been used for approximations of point processes functionals [7, 8, 20]. But to the best of our knowledge, up to the notable exception of [2], all these investigations consider finite dimensional Gaussian random variables. We here develop the framework for a Stein theory on Hilbert spaces. It requires two types of Malliavin *gradients* : One used to characterized the target (Gaussian) measure, one built on the probability space of the measure to be compared to the Gaussian measure, used to perform the necessary integration by parts. We show that our method is applicable in three different settings: Whenever the alea on which the approximate process is built upon is either the Poisson space, the Rademacher space or the Wiener space.

Since [1, 13], it is well known that Stein method can also lead to expansions of higher order by pursuing the development. We here generalize this method to infinite dimension and give the exact values of the bound coefficients.

This paper is organized as follows. We first construct the Wiener measure on the Besov-Liouville spaces. In Section 3, we transfer this structure to  $l^2(\mathbf{N})$  for the ease of computations. Section 4 is devoted to the development of the abstract version of the Stein method for Hilbert valued random variables. In Section 5 to Section 7, we exemplify this general scheme of reasoning successively for the Poisson approximation of the Brownian motion, for the linear interpolation of the Brownian motion and for the Donsker theorem. For the former two examples, we iterate the procedure and derive an Edgeworth expansion with explicit error bounds. In Section 8, we show that by a transfer principle, similar results can be obtained for other Gaussian processes like the fractional Brownian motion, extending some earlier result [9]. Technical proofs are deported to Section 9.

## 2. GAUSSIAN STRUCTURE ON BESOV-LIOUVILLE SPACES

In order to compare quantitatively the distribution of a piecewise differentiable process with that of a Brownian motion, we need to consider a functional space to which the sample-paths of both processes belong to. Ordinary Brownian motion is known to have sample-paths Hölder continuous of any order smaller than  $1/2$ . Thus, Theorem [A.1] ensures that its sample-paths belongs to  $\mathcal{I}_{\beta,\infty} \subset \mathcal{I}_{\beta,2}$  for any  $\beta < 1/2$ .

Moreover, a simple calculation shows that  $\mathbf{1}_{[a,+\infty)} = I_{0+}^1(\epsilon_a)$ . Since for any  $\epsilon > 0$ ,  $\mathcal{I}_{1/2+\epsilon,2}^+$  is embedded into  $\text{Hol}(\epsilon)$  and since  $\epsilon_a$  belongs to the dual of  $\text{Hol}(\epsilon)$ , it follows that  $\epsilon_a$  belongs to  $\mathcal{I}_{1/2+\epsilon,2}^* = \mathcal{I}_{-1/2-\epsilon,2}$ . Hence  $\mathbf{1}_{[a,+\infty)}$  belongs to  $\mathcal{I}_{1/2-\epsilon,2}$  for any  $\epsilon > 0$ . This implies that random step functions belong to  $\mathcal{I}_{\beta,2}$  for any  $\beta < 1/2$ . The space of choice may thus be any space  $\mathcal{I}_{\beta,2}$  for any  $\beta < 1/2$ . The closer to  $1/2$   $\beta$  is, the most significant the distance is but the the greater the error bound is.

To construct the Wiener measure on  $\mathcal{I}_{\beta,2}$ , we start from the Itô-Nisio theorem. Let  $(X_n, n \geq 1)$  be a sequence of independant centered Gaussian random variables of variance 1 defined on a common probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Let  $(e_n, n \geq 1)$  be a complete orthonormal basis of  $\mathcal{L}^2([0, 1])$ . Then,

$$B(t) := \sum_{n \geq 1} X_n I_{0+}^1(e_n)(t)$$

converges almost-surely for any  $t \in [0, 1]$ . From [14], we already know that the convergence holds uniformly with respect to  $t$  and thus that  $B$  is continuous. To prove that the convergence holds in  $L^2(\Omega; \mathcal{I}_{\beta,2})$ , it suffices to show that

$$(1) \quad \sum_{n \geq 1} \|I_{0+}^1 e_n\|_{\mathcal{I}_{\beta,2}}^2 = \sum_{n \geq 1} \|I_{0+}^{1-\beta} e_n\|_{\mathcal{L}^2}^2 = \|I_{0+}^{1-\beta}\|_{\text{HS}} < \infty.$$

From [24], we know that  $I^{1-\beta}$  is an Hilbert-Schmidt operator from  $\mathcal{L}^2$  into itself and only if  $1 - \beta > 1/2$ , i.e.  $\beta < 1/2$ . Thus, for  $\beta < 1/2$ , the distribution of  $B$  defines the Wiener measure on  $\mathcal{I}_{\beta,2}$ . We denote this measure by  $\mu_\beta$ . Note that (1) implies that the embedding from  $\mathcal{I}_{1-\beta,2}$  into  $\mathcal{L}^2$  is also Hilbert-Schmidt and that its Hilbert-Schmidt norm is  $\|I_{0+}^{1-\beta}\|_{\text{HS}}$ . For latter purposes, we set

$$c_\beta = \|I_{0+}^{1-\beta}\|_{\text{HS}} = \frac{1}{2\Gamma(1-\beta)} \left( \frac{1}{(1-\beta)(1/2-\beta)} \right)^2,$$

according to Theorem A.3. By the very definition of the scalar product on  $\mathcal{I}_{\beta,2}$ , for  $\eta \in \mathcal{I}_{\beta,2}$ , we have

$$\begin{aligned} \mathbb{E}_{\mu_\beta} [\exp(i\langle \eta, \omega \rangle_{\mathcal{I}_{\beta,2}})] &= \mathbb{E}_{\mathbb{P}} \left[ \exp\left(i \sum_{n \geq 1} \int_0^1 (I_{1-}^{1-\beta} \circ I_{0+}^{-\beta}) \eta(s) e_n(s) ds X_n\right) \right] \\ &= \exp\left(-\frac{1}{2} \sum_{n \geq 1} \left( \int_0^1 (I_{1-}^{1-\beta} \circ I_{0+}^{-\beta}) \eta(s) e_n(s) ds \right)^2\right) \\ &= \exp\left(-\frac{1}{2} \|(I_{1-}^{1-\beta} \circ I_{0+}^{-\beta}) \eta\|_{\mathcal{L}^2([0,1])}^2\right) \\ &= \exp\left(-\frac{1}{2} \int_0^1 (I_{0+}^{1-\beta} \circ I_{1-}^{1-\beta}) \dot{\eta}(s) \dot{\eta}(s) ds\right), \end{aligned}$$

where  $\dot{\eta}$  is the unique element of  $\mathcal{L}^2$  such that  $\eta = I_{0+}^\beta \dot{\eta}$ . Thus,  $\mu_\beta$  is a Gaussian measure on  $\mathcal{I}_{\beta,2}$  of covariance operator given by

$$V_\beta = I_{0+}^\beta \circ I_{0+}^{1-\beta} \circ I_{1-}^{1-\beta} \circ I_{0+}^{-\beta}.$$

This means that

$$\mathbb{E}_{\mu_\beta} [\exp(i\langle \eta, \omega \rangle_{\mathcal{I}_{\beta,2}})] = \exp(-\frac{1}{2}\langle V_\beta \eta, \eta \rangle_{\mathcal{I}_{\beta,2}}).$$

We could thus in principle make all the computations in  $\mathcal{I}_{\beta,2}$ . It turns out that we were not able to be explicit in the computations of some traces of some involved operators the expressions of which turned to be rather straightforward in  $l^2(\mathbf{N})$ . This is why we transfer all the structure to  $l^2(\mathbf{N})$ . This is done at no loss of generality nor precision since there exists a bijective isometry between  $\mathcal{I}_{\beta,2}$  and  $l^2(\mathbf{N})$ .

### 3. GAUSSIAN STRUCTURE ON $l^2(\mathbf{N})$

Actually, the canonical isometry is given by the Fourier expansion of the  $\beta$ -th derivative of an element of  $\mathcal{I}_{\beta,2}$ . As is, that would not be explicit enough for the computations to come to be tractable. We take benefit from the dual aspect of a time indexed point process. On the one hand, as mentioned above, the sample-path of a point process is of the form

$$t \mapsto \sum_{n \geq 1} \mathbf{1}_{[t_n, 1]}(t)$$

where  $(t_n, n \geq 1)$  is a strictly increasing sequence of reals, all but a finite number greater than 1, and thus belongs to  $\mathcal{I}_{\beta,2}$  for any  $\beta < 1/2$  as shown above. On the other hand, it can be seen as a locally finite point measure defined by

$$f \in \mathcal{L}^2([0, 1]) \mapsto \sum_{n \geq 1} f(t_n).$$

Said otherwise, we have the following definition.

**Definition 1.** For  $(h, \omega) \in \mathcal{I}_{1-\beta,2}^- \times \mathcal{I}_{\beta,2}^+$

$$\int_0^1 h(s) d\omega_s := \langle h, I_{0+}^{-1}(\omega) \rangle_{\mathcal{I}_{1-\beta,2}^-, \mathcal{I}_{\beta-1,2}^+} = \langle I_{1-}^{\beta-1}(h), I_{0+}^{-\beta}(\omega) \rangle_{\mathcal{L}^2}.$$

Recall that  $(e_n, n \in \mathbf{N})$  is a complete orthonormal basis of  $\mathcal{L}^2([0, 1])$  and set  $h_n = I_{1-}^{1-\beta}(e_n)$ . Then  $(h_n, n \in \mathbf{N})$  is a complete orthonormal basis of  $\mathcal{I}_{1-\beta,2}^-$ . Consider the map  $\mathfrak{J}_\beta$  defined by:

$$\begin{aligned} \mathfrak{J}_\beta : \mathcal{I}_{\beta,2}^+ &\longrightarrow l^2(\mathbf{N}) \\ \omega &\longmapsto \left( \int_0^1 h_n(s) d\omega(s) = \langle e_n, \dot{\omega} \rangle_{\mathcal{L}^2([0, 1])}, n \geq 1 \right). \end{aligned}$$

**Theorem 3.1.** The map  $\mathfrak{J}_\beta$  is a bijective isometry from  $\mathcal{I}_{\beta,2}$  into  $l^2(\mathbf{N})$ . Its inverse is given by:

$$\begin{aligned} \mathfrak{J}_\beta^{-1} : l^2(\mathbf{N}) &\longrightarrow \mathcal{I}_{\beta,2} \\ (\alpha_n, n \in \mathbf{N}) &\longmapsto \sum_{n \geq 0} \alpha_n I_{0+}^\beta(e_n). \end{aligned}$$

We thus have the commutative diagram.

$$\begin{array}{ccc} \mathcal{I}_{\beta,2} & \xrightarrow{\mathfrak{J}_\beta} & l^2(\mathbf{N}) \\ V_\beta \downarrow & & \downarrow S_\beta := \mathfrak{J}_\beta^{-1} \circ V_\beta \circ \mathfrak{J}_\beta \\ \mathcal{I}_{\beta,2} & \xrightarrow{\mathfrak{J}_\beta} & l^2(\mathbf{N}) \end{array}$$

According to the properties of Gaussian measure (see [15]), we have the following result.

**Theorem 3.2.** *Let  $\mu_\beta$  denote the Wiener measure on  $\mathcal{I}_{\beta,2}$ . Then  $\mathfrak{J}_\beta^* \mu_\beta = m_\beta$ , where  $m_\beta$  is the Gaussian measure on  $l^2(\mathbf{N})$  such that for any  $v \in l^2(\mathbf{N})$ ,*

$$\int_{l^2(\mathbf{N})} \exp(i v \cdot u) dm_\beta(u) = \exp\left(-\frac{1}{2} S_\beta v \cdot v\right)$$

with the following notations.

$$\|x\|_{l^2(\mathbf{N})}^2 = \sum_{n=1}^{\infty} |x_n|^2 \text{ and } x \cdot y = \sum_{n=1}^{\infty} x_n y_n, \text{ for all } x, y \in l^2(\mathbf{N}).$$

In view of Theorem [3.1], it is straightforward that the map  $S_\beta$  admits the representation:

$$\begin{aligned} S_\beta : l^2(\mathbf{N}) &\longrightarrow l^2(\mathbf{N}) \\ u = (u_n, n \in \mathbf{N}) &\longmapsto \left( \sum_{j \geq 1} \langle h_n, h_j \rangle_{\mathcal{L}^2} u_j, n \in \mathbf{N} \right). \end{aligned}$$

By  $\mathcal{C}_b^k(l^2(\mathbf{N}); X)$ , we denote the space of  $k$ -times Fréchet differentiable functions from  $l^2(\mathbf{N})$  into an Hilbert space  $X$  with bounded derivatives: A function  $F$  belongs to  $\mathcal{C}_b^k(l^2(\mathbf{N}); X)$  whenever

$$\|F\|_{\mathcal{C}_b^k(l^2(\mathbf{N}); X)} := \sup_{j=1, \dots, k} \text{ess-sup}_{x \in l^2(\mathbf{N})} \|\nabla^{(j)} F(x)\|_{X \otimes l^2(\mathbf{N})^{\otimes j}} < \infty.$$

A simple calculation with characteristic functions shows that a probability measure  $m$  on  $l^2(\mathbf{N})$  is equal to  $m_\beta$  if and only if for any  $F \in \mathcal{C}^2(l^2(\mathbf{N}); X)$ ,

$$\int_{l^2(\mathbf{N})} A^\beta F(x) dm(x) = 0 \text{ where } A^\beta F(x) = x \cdot \nabla F(x) - \text{trace}(S_\beta \circ \nabla^{(2)} F(x)).$$

We are thus led to introduce the Markov process the infinitesimal generator of which is  $A^\beta$ . More precisely, we consider its semi-group, usually called Ornstein-Uhlenbeck semi-group as its stationary measure is that of the Brownian motion (in fact the image of this measure on  $l^2(\mathbf{N})$ ).

**Definition 2.** *The Ornstein-Uhlenbeck semi-group on  $(l^2(\mathbf{N}), m_\beta)$  is defined for any  $F \in L^2(l^2(\mathbf{N}), \mathcal{F}, m_\beta; X)$  by*

$$P_t^\beta F(u) = \int_{l^2(\mathbf{N})} F(e^{-t}u + \sqrt{1 - e^{-2t}}v) dm_\beta(v),$$

where the integral is a Bochner integral. It is well known that this semi-group is ergodic in the sense that

$$P_t F(u) \xrightarrow{t \rightarrow \infty} \int f dm_\beta, \text{ } m_\beta - \text{a.s.}$$

Moreover, if  $F \in C_b^k(l^2(\mathbf{N}); X)$ , the function  $(u \mapsto P_t F(u))$  is  $(k+1)$ -times differentiable and we have (see [15]):

$$(2) \quad \nabla^{(k+1)}(P_t F)(u) = \frac{e^{-(k+1)t}}{\sqrt{1-e^{-2t}}} \int_{l^2(\mathbf{N})} (\nabla^{(k)} F)(e^{-t}u + \sqrt{1-e^{-2t}}v) \otimes v \, dm_\beta(v).$$

**Lemma 3.1.** *If  $F$  belongs to  $C_b^k(l^2(\mathbf{N}); X)$ , then, for any integer  $k$ ,*

$$\int_0^\infty \text{ess-sup}_{x \in l^2(\mathbf{N})} \|\nabla^{(k+1)}(P_t F)(u)\|_{l^2(\mathbf{N})^{\otimes(k+1)} \otimes X} \, dt \leq \beta_{k+1} \|F\|_{C_b^k(l^2(\mathbf{N}); X)},$$

where

$$\beta_k := \frac{(1-2\beta)^{-1/2}(2-2\beta)^{-1/2}}{2\Gamma(1-\beta)} B\left(\frac{k}{2}, \frac{1}{2}\right).$$

and  $B$  is the Beta function.

In the sequel, we need to apply  $A^\beta$  to Hilbert valued functions. We thus introduce the notion of partial trace. For any vector space  $X$ ,  $\text{Lin}(X)$  is the set of linear operator from into itself. For  $X$  and  $Y$  two Hilbert spaces, the partial trace operator along  $X$  can be defined as follows: it is the unique linear operator

$$\text{trace}_X : \text{Lin}(X \otimes Y) \longrightarrow \text{Lin}(Y)$$

such that for any  $R \in \text{Lin}(Y)$ , for any trace class operator  $S$  on  $X$ ,

$$\text{trace}_X(S \otimes R) = \text{trace}_X(S) R.$$

For Hilbert valued functions, we define  $A^\beta$  as follows.

**Lemma 3.2.** *The domain of  $A^\beta$  contains  $C_b^2(l^2(\mathbf{N}); X)$  and for  $F$  in this space, we have:*

$$(A^\beta F)(u) = u \cdot (\nabla F)(u) - \text{trace}_{l^2(\mathbf{N})}(S_\beta \nabla^2 F(u)), \text{ for all } x \in l^2(\mathbf{N}).$$

#### 4. STEIN METHOD

For  $\nu^*$  and  $\mu^*$  two probability measures on  $\mathbf{R}^{\mathbf{N}}$ , we define a distance by

$$\rho_{\mathfrak{T}}(\nu^*, \mu^*) = \sup_{\|F\|_{\mathfrak{T}} \leq 1} \int F \, d\nu^* - \int F \, d\mu^*.$$

where  $\mathfrak{T}$  is a normed space of test functions (the norm of which is denoted by  $\|\cdot\|_{\mathfrak{T}}$ ). If  $\mathfrak{T}$  is the set 1-Lipschitz functions on  $l^2(\mathbf{N})$ , then  $\rho_{\mathfrak{T}}$  corresponds to the optimal transportation problem for the cost function  $c(x, y) = \|x - y\|_{l^2(\mathbf{N})}$ ,  $x, y \in \mathbf{R}^{\mathbf{N}}$  (see [25]). For technical reasons (as in [21]) mainly due to the infinite dimension, we must restrict the space  $\mathfrak{T}$  to smaller subsets. We thus introduce the distances  $\rho_j$  for  $j \geq 1$  as

$$\rho_j(\nu^*, \mu^*) = \sup_{\|F\|_{C_b^j(l^2(\mathbf{N}); \mathbf{R})} \leq 1} \int F \, d\nu^* - \int F \, d\mu^*.$$

However, these weaker distances still metrize the space of weak convergence of probability measures on  $l^2(\mathbf{N})$ .

**Theorem 4.1.** *Let  $(\nu_n^*, n \geq 1)$  be a sequence of probability measures on  $l^2(\mathbf{N})$  such that some  $j \geq 1$ ,*

$$\rho_j(\nu_n^*, \mu^*) \xrightarrow{n \rightarrow \infty} 0.$$

Then,  $(\nu_n^*, n \geq 1)$  converges weakly to  $\mu^*$  in  $l^2(\mathbf{N})$ :

$$\int F d\nu_n^* \xrightarrow{n \rightarrow \infty} \int F d\mu^*,$$

for any  $F \in C_b^0(l^2(\mathbf{N}), \mathbf{R})$ .

*Proof.* As Hilbert spaces admit arbitrarily smooth partition of unity [16], for  $j \geq 2$ , one can mimick the proof of [10, page 396] (see also [4]) which corresponds to  $\rho_1$ .  $\square$

Say that  $\mu^* = m_\beta$  is our reference measure, that is the measure we want the other measures to be compared to. Stein method relies on the characterization of  $m_\beta$  as the unique measure on  $l^2(\mathbf{N})$  such that any  $F$  sufficiently regular,

$$\int_{l^2(\mathbf{N})} A^\beta F(x) dm(x) = 0.$$

It follows that the Markov process of infinitesimal generator  $A^\beta$  has  $m_\beta$  as stationary measure. Since its semi-group  $(P_t^\beta, t \geq 0)$  is ergodic, by the well known properties of Markov processes [11], one can write, for  $F$  sufficiently regular,

$$\int_{l^2(\mathbf{N})} F(x) dm_\beta(x) - F(x) = \int_0^\infty A^\beta P_t^\beta F(x) dt, \nu^*\text{-a.s.}$$

Thus,

$$\rho_2(\nu^*, m_\beta) = \sup_{\|F\|_{C_b^3(l^2(\mathbf{N}), \mathbf{R})} \leq 1} \int_{l^2(\mathbf{N})} \int_0^\infty A^\beta P_t F(x) dt d\nu^*(x).$$

Thanks to the integration by parts induced by Malliavin calculus, we can control the right-hand-side integrand and obtain bounds on  $\rho_2(\nu^*, m_\beta)$ . To be more illustrative, the Stein method works as follows: construct a process  $(t \mapsto \mathfrak{X}(x, t))$  constant in distribution if its initial condition  $x$  is distributed according to  $m_\beta$ . Moreover, for any initial distribution, the law of  $\mathfrak{X}(x, t)$  tends to  $m_\beta$  as  $t$  goes to infinity. Stein method then consists in going back in time, from infinity to 0, controlling along the way the derivative of the changes, yielding a bound on the distance between the two initial measures. Other versions (coupling, size-bias, etc) are just other ways to construct another process  $\mathfrak{X}$ . In these approaches, for every  $\nu^*$ , the couplings are ad-hoc whereas Malliavin calculus gives a certain kind of universality as it depends only on the underlying alea. Malliavin structures are well established for sequences of Bernoulli random variables, Poisson processes, Gaussian processes and several other spaces (see [22]). In what follows, we show an example of the machinery for each of these three examples.

The core of the method can be summarized in the following theorem.

**Hypothesis I.** For  $X$  a Hilbert space,  $H \in l^2(\mathbf{N}) \otimes X$  and  $\alpha$  a non-negative real, we say that  $\text{Hyp}(X, H, \alpha)$  holds whenever for any  $G \in C_b^2(l^2(\mathbf{N}); l^2(\mathbf{N}))$

$$(3) \quad \left| \mathbb{E}_{\nu^*} [\text{Id} \cdot G] - \mathbb{E}_{\nu^*} [\text{trace}(\text{trace}_X(H \otimes H) \circ \nabla G)] \right| \leq \alpha \|G\|_{C_b^2(l^2(\mathbf{N}); l^2(\mathbf{N})^{\otimes 3})} \|H\|_{l^2(\mathbf{N}) \otimes X}^3.$$

**Theorem 4.2** (Stein method). Assume that  $\text{Hyp}(X, H, \alpha)$  holds, then,

$$(4) \quad \rho_2(\nu^*, m_\beta) \leq \beta_2 \left| \text{trace}(\text{trace}_X(H \otimes H) - S_\beta) \right| + \alpha \beta_3 \|H\|_{l^2(\mathbf{N}) \otimes X}^3.$$

**Remark 4.1.** *The two terms in the right-hand-side of (4) are of totally different nature. The trace term really measures the effect of the approximation scheme whereas the second term comes from a sort of curvature of the space on which is built the approximate process. As will become evident in the examples below, this term is zero when the Malliavin gradient satisfies the chain rule formula and non-zero otherwise.*

## 5. NORMAL APPROXIMATION OF POISSON PROCESSES

It is well known that for a Poisson process  $N_\lambda$  of intensity  $\lambda$ , the process

$$N^\lambda(t) = \frac{1}{\sqrt{\lambda}} (N_\lambda(t) - \lambda t)$$

converges in distribution on  $\mathfrak{D}$  to a Brownian motion as  $\lambda$  goes to infinity. For any  $\beta < 1/2$ , we want to precise the rate of convergence for the distance  $d_\beta$ , where for two measures  $\mu$  and  $\nu$  supported on  $\mathcal{I}_{\beta, 2}$ ,

$$d_\beta(\mu, \nu) = \sup_{\|F\|_{C_b^2(\mathcal{I}_{\beta, 2}; \mathbf{R})} \leq 1} \int F d\nu - \int F d\mu.$$

By isometry, this turns out to be equivalent to estimate  $\rho_2(\mathfrak{J}_\beta \nu_\lambda, m_\beta)$  where  $\nu_\lambda$  is the distribution of  $N_\lambda$  on  $\mathcal{I}_{\beta, 2}$ .

Let  $\chi_{[0, 1]}$  the space of locally finite measures on  $[0, 1]$  equipped with the vague topology. We identify a point measure  $\omega = \sum_{n \in \mathbf{N}} \delta_{t_n}$  with the one dimensional process

$$N : t \in [0, 1] \mapsto \int_0^t d\omega(s) = \sum_{n \in \mathbf{N}} \mathbf{1}_{[0, t]}(t_n).$$

The measure  $\nu_\lambda$  is the only measure on  $(\chi_{[0, 1]}, \mathfrak{B}(\chi_{[0, 1]}))$  such that the canonical process  $N$  is a Poisson process of intensity  $\lambda d\tau$ . It is customary to define the discrete gradient as

$$D_\tau F(N) = F(N + \epsilon_\tau) - F(N), \text{ for any } \tau \in [0, 1],$$

where  $N + \epsilon_\tau$  is the point process  $N$  with an extra atom at time  $\tau$ . We denote by  $\mathbb{D}_{2,1}$  the set of square integrable functionals  $F$  such that  $\mathbb{E} \left[ \int_0^1 |D_\tau F(N)|^2 d\tau \right]$  is finite. We then have the following relationship:

$$(5) \quad \mathbb{E}_{\nu_\lambda} \left[ F \int_0^1 g(\tau) (dN(\tau) - \lambda d\tau) \right] = \lambda \mathbb{E}_{\nu_\lambda} \left[ \int_0^1 D_\tau F g(\tau) d\tau \right],$$

for any  $g \in \mathcal{L}^2([0, 1])$  and any  $F \in \mathbb{D}_{2,1}$ . Moreover,

$$D_\tau \left( \int_0^1 g(s) (dN(s) - \lambda ds) \right) = g(\tau).$$

It is clear that

$$\mathfrak{J}_\beta N_\lambda = \left( \frac{1}{\sqrt{\lambda}} \int_0^1 h_n(s) (dN(s) - \lambda ds), n \in \mathbf{N} \right).$$

For the sake of notations, we introduce

$$R_\lambda = \mathfrak{J}_\beta N_\lambda \text{ and } H_\lambda = \left( \frac{1}{\sqrt{\lambda}} h_j, j \in \mathbf{N} \right),$$

so that  $\mathfrak{J}_\beta(N_\lambda + \epsilon_\tau) = \mathfrak{J}_\beta N_\lambda + H_\lambda(\tau) = R_\lambda + H_\lambda(\tau)$ .

**Theorem 5.1.** *We denote by  $\nu_\lambda^*$  the distribution of  $\mathfrak{J}_\beta N_\lambda$  in  $l^2(\mathbf{N})$ . The measure  $\nu_\lambda^*$  satisfies  $\text{Hyp}(\mathcal{L}^2([0, 1]), H_1, 1/2\sqrt{\lambda})$ . Hence,*

$$(6) \quad \rho_2(\nu_\lambda^*, m_\beta) \leq \frac{\beta_3}{2\sqrt{\lambda}} c_\beta^3.$$

**Remark 5.1.** *It is remarkable that by homogeneity, the partial trace of  $H_1 \otimes H_1$  is equal to  $S_\beta$ . The only remaining term in Theorem [4.2] comes from the fact that the discrete gradient does not satisfy the chain rule.*

*One could also remark that the choice of the space in which we embed the Poisson and Brownian sample-paths (i.e. the choice of the value of  $\beta$ ) modifies only the constant but not the order of convergence, which remains proportional to  $\lambda^{-1/2}$ .*

As is clear from the proof, we may expect to obtain a refined approximation if we go further in the Taylor expansion of the discrete gradient. By doing this, we obtain the so-called Edgeworth expansion.

In what follows, we make the convention that a sum like  $\sum_{r=1}^0 \dots$  is zero. Let  $(\alpha(r, k), r \geq 0, k \geq 0)$  be a family of real numbers such that

$$\alpha(0, 0) = 1, \quad \alpha(r, k) = 0 \text{ if } k - r \notin 2\{1, \dots, r\}$$

and for any  $r \in \mathbf{N}$ , any  $m \in \{1, \dots, r\}$

$$(7) \quad \alpha(r, r + 2m) = \frac{1}{r + 2m} \sum_{j=1}^r \frac{1}{(j+1)!} \alpha(r-j, r-j+2(m-1)).$$

Consider also the sequence  $(\gamma_s, s \geq 0)$  given by the recursion formula:

$$(8) \quad \gamma_s = \sum_{j=2}^{s+1} \frac{1}{j!} \gamma_{s+1-j} \beta_{2s+5-j} + \frac{c_\beta^{s+3}}{(s+2)!} \beta_{s+3}.$$

**Theorem 5.2.** *Let  $s$  be an integer. For  $F \in \mathcal{C}_b^{2(s+1)+1}(l^2(\mathbf{N}); \mathbf{R})$ , we have*

$$(9) \quad \int_{l^2(\mathbf{N})} F(u) d\nu_\lambda^*(u) = \int_{l^2(\mathbf{N})} F(u) dm_\beta(u) \\ + \sum_{r=1}^s \left( \frac{1}{\sqrt{\lambda}} \right)^r \sum_{k=1}^r \alpha(r, r+2k) \int_{l^2(\mathbf{N})} \nabla^{(r+2k)} F(u) \cdot \int_0^1 H_1(\tau)^{\otimes(r+2m)} d\tau \, dm_\beta(u) \\ + \text{Rem}(s, F, \lambda)$$

where the remainder term can be bounded as

$$|\text{Rem}(s, F, \lambda)| \leq \gamma_s \lambda^{-(s+1)/2} \|F\|_{\mathcal{C}_b^{2(s+1)}(l^2(\mathbf{N}); \mathbf{R})}.$$

**Corollary 5.1.** *Let  $s$  be an integer. For  $F \in \mathcal{C}_b^{2s+1}(l^2(\mathbf{N}); \mathbf{R})$ , we have*

$$\int_{l^2(\mathbf{N})} F(u) d\nu_\lambda^*(u) = \int_{l^2(\mathbf{N})} F(u) dm_\beta(u) \\ + \sum_{r=1}^s \left( \frac{1}{\sqrt{\lambda}} \right)^r \sum_{k=1}^r \alpha(r, r+2k) \int_{l^2(\mathbf{N})} F(u) \mathcal{H}_{r+2m} \left( \int_0^1 H_1(\tau) d\tau \cdot u \right) dm_\beta(u) \\ + \text{Rem}(s, F, \lambda),$$

where  $(\mathcal{H}_k, k \geq 0)$  is the family of Hermite polynomials defined by

$$\mathcal{H}_k(x)\Phi(x) = (-1)^{k-1} \frac{d^k \Phi}{dx^k}(x),$$

with  $\Phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ .

**Remark 5.2.** In [3, 13], the following polynomials are introduced. For any  $u \in \mathbf{R}$ , there exists a family of real polynomials  $(Q_r(\cdot), r \geq 1)$  such that

$$(10) \quad \exp\left(\sum_{r=1}^{\infty} \frac{t^{r+2}}{(r+2)!} u^r\right) = \sum_{r=0}^{\infty} u^r Q_r(t).$$

In particular,  $Q_0 \equiv 1$ . For  $r > 0$ , it is well known (see [3]) that  $Q_r$  has degree  $3r$ , we denote  $(q_r(k), k = 0, \dots, 3r)$  its coefficients. For a function  $F$ , we introduce the formal polynomial  $Q_r(F)$  defined by substituting  $t^k$  by  $\int F(u)\mathcal{H}_k(u) dm_\beta(u)$  in the expansion of  $Q_r(t)$ . According to [3, 13], if we choose as functional  $F(N_\lambda) = \int_0^1 \phi(s) dN_\lambda(s)$  where  $\phi \in \mathcal{C}^\infty([0, 1]; \mathbf{R})$  and  $\int_0^1 \phi(s)^2 ds = 1$ , their asymptotic Edgeworth expansion coincides with (9) for any order. By identification, it follows that  $\alpha(r, k) = q_r(k)$  for any  $r$  and  $k$ . Thus, Equation (7) gives a recursive but simple method to compute the Edgeworth coefficients up to any order.

## 6. LINEAR INTERPOLATION OF THE BROWNIAN MOTION

For  $m \geq 1$ , the linear interpolation  $B_m^\dagger$  of a Brownian motion  $B^\dagger$  is defined by

$$B_m^\dagger(0) = 0 \text{ and } dB_m^\dagger(t) = m \sum_{j=0}^{m-1} (B^\dagger(i+1/m) - B^\dagger(i/m)) \mathbf{1}_{[i/m, (i+1)/m)}(t) dt.$$

Thus,  $\mathfrak{J}_\beta B_m^\dagger$  is given by

$$\mathfrak{J}_\beta B_m^\dagger = \left( m \sum_{j=0}^{m-1} (B^\dagger(i+1/m) - B^\dagger(i/m)) \int_{i/m}^{(i+1)/m} h_n(t) dt, n \geq 1 \right).$$

Consider the  $\mathcal{L}^2([0, 1])$ -orthonormal functions

$$e_j^m(s) = \sqrt{m} \mathbf{1}_{[i/m, (i+1)/m)}(s), j = 0, \dots, m-1, s \in [0, 1]$$

and  $F_m^\dagger = \text{span}(e_j^m, j = 0, \dots, m-1)$ . We denote by  $p_{F_m^\dagger}$  the orthogonal projection over  $F_m^\dagger$ . Since  $B_m^\dagger$  is constructed as a function of a standard Brownian motion, we work on the canonical Wiener space  $(\mathcal{C}^0([0, 1]; \mathbf{R}), \mathcal{I}_{1,2}, m^\dagger)$ . The gradient we consider,  $D^\dagger$ , is the derivative of the usual gradient on the Wiener space and the integration by parts formula reads as:

$$(11) \quad \mathbb{E}_{m^\dagger} \left[ F \int_0^1 u(s) dB^\dagger(s) \right] = \mathbb{E}_{m^\dagger} \left[ \int_0^1 D_s^\dagger F u(s) ds \right]$$

for any  $u \in \mathcal{L}^2([0, 1])$ .

**Theorem 6.1.** Let  $\nu_m^\dagger$  be the law of  $\mathfrak{J}_\beta B_m^\dagger$  on  $l^2(\mathbf{N})$ . The measure  $\nu_m^\dagger$  satisfies  $\text{Hyp}(\mathcal{L}^2, H_m^\dagger, 0)$  where

$$H_m^\dagger = (p_{F_m^\dagger} h_n, n \geq 1).$$

Moreover,

$$\rho_2(\nu_m^\dagger, m_\beta) \leq \frac{\zeta_\beta}{3/2 - \beta} \frac{\beta_3 c_\beta^2}{m^{1/2-\beta}},$$

where

$$\zeta_\beta = \sup_{\|f\|_{\mathcal{X}_{1-\beta,2}}=1} \|f\|_{\text{Hol}_0(1/2-\beta)}.$$

**Remark 6.1.** For this example, the constant  $\alpha$  of  $\text{Hyp}(X, H, \alpha)$  is zero because  $D^\dagger$  satisfies the chain rule for derivation. It follows that the Edgeworth expansion has a completely different flavor from that established for Poisson approximation.

Let

$$\gamma_0^\dagger = \frac{\zeta_\beta \beta_3 c_\beta^2}{3/2 - \beta} \text{ and } \gamma_s^\dagger = \frac{(\gamma_0^\dagger)^s}{2^s (s+1)!}.$$

**Theorem 6.2.** For any integer  $s$ , for any  $F \in \mathcal{C}_b^{2s+2}(l^2(\mathbf{N}); \mathbf{R})$ , we have the following expansion:

$$\begin{aligned} \mathbb{E}_{\nu_m^\dagger} [F] &= \sum_{j=0}^s \frac{1}{2^j j!} \int_{l^2(\mathbf{N})} \langle \nabla^{(2j)} F(u), (S_m^\dagger - S_\beta)^{\otimes j} \rangle_{l^2(\mathbf{N})^{\otimes 2j}} dm_\beta(u) \\ &\quad + \text{Rem}^\dagger(F, s, m), \end{aligned}$$

where  $S_m^\dagger = \text{trace}_{\mathcal{L}^2}(H_m^\dagger \otimes H_m^\dagger)$  and  $\text{Rem}^\dagger(F, s, m)$  can be bounded by

$$\left| \text{Rem}^\dagger(F, s, m) \right| \leq \frac{\gamma_s^\dagger \|F\|_{\mathcal{C}_b^{2s+2}(l^2(\mathbf{N}); \mathbf{R})}}{m^{(s+1)(1/2-\beta)}}.$$

## 7. DONSKER THEOREM

The same approach can be applied to have precise asymptotics for the Donsker theorem. Let  $X = (X_n, n \in \mathbf{N})$  be a sequence of independent and identically distributed Rademacher random variables, i.e.  $P(X_n = \pm 1) = 1/2$  for any  $n$ . For any  $k$  in  $\mathbf{N}$ , we set

$$\begin{aligned} X_k^+ &= (X_1, \dots, X_{k-1}, 1, X_{k+1}, \dots) \\ \text{and } X_k^- &= (X_1, \dots, X_{k-1}, -1, X_{k+1}, \dots). \end{aligned}$$

The discrete gradient on this probability space is given by

$$D_k^\sharp F(X) = \frac{1}{2}(F(X_k^+) - F(X_k^-)).$$

Then, the integration by parts formula reads as

$$\mathbb{E} \left[ \sum_{k \in \mathbf{N}} u_k D_k^\sharp F(X) \right] = \mathbb{E} \left[ F(X) \sum_{k \in \mathbf{N}} u_k X_k \right]$$

for any  $u = (u_k, k \in \mathbf{N})$  which belongs to  $l^2(\mathbf{N})$ .

Consider

$$B_m^\sharp(t) = \frac{1}{\sqrt{m}} \sum_{k=1}^{[mt]} X_k = \sum_{k=1}^m \frac{1}{\sqrt{m}} X_k \mathbf{1}_{[k/m, 1]}(t).$$

Since the derivative of  $B_m^\sharp$  with respect to  $t$  is a sum of weighted Dirac measures, we have

$$\mathfrak{J}_\beta B_m^\sharp = \left( \sum_{k=1}^m \frac{1}{\sqrt{m}} X_k h_n(k/m), n \geq 1 \right).$$

**Theorem 7.1.** *We denote by  $\nu_m^\sharp$  the distribution of  $\mathfrak{J}_\beta B_m^\sharp$  on  $l^2(\mathbf{N})$ . The measure  $\nu_m^\sharp$  satisfies  $\text{Hyp}(l^2(\mathbf{N}), H_m^\sharp, 1)$  where*

$$H_m^\sharp = \left( \frac{1}{\sqrt{m}} h_n(k/m) \mathbf{1}_{[1, n]}(k), n \in \mathbf{N}, k \in \mathbf{N} \right).$$

Furthermore, for any  $\epsilon > 0$ , there exists  $m_0$  such that for  $m \geq m_0$ ,

$$\rho_2(\nu_m^\sharp, m_\beta) \leq (1 + \epsilon) \frac{2\zeta_\beta^2}{3/2 - \beta} m^{\beta-1/2}.$$

**Remark 7.1.** *For this example, the Edgeworth expansion would be the sum of the previous Edgeworth expansion as neither the trace term nor the constant  $\alpha$  are null.*

## 8. TRANSFER PRINCIPLE

For  $X$  and  $Y$  two Hilbert spaces and  $\Theta$  a continuous linear map from  $X$  to  $Y$ . Let  $\mu$  and  $\nu$  two probability measures on  $X$  and  $\mu_Y$  (respectively  $\nu_Y$ ) their image measure with respect to  $\Theta$ . Since  $\Theta$  is linear and continuous, for  $F \in C_b^k(Y, \mathbf{R})$ ,  $F \circ \Theta$  belongs to  $C_b^k(X, \mathbf{R})$ , hence, we have

$$\begin{aligned} \sup_{F \in C_b^k(Y, \mathbf{R})} \int F d\mu_Y - \int F d\nu_Y &= \sup_{F \in C_b^k(Y, \mathbf{R})} \int F \circ \Theta d\mu - \int F \circ \Theta d\nu \\ &\leq \sup_{F \in C_b^k(X, \mathbf{R})} \int F d\mu - \int F d\nu. \end{aligned}$$

As an application, we can precise the convergence established in [9]. Note that in this paper, the key tool was also a matter of Hilbert-Schmidt property of some operator.

The fractional Brownian motion of Hurst index  $H \in [0, 1]$  may be defined (see [6]) by

$$B^H(t) = \int_0^t K_H(t, s) dB(s),$$

where

$$K_H(t, r) := \frac{(t-r)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{r}\right) \mathbf{1}_{[0, t)}(r).$$

The Gauss hyper-geometric function  $F(\alpha, \beta, \gamma, z)$  (see [17]) is the analytic continuation on  $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \setminus \{-1, -2, \dots\} \times \{z \in \mathbb{C}, \text{Arg}|1-z| < \pi\}$  of the power series

$$\sum_{k=0}^{+\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k,$$

and

$$(a)_0 = 1 \text{ and } (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1).$$

Furthermore, according to [23],  $K_H$  is a continuous map from  $\mathcal{L}^2$  in to  $\mathcal{I}_{H+1/2, 2}$  hence the map  $\Theta_H = K_H \circ I_{0+}^{-1}$  can be defined continuously from  $\mathcal{I}_{\beta, 2}$  to  $\mathcal{I}_{H-(1/2-\beta), 2}$ . Since  $\Theta_H B = B^H$ , we have the following result.

**Theorem 8.1.** *For any  $H \in [0, 1]$ , for any  $1/2 > \epsilon > 0$ ,*

$$\rho_2 \left( \mathfrak{J}_{H-\epsilon} \left( \int_0^\cdot K_H(t, s) dN^\lambda(s) \right), \mathfrak{J}_{H-\epsilon}(B^H) \right) \leq \frac{\beta_3}{2\sqrt{\lambda}} c_{1/2-\epsilon}^3.$$

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## 9. PROOFS

**Proof of Theorem [3.1].** The norms on  $\mathcal{I}_{\beta,2}^+$  and  $\mathcal{I}_{\beta-1,2}^+$  are defined such that

$$\|\omega\|_{\mathcal{I}_{\beta,2}^+} = \|\dot{\omega}\|_{\mathcal{I}_{\beta-1,2}^+}.$$

It follows that  $\mathcal{I}_{\beta,2}^+$  and  $\mathcal{I}_{\beta-1,2}^+$  are isometrically isomorphic. Since  $\mathcal{I}_{\beta-1,2}^+$  is an Hilbert space, Parseval equality implies that

$$\|\dot{\omega}\|_{\mathcal{I}_{\beta-1,2}^+}^2 = \sum_{n \geq 1} \langle h_n, \dot{\omega} \rangle_{\mathcal{I}_{1-\beta,2}^-, \mathcal{I}_{\beta-1,2}^+}^2,$$

for any CONB  $(h_n, n \geq 1)$  of  $\mathcal{I}_{1-\beta,2}^-$  and  $\Theta_\beta$  is an isometry. The inverse of  $\mathfrak{J}_\beta$  is clearly given by

$$\begin{aligned} \Theta_\beta^{-1} : l^2(\mathbf{N}) &\longrightarrow \mathcal{I}_{\beta,2}^+ \\ (\alpha_n, n \in \mathbf{N}) &\longmapsto \sum_{n \geq 0} \alpha_n I_{0+}^\beta(h_n). \end{aligned}$$

The proof is thus complete.

**Proof of Corollary [3.1].** By a change of variables, it is straightforward that

$$\int_0^\infty \frac{e^{-kt}}{\sqrt{1-e^{-2t}}} dt = \frac{1}{2} B\left(\frac{k}{2}, \frac{1}{2}\right),$$

where  $B$  is the Beta function. Furthermore, since  $\mathfrak{J}_\beta$  is an isometry, according to Theorem [3.2], we have

$$\begin{aligned} \int_{l^2(\mathbf{N})} \|y\| dm_\beta(y) &= \int_{\mathcal{I}_{\beta,2}^+} \|I^{-\beta}\omega\|_{\mathcal{L}^2([0,1])} d\mu_\beta(\omega) \\ &= \frac{1}{\Gamma(1-\beta)} \mathbb{E} \left[ \left( \int_0^1 \left( \int_0^t (t-s)^{-\beta} dB(s) \right)^2 dt \right)^{1/2} \right] \\ &\leq \frac{1}{\Gamma(1-\beta)} \left( \int_0^1 \int_0^t (t-s)^{-2\beta} ds dt \right)^{1/2} \\ &= \frac{1}{\Gamma(1-\beta)} (1-2\beta)^{-1/2} (2-2\beta)^{-1/2}. \end{aligned}$$

The result then follows from (2).

**Proof of Theorem [4.2].** For  $F \in \mathcal{C}_b^3$ , we have

$$(12) \quad \mathbb{E}_{\nu^*} [F] - \mathbb{E}_{m_\beta} [F] = - \int_{l^2(\mathbf{N})} \int_0^\infty x \cdot \nabla P_t^\beta F(x) - \text{trace} \left( S_\beta \nabla^{(2)} P_t^\beta F(x) \right) dt d\nu^*(x).$$

Applying Hyp( $X, H, \alpha$ ) to  $G = \nabla P_t^\beta F$ , we have

$$\begin{aligned} |\mathbb{E}_{\nu^*} [F] - \mathbb{E}_{m_\beta} [F]| &\leq \left| \mathbb{E}_{\nu^*} \left[ \int_0^\infty \text{trace}(\text{trace}_X(H \otimes H) - S_\beta) \circ \nabla^{(2)} P_t^\beta F dt \right] \right| \\ &\quad + \alpha \|H\|_{l^2(\mathbf{N}) \otimes X}^3 \mathbb{E}_{\nu^*} \left[ \int_0^\infty \|\nabla^{(3)} P_t^\beta F\|_\infty dt \right] \\ &\leq \beta_2 \|\nabla F\|_\infty |\text{trace}(\text{trace}_X(H \otimes H) - S_\beta)| + \alpha \beta_3 \|H\|_{l^2(\mathbf{N}) \otimes X}^3 \|\nabla^{(2)} F\|_\infty, \end{aligned}$$

according to Lemma [3.1] and to the relation (see [15])  $|\text{trace}(A \circ C)| \leq |\text{trace}(A)| \|C\|$ .

**Proof of Theorem [5.1].** For  $G \in \mathcal{C}_b^2(l^2(\mathbf{N}))$ , we have

$$\mathbb{E}_{\nu_\lambda^*} [R_\lambda \cdot G] = \mathbb{E}_{\nu_\lambda^*} \left[ \int_0^1 \langle H_\lambda(\tau), D_\tau G(R_\lambda) \rangle_{l^2(\mathbf{N})} \lambda \, d\tau \right].$$

According to the Taylor formula,

$$\begin{aligned} D_\tau G(R_\lambda) &= G(R_\lambda + H_\lambda(\tau)) - G(R_\lambda) \\ &= (\nabla G)(R_\lambda) \cdot H_\lambda(\tau) + \int_0^1 (1-r) \langle \nabla^2 G(R_\lambda + rH_\lambda), H_\lambda(\tau) \otimes H_\lambda(\tau) \rangle_{l^2(\mathbf{N}) \otimes 2} \, dr \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}_{\nu_\lambda^*} [R_\lambda \cdot G] &= \lambda \mathbb{E}_{\nu_\lambda^*} \left[ \int_0^1 \langle (\nabla G)(R_\lambda), H_\lambda(\tau) \otimes H_\lambda(\tau) \rangle_{l^2(\mathbf{N}) \otimes 2} \, d\tau \right] \\ &\quad + \lambda \mathbb{E}_{\nu_\lambda^*} \left[ \int_0^1 \int_0^1 (1-r) \langle \nabla^2 G(R_\lambda + rH_\lambda), H_\lambda(\tau)^{\otimes 3} \rangle_{l^2(\mathbf{N}) \otimes 3} \, dr \, d\tau \right] \end{aligned}$$

On the one hand, we have

$$\int_0^1 H_\lambda(\tau) \otimes H_\lambda(\tau) \, d\tau = \left( \frac{1}{\lambda} \int_0^1 h_i(s) h_j(s) \, ds, \, i, j \in \mathbf{N} \right).$$

On the other hand, since  $\nabla^2 G$  is bounded, we have

$$\begin{aligned} \left| \mathbb{E}_{\nu_\lambda^*} \left[ \int_0^1 \int_0^1 (1-r) \langle \nabla^2 G(R_\lambda + rH_\lambda), H_\lambda(\tau)^{\otimes 3} \rangle_{l^2(\mathbf{N}) \otimes 3} \, dr \, d\tau \right] \right| \\ \leq \frac{1}{2\lambda^3} \|\nabla^2 G\|_\infty \|H_1\|_{l^2(\mathbf{N}) \otimes X}^3. \end{aligned}$$

Hence,

$$|\mathbb{E}_{\nu_\lambda^*} [R_\lambda \cdot G] - \mathbb{E}_{\nu_\lambda^*} [\text{trace}(V \nabla G)]| \leq \frac{1}{2\sqrt{\lambda}} \|\nabla^2 G\|_\infty \|H_1\|_{l^2(\mathbf{N}) \otimes X}^3,$$

which is Equation (3) with  $\alpha = 1/2\sqrt{\lambda}$ . The formula (6) then follows from the remarks that  $\text{trace}_{\mathcal{L}^2}(H_1 \otimes H_1) = S_\beta$  and that  $\|H_1\|_{l^2(\mathbf{N})} = c_\beta$ .

**Proof of Theorem [5.2].** We proceed by induction on  $s$ . Theorem [5.1] boils down to (9) for  $s = 0$ . Assume that the results holds for  $s$ . According to the Taylor

formula, we have

$$\begin{aligned}
\mathbb{E}_{\nu_\lambda^*} [F(N_\lambda)] &= \int_{l^2(\mathbf{N})} F(u) \, dm_\beta(u) \\
&+ \sum_{j=2}^{s+2} \frac{1}{j!} \left( \frac{1}{\sqrt{\lambda}} \right)^{j-1} \int_{l^2(\mathbf{N})} \int_0^\infty \int_0^1 \nabla^{(j+1)} P_t F(u) \cdot (H_1(\tau))^{\otimes(j+1)} \, d\tau \, dt \, d\nu_\lambda^*(u) \\
&+ \frac{\lambda^{-(s+2)/2}}{(s+2)!} \times \\
&\int_{l^2(\mathbf{N})} \int_0^\infty \int_0^1 \int_0^1 (1-\theta)^{s+2} \nabla^{(s+4)} P_t F(u + \theta H_1(\tau)) \cdot (H_1(\tau))^{\otimes(s+4)} \, d\theta \, d\tau \, dt \, d\nu_\lambda^*(u) \\
&= \int_{l^2(\mathbf{N})} F(u) \, dm_\beta(u) + A_1 + A_2.
\end{aligned}$$

In view of the recursion hypothesis, we have

$$\begin{aligned}
(13) \quad A_1 &= \sum_{j=2}^{s+2} \frac{\lambda^{-(j-1)/2}}{j!} \sum_{r=0}^{s-j+2} \left( \frac{1}{\sqrt{\lambda}} \right)^r \sum_{k=0}^r \alpha(r, r+2k) \times \\
&\int_{l^2(\mathbf{N})} \int_0^\infty \int_0^1 \nabla^{(j+1+r+2k)} P_t F(u) \cdot (H_1(\tau))^{\otimes(j+1+r+2k)} \, d\tau \, dt \, dm_\beta(u) \\
&+ \sum_{j=2}^{s+2} \frac{\lambda^{-(j-1)/2}}{j!} \int_0^\infty \text{Rem}(s+2-j, \nabla^{(j+1)} P_t F, \lambda) \, dt = B_1 + B_2
\end{aligned}$$

Now then, according to the very definition of the Ornstein-Uhlenbeck semi-group, for any integer  $l$ ,  $\nabla^{(l)} P_t F = e^{-lt} P_t \nabla^{(l)} F$ . Moreover, since  $m_\beta$  is  $P_t$  invariant, for any integer  $l$ ,

$$\begin{aligned}
&\int_{l^2(\mathbf{N})} \int_0^\infty \int_0^1 \nabla^{(l)} P_t F(u) \cdot (H_1(\tau))^{\otimes(l)} \, d\tau \, dt \, dm_\beta(u) \\
&= \frac{1}{l} \int_{l^2(\mathbf{N})} \int_0^1 \nabla^{(l)} F(u) \cdot (H_1(\tau))^{\otimes(l)} \, d\tau \, dm_\beta(u).
\end{aligned}$$

We proceed to the change of variables  $r \leftarrow r+j-1$ ,  $j \leftarrow j-1$ ,  $m \leftarrow k+(j+1)/2$  in (13) to obtain

$$\begin{aligned}
B_1 &= \sum_{r=1}^{s+1} \left( \frac{1}{\sqrt{\lambda}} \right)^r \sum_{m=1}^r \frac{1}{r+2m} \left( \sum_{j=1}^r \frac{1}{(j+1)!} \alpha(r-j, r-j+2(m-1)) \right) \times \\
&\int_{l^2(\mathbf{N})} \int_0^1 \nabla^{(r+2m)} F(u) \cdot (H_1(\tau))^{\otimes(r+2m)} \, d\tau \, dm_\beta(u).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\text{Rem}(s+1, F, \lambda) &= \sum_{j=2}^{s+2} \frac{1}{j!} \lambda^{-(j-1)/2} \int_0^\infty \text{Rem}(s+2-j, \nabla^{(j+1)} P_t F, \lambda) dt + A_2 \\
&\leq \sum_{j=2}^{s+2} \frac{\lambda^{-(s+2)/2}}{j!} \gamma_{s+2-j} \text{ess-sup}_{u \in l^2(\mathbf{N})} \int_0^\infty \|\nabla^{(2s+7-j)} P_t F(u)\| dt \\
&+ \frac{\lambda^{-(s+2)/2}}{(s+3)!} \left\| \int_0^1 H_1(\tau) d\tau \right\|_{l^2(\mathbf{N})}^{s+4} \text{ess-sup}_{u \in l^2(\mathbf{N})} \int_0^\infty \|\nabla^{(s+4)} P_t F(u)\| dt \\
&\leq \sum_{j=2}^{s+2} \frac{\lambda^{-(s+2)/2}}{j!} \gamma_{s+2-j} \beta_{2s+7-j} \|F\|_{C_b^{2s-j+6}} \\
&\quad + \frac{\lambda^{-(s+2)/2}}{(s+3)!} \left\| \int_0^1 H_1(\tau) d\tau \right\|_{l^2(\mathbf{N})}^{s+4} \beta_{s+4} \|F\|_{C_b^{s+3}}.
\end{aligned}$$

We remark that for  $s \geq 0$ ,

$$s+3 \leq \max_{j=2, \dots, s+2} (2s-j+6) = 2s+4 = 2((s+1)+1).$$

Furthermore,

$$\left\| \int_0^1 H_1(\tau) d\tau \right\|_{l^2(\mathbf{N})}^2 = \sum_{n \in \mathbf{N}} \left( \int_0^1 h_n(\tau) d\tau \right)^2 \leq \sum_{n \in \mathbf{N}} \int_0^1 h_n^2(\tau) d\tau = \|c_\beta^2\|.$$

According to Lemma [3.1], it follows that

$$\begin{aligned}
\text{Rem}(s+1, F, \lambda) &\leq \lambda^{-(s+2)/2} \left( \sum_{j=2}^{s+2} \frac{1}{j!} \gamma_{s+2-j} \beta_{2s+7-j} + \frac{c_\beta^{s+4}}{(s+3)!} \beta_{s+4} \right) \|F\|_{C_b^{2s+4}}.
\end{aligned}$$

In view of (8), the proof is complete.

**Proof of Corollary [5.1].** We need to introduce an auxiliary space, namely  $l_\beta^2(\mathbf{N}) := V_\beta^{-1/2} l^2(\mathbf{N})$  equipped with the scalar product

$$\langle u, v \rangle_{l_\beta^2(\mathbf{N})} = V_\beta^{-1/2} u \cdot V_\beta^{-1/2} v.$$

Following [15], the couple  $(l^2(\mathbf{N}), l^\beta(\mathbf{N}))$  can be viewed as an abstract Wiener space where  $l_\beta^2(\mathbf{N})$  plays the role of the Cameron-Martin space.

Let  $D^\beta$  denote the gradient on this space: For any Hilbert space  $X$ , for any  $v \in l_\beta^2(\mathbf{N})$ , any  $x$  in  $X$ , any  $w_1, \dots, w_n$  in  $l_\beta^2(\mathbf{N})$ , any  $u \in l^2(\mathbf{N})$ ,

$$\begin{aligned}
D^\beta f(\langle w_1, u \rangle_{l_\beta^2}, \dots, \langle w_1, u \rangle_{l_\beta^2}) \otimes x \\
= \sum_{j=1}^n \partial_j f(\langle w_1, u \rangle_{l_\beta^2}, \dots, \langle w_1, u \rangle_{l_\beta^2}) w_1 \otimes x \in l_\beta^2(\mathbf{N}) \otimes X.
\end{aligned}$$

We denote by  $\mathbb{D}_{2,1}^\beta(X)$ , the completion of the set of such cylindric functions with respect to the norm

$$\|F\|_{2,1}^2 = \int_{l^2(\mathbf{N})} F^2(u) dm_\beta(u) + \int_{l^2(\mathbf{N})} \|D^\beta F(u)\|_{l_\beta^2(\mathbf{N}) \otimes X}^2 dm_\beta(u).$$

The  $n$ -th power of  $D^\beta$  and  $\mathbb{D}_{2,k}(X)$  are defined inductively. Note that if  $F$  belongs to  $\mathcal{C}_b^k(l^2(\mathbf{N}), \mathbf{R})$ , it belongs to the domain of  $\mathbb{D}_{2,k}(\mathbf{R})$ . Moreover, it is well known (see [19, 22]) that

$$\mathbb{E}_{m_\beta} \left[ \langle (D^\beta)^{(k)} F, v^{\otimes k} \rangle_{(l_\beta^2)^{\otimes k}} \right] = \int_{l^2(\mathbf{N})} F(u) \mathcal{H}_k(u.v) dm_\beta(u)$$

for any  $v \in l_\beta^2(\mathbf{N})$ . By a density argument, if  $v \in l^2(\mathbf{N})$  and  $F \in \mathbb{D}_{2,k}(\mathbf{R})$ , then we have

$$\mathbb{E}_{m_\beta} \left[ \langle (D^\beta)^{(k)} F, v^{\otimes k} \rangle_{l^2(\mathbf{N})^{\otimes k}} \right] = \int_{l^2(\mathbf{N})} F(u) \mathcal{H}_k(u.v) dm_\beta(u),$$

hence the result.

**Proof of Theorem [6.1].** For  $G$  sufficiently regular, according to the definition of  $B^m$  and to (11), we have

$$\begin{aligned} (14) \quad \mathbb{E}_{\nu_m^\dagger} [\text{Id} . G] &= \mathbb{E}_{m^\dagger} \left[ \sum_{n \in \mathbf{N}} m \sum_{i=0}^{m-1} (B(i+1/m) - B(i/m)) \int_{i/m}^{(i+1)/m} h_n(t) dt G_n(W^m) \right] \\ &= m \sum_{n \in \mathbf{N}} \sum_{i=0}^{m-1} \int_{i/m}^{(i+1)/m} h_n(t) dt \mathbb{E}_{m^\dagger} \left[ \int_{i/m}^{(i+1)/m} D_s^\dagger G_n(W^m) ds \right] \\ &= \int_0^1 H_m^\dagger(t) dt . \mathbb{E}_{m^\dagger} \left[ \int_{i/m}^{(i+1)/m} D_s^\dagger G_n(W^m) ds \right]. \end{aligned}$$

Since  $D^\dagger$  obeys the chain rule formula,

$$\begin{aligned} (15) \quad D_s^\dagger G_n &= \sum_{k \in \mathbf{N}} \nabla_k G_n(W^m) D_s^\dagger W_k^m \\ &= \sum_{k \in \mathbf{N}} \nabla_k G_n(W^m) (m \sum_{l=0}^{m-1} \mathbf{1}_{[l/m, (l+1)/m)}(s) \int_{l/m}^{(l+1)/m} h_k(s) ds) \\ &= \nabla G_n(W^m) . H_m^\dagger(s). \end{aligned}$$

Combining (14) and (15), we get

$$\begin{aligned} \mathbb{E}_{\nu_m^\dagger} [\text{Id} . G] &= m \mathbb{E}_{\nu_m^\dagger} \left[ \sum_{k \in \mathbf{N}} \sum_{n \in \mathbf{N}} \sum_{i=0}^{m-1} \nabla_k G_n \int_{i/m}^{(i+1)/m} h_n(t) dt \int_{i/m}^{(i+1)/m} h_k(s) ds \right] \\ &= \mathbb{E}_{\nu_m^\dagger} [\text{trace}(\text{trace}_{\mathcal{L}^2}(H_m^\dagger \otimes H_m^\dagger) \circ \nabla G)]. \end{aligned}$$

It follows that  $\nu_m^\dagger$  satisfies  $\text{Hyp}(\mathcal{L}^2, H_m^\dagger, 0)$ . To conclude, it remains to estimate

$$\text{trace}(\text{trace}_{\mathcal{L}^2}(H_m^\dagger \otimes H_m^\dagger) - S_\beta).$$

In view of the definition  $\zeta_\beta$ , for any integer  $n$ ,  $h_n$  is of norm 1 in  $\mathcal{I}_{1-\beta, 2}$ , we thus have

$$|h_n(t) - h_n(s)| \leq \zeta_\beta |t - s|^{1/2-\beta}, \text{ for any } t, s \in [0, 1].$$

Thus,

$$\begin{aligned}
& \left| \text{trace}(\text{trace}_{\mathcal{L}^2}(H_m^\dagger \otimes H_m^\dagger) - S_\beta) \right| \\
& \leq \sum_{n \in \mathbf{N}} \left| \int_0^1 h_n(s)^2 ds - m \sum_{i=0}^{m-1} \left( \int_{i/m}^{(i+1)/m} h_n(s) ds \right)^2 \right| \\
& = \sum_{n \in \mathbf{N}} \sum_{i=0}^{m-1} \int_{i/m}^{(i+1)/m} \left| h_n(s)(h_n(s) - m \int_{i/m}^{(i+1)/m} h_n(r) dr) \right| ds \\
& \leq \frac{\zeta_\beta}{(3/2 - \beta)m^{1/2-\beta}} \sum_{n \in \mathbf{N}} \|h_n\|_{\mathcal{L}^2}^2.
\end{aligned}$$

The result follows from the definition of  $c_\beta$ .

**Proof of Theorem [6.2].** For  $s = 0$ , the result boils down to Theorem [6.1]. We proceed by induction on  $s$ . According to the induction hypothesis, for  $F$  sufficiently regular,

$$\begin{aligned}
& \int_{l^2(\mathbf{N})} F(u) d\nu_m^\dagger(u) - \int_{l^2(\mathbf{N})} F(u) dm_\beta(u) \\
& = \int_{l^2(\mathbf{N})} \int_0^\infty \langle \nabla^{(2)} P_t F(u), (S_m^\dagger - S_\beta) \rangle_{l^2(\mathbf{N}) \otimes 2} dt d\nu_m^\dagger(u) \\
& = \sum_{j=0}^s \frac{1}{2^j j!} \int_{l^2(\mathbf{N})} \langle \nabla^{(2j+2)} P_t F(u), (S_m^\dagger - S_\beta)^{\otimes j+2} \rangle_{l^2(\mathbf{N}) \otimes 2j+2} dm_\beta(u) \\
& \quad + \int_{l^2(\mathbf{N})} \int_0^\infty \text{Rem}^\dagger(\nabla^{(2)} P_t F, s, m) dt d\nu_m^\dagger(u).
\end{aligned}$$

Since  $\nabla^{(j)} P_t F(u) = e^{-jt} P_t \nabla^{(j)} F(u)$  and since  $m_\beta$  is invariant under the action of  $P_t$ , we obtain

$$\begin{aligned}
& \int_{l^2(\mathbf{N})} F(u) d\nu_m^\dagger(u) = \int_{l^2(\mathbf{N})} F(u) dm_\beta(u) \\
& \quad + \sum_{j=0}^s \frac{1}{2^{j+1} (j+1)!} \int_{l^2(\mathbf{N})} \langle \nabla^{(2j+2)} F(u), (S_m^\dagger - S_\beta)^{\otimes j+2} \rangle_{l^2(\mathbf{N}) \otimes 2j+2} dm_\beta(u) \\
& \quad + \int \int_0^\infty \langle \text{Rem}^\dagger(\nabla^{(2)} P_t F, s, m), (S_m^\dagger - S_\beta)^{\otimes 2} \rangle_{l^2(\mathbf{N}) \otimes 2} dt d\nu_m^\dagger(u).
\end{aligned}$$

By a change of index in the sum, we obtain the main part of the expansion for the rank  $s + 1$ . Moreover,

$$\begin{aligned}
| \text{Rem}^\dagger(F, s+1, m) | & \leq \gamma_s^\dagger \int_0^\infty \| \nabla^{(2s+4)} P_t F \|_\infty \| S_m^\dagger - S_\beta \|_{l^2(\mathbf{N}) \otimes 2} dt \\
& \leq \frac{\gamma_s^\dagger \gamma_0^\dagger}{2((s+1)+1)} \frac{\| F \|_{\mathcal{C}_b^{2((s+1)+1)}(l^2(\mathbf{N}); \mathbf{R})}}{m^{1/2-\beta}},
\end{aligned}$$

hence the result.

**Proof of Theorem [7.1].**

$$\mathbb{E} [\mathfrak{J}_\beta B_m^\# \cdot G] = \frac{1}{\sqrt{m}} \mathbb{E} \left[ \sum_{k=1}^m h_n(k/m) D_k^\# G(\mathfrak{J}_\beta B_m^\#) \right].$$

Let

$$H_m^\#(k) = \left( \frac{1}{\sqrt{m}} h_n(k/m), n \in \mathbf{N} \right).$$

According to the Taylor formula,

$$\begin{aligned} D_k^\# G(\mathfrak{J}_\beta B_m^\#) &= \frac{1}{2} (G(\mathfrak{J}_\beta B_m^\# + H_m^\#(k)) - G(\mathfrak{J}_\beta B_m^\# - H_m^\#(k))) \\ &= \langle (\nabla G)(\mathfrak{J}_\beta B_m^\#), H_m^\#(k) \rangle_{l^2(\mathbf{N})} \\ &\quad + \int_0^1 (1-r) \langle \nabla^2 G(\mathfrak{J}_\beta B_m^\# + rH_m^\#(k)), H_m^\#(k) \otimes H_m^\#(k) \rangle_{l^2(\mathbf{N})^{\otimes 2}} dr \\ &\quad + \int_0^1 (1-r) \langle \nabla^2 G(\mathfrak{J}_\beta B_m^\# - rH_m^\#(k)), H_m^\#(k) \otimes H_m^\#(k) \rangle_{l^2(\mathbf{N})^{\otimes 2}} dr. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E} [\mathfrak{J}_\beta B_m^\# \cdot G] &= \mathbb{E} \left[ \sum_{k \in \mathbf{N}} \langle (\nabla G)(\mathfrak{J}_\beta B_m^\#), H_m^\#(k) \otimes H_m^\#(k) \rangle_{l^2(\mathbf{N})^{\otimes 2}} \right] \\ &\quad + \sum_{z=\pm 1} \mathbb{E} \left[ \sum_{k \in \mathbf{N}} \int_0^1 (1-r) \langle \nabla^2 G(\mathfrak{J}_\beta B_m^\# + zrH_m^\#(k)), H_m^\#(k)^{\otimes 3} \rangle_{l^2(\mathbf{N})^{\otimes 3}} dr \right] \end{aligned}$$

According to the definition of  $V_{\nu_m^\#}$ , we get

$$\left| \mathbb{E} [\mathfrak{J}_\beta B_m^\# \cdot G] - \mathbb{E} [\text{trace}(V_{\nu_m^\#} \nabla^2 G)] \right| \leq \|\nabla^2 G\|_\infty \|u\|_{l^2(\mathbf{N})^{\otimes 2}}^3,$$

which is (3) with  $\alpha = 1$ . Furthermore, we have

$$\begin{aligned} |\text{trace}(\text{trace}_{l^2(\mathbf{N})}(H_m^\# \otimes H_m^\#) - S_\beta)| &\leq \sum_{i=1}^m \left( \int_0^1 h_i(s)^2 ds - \frac{1}{m} \sum_{k=1}^m h_i(k/m)^2 \right) \\ &\quad + \sum_{i>m} \|h_i\|_{\mathcal{L}^2}^2. \end{aligned}$$

By the very definition of  $\zeta_\beta$ , we have  $\|h_n\|_\infty \leq \zeta_\beta$  for any  $n \in \mathbf{N}$ . Thus,

$$\begin{aligned} \left| \int_0^1 h_i(s)^2 ds - \frac{1}{m} \sum_{k=1}^m h_i(k/m)^2 \right| &\leq \sum_{k=0}^{m-1} \int_{k/m}^{(k+1)/m} |h_i(s)^2 - h_i(k/m)^2| ds \\ &\leq 2\zeta_\beta \sum_{k=0}^{m-1} \int_{k/m}^{(k+1)/m} |h_i(s) - h_i(k/m)| ds \\ &\leq 2\zeta_\beta^2 \sum_{k=0}^{m-1} \int_{k/m}^{(k+1)/m} |s - k/m|^{1/2-\beta} ds \\ &\leq \frac{2\zeta_\beta^2}{3/2-\beta} m^{\beta-1/2}. \end{aligned}$$

We can choose  $h_n = \sqrt{2}I_{0+}^{1-\beta}(\cos(2\pi n \cdot))$  as a CONB of  $\mathcal{I}_{1-\beta,2}$ . According to Parseval formula,

$$\|h_n\|_{\mathcal{L}^2}^2 = 2 \sum_{l=0}^{\infty} \left( \int_0^1 h_n(s) \cos(2\pi ls) ds \right)^2.$$

Recall that for any  $\beta > 0$ ,

$$\sup_{x \in \mathbf{R}^+} \int_0^x s^{-\beta} \cos(2\pi s) ds < \infty.$$

Since,  $I^{1-\beta}f$  can be expressed as a convolution product, there exists  $c > 0$  such that

$$\begin{aligned} \left| \int_0^1 h_n(s) \cos(2\pi ls) ds \right| &\leq c \left| \int_0^1 s^{-\beta} \cos(2\pi ls) ds \cdot \int_0^1 \cos(2\pi ns) \cos(2\pi ls) ds \right| \\ &= c l^{\beta-1} \left| \int_0^l s^{-\beta} \cos(2\pi s) ds \right| \mathbf{1}_{n=l} \leq c l^{\beta-1} \mathbf{1}_{n=l}. \end{aligned}$$

Hence,

$$\sum_{n>m} \|h_n\|_{\mathcal{L}^2}^2 \leq c \sum_{n>m} n^{2\beta-2} \leq c m^{2\beta-1}.$$

On the other hand,

$$\begin{aligned} \|H_m^\dagger\|_{l^2(\mathbf{N}) \otimes l^2(\mathbf{N})} &= \frac{1}{\sqrt{m}} \left( \sum_{i \in \mathbf{N}} \frac{1}{\sqrt{m}} \frac{1}{m} \sum_{k=1}^m h_i^2(k/m) \right)^{3/2} \\ &\leq \frac{\zeta_\beta^2}{\sqrt{m}} \left( \frac{1}{m} \sum_{k=0}^{m-1} (k/m)^{1-\beta} \right)^3 \sum_{i \in \mathbf{N}} \|h_i\|_{\mathcal{L}^2}^2 \\ &\leq \frac{\zeta_\beta^2 c_\beta^2}{(2-2\beta)^3} \frac{1}{\sqrt{m}}. \end{aligned}$$

Thus, as  $m$  goes to infinity, the dominating term is that corresponding to the power  $m^{\beta-1/2}$ , hence the result.

#### APPENDIX A. BESOV-LIOUVILLE SPACES

This part is devoted to the presentation of the so-called Besov-Liouville spaces. A complete exposition can be found in [23].

For  $f \in \mathcal{L}^1([0, 1]; dt)$ , (denoted by  $\mathcal{L}^1$  for short) the left and right fractional integrals of  $f$  are defined by :

$$\begin{aligned} (I_{0+}^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1} dt, \quad x \geq 0, \\ (I_{1-}^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^1 f(t)(t-x)^{\alpha-1} dt, \quad x \leq 1, \end{aligned}$$

where  $\alpha > 0$  and  $I_{0+}^0 = I_{1-}^0 = \text{Id}$ . For any  $\alpha \geq 0$ ,  $p, q \geq 1$ , any  $f \in \mathcal{L}^p$  and  $g \in \mathcal{L}^q$  where  $p^{-1} + q^{-1} \leq \alpha + 1$ , we have :

$$(16) \quad \int_0^1 f(s)(I_{0+}^\alpha g)(s) ds = \int_0^1 (I_{1-}^\alpha f)(s)g(s) ds.$$

For  $p \in [1, +\infty]$ , the Besov-Liouville space  $I_{0+}^\alpha(\mathcal{L}^p) := \mathcal{I}_{\alpha,p}^+$  is usually equipped with the norm :

$$(17) \quad \|I_{0+}^\alpha f\|_{\mathcal{I}_{\alpha,p}^+} = \|f\|_{\mathcal{L}^p}.$$

Analogously, the Besov-Liouville space  $I_{1-}^\alpha(\mathcal{L}^p) := \mathcal{I}_{\alpha,p}^-$  is usually equipped with the norm :

$$\|I_{1-}^\alpha f\|_{\mathcal{I}_{\alpha,p}^-} = \|f\|_{\mathcal{L}^p}.$$

We then have the following continuity results (see [12, 23]) :

- Theorem A.1.** *i. If  $0 < \alpha < 1$ ,  $1 < p < 1/\alpha$ , then  $I_{0+}^\alpha$  is a bounded operator from  $\mathcal{L}^p$  into  $\mathcal{L}^q$  with  $q = p(1 - \alpha p)^{-1}$ .*  
*ii. For any  $0 < \alpha < 1$  and any  $p \geq 1$ ,  $\mathcal{I}_{\alpha,p}^+$  is continuously embedded in  $\text{Hol}_0(\alpha - 1/p)$  provided that  $\alpha - 1/p > 0$ .  $\text{Hol}_0(\nu)$  denotes the space of  $\alpha$  Hölder-continuous functions, null at time 0, equipped with the usual norm.*  
*iii. For any  $0 < \alpha < \beta < 1$ ,  $\text{Hol}_0(\beta)$  is compactly embedded in  $\mathcal{I}_{\alpha,\infty}$ .*  
*iv. By  $I_{0+}^{-\alpha}$ , respectively  $I_{1-}^{-\alpha}$ , we mean the inverse map of  $I_{0+}^\alpha$ , respectively  $I_{1-}^\alpha$ . The relation  $I_{0+}^\alpha I_{0+}^\beta f = I_{0+}^{\alpha+\beta} f$ , respectively  $I_{1-}^\alpha I_{1-}^\beta f = I_{1-}^{\alpha+\beta} f$ , holds whenever  $\beta > 0$ ,  $\alpha + \beta > 0$  and  $f \in \mathcal{L}^1$ .*  
*v. For  $\alpha p < 1$ , the spaces  $\mathcal{I}_{\alpha,p}^+$  and  $\mathcal{I}_{\alpha,p}^-$  are canonically isomorphic. We will thus use the notation  $\mathcal{I}_{\alpha,p}$  to denote any of this spaces.*

We now recall the definition and properties of Besov-Liouville spaces of negative orders. The proofs can be found in [6].

Denote by  $\mathcal{D}_+$  the space of  $\mathcal{C}^\infty$  functions defined on  $[0, 1]$  and such that  $\phi^{(k)}(0) = 0$ ,  $\forall k \in \mathbf{N}$ . Analogously, set  $\mathcal{D}_-$  the space of  $\mathcal{C}^\infty$  functions defined on  $[0, 1]$  and such that  $\phi^{(k)}(1) = 0$ ,  $\forall k \in \mathbf{N}$ . They are both equipped with the projective topology induced by the semi-norms  $p_k(\phi) = \sum_{j \leq k} \|\phi^{(j)}\|_\infty$ ,  $\forall k \in \mathbf{N}$ . Let  $\mathcal{D}'_+$ , resp.  $\mathcal{D}'_-$ , be their strong topological dual. It is straightforward that  $\mathcal{D}_+$  is stable by  $I_{0+}^\beta$  and  $\mathcal{D}_-$  is stable  $I_{1-}^\beta$ , for any  $\beta \in \mathbf{R}^+$ . Hence, guided by (16), we can define the fractional integral of any distribution (i.e., an element of  $\mathcal{D}'_-$  or  $\mathcal{D}'_+$ ):

$$\begin{aligned} \text{For } T \in \mathcal{D}'_-; I_{0+}^\beta T : \phi \in \mathcal{D}_- \mapsto \langle T, I_{1-}^\beta \phi \rangle_{\mathcal{D}'_-, \mathcal{D}_-}, \\ \text{For } T \in \mathcal{D}'_+; I_{1-}^\beta T : \phi \in \mathcal{D}_+ \mapsto \langle T, I_{0+}^\beta \phi \rangle_{\mathcal{D}'_+, \mathcal{D}_+}. \end{aligned}$$

We introduce now our Besov-Liouville spaces of negative order as follows.

**Definition 3.** *For  $\beta > 0$  and  $r > 1$ ,  $\mathcal{I}_{-\beta,r}^+$  (resp.  $\mathcal{I}_{-\beta,r}^-$ ) is the space of distributions  $T \in \mathcal{D}_-$  (resp.  $T \in \mathcal{D}_+$ ) such that  $I_{0+}^\beta T$  (resp.  $I_{1-}^\beta T$ ) belongs to  $\mathcal{L}^r$ . The norm of an element  $T$  in this space is the norm of  $I_{0+}^\beta T$  in  $\mathcal{L}^r$  (resp. of  $I_{1-}^\beta T$ ).*

**Theorem A.2.** *For  $\beta > 0$  and  $r > 1$ , the dual space of  $\mathcal{I}_{\beta,r}^+$  (resp.  $\mathcal{I}_{\beta,r}^-$ ) is canonically isometrically isomorphic to  $I_{1-}^{-\beta}(\mathcal{L}^{r^*})$  (resp.  $I_{0+}^{-\beta}(\mathcal{L}^{r^*})$ ), where  $r^* = r(r-1)^{-1}$ .*

*Moreover, for  $\beta \geq \alpha \geq 0$  and  $r > 1$ ,  $I_{1-}^\beta$  is continuous from  $\mathcal{I}_{-\alpha,r}^-$  into  $\mathcal{I}_{\beta-\alpha,r}^-$ .*

The first part of the next theorem is a deep result which can be found in [24]. We complement it by the computation of the Hilbert-Schmidt norm of the canonical embedding  $\kappa_\gamma$  from  $\mathcal{I}_{\gamma,2}^+$  into  $\mathcal{L}^2$ .

**Theorem A.3.** *The canonical embedding  $\kappa_\gamma$  from  $\mathcal{I}_{\gamma,2}^+$  into  $\mathcal{L}^2$  is Hilbert-Schmidt if and only if  $\gamma > 1/2$ . Moreover,*

$$\|\kappa_\gamma\|_{HS} = \frac{1}{2\Gamma(\gamma)} \left( \frac{1}{\gamma(\gamma - 1/2)} \right)^{1/2}.$$

*Proof.* Let  $(e_n, n \geq 1)$  be a CONB of  $\mathcal{L}^2$  then  $(h_n = I_{0+}^\gamma(e_n), n \in \mathbf{N})$  is a CONB of  $\mathcal{I}_{\gamma,2}^+$  and

$$\begin{aligned} \|\kappa_\gamma\|_{HS}^2 &= \sum_n \|h_n\|_{\mathcal{L}^2}^2 = \sum_n \|I_{0+}^\gamma(e_n)\|_{\mathcal{L}^2}^2 = \|I_{0+}^\gamma\|_{HS}^2 \\ &= \frac{1}{\Gamma(\gamma)^2} \iint_{[0,1]^2} (t-s)^{2\gamma-2} ds dt, \end{aligned}$$

and the result follows by straightforward quadratures.  $\square$

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