A cohomological formula for the Atiyah-Patodi-Singer index on manifolds with boundary

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Abstract

We compute a cohomological formula for the index of a fully elliptic pseudodifferential operator on a manifold with boundary. As in the classic case of Atiyah-Singer, we use an embedding into an euclidean space to express the index as the integral of a cohomology class depending in this case on a noncommutative symbol, the integral being over a C^∞-manifold called the singular normal bundle associated to the embedding. The formula is based on a K-theoretical Atiyah-Patodi-Singer theorem for manifolds with boundary that is drawn from Connes' tangent groupoid approach.

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1 Introduction

In the early 60's, Atiyah and Singer gave a positive answer to a problem posed by Gelfand about investigating the relationship between topological and analytical invariants of elliptic (pseudo)differential operators on closed smooth manifolds without boundary, [4, 5]. In a series of papers, Atiyah-Singer not only gave a general cohomological formula for the

index of an elliptic (pseudo)differential operator on a closed smooth manifold, they also gave several applications and more important they opened a entire new way of study index problems. Since then, index theory has been at the core of interest of several domains in mathematics and mathematical physics.

To be more descriptive, let M be a closed smooth manifold, let D be a elliptic (pseudo)differential operator with principal symbol σ_D. The Atiyah-Singer index formula states

$$ind\, D = \int_{T^*M} ch([\sigma_D])\mathcal{T}(M) \tag{1.1}$$

where $[\sigma_D] \in K^0(T^*M)$ is the principal symbol class in K-theory, $ch([\sigma_D]) \in H^{ev}_{dR}(T^*M)$ its Chern character and $\mathcal{T}(M) \in H^{ev}_{dR}(M)$ the Todd class of (the complexified) T^*M.

A fundamental step in order to achieve such a formula was to realize that the map $D \mapsto ind\, D$ is completely encoded by a group morphism $K^0(T^*M) \longrightarrow \mathbb{Z}$, called the analytic index of M. That is, if $Ell(M)$ denotes the set of elliptic pseudodifferential operators over M, then the following diagram is commutative:

$$\begin{array}{ccc} Ell(M) & \xrightarrow{ind} & \mathbb{Z} \\ {\scriptstyle\sigma}\downarrow & \nearrow_{ind_{a,M}} & \\ K^0(T^*M) & & \end{array}, \tag{1.2}$$

where $Ell(M) \xrightarrow{\sigma} K^0(T^*M)$ is the surjective map that associates the class of the principal symbol of the operator in $K^0(T^*M)$. The use of K-theory was a breakthrough in the approach by Atiyah-Singer, indeed, they could use its (generalized) cohomological properties to decompose the analytic index morphism in a composition of topologic (and hence computable) morphisms. The idea is as follows: Consider an embedding $M \hookrightarrow \mathbb{R}^N$ (assume N even for the purpose of this exposition) and the correspondant normal bundle $N(M)$, Atiyah-Singer showed that the analytic index decomposes as the composition of

- The Thom isomorphism[1]

$$K^0(T^*M) \xrightarrow{\mathscr{T}} K^0(N(M))$$

  followed by

- the canonical morphism

$$K^0(N(M)) \xrightarrow{j!} K^0(\mathbb{R}^N)$$

  induced from a identification of the normal bundle as an open subset of $\mathbb{R}^N$, and followed by

- the Bott isomorphism

$$K^0(\mathbb{R}^N) \xrightarrow{B} K^0(\{pt\}) \approx \mathbb{Z}.$$

In particular, modulo the Thom and Bott isomorphisms, the analytic index is transformed in a very simple shriek map: $K^0(N(M)) \xrightarrow{j!} K^0(\mathbb{R}^N)$. The formula (1.1) is then obtained as an algebraic topology exercice of comparison between K-theory and cohomology, [5].

For the purposes of the present paper remark that the formula (1.1) can be also written as follows:

$$ind\, D = \int_{N(M)} ch(\mathscr{T}([\sigma_D])). \tag{1.3}$$

[1]The composition (with obvious notations) $T^*M \xrightarrow{\pi} M \xrightarrow{i_0} N(M)$ is K-oriented.

For passing the discussion to the subject of the present article we need to start taking about groupoids. In his book, [9] section II.5, Connes sketches a (conceptually) simple proof of the K-theoretical Atiyah-Singer Index theorem for closed smooth manifolds using tangent groupoid techniques. This proof is in fact a groupoid translation of Atiyah-Singer's original proof.

Let M be a closed smooth manifold, $G_M = M \times M$ its pair groupoid. For the readers not familiar with groupoids, one can think on the kernel algebra (convolution algebra of the groupoid), that is, for the pair groupoid: on the algebra of smooth complex valued functions on $M \times M$ with kernel convolution product.

Consider the tangent groupoid

$$^TG_M := TM \times \{0\} \bigsqcup M \times M \times (0,1] \rightrightarrows M \times [0,1].$$

It is nowadays well known that the index morphism provided by this deformation groupoid is precisely the analytic index of Atiyah-Singer, [9, 24]. In other words, there is a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(M \times M \times (0,1]) \longrightarrow C^*(^TG_M) \xrightarrow{e_0} C_0(T^*M) \longrightarrow 0 \tag{1.4}$$

and since $C^*(M \times M \times (0,1])$ is contractible, the morphism induced in K-theory by e_0 is invertible. The analytic index of M is the morphism

$$K^0(TM) \xrightarrow{(e_0)_*^{-1}} K^0(^TG_M) \xrightarrow{(e_1)_*} K^0(M \times M) := K_0(\mathscr{K}(L^2(M))) \approx \mathbb{Z}, \tag{1.5}$$

where e_t are the obvious evaluation morphisms at t, and where we denote for a general groupoid $\mathscr{G}$: $K^0(\mathscr{G}) := K_0(C_r^*(\mathscr{G}))$. Actually, all groupoids considered in this work are amenable, so the distinction between the reduced and envelopping C^*-algebras is not even necessary.

As discussed by Connes, if the groupoids appearing in this interpretation of the index were equivalent to spaces then we would immediately have a geometric interpretation of the index. Now, $M \times M$ is equivalent to a point (hence to a space), but the other fundamental groupoid playing a role in the previous discussion is not, that is, TM is a groupoid whose fibers are the groups T_xM, which are not equivalent (as groupoids) to spaces. The idea of Connes is to use an appropriate action of the tangent groupoid in some $\mathbb{R}^N$ in order to translate the index (via a Thom isomorphism) in an index associated to a deformation groupoid which will be equivalent to some space.

The case of manifolds with boundary:

Atiyah, Patodi and Singer wrote a remarkable series of papers on non-local elliptic boundary value problems which have been also intensively studied ever since in several domains of mathematics and mathematical physics, [1, 2, 3]. They showed that under some boundary conditions (the now so called APS boundary conditions), Dirac operators become Fredholm on suitable spaces and computed the index. To the characteristic form from the closed smooth case they added a correction term, called the eta invariant determined by an appropriate restriction of the operator to the boundary and that is a spectral invariant measuring the asymmetry of the spectrum. Atiyah-Patodi-Singer's eta invariant and index formulae had a big impact in the late 20th's century geometrical analysis and are still nowadays the subject of different kind of studies in several branches of mathematics. However a cohomological formula expressing the APS index as an integration of some characteristic form was not available. The main problem is that the boundary condition is non local, thus applying directly the methods of Atiyah-Singer would not be a good choice. We will see that allowing some noncommutative spaces will bring us to

the possibility to understand the computation of the APS index using "classic" algebraic topology methods.[2]

In this paper, we will follow Connes groupoid approach to obtain a cohomological formula for the index of a fully elliptic (pseudo)differential operator on a closed manifold with boundary. For the case of such a manifold, the pair groupoid does not give the same information as the smooth case. In fact the case of boundary becomes more interesting since different boundary conditions can be considered and each of these give different index problems. In this paper we will be interested in the so-called Atiyah-Patodi-Singer boundary condition. For the moment we will not recall what this condition is, in fact we rather describe the groupoid whose pseudodifferential calculus gives rise to the index theory related to such a condition.

Let X be a manifold with boundary. We denote, as usual, $\overset{\circ}{X}$ the interior which is a smooth manifold and ∂X its boundary. Let

$$\Gamma(X) \rightrightarrows X \tag{1.6}$$

be the groupoid of the b-calculus, where

$$\Gamma(X) = \overset{\circ}{X} \times \overset{\circ}{X} \bigsqcup \partial X \times \partial X \times \mathbb{R},$$

with groupoid structure given as a family of pair groupoids and the (additive) group $\mathbb{R}$. It is a continuous family groupoid with the topology explicity described in [23] (see beginning of section 3.1 below).

Consider

$$\Gamma(X)^{tan} = A(\Gamma(X)) \bigsqcup \Gamma(X) \times (0,1] \rightrightarrows X \times [0,1]$$

its tangent groupoid.

Take now the open subgroupoid of $\Gamma(X)^{tan}$ obtained by restriction to $X_{\mathcal{F}} := X \times [0,1] \setminus (\partial X \times \{1\})$

$$\Gamma(X)_{\mathcal{F}} = A(\Gamma(X)) \times \{0\} \bigsqcup \overset{\circ}{X} \times \overset{\circ}{X} \times (0,1] \bigsqcup \partial X \times \partial X \times \mathbb{R} \times (0,1) \rightrightarrows X_{\mathcal{F}}.$$

By definition $\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1]$ is a saturated, open subgroupoid of $\Gamma(X)_{\mathcal{F}}$. This leads to a complementary closed subgroupoid of $\Gamma(X)_{\mathcal{F}}$:

$$\mathcal{T}_{nc}X \rightrightarrows X_{\partial}, \tag{1.7}$$

where $X_{\partial} := X_{\mathcal{F}} \setminus \overset{\circ}{X} \times (0,1] = X \underset{\partial X \times \{0\}}{\cup} \partial X \times [0,1]$.

The groupoid $\mathcal{T}_{nc}X$, called *"The noncommutative tangent space of X"*, was introduced in [11], where it was proved to be the K-Poincaré dual of the conic pseudomanifold associated to X.

Deformation groupoids like $\Gamma(X)_{\mathcal{F}}$ induce index morphisms. Indeed, its algebra comes equipped with a restriction morphism to the algebra of $\mathcal{T}_{nc}X$ and an evaluation morphism to the algebra of $\overset{\circ}{X} \times \overset{\circ}{X}$ (for $t = 1$). Indeed, we have a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1]) \longrightarrow C^*(\Gamma(X)_{\mathcal{F}}) \xrightarrow{e_0} C^*(\mathcal{T}_{nc}X) \longrightarrow 0 \tag{1.8}$$

where the algebra $C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1])$ is contractible. Hence applying the K-theory functor to this sequence we obtain an index morphism

$$ind_{\mathcal{F}} = (e_1)_* \circ (e_0)_*^{-1} : K^0(\mathcal{T}_{nc}X) \longrightarrow K^0(\overset{\circ}{X} \times \overset{\circ}{X}) \approx \mathbb{Z}. \tag{1.9}$$

[2] if one allows for instance groupoid's K-theory and cohomology in this framework.

This index computes indeed the Fredholm index of those elliptic operators on X satisfying the APS boundary condition, and hence we call it *The Fredholm index morphism of X*. To be more explicit, the statement is the following:

Proposition 1.1 *[16, 18, 10]. For any fully elliptic operator D on X, there is a naturally associated "non commutative symbol" $[\sigma_D] \in K^0(\mathcal{T}_{nc}X)$ and*

$$ind_{\mathcal{F}}([\sigma_D]) = Index_{APS}(D), \tag{1.10}$$

where $Index_{APS}(D)$ is the Fredholm index of D. Moreover, every element in $K^0(\mathcal{T}_{nc}X)$ can be realized in this way.

A first task in order to follow the Atiyah-Singer approach would be to compute the morphism $ind_{\mathcal{F}}$ by topological means. For instance, using an appropriate embedding into a space in which the computation could follow in an easier way. This idea has been already followed up in [12] in the framework of manifolds with conical singularities, using a KK-equivalent version of the noncommutative tangent space $\mathcal{T}_{nc}X$. There, the authors use embeddings into euclidean spaces to extend the construction of the Atiyah-Singer topological index map, thanks to a "Thom isomorphism" between the noncommutative tangent spaces of the singular manifold and of its singular normal bundle, and then get an index theorem in the framework of K-theory. Here, we follow a different approach and we are going to extend the Atiyah-Singer topological index map using Connes' ideas on tangent groupoid actions on euclidean spaces; moreover we investigate the cohomological counterpart of the K-theoritic statement of the index theorem. Note also that the index map considered here coincide, through KK-equivalences, with the index maps considered in [16] and in [12].

We start in section 3.2 by considering an embedding

$$i : X \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+ \tag{1.11}$$

of X as a neat submanifold of $\mathbb{R}^{N-1} \times \mathbb{R}_+$ (*i.e.*, respecting the boundary). As Connes, we use it to define a continuous families groupoid morphism (see 3.4 for the explicit definition)

$$h : \Gamma(X) \to \mathbb{R}^N \tag{1.12}$$

where we see $\mathbb{R}^N$ as an additive group and we assume N even. This morphism induces an action of $\Gamma(X)$ on $X \times \mathbb{R}^N$ and an induced deformation action of $\Gamma(X)_{\mathcal{F}}$ on $X_{\mathcal{F}} \times \mathbb{R}^N$ (coming from an induced morphism $h_{\mathcal{F}}$). The main task in section 3.2 is to prove the following result (proposition 3.6 below):

Proposition 1.2 *The crossed product groupoid $(\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}} := \Gamma(X)_{\mathcal{F}} \rtimes_{h_{\mathcal{F}}} \mathbb{R}^N$ is a free proper groupoid.*

In section 2.3 we explain how the Connes-Thom isomorphism links the K-theory of a groupoid with the K-theory of a crossed product as above. For instance, the new groupoid $(\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ defines as well an index morphism and this one is linked with the index (1.9) by a natural isomorphism, the so called Connes-Thom isomorphism, thus giving the following commutative diagram

$$\begin{array}{ccccc}
 & & \overset{ind_{\mathcal{F}}}{\frown} & & \\
K^0(\mathcal{T}_{nc}X) & \xleftarrow[\approx]{e_0} & K^0(\Gamma(X)_{\mathcal{F}}) & \xrightarrow{e_1} & K^0(\overset{\circ}{X} \times \overset{\circ}{X}) \approx \mathbb{Z} \\
\downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx & & \downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx & & \downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx \\
K^0((\mathcal{T}_{nc}X)_{h_0}) & \xleftarrow[\approx]{e_0} & K^0((\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}}) & \xrightarrow{e_1} & K^0((\overset{\circ}{X} \times \overset{\circ}{X})_{h_1}) \\
 & & \underset{ind_{h_{\mathcal{F}}}}{\smile} & &
\end{array} \tag{1.13}$$

where h_0 and h_1 denotes the respective restricted actions to $\mathcal{T}_{nc}X$ and $\overset{\circ}{X} \times \overset{\circ}{X}$ on $\mathbb{R}^N$.

Now, the proposition above tells us that the orbit space of $(\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ is a nice space and moreover that this crossed product groupoid is Morita equivalent to this orbit space. This means that the index morphism $ind_{h_{\mathcal{F}}}$ can be computed, modulo Morita equivalences, as the deformation index of some space, more precisely, denoting by $\mathscr{B}_{h_{\mathcal{F}}}$ the orbit space of $(\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$, by $\mathscr{B}_{h_0}$ the orbit space of $(\mathcal{T}_{nc}X)_{h_0}$ and by $\mathscr{B}_{h_1}$ the orbit space of $(\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ we have an index morphism between K-theories of spaces (topological K-groups, no more C^*-algebras if one does not like it!)

$$ind_{\mathscr{B}_{h_{\mathcal{F}}}} : \qquad K^0(\mathscr{B}_{h_0}) \xleftarrow[\approx]{e_0} K^0(\mathscr{B}_{h_{\mathcal{F}}}) \xrightarrow{e_1} K^0(\mathscr{B}_{h_1}) \tag{1.14}$$

from which we would be able to compute the Fredholm index. This is what we achieve next, indeed, in section 3.2.2 we are able to explicitly identify these orbit spaces.

In order to describe them we need to introduce a new player, but let us first motivate this by looking at the situation when $\partial X = \emptyset$ (following [9] II.5), in this case, the orbit space of $(\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ can be identified with the deformation to the normal cone (see appendix A below for the C^∞-structure of such deformations) of the embedding $X \hookrightarrow \mathbb{R}^N$, that is, it is a C^∞-cobordism between the normal bundle to X in $\mathbb{R}^N$ and $\mathbb{R}^N$ itself:

$$\mathscr{B}_{AS} := \mathscr{N}(X) \bigsqcup (0,1] \times \mathbb{R}^N. \tag{1.15}$$

In this picture we also see the orbit space of $(TX)_{h_0}$ which identifies with $N(X)$ and the orbit space of $(X \times X)_{h_1}$ which identifies with $\mathbb{R}^N$.

Still, in this boundaryless case ($\partial X = \emptyset$), this space $\mathscr{B}_{AS}$ gives in K-theory a deformation index

$$ind_{\mathscr{B}_{AS}} : \qquad K^0(N(X)) \xleftarrow[e_0]{\approx} K^0(\mathscr{B}_{AS}) \xrightarrow{e_1} K^0(\mathbb{R}^N)$$

which is easily seen to be the shriek map associated to the identification of $N(X)$ as an open subset of $\mathbb{R}^N$.

In the boundary case, the normal bundle is not the right player, we know for instance that the APS index cannot be computed by an integration over this space by the non-locality of the APS boundary condition. One has then to compute the orbit spaces, in fact the orbit space of $(\mathcal{T}_{nc}X)_{h_0}$ identifies (lemma 3.9) with the singular normal bundle:

$$\mathscr{N}_{sing}(X) := N(X) \times \{0\} \bigsqcup \mathbb{R}^{N-1} \times (0,1) \tag{1.16}$$

is the C^∞-manifold obtained by gluing $N(X)$ and

$$D_\partial := N(\partial X) \times \{0\} \bigsqcup \mathbb{R}^{N-1} \times (0,1)$$

the deformation to the normal cone associated to the embedding $\partial X \hookrightarrow \mathbb{R}^{N-1}$, along their common boundary (the gluing depending on a choice of a defining function of the boundary of X). The orbit space of $(\overset{\circ}{X} \times \overset{\circ}{X})_{h_1}$ is easily identified with $\mathbb{R}^N$ (lemma 3.8).

Finally the orbit space of $(\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ is homeomorphic to a space (section 3.2.2) looking as

$$\mathscr{B}_{\mathscr{F}} := \mathscr{N}_{sing}(X) \bigsqcup (0,1] \times \mathbb{R}^N, \tag{1.17}$$

where more precisely we prove the following (proposition 3.11)

Proposition 1.3 *The locally compact space $\mathscr{B}_{\mathscr{F}}$ admits an oriented C^∞-manifold with boundary structure of dimension $N+1$.*

The last proposition is essential to explicitly compute the index (1.14) above once the explicit identifications performed, indeed $\mathscr{B}_{\mathscr{F}}$ is an oriented cobordism from $\mathscr{N}_{sing}(X)$ to $\mathbb{R}^N$, we can hence apply a Stoke's theorem argument to obtain the following (proposition 3.12):

Proposition 1.4 *The following diagram is commutative*

$$\begin{array}{ccccc} K^0(\mathscr{N}_{sing}(X)) & \xleftarrow[\approx]{(e_0)_*} & K^0(\mathscr{B}_{\mathscr{F}}) & \xrightarrow{(e_1)_*} & K^0(\mathbb{R}^N) \\ & \searrow^{\int_{\mathscr{N}_{sing}(X)} ch(\cdot)} & & \swarrow_{\int_{\mathbb{R}^N} ch(\cdot)} & \\ & & \mathbb{R} & & \end{array}$$

Before enouncing the index theorem, we mentioned that there is a Connes-Thom isomorphism and a Morita equivalence

$$K^0(\mathcal{T}_{nc}X) \xrightarrow{\mathscr{C}\mathscr{T}} K^0((\mathcal{T}_{nc}X)_{h_0}) \xrightarrow{Morita} K^0(\mathscr{N}_{sing}(X)).$$

In section 2.3 we develop Connes-Thom using deformation groupoids, this allowed us to perform an explicit computation of the above morphism for "noncommutative symbols" $[\sigma_D] \in K^0(\mathcal{T}_{nc}X)$.

The index theorem is the following:

Theorem 1.5 *[K-theoretic APS] Let X be a manifold with boundary, consider an embedding of X in $\mathbb{R}^N$ as in 1.11. The Fredholm index morphism $ind_{\mathcal{F}} : K^0(T_{nc}X) \to \mathbb{Z}$ decomposes as the composition of the following three morphisms*

1. *A Connes-Thom isomorphism $\mathscr{C}\mathscr{T}$:*

$$K^0(\mathcal{T}_{nc}X) \xrightarrow{\mathscr{C}\mathscr{T}} K^0(\mathscr{N}_{sing}(X)),$$

2. *The index morphism of the deformation space $\mathscr{B}_{\mathcal{F}}$:*

$$K^0(\mathscr{N}_{sing}(X)) \xleftarrow[\approx]{(e_0)_*} K^0(\mathscr{B}_{\mathcal{F}}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)$$

3. *the usual Bott periodicity isomorphism:*

$$K^0(\mathbb{R}^N) \xrightarrow{Bott} \mathbb{Z}.$$

As discussed above, the three morphisms of the theorem above are computable, and then, exactly as in the classic Atiyah-Singer theorem the last theorem allows to conclude that, given an embedding $i : X \hookrightarrow \mathbb{R}^N$ as above, any fully elliptic operator D on X with "non commutative symbol" $[\sigma_D] \in K^0(\mathcal{T}_{nc}X)$ gives rise to the following formula:
Cohomological formula for the APS index (corollary 5.1)

$$Index_{APS}(D) = \int_{\mathscr{N}_{sing}(X)} Ch(\mathscr{C}\mathscr{T}([\sigma_D])) \tag{1.18}$$

where $\int_{\mathscr{N}_{sing}(X)}$ is the integration with respect to the fundamental class of $\mathscr{N}_{sing}(X)$. In section 5.3 we perform an explicit description for $\mathscr{C}\mathscr{T}([\sigma_D]) \in K^0(\mathscr{N}_{sing}(X))$.

The manifold $\mathscr{N}_{sing}(X)$ (see (1.16) above)already reflects an interior contribution and a boundary contribution. In particular, picking up a differential form ω_D on $\mathscr{N}_{sing}(X)$ representing $Ch(\mathscr{C}\mathscr{T}([\sigma_D]))$, we obtain:

$$Index_{APS}(D) = \int_{\mathscr{N}(X)} \omega_D + \int_{D_\partial} \omega_D. \tag{1.19}$$

The first integral above involves the restriction of ω_D to $\mathscr{N}(X)$, which is related to the ordinary principal symbol of D. More precisely, the principal symbol $\sigma_{pr}(D)$ of D provides a K-theory class of $C^*(A^*(\Gamma(X)))$, that is a compactly supported K-theory class of the dual of the Lie algebroid of $\Gamma(X)$ or in other words of the b-cotangent bundle ${}^bT^*X$, and by functoriality of both the Chern character and Thom-Connes maps, we have

$$[(\omega_D)|_{\mathscr{N}(X)}] = Ch(\mathscr{C}\mathscr{T}([\sigma_{pr}(D)]).$$

The second integral can thus be viewed as a correction term, which contains the eta invariant appearing in APS formula and which also depends on the choice of the representative $\omega_D \in Ch(\mathscr{C}\mathscr{T}([\sigma_D]))$.

Further developments: The same methods as above can be applied for manifolds with corners for which we already count with the appropriate b-groupoids ([23]). The generalization of the formula is not however immediate. Indeed, we need to explicitly compute the orbit spaces, which will not be anymore smooth manifolds.

In another direction, as we mentioned above, there is a relation between the second integral on (1.19) and the so called eta invariant. For deeply understanding this, we need to explicitly describe the Chern character of the $\mathscr{C}\mathscr{T}([\sigma_D] \in K^0(\mathscr{N}_{sing}(X))$, for which one might need to use the Chern character computations done in [7] for instance. Also, the second integral comes from the part of the b-groupoid corresponding to the boundary,

$$\partial X \times \partial X \times \mathbb{R} \times (0,1),$$

and this groupoid's algebra is related with the suspended algebra of Melrose ([21]), a relation between this integral and this work of Melrose becomes then very interesting to study.

2 Groupoids

2.1 Preliminaries

Let us recall some preliminaries on groupoids:

Definition 2.1 *A groupoid consists of the following data: two sets $\mathscr{G}$ and $\mathscr{G}^{(0)}$, and maps*

(1) $s, r : \mathscr{G} \to \mathscr{G}^{(0)}$ called the source and range (or target) map respectively,

(2) $m : \mathscr{G}^{(2)} \to \mathscr{G}$ called the product map (where $\mathscr{G}^{(2)} = \{(\gamma, \eta) \in \mathscr{G} \times \mathscr{G} : s(\gamma) = r(\eta)\}$),

such that there exist two maps, $u : \mathscr{G}^{(0)} \to \mathscr{G}$ (the unit map) and $i : \mathscr{G} \to \mathscr{G}$ (the inverse map), which, if we denote $m(\gamma, \eta) = \gamma \cdot \eta$, $u(x) = x$ and $i(\gamma) = \gamma^{-1}$, satisfy the following properties:

(i). $r(\gamma \cdot \eta) = r(\gamma)$ and $s(\gamma \cdot \eta) = s(\eta)$.

(ii). $\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta$, $\forall \gamma, \eta, \delta \in \mathscr{G}$ when this is possible.

(iii). $\gamma \cdot x = \gamma$ and $x \cdot \eta = \eta$, $\forall \gamma, \eta \in \mathscr{G}$ with $s(\gamma) = x$ and $r(\eta) = x$.

(iv). $\gamma \cdot \gamma^{-1} = u(r(\gamma))$ and $\gamma^{-1} \cdot \gamma = u(s(\gamma))$, $\forall \gamma \in \mathscr{G}$.

Generally, we denote a groupoid by $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$. A morphism f from a groupoid $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ to a groupoid $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ is given by a map f from $\mathscr{G}$ to $\mathscr{H}$ which preserves the groupoid structure, i.e. f commutes with the source, target, unit, inverse maps, and respects the groupoid product in the sense that $f(h_1 \cdot h_2) = f(h_1) \cdot f(h_2)$ for any $(h_1, h_2) \in \mathscr{H}^{(2)}$.

For A, B subsets of $\mathscr{G}^{(0)}$ we use the notation $\mathscr{G}_A^B$ for the subset

$$\{\gamma \in \mathscr{G} : s(\gamma) \in A,\, r(\gamma) \in B\}.$$

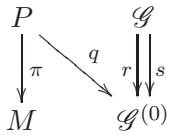
A groupoid can be endowed for a structure of topological space, or manifold, for instance. In the case when $\mathscr{G}$ and $\mathscr{G}^{(0)}$ are smooth manifolds, and s, r, m, u are smooth maps (with s and r subimmersions), then $\mathscr{G}$ is a Lie groupoid. In the case of manifolds with boundary, or with corners, this notion can be generalized to that of continuous family groupoids, see [27]).

One can endow a groupoid with a topological structure. A strict morphism of locally compact groupoids is a groupoid morphisms which is continuous. Locally compact groupoids form a category with strict morphisms of groupoids. It is now classical in groupoid's theory that the right category to consider is the one in which Morita equivalences correspond precisely to isomorphisms. For more details about the assertions about generalized morphisms written in this section, the reader can read [31] section 2.1, or [15, 25, 22].

We need to introduce some basic definitions, classical when dealing with principal bundles for groups over spaces.

We recall first the notion of groupoid action. Given a l.c. groupoid $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$, a right $\mathscr{G}$-bundle over a manifold M is a manifold P such that:

- P is endowed with maps p and q as in



- P is endowed with a continuous right action $\mu : P \times_{(q,r)} \mathscr{G} \to P$, such that if we denote $\mu(p, \gamma) = p\gamma$, one has $\pi(p\gamma) = \pi(p)$ and $p(\gamma_1 \cdot \gamma_2) = (p\gamma_1)\gamma_2$ for any $(\gamma_1, \gamma_2) \in \mathscr{G}^{(2)}$. Here $P \times_{(q,r)} \mathscr{G}$ denotes the fiber product of $q : P \to \mathscr{G}^{(0)}$ and $r : \mathscr{G} \to \mathscr{G}^{(0)}$.

A $\mathscr{G}$-bundle P is called principal if

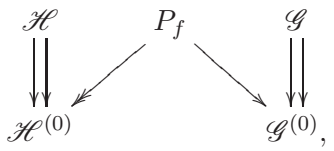
(i) π is a surjective submersion, and

(ii) the map $P \times_{(q,r)} \mathscr{G} \to P \times_M P$, $(p, \gamma) \mapsto (p, p\gamma)$ is a homeomorphism.

We can now define generalized morphism between two Lie groupoids.

Definition 2.2 (Generalized morphism) *Let $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ and $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ be two Lie groupoids. A generalized morphism (or Hilsum-Skandalis morphism) from $\mathscr{G}$ to $\mathscr{H}$, $f : \mathscr{H} - - \succ \mathscr{G}$, is given by the isomorphism class of a right $\mathscr{G}-$principal bundle over $\mathscr{H}$, that is, the isomorphism class of:*

- *A right principal $\mathscr{G}$-bundle P_f over $\mathscr{H}^{(0)}$ which is also a left $\mathscr{H}$-bundle such that the two actions commute, formally denoted by*



First of all, usual morphisms between groupoids are also generalized morphisms. Next, as the word suggests it, generalized morphism can be composed. Indeed, if P and P' are generalized morphisms from $\mathscr{H}$ to $\mathscr{G}$ and from $\mathscr{G}$ to $\mathscr{L}$ respectively, then

$$P \times_{\mathscr{G}} P' := P \times_{\mathscr{G}^{(0)}} P'/(p, p') \sim (p \cdot \gamma, \gamma^{-1} \cdot p')$$

is a generalized morphism from $\mathscr{H}$ to $\mathscr{L}$. The composition is associative and thus we can consider the category $Grpd_{HS}$ with objects l.c. groupoids and morphisms given by generalized morphisms. There is a functor

$$Grpd \longrightarrow Grpd_{HS} \tag{2.1}$$

where $Grpd$ is the category of groupoids with usual morphisms.

Definition 2.3 (Morita equivalence) *Two groupoids are Morita equivalent if they are isomorphic in $Grpd_{HS}$.*

2.2 Free proper groupoids

Definition 2.4 (Free proper groupoid) *Let $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ be a locally compact groupoid. We will say that it is free and proper if it has trivial isotropy groups and it is proper.*

Given a groupoid $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$, its orbit space is $\mathscr{O}(\mathscr{H}) := \mathscr{H}^{(0)}/\sim$, where $x \sim y$ iff there is $\gamma \in \mathscr{H}$ such that $s(\gamma) = x$ and $r(\gamma) = y$.

The following fact is well known. In particular in can be deduced from propositions 2.11, 2.12 and 2.29 in [30].

Proposition 2.5 *If $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ is a free proper (Lie) groupoid, then $\mathscr{H}$ is Morita equivalent to the locally compact space (manifold) $\mathscr{O}(\mathscr{H})$.*

In fact the Morita bibundle which gives the Morita equivalence is the unit space $\mathscr{H}^{(0)}$:

$$\begin{array}{ccccc} \mathscr{H} & & \mathscr{H}^{(0)} & & \mathscr{O}(\mathscr{H}) \\ \Downarrow & \overset{Id}{\swarrow} & & \overset{\pi}{\searrow} & \Downarrow \\ \mathscr{H}^{(0)} & & & & \mathscr{O}(\mathscr{H}), \end{array} \tag{2.2}$$

It is obvious that $\mathscr{H}$ acts on its units freely and properly if $\mathscr{H}$ is free and proper, and for the same reason $\mathscr{O}(\mathscr{H})$ is a nice locally compact space (even a manifold if the groupoid is Lie).

In particular there is an invertible Hilsum-Skandalis generalized morphism

$$\mathscr{O}(\mathscr{H}) - -- > \mathscr{H}, \tag{2.3}$$

that can also be given as a 1-cocycle from $\mathscr{O}(\mathscr{H})$ with values in $\mathscr{H}$. This point of view will be very useful for us in the sequel.

2.3 Crossed product groupoids by homomorphisms on $\mathbb{R}^N$ and Connes-Thom isomorphism

Let $\mathscr{G} \rightrightarrows M$ be a locally compact groupoid.

We consider $\mathbb{R}^N$ as an additive group, hence a one unit groupoid. Suppose we have an homomorphism of groupoids

$$\mathscr{G} \xrightarrow{h} \mathbb{R}^N. \tag{2.4}$$

This gives rise to an action (say, a right one) of $\mathscr{G}$ on the space $M \times \mathbb{R}^N$ and thus to a new groupoid:

$$\mathscr{G}_h := \mathscr{G}_h \ltimes \mathbb{R}^N : \mathscr{G} \times \mathbb{R}^N \rightrightarrows M \times \mathbb{R}^N \tag{2.5}$$

which has the following structural maps:

- The source and target maps are given by

$$s(\gamma, X) = (s(\gamma), X + h(\gamma)) \text{ and } r(\gamma, X) = (r(\gamma), X)$$

- The multiplication is defined on composable arrows by the formula

$$(\gamma, X) \cdot (\eta, X + h(\gamma)) := (\gamma \cdot \eta, X).$$

Then it is obviously a groupoid with unit map $u(m, X) = (m, X)$ ($h(m) = 0$ since h is an homomorphism), and inverse given by $(\gamma, X)^{-1} = (\gamma^{-1}, X + h(\gamma))$ (again since we have an homomorphism, $h(\gamma) + h(\gamma^{-1}) = 0$).

Remark 2.6 *For the trivial homomorphism $h_0 = 0$, the associated groupoid is just the product groupoid*

$$\mathscr{G} \times \mathbb{R}^N \rightrightarrows M \times \mathbb{R}^N.$$

At the level of C^*-algebras, $C^*(G_h)$ can be seen as the crossed product algebra $C^*(G) \rtimes_h \mathbb{R}^N$ where $\mathbb{R}^N$ acts on $C^*(G)$ by automorphisms by the formula: $\alpha_X(f)(\gamma) = e^{i\cdot(X\cdot h(\gamma))} f(\gamma)$, $\forall f \in C_c(G)$, (see [9], propostion II.5.7 for details). In particular, in the case N is even, we have a Connes-Thom isomorphism in K-theory ([9], II.C)

$$K^0(G) \xrightarrow[\approx]{\mathscr{C}\mathscr{T}} K^0(G_h) \tag{2.6}$$

which generalizes the classical Thom isomorphism, and which is natural with respect to morphisms of algebras.

Since we will need to compute explicitly the morphism induced by the homomorphism we propose an alternative construction of Connes-Thom which can be computed in our context. More precisely we want to work directly with the groupoid algebras $C^*(G_h)$ without passing through the isomorphism with $C^*(G) \rtimes_h \mathbb{R}^N$.

Given the morphism $\mathscr{G} \xrightarrow{h} \mathbb{R}^N$ we consider the product groupoid $\mathscr{G} \times [0,1] \rightrightarrows M \times [0,1]$ of the groupoid $\mathscr{G}$ with the *space* $[0,1]$ and we define

$$H : \mathscr{G} \times [0,1] \longrightarrow \mathbb{R}^N,$$

the homomorphism given by

$$H(\gamma, t) := t \cdot h(\gamma).$$

This homomorphism gives a deformation between the trivial homomorphism and h.

Consider the semi-direct groupoid associated to the homomorphism H. Since the action is trivial at 0, it can be identified as

$$\mathscr{G}_H = \mathscr{G} \times \mathbb{R}^N \times \{0\} \bigsqcup \mathscr{G}_h \times (0,1]. \tag{2.7}$$

This decomposition in an open, saturated subgroupoid and a closed one gives rise to a short exact sequence of C^*-algebras ([15, 29]):

$$0 \to C^*(\mathscr{G}_h \times (0,1]) \to C^*(\mathscr{G}_H) \xrightarrow{e_0} C^*(\mathscr{G} \times \mathbb{R}^N) \to 0, \tag{2.8}$$

where e_0 is induced by the evaluation at zero. This defines a deformation index

$$\mathscr{D}_h : K^*(\mathscr{G} \times \mathbb{R}^N) \to K^*(\mathscr{G}_h). \tag{2.9}$$

For another definition of the deformation index above the reader might look at theorem 3.1 in [13].

Definition 2.7 *Let $(\mathscr{G}, h)$ be a groupoid together with a homomorphism on $\mathbb{R}^N$ (with N even). Consider the morphism in K-theory*

$$K^*(\mathscr{G}) \xrightarrow{\mathscr{C}\mathscr{T}_h} K^*(\mathscr{G}_h), \tag{2.10}$$

given by the composition of the Bott morphism

$$K^*(\mathscr{G}) \xrightarrow{B} K^*(\mathscr{G} \times \mathbb{R}^N),$$

and the deformation index

$$K^*(\mathscr{G} \times \mathbb{R}^N) \xrightarrow{\mathscr{D}_h} K^*(\mathscr{G}_h).$$

We will refer to this morphism as the Connes-Thom map associated to h.

In fact, Elliot, Natsume and Nest proved that this morphism coincides with the usual Connes-Thom isomorphism, theorem 5.1 in [13] and its proof. We can state the result as follows:

Proposition 2.8 (Elliot-Natsume-Nest) *Let $(\mathscr{G}, h)$ be a groupoid together with a homomorphism on $\mathbb{R}^N$ (N even). Then the morphism $\mathscr{C}\mathscr{T}_h : K^*(\mathscr{G}) \to K^*(\mathscr{G}_h)$ coincides with the Connes-Thom isomorphism. In particular, it satisfies the following properties:*

1. *Naturality.*
2. *If $\mathscr{G}$ is a space (the groupoid equals its set of units), then $\mathscr{C}\mathscr{T}_h$ is the Bott morphism.*

3 Noncommutative spaces for manifolds with boundary

3.1 The noncommutative tangent space of a manifold with boundary

Let X be a manifold with boundary. We denote, as usual, $\overset{\circ}{X}$ the interior which is a smooth manifold and ∂X its boundary. Let

$$\Gamma(X) \rightrightarrows X \tag{3.1}$$

be the groupoid of the b-calculus ([24, 16, 23]). This groupoid has a pseudodifferential calculus which coincides with Melrose's b-calculus. There is a canonical definition, but in our case we need to choose a defining function of the boundary, making the definition simpler. A defining function of the boundary is a smooth function $\rho : X \to \mathbb{R}_+$ which is zero on the boundary and only there, and whose differential is non zero on the boundary.

Definition 3.1 *The b-groupoid of X is*

$$\Gamma(X) = \{(x, y, \alpha) \in X \times X \times \mathbb{R}, \rho(x) = e^{\alpha}\rho(y)\}.$$

this implies that

$$\Gamma(X) = \overset{\circ}{X} \times \overset{\circ}{X} \bigsqcup \partial X \times \partial X \times \mathbb{R} \rightrightarrows X,$$

with groupoid structure compatible with those of $\overset{\circ}{X} \times \overset{\circ}{X}$ and $\partial X \times \partial X \times \mathbb{R}$ ($\mathbb{R}$ as an additive group). It is a continuous family groupoid, see [23, 16] for details. For instance,

$$(x_n, y_n) \to (x, y, \alpha)$$

if and only if

$$x_n \to x,\ y_n \to y \text{ and } log(\frac{\rho(x_n)}{\rho(y_n)}) \to \alpha.$$

For such a groupoid it is possible to construct an algebra of pseudodifferential operators. Although we do not need it in this article, we recall this background to help the reader relate our work to usual index theory. See [24, 26, 23, 16, 32] for a detailed presentation of pseudodifferential calculus on groupoids.

A pseudodifferential operator on a Lie groupoid (or more generally a continuous family groupoid) G is a family of peudodifferential operators on the fibers of G (which are smooth manifolds without boundary), the family being equivariant under the natural action of G.

Compactly supported pseudodifferential operators form an algebra, denoted by $\Psi^\infty(G)$. The algebra of order 0 pseudodifferential operators can be completed into a C^*-algebra, $\overline{\Psi^0}(G)$. There exists a symbol map, σ, whose kernel is $C^*(G)$. This gives rise to the following exact sequence:

$$0 \to C^*(G) \to \overline{\Psi^0}(G) \to C_0(S^*(G))$$

where $S^*(G)$ is the cosphere bundle of the Lie algebroid of G.

In the general context of index theory on groupoids, there is an analytic index which can be defined in two ways. The first way, classical, is to consider the boundary map of the 6-terms exact sequence in K-theory induced by the short exact sequence above:

$$ind_a : K_1(C_0(S^*(G))) \to K_0(C^*(G)).$$

Actually, an alternative is to define it through the tangent groupoid of Connes, which was originally defined for the groupoid of a smooth manifold and later extended to the case of continuous family groupoids ([24, 16]). In general,

$$G^{tan} = A(G) \bigsqcup G \times (0,1] \rightrightarrows G^{(0)} \times [0,1]$$

Using the evaluation maps, one has two K-theory maps, $e_0 : K^*(G^{tan}) \to K_*(A(G))$ which is an isomorphism (since $K_*(G \times (0,1]) = 0$), and $e_1 : K^*(G^{tan}) \to K^*(G)$. The analytic index can be defined as

$$ind_a = e_1 \circ e_0^{-1} : K_*(A(G)) \to K^*(G).$$

It is thus possible to work on index problems without using the algebra of pseudodifferential operators. In the rest of this article, we will only use deformation groupoids like the tangent groupoid, and not pseudodifferential algebras.

But in general, this analytic index is not the Fredholm index. In certain cases, it is possible to define the latter using a refinement of the tangent groupoid.

In order to use explicitly the tangent groupoid in our case, a little discussion on the algebroid is needed. For this, remark that we can define the vector bundle TX over X as the restriction of the tangent space of a smooth manifold $\tilde{X}$, for example we will use the double of X (gluing along the boundary with the defining function ρ), on which X is included. So, $TX := T_X\tilde{X}$ and we can use the defining function to trivialize the normal bundle of ∂X in $\tilde{X}$ to identify $T_X\tilde{X}$ with the bundle with fibers

$$T_X\tilde{X}_x = T_x\overset{\circ}{X} \text{ if } x \in \overset{\circ}{X} \text{ and } T_X\tilde{X}_x = T_x\partial X \times \mathbb{R} \text{ if } x \in \partial X.$$

Now, $\Gamma(X)$ is a continuous families groupoid and at such one can define $T(\Gamma(X))$, see [24, 16, 28] for more details. What it is important for us is its restriction to X, where $X \hookrightarrow \Gamma(X)$ as the groupoid's units. This vector bundle $T_X(\Gamma(X))$ over X has fibers

$$T_X(\Gamma(X))_x = T_{(x,x)}(\overset{\circ}{X} \times \overset{\circ}{X}) \text{ if } x \in \overset{\circ}{X} \text{ and}$$

$$T_X\Gamma(X)_x = T_{(x,x)}(\partial X \times \partial X) \times \mathbb{R} \times \mathbb{R} \text{ if } x \in \partial X.$$

The algebroid $A(\Gamma(X)) \to X$ is defined as the restriction to the unit space X of the vector bundle $T\Gamma(X) = \cup_{x\in X} T\Gamma(X)_x$ over the groupoid $\Gamma(X)$, in particular:

$$A(\Gamma(X))_x \simeq T_x \overset{\circ}{X} \text{ if } x \in \overset{\circ}{X} \text{ and } A(\Gamma(X))_x \simeq T_x \partial X \times \mathbb{R} \text{ if } x \in \partial X.$$

Take the tangent groupoid associated to $\Gamma(X)$:

$$\Gamma(X)^{tan} = A(\Gamma(X)) \bigsqcup \Gamma(X) \times (0,1] \rightrightarrows X \times [0,1]$$

Let us now consider the open subgroupoid of $\Gamma(X)^{tan}$ given by the restriction to $X_{\mathcal{F}} := X \times [0,1] \setminus (\partial X \times \{1\})$:

$$\Gamma(X)_{\mathcal{F}} := A(\Gamma(X)) \bigsqcup \partial X \times \partial X \times \mathbb{R} \times (0,1) \bigsqcup \overset{\circ}{X} \times \overset{\circ}{X} \times (0,1] \rightrightarrows X_{\mathcal{F}}.$$

As we will discuss below, the deformation index associated to this groupoid gives precisely the Fredholm index. But let us continue with our construction of noncommutative spaces.

By definition

$$\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1] \subset \Gamma(X)_{\mathcal{F}}$$

as a saturated open subgroupoid. We can then obtain a complementary closed subgroupoid of $\Gamma(X)_{\mathcal{F}}$:

$$\mathcal{T}_{nc} X \rightrightarrows X_{\partial}, \tag{3.2}$$

where $X_{\partial} := X_{\mathcal{F}} \setminus \overset{\circ}{X} \times (0,1]$.

To be more descriptive, the groupoid looks like

$$\mathcal{T}_{nc} X = A(\Gamma(X)) \bigsqcup \partial X \times \partial X \times \mathbb{R} \times (0,1).$$

Definition 3.2 *The groupoid $\mathcal{T}_{nc}X$ will be called "The noncommutative tangent space of X". We will also refer to $\Gamma(X)_{\mathcal{F}}$ as "The Fredholm tangent groupoid of X". See also [11] for a KK-equivalent description of the groupoid.*

3.2 Embeddings of manifolds with boundary and proper free groupoids

To define a homomorphism $\Gamma(X)_{\mathcal{F}} \xrightarrow{h} \mathbb{R}^N$ we will need as in the nonboundary case an appropiate embedding. Consider an embedding

$$i_{\partial} : X \hookrightarrow \mathbb{R}^{N-1},$$

in which N is an even integer. The embedding we are going to use is

$$i : X \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+ \tag{3.3}$$

where

$$i(x) = (i_{\partial}(x), \rho(x)).$$

We can then use it to define a homomorphism $h : \Gamma(X) \to \mathbb{R}^N$ as follows.

$$h : \begin{cases} h(x,y) = (i_{\partial}(x) - i_{\partial}(y),\ log(\frac{\rho(x)}{\rho(y)})) \text{ on } \overset{\circ}{X} \times \overset{\circ}{X} \\ h(x,y,\alpha) = (i_{\partial}(x) - i_{\partial}(y), \alpha) \text{ on } \partial X \times \partial X \times \mathbb{R} \end{cases} \tag{3.4}$$

By the definition of the topology on $\Gamma(X)$, it is clearly a continuous family groupoid morphism.

The interest of defining a good morphism h is that the induced groupoid will be free and proper, like in the case of a smooth manifold, as it was shown by Connes. The freeness

follows from a general principle: if we have a homomorphism $G \xrightarrow{h} \mathbb{R}^N$ then the isotropy groups of the induced groupoid G_h are

$$(G_h)^{(x,X)}_{(x,X)} = \{(\gamma, X) \in G \times \mathbb{R}^N / \gamma \in G^x_x,\, h(\gamma) = 0\}.$$

In particular, if h is a monomorphism ($h(\gamma) = 0$ iff γ is a unit), it implies that the isotropy groups described above are trivial, $(G_h)^{(x,X)}_{(x,X)} = \{(x, X)\}$.

We will now check the properness of the several groupoids G_h we will be using. For that we will use the general properness condition (ii) of proposition 2.14 in [30] which tell us that we need to check two things:

(A) The map

$$G_h \xrightarrow{(t,s)} (G^{(0)} \times \mathbb{R}^N) \times (G^{(0)} \times \mathbb{R}^N)$$

is closed, and

(B) For every $(a, X) \in G^{(0)} \times \mathbb{R}^N$ the stabilizers $(G_h)_{(a,X)} := \{\gamma \in G : t(\gamma) = a = s(\gamma)\, and\, X = X + h(\gamma)\}$ are quasi-compact.

In our case property (B) is immediately verified, indeed, as we mentioned above, h is a monomorphism so the stabilizers are trivial, $(\Gamma(X)_{\mathscr{F}})_{(a,X)} = \{(a, X)\}$, hence quasi-compact.

Lemma 3.3 *The induced crossed product groupoid $\Gamma(X)_h$ is a free proper groupoid.*

Proof : As mentioned above we have to verify only property (A), that is, we have to check that the map

$$\Gamma(X) \times \mathbb{R}^N \xrightarrow{(t,s)} (X \times \mathbb{R}^N) \times (X \times \mathbb{R}^N)$$

given by

$$(t, s) : \begin{cases} (x, y, X) \mapsto ((x, X), (y, X + (i_\partial(x) - i_\partial(y),\, log(\frac{\rho(x)}{\rho(y)})))\ \text{on } \overset{\circ}{X} \times \overset{\circ}{X} \times \mathbb{R}^N \\ (x, y, \alpha, X) \mapsto ((x, X), (y, X + (i_\partial(x) - i_\partial(y), \alpha))\ \text{on } \partial X \times \partial X \times \mathbb{R} \times \mathbb{R}^N \end{cases} \tag{3.5}$$

is closed.

Let $(A_n)_n := (\gamma_n, X_n)_n$ a sequence in $\Gamma(X) \times \mathbb{R}^N$ such that

$$lim_{n\to\infty}(t, s)(\gamma_n, X_n) = P \tag{3.6}$$

with P a point in $(X \times \mathbb{R}^N) \times (X \times \mathbb{R}^N)$. It is enough to justify that there is a subsequence of $(A_n)_n$ converging to an antecedent of P: We will separate the analysis in four cases

(a) Suppose $P = ((x, X), (y, Y))$ with $x \in \overset{\circ}{X}$ and $y \in \partial X$. Since $\overset{\circ}{X}$ is open in X we have that $\gamma_n \in \overset{\circ}{X} \times \overset{\circ}{X}$ from a certain large enough n, that is, we might suppose that for each n, $\gamma_n = (x_n, y_n)$ for some $x_n, y_n \in \overset{\circ}{X}$ and in particular

$$(t, s)(\gamma_n, X_n) = ((x_n, X_n), (y_n, X_n + (i_\partial(x_n) - i_\partial(y_n),\, log(\frac{\rho(x_n)}{\rho(y_n)})))).$$

Now, the limit (3.6) above implies the following convergences: $x_n \to x$, $y_n \to y$, $X_n \to X$ and $X_n + (i_\partial(x_n) - i_\partial(y_n),\, log(\frac{\rho(x_n)}{\rho(y_n)})) \to Y$. Hence we also have that $log(\frac{\rho(x_n)}{\rho(y_n)})$ converges, but this is impossible since $\rho(x_n) \to \rho(x) > 0$ and $\rho(y_n) \to \rho(y) = 0$. This case is thus not possible.

(*b*) Suppose $P = ((x, X), (y, Y))$ with $x \in \partial X$ and $y \in \overset{\circ}{X}$. This case is symmetric to the precedent one, the same analysis shows is empty.

(*c*) Suppose $P = ((x, X), (y, Y))$ with $x \in \overset{\circ}{X}$ and $y \in \overset{\circ}{X}$. We might suppose again that for each n, $\gamma_n = (x_n, y_n)$ for some $x_n, y_n \in \overset{\circ}{X}$ and in particular

$$(t, s)(\gamma_n, X_n) = ((x_n, X_n), (y_n, X_n + (i_\partial(x_n) - i_\partial(y_n), \, log(\frac{\rho(x_n)}{\rho(y_n)})))).$$

Let $A = (x, y, X)$, $A \in \Gamma(X) \times \mathbb{R}^N$ and $(t, s)(A) = P$. The limit (3.6) implies that we have the convergence $A_n \to A$.

(*d*) Finally, suppose $P = ((x, X), (y, Y))$ with $x \in \partial X$ and $y \in \partial X$. In this case we have two possibilities[3], (d_1) : either A_n has a subsequence completely contained in $\overset{\circ}{X} \times \overset{\circ}{X} \times \mathbb{R}^N$, or (d_2) : A_n has a subsequence completely contained in $\partial X \times \partial X \times \mathbb{R} \times \mathbb{R}^N$.

In the case (d_1) we might suppose again that for each n, $\gamma_n = (x_n, y_n)$ for some $x_n, y_n \in \overset{\circ}{X}$. The limit (3.6) above implies the following convergences: $x_n \to x$, $y_n \to y$, $X_n \to X$ and $X_n + (i_\partial(x_n) - i_\partial(y_n), \, log(\frac{\rho(x_n)}{\rho(y_n)})) \to Y$. In particular we also have $log(\frac{\rho(x_n)}{\rho(y_n)})$ converges to a certain $\alpha \in \mathbb{R}$. Hence, letting $A = (x, y, \alpha, X)$ we have that $(t, s)(A) = P$ and A_n converges to A.

In the case (d_2), we might suppose that for each n, $\gamma_n = (x_n, y_n, \alpha_n)$ with $x_n, y_n \in \partial X$ and $\alpha_n \in \mathbb{R}$. The limit (3.6) above implies the following convergences: $x_n \to x$, $y_n \to y$, $X_n \to X$ and $X_n + (i_\partial(x_n) - i_\partial(y_n), \alpha_n) \to Y$. Thus α_n converges too to a $\alpha \in \mathbb{R}$. Hence, letting $A = (x, y, \alpha, X)$ we have that $(t, s)(A) = P$ and A_n converges to A.

□

Now, the morphism h induces a morphism between the algebroids

$$A(h) : A(\Gamma(X)) \to A(\mathbb{R}^N) = \mathbb{R}^N.$$

With the identification that we have for the algebroid, we explicitly have

$$A(h)(x, V) = d_x i^+(V) \tag{3.7}$$

if $x \in \overset{\circ}{X}$ and $V \in T_x\overset{\circ}{X}$, where $i^+ : \overset{\circ}{X} \to \mathbb{R}^N$ is defined as $i^+(x) := (i_\partial(x), log(\rho(x)))$; and

$$A(h)((x, \xi), \alpha) = (d_x i_\partial(\xi), \alpha) \tag{3.8}$$

if $x \in \partial X$, $\xi \in T_x\partial X$ and $\alpha \in \mathbb{R}$, where i_∂ is the restriction of i_∂ to ∂X. We also have the properness of the respective action.

Lemma 3.4 *The induced crossed product groupoid $A(\Gamma(X))_{A(h)}$ is a free proper groupoid.*

Proof : Again, we have to verify only property (A), that is, we have to check that the map

$$A(\Gamma(X)) \times \mathbb{R}^N \xrightarrow{(t,s)} (X \times \mathbb{R}^N) \times (X \times \mathbb{R}^N)$$

given by

$$(t, s) : \begin{cases} ((x, V), X) \mapsto ((x, X), (x, X + d_x i^+(V)) \text{ on } T\overset{\circ}{X} \times \mathbb{R}^N \\ ((x, \xi), X) \mapsto ((x, X), (x, X + (d_x i_\partial(\xi), \alpha)) \text{ on } T\partial X \times \mathbb{R} \times \mathbb{R}^N \end{cases} \tag{3.9}$$

[3]the two options may coexist.

is closed.

Let $(A_n)_n := (\gamma_n, X_n)_n$ a sequence in $A(\Gamma(X)) \times \mathbb{R}^N$ such that

$$lim_{n\to\infty}(t,s)(\gamma_n, X_n) = P \tag{3.10}$$

with P a point in $(X \times \mathbb{R}^N) \times (X \times \mathbb{R}^N)$. It is enough to justify that there is a subsequence of $(A_n)_n$ converging to an antecedent of P: We will separate the analysis in four cases

(a) Suppose $P = ((x, X), (y, Y))$ with $x \in \overset{\circ}{X}$ and $y \in \partial X$. Since $\overset{\circ}{X}$ is open in X we have that $\gamma_n \in T\overset{\circ}{X}$ from a certain large enough n, that is, we might suppose that for each n, $\gamma_n = (x_n, V_n)$ for some $V_n \in T_{x_n}\overset{\circ}{X}$ and in particular

$$(t,s)(\gamma_n, X_n) = ((x_n, X_n), (x_n, X_n + d_{x_n} i^+(V_n))).$$

Now, the limit (3.10) above implies in particular the following convergences: $x_n \to x$ and $x_n \to y$. This case is thus not possible.

(b) The case $P = ((x, X), (y, Y))$ with $x \in \partial X$ and $y \in \overset{\circ}{X}$ is empty, the argument of (a) above applies as well.

(c) Suppose $P = ((x, X), (y, Y))$ with $x \in \overset{\circ}{X}$ and $y \in \overset{\circ}{X}$. We might suppose again that for each n, $\gamma_n = (x_n, V_n)$ for some $\xi_n \in T_{x_n}\overset{\circ}{X}$ and in particular

$$(t,s)(\gamma_n, X_n) = ((x_n, X_n), (x_n, X_n + d_{x_n} i^+(V_n))).$$

The limit (3.10) implies that $x_n \to x$, $X_n \to X$ and $X_n + d_{x_n} i^+(V_n) \to Y$, in particular $(d_{x_n} i^+(V_n))_n$ converges in $\mathbb{R}^N$ too. Now, since i^+ is an embedding we have that di^+ is a closed embedding, in other words there is a $V \in T_x\overset{\circ}{X}$ such that $d_x i^+(V)$ is the limit of $(d_{x_n} i^+(V_n))_n$. Hence, letting $A = ((x, V), X) \in T\overset{\circ}{X} \times \mathbb{R}^N$ we have that $(t,s)(A) = P$ and A_n converges to A.

(d) Finally, suppose $P = ((x, X), (y, Y))$ with $x \in \partial X$ and $y \in \partial X$. In this case we have two possibilities, (d_1) : either A_n has a subsequence completely contained in $T\overset{\circ}{X} \times \mathbb{R}^N$, or ($d_2$) : A_n has a subsequence completely contained in $T\partial X \times \mathbb{R} \times \mathbb{R}^N$.

In the case (d_1) we might suppose again that for each n, $\gamma_n = (x_n, V_n)$ for some $V_n \in T_{x_n}\overset{\circ}{X}$. The limit (3.10) above implies the following convergences: $x_n \to x$, $X_n \to X$ and $X_n + d_{x_n} i^+(V_n) \to Y$. In particular we also have $d_{x_n} i^+(V_n)$ converges in $\mathbb{R}^N$. Again, there is then a $V \in T_x\overset{\circ}{X}$ such that $d_x i^+(V)$ is the limit of $(d_{x_n} i^+(V_n))_n$. Hence, letting $A = ((x, V), X)$ we have that $(t,s)(A) = P$ and A_n converges to A.

In the case (d_2), we might suppose that for each n, $\gamma_n = ((x_n, \xi_n), \alpha_n)$ with $\xi_n \in T_{x_n}\partial X$ and $\alpha_n \in \mathbb{R}$. The limit (3.10) above implies the following convergences: $x_n \to x$, $X_n \to X$ and $X_n + (d_{x_n} i_\partial(\xi_n), \alpha_n) \to Y$. Thus $d_{x_n} i_\partial(\xi_n)$ and α_n converge too in $\mathbb{R}^{N-1}$ and in $\mathbb{R}$ respectively. Again, because di_∂ is a closed embedding, there is a $\xi \in T_x\partial X$ such that $d_x i_\partial(\xi)$ is the limit of $(d_{x_n} i_\partial(\xi_n))_n$. Letting $A = ((x, \xi), \alpha, X) \in T_\partial X \times \mathbb{R} \times \mathbb{R}^N$ with α the limit of $(\alpha_n)_n$, we have that $(t,s)(A) = P$ and A_n converges to A.

□

Let us now apply to h the tangent groupoid functor to obtain a continuous family groupoid morphism

$$h^{tan} : \Gamma(X)^{tan} \to (\mathbb{R}^N)^{tan},$$

explicitly given by

$$h^{tan} : \begin{cases} h^{tan}(\gamma, \varepsilon) = (h(\gamma), \varepsilon) \text{ on } \Gamma(X)^{tan} \times (0,1] \\ h^{tan}(x, \xi) = A(h)(x, \xi) \text{ on } A(\Gamma(X)) \end{cases} \tag{3.11}$$

Remember now that the tangent groupoid of $\mathbb{R}^N$ (as an additive group) is diffeomorphic to $\mathbb{R}^N \times [0,1]$ by the diffeormorphism $(\mathbb{R}^N)^{tan} \to \mathbb{R}^N \times [0,1]$ given by

$$\begin{cases} (X, 0) \mapsto (X, 0) \text{ on } \mathbb{R}^N \times \{0\} \\ (X, \varepsilon) \mapsto (\frac{X}{\varepsilon}, \varepsilon) \text{ on } \mathbb{R}^N \times (0,1] \end{cases} \tag{3.12}$$

As a corollary of proposition A.5 and the two lemmas above we have

Corollary 3.5 *Consider the continuous family groupoids morphism $\Gamma(X)^{tan} \stackrel{h^T}{\to} \mathbb{R}^N$ given as the composition of h^{tan} composed with the diffeomorphism $(\mathbb{R}^N)^{tan} \approx \mathbb{R}^N \times [0,1]$ and finally with the projection on $\mathbb{R}^N$.*

Then the crossed product groupoid $(\Gamma(X)^{tan})_{h^T}$ is a free proper groupoid.

We consider finally the morphism on the Fredholm groupoid

$$h_{\mathcal{F}} : \Gamma(X)_{\mathcal{F}} \to \mathbb{R}^N \tag{3.13}$$

given by the restriction of h^T to $\Gamma(X)_{\mathcal{F}}$.

We have obtained in particular the following result:

Proposition 3.6 *$h_{\mathcal{F}} : \Gamma(X)_{\mathcal{F}} \to \mathbb{R}^N$ defines a homomorphism of continuous family groupoids and the groupoid $(\Gamma(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ is a free proper groupoid.*

Remark 3.7 *As an immediate consequence of the proposition above, the groupoid $(\Gamma(X)_{\mathcal{F}})_h$ is Morita equivalent to its space of orbits, (see proposition 2.5).*

The index morphism of the orbits space of $(\Gamma(X)_{\mathcal{F}})_h$

Let us denote by $\mathscr{B}_h = X_{\mathcal{F}} \times \mathbb{R}^N / \sim_h$ the space of orbits of $(\Gamma(X)_{\mathcal{F}})_h$. Remember that to define the deformation index associated to the Fredholm groupoid we considered the saturated open subgroupoid $\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1] \rightrightarrows \overset{\circ}{X} \times (0,1]$ and its closed complement $\mathcal{T}_{nc}(X) \rightrightarrows X_\partial$. The restrictions of h to these two subgroupoids have the same properties as h, the induced actions are free and proper. Moreover, since we are dealing with saturated subgroupoids, we have a good behaviour at the level of orbit spaces. That is, denoting by $\mathscr{B}_{h_0} = X_\partial \times \mathbb{R}^N / \sim_{h_0}$ and $\mathscr{B}_{h_1} = \overset{\circ}{X} \times \mathbb{R}^N / \sim_{h_1}$ the orbit spaces of $(\mathcal{T}_{nc}(X))_{h_0}$ and $(\overset{\circ}{X} \times \overset{\circ}{X})_{h_1}$ respectively, we have an index morphism

$$K^0(\mathscr{B}_{h_0}) \xleftarrow[\approx]{(e_0)_*} K^0(\mathscr{B}_h) \xrightarrow{(e_1)_*} K^0(\mathscr{B}_{h_1}) \tag{3.14}$$

by considering the open saturated subset $\mathscr{B}_{h_{(0,1]}} = \overset{\circ}{X} \times (0,1] \times \mathbb{R}^N / \sim$ of $\mathscr{B}_h$.

We want next to fully understand this index morphism. Since we are now dealing with spaces this should be possible.

Lemma 3.8 *We have an homeomorphism between the open dense subset $\mathscr{B}_{h_{(0,1]}} = \overset{\circ}{X} \times (0,1] \times \mathbb{R}^N / \sim$ of $\mathscr{B}_h$ and $(0,1] \times \mathbb{R}^N$. More explicitly, the map*

$$(x, \varepsilon, X) \mapsto (\varepsilon, \varepsilon \cdot X + i(x))$$

passes to the quotient into an homeomorphism

$$\overset{\sim}{\circ}{q} : \overset{\circ}{X} \times (0,1] \times \mathbb{R}^N / \sim \longrightarrow (0,1] \times \mathbb{R}^N$$

Proof : The map

$$\overset{\circ}{q} : (x, \varepsilon, X) \mapsto (\varepsilon, \varepsilon \cdot X + i(x))$$

is obviously a continuous open surjection from $\overset{\circ}{X} \times (0,1] \times \mathbb{R}^N$ to $(0,1] \times \mathbb{R}^N$. Moreover, by definition

$$\overset{\circ}{q}(x, \varepsilon, X) = \overset{\circ}{q}(y, \varepsilon', Y)$$

if and only if

$$\varepsilon = \varepsilon' \qquad and \qquad Y = X + \frac{x-y}{\varepsilon}$$

that is, if and only if

$$(x, \varepsilon, X) \sim_h (y, \varepsilon', Y).$$

The conclusion follows now immediately.

□

3.2.1 Singular normal bundle

We will need to describe as well the closed complement of the open subset considered above.

Let us consider

$$\mathscr{N}_{sing}(X) := N(X) \times \{0\} \bigsqcup \mathbb{R}^{N-1} \times (0,1) \tag{3.15}$$

with the topology such that $N(X)$ is a closed subset and $\mathbb{R}^{N-1} \times (0,1)$ is an open subset whose closure looks like

$$D_\partial := N(\partial X) \times \{0\} \bigsqcup \mathbb{R}^{N-1} \times (0,1)$$

the deformation to the normal cone associated to the embedding $\partial X \hookrightarrow \mathbb{R}^{N-1}$. In fact we will be more precise in this statement, we will describe $\mathscr{N}_{sing}(X)$ as a C^∞-manifold of dimension N. The structure is such that $N(X)|_{\overset{\circ}{X}}$ and $\mathbb{R}^{N-1} \times (0,1)$ are two open submanifolds. We have then to describe the structure around $N(X)|_{\partial X}$:

Let $V \subset \mathbb{R}^{dim \partial X}$ be an open subset, consider

$$W_V^- := V \times \mathbb{R}^{N-1-dim\partial X} \times (-1, 0].$$

Take now, $U := V \times (-1,1)^{N-1-dim\partial X} \subset \mathbb{R}^{N-1}$ and consider

$$W_V^+ := \{(a, Y, \varepsilon) \in \mathbb{R}^{dim\partial X} \times \mathbb{R}^{N-1-dim\partial X} \times [0,1) : a + \varepsilon Y \in U\}.$$

In particular remark that $W_V^+ \bigcap \mathbb{R}^{dim\partial X} \times \mathbb{R}^{N-1-dim\partial X} \times \{0\} = V \times \{0\} \times \{0\}$

We can consider the open subset of $\mathbb{R}^N$:

$$W_V := W_V^- \bigcup W_V^+.$$

We define charts

$$W_V \xrightarrow{\Psi} \mathcal{W}_\mathcal{V} \subset \mathscr{N}_{sing}(X) \tag{3.16}$$

around $N(X)|_{\partial X}$ by taking charts $V \overset{\phi}{\approx} \mathcal{V}$ covering ∂X and such that we have trivializations

$$V \times \mathbb{R}^{N-1-dim\partial X} \times (-1,0] \approx \mathcal{V} \times \mathbb{R}^{N-1-dim\partial X} \times (-1,0] \approx N(X)|_{\mathcal{V} \times (-1,0]}$$

Thus obtaining in this way

$$W_V^- \xrightarrow{\Psi^-} \mathcal{W}_\mathcal{V}^- := N(X)|_{\mathcal{V} \times (-1,0]}. \tag{3.17}$$

For W_V^+, we might suppose that ϕ gives a slice chart of ∂X in $\mathbb{R}^{N-1}$ diffeomorphic to

$$U := V \times (-1,1)^{N-1-dim\partial X}.$$

Then the W_V^+ are precisely the open subsets Ω_V^U considered in [8] section 3. We consider the deformation to the normal cone charts explicitly described in [8] section 3, proposition 3.1 (see also [9] section II.5).

$$W_V^+ \overset{\Psi^+}{\longrightarrow} \mathcal{W}_{\mathcal{V}}^+ \subset D_\partial. \tag{3.18}$$

Locally they look like:

$$\Omega_V^U \to D_V^U \approx D_{\mathcal{V}}^{\mathcal{U}} \subset D_\partial$$
$$(a, Y, 0) \mapsto (a, Y, 0), \; and$$
$$(a, Y, \varepsilon) \mapsto (a + \varepsilon Y, \varepsilon) \; for \; \varepsilon \neq 0.$$

The fact that $\{(\mathcal{W}_{\mathcal{V}}, \Psi)\}$ are compatible with $N(X)|_{\overset{\circ}{X}}$ is immediate and the fact that they are compatible with $\mathbb{R}^{N-1} \times (0,1)$ follows from proposition 3.1 in [8].

Lemma 3.9 *We have an homeomorphism between the closed subset $\mathscr{B}_{h_0} = X_\partial \times \mathbb{R}^N / \sim$ of $\mathscr{B}_h$ and $\mathscr{N}_{sing}(X)$. More explicitly, the map*

$$\begin{cases} (x, \varepsilon, X) \mapsto (\varepsilon, \varepsilon \cdot X_{N-1} + i_\partial(x)) \\ (x, 0, X) \mapsto [X]_x \in N_x(X) \end{cases} \tag{3.19}$$

passes to the quotient into an homeomorphism

$$\widetilde{q_\partial} : X_\partial \times \mathbb{R}^N / \sim \longrightarrow \mathscr{N}_{sing}(X)$$

Proof : Let us denote by $q_\partial : X_\partial \times \mathbb{R}^N \to \mathscr{N}_{sing}(X)$ the map (3.19) above. We will show that it is an open continuous surjection. The fact that is surjective is immediate. For proving that it is open and continuous, it is enough to check that for every point $p \in X_\partial \times \mathbb{R}^N$ there is an open neighborhood $\mathcal{V}_p$ such that $q_\partial(\mathcal{V}_p)$ is open and $q_\partial : \mathcal{V}_p \to q_\partial(\mathcal{V}_p)$ is continuous.

For points in $\partial X \times (0,1) \times \mathbb{R}^N$, the restriction

$$\partial X \times (0,1) \times \mathbb{R}^N \to q_\partial(\partial X \times (0,1) \times \mathbb{R}^N) = (0,1) \times \mathbb{R}^{N-1}$$

is given by $(x, \varepsilon, X) \mapsto (\varepsilon, \varepsilon \cdot X_{N-1} + i_\partial(x))$ which is open and continuous (the proof is completely analog to lemma 3.8).

For points in $\overset{\circ}{X} \times \{0\} \times \mathbb{R}^N$, the restriction

$$\overset{\circ}{X} \times \{0\} \times \mathbb{R}^N \to q_\partial(\overset{\circ}{X} \times \{0\} \times \mathbb{R}^N) = N(X)|_{\overset{\circ}{X}}$$

is given by $(x, 0, X) \mapsto [X]_x \in N_x(X)$ which is clearly open and continuous.

Finally, for points in $\partial X \times \{0\} \times \mathbb{R}^N$ we need to have more careful. Let $x_0 \in \partial X$, we take a chart $V \overset{\phi}{\approx} \mathcal{V}$ as above, together with the correspondant slice chart $U \in \mathbb{R}^{N-1}$. Around a point $(x, 0, X_0) \in X_{\mathscr{F}} \times \mathbb{R}^N$ we can consider a chart (with boundary) diffeomorphic to

$$\mathcal{U} := \{(x, t, \varepsilon, X) \in V \times (-1, 0] \times [0, 1) \times \mathbb{R}^N : x + \varepsilon X_{N-1-dim\partial X} \in U\}.$$

Take $\mathcal{U}_\partial$ the corresponding intersection with $X_\partial \times \mathbb{R}^N$. In these local coordinates, the map q_∂ looks like:

$$\mathcal{U} \to W_V$$

given by

$$(x, 0, \varepsilon, X) \mapsto (x, X_{N-1-dim\partial X}, \varepsilon) \in W_V^+$$

and

$$(x, t, 0, X) \mapsto (x, X_{N-1-dim\partial X}, t) \in W_V^-.$$

It is evidently an open continuous map. Hence the map q_∂ is an open continuous surjection for which $q_\partial(a) = q_\partial(b)$ if and only if $a \sim_h b$ in $X_\partial \times \mathbb{R}^N$. We conclude then that the induced map is an homeomorphism.

□

3.2.2 The APS classifying space

Consider

$$\mathscr{B}_{\mathscr{F}} := \mathscr{N}_{sing}(X) \bigsqcup (0, 1] \times \mathbb{R}^N. \tag{3.20}$$

By the two precedent lemmas we can conclude that there is an unique locally compact topology on $\mathscr{B}_{\mathscr{F}}$ such that the bijection

$$\widetilde{q_{\mathscr{F}}} := \widetilde{q_\partial} \bigsqcup \overset{\sim}{\underset{\circ}{q}} : \mathscr{B}_h \longrightarrow \mathscr{B}_{\mathscr{F}}$$

induced by $\widetilde{q_\partial}$ and $\overset{\sim}{\underset{\circ}{q}}$ is an homeomorphism.

With the new identifications, the index morphism (3.14) takes the following form:

$$K^0(\mathscr{N}_{sing}(X)) \xleftarrow[\approx]{(e_0)_*} K^0(\mathscr{B}_{\mathscr{F}}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N). \tag{3.21}$$

The APS index theorem proposed below, will be useful only if we can compute this index morphism. In the case $\partial X = \emptyset$ the topology of $\mathscr{B}_{\mathscr{F}}$ gives immediately that the above index is just the shriek map induced by an open inclusion. In the case with boundary, $\mathscr{N}_{sing}(X)$ is not quite an open subset of $\mathbb{R}^N$, but still we can prove the following by analyzing the topology of $\mathscr{B}_{\mathscr{F}}$:

To describe the topology of this space we will describe three big open subsets that cover $\mathscr{B}_{\mathscr{F}}$ and that generate the entire topology (see proposition below). Let us first define the subjacent subsets $U^0_{\mathscr{F}}$, $U^1_{\mathscr{F}}$ and $U^2_{\mathscr{F}}$:

$$U^0_{\mathscr{F}} = \mathbb{R}^N \times (0, 1],$$

$$U^1_{\mathscr{F}} = (\mathbb{R}^{N-1} \times (0, 1)) \bigsqcup (\mathbb{R}^{N-1} \times (0, 1) \times (0, 1)),$$

where the first component of the disjoint union belongs to $\mathscr{N}_{sing}(X)$ and the second one is the open subset $(\mathbb{R}^{N-1} \times (0, 1)) \times (0, 1)$ of $\mathbb{R}^N \times (0, 1)$. For introduce $U^2_{\mathscr{F}}$, we will need to chose a (good) tubular neighborhood of X in $\mathbb{R}^N$, that is a diffeomorphism

$$\begin{array}{ccc} N(X) & \xrightarrow[\approx]{f} & W \subset \mathbb{R}^{N-1} \times \mathbb{R}_- \\ \uparrow & & \uparrow \\ X & \xrightarrow{\quad id \quad} & X \end{array} \tag{3.22}$$

from $N(X)$ to W an open neighborhood of X (or $i_\partial(X)$ to be formal) in $\mathbb{R}^N$ such that f is the identity in X (identifying X with the zero section), and the restriction to ∂X gives a tubular neighborhood of ∂X in $\mathbb{R}^{N-1}$:

$$\begin{array}{ccc} N(X)|_{\partial X} & \xrightarrow[\approx]{f_\partial} & W_\partial \subset \mathbb{R}^{N-1} \times \{0\} \\ \uparrow & & \uparrow \\ \partial X & \xrightarrow{\quad id \quad} & \partial X \end{array} \tag{3.23}$$

where $W_\partial = W \bigcap \mathbb{R}^{N-1} \times \{0\}$. Consider the open subset of $\mathbb{R}^N$ consisting of putting a collar to W in the positive direction:

$$W_{sing} := W \bigcup W_\partial \times [0,1)$$

We will introduce the subset $U^2_{\mathscr{F}}$ by its intersections with $\mathscr{N}_{sing}(X)$ and with $\mathbb{R}^N \times (0,1]$:

$$U^2_{\mathscr{F}} \bigcap \mathscr{N}_{sing}(X) = N(X) \times \{0\} \bigsqcup W_\partial \times (0,1),$$

and

$$U^2_{\mathscr{F}} \bigcap \mathbb{R}^N \times (0,1] = W_{sing} \times (0,1).$$

Lemma 3.10 *The subsets $U^0_{\mathscr{F}}$, $U^1_{\mathscr{F}}$ and $U^2_{\mathscr{F}}$ of $\mathscr{B}_{\mathscr{F}}$ are open.*

Proof : By definition $\mathscr{B}_{\mathscr{F}}$ induced from the quotient topology on $\mathscr{B}_h$. Hence if we consider the map

$$q_{\mathscr{F}} : X_{\mathscr{F}} \times \mathbb{R}^N \to \mathscr{B}_{\mathscr{F}}$$

given by $q_{\mathscr{F}} := q_\partial \bigsqcup \overset{\circ}{q}$ (with the notations of lemmas 3.8 and 3.9), it is enough to observe that $q^{-1}_{\mathscr{F}}(V)$ is an open subset of $X_{\mathscr{F}} \times \mathbb{R}^N$ to conclude that V is open in $\mathscr{B}_{\mathscr{F}}$. In the case we are dealing with, we have:

- $q^{-1}_{\mathscr{F}}(U^0_{\mathscr{F}}) = \overset{\circ}{X} \times (0,1] \times \mathbb{R}^N$, which is clearly open in $X_{\mathscr{F}} \times \mathbb{R}^N$.
- $q^{-1}_{\mathscr{F}}(U^1_{\mathscr{F}}) = X \times (0,1) \times \mathbb{R}^N$, which is clearly open in $X_{\mathscr{F}} \times \mathbb{R}^N$.
- For the last one, an explicit description of the inverse image allows to verify it is open by a direct analysis at each point, we leave this as direct excercice computation,

$$q^{-1}_{\mathscr{F}}(U^2_{\mathscr{F}}) = X \times \{0\} \times \mathbb{R}^N$$

$$\bigsqcup \{(x,\varepsilon,X) \in \partial X \times (0,1) \times \mathbb{R}^N : i_\partial(x) + \varepsilon \cdot X_{N-1} \in W_\partial\}$$

$$\bigsqcup \{(x,\varepsilon,X) \in X \times (0,1) \times \mathbb{R}^N : i(x) + \varepsilon \cdot X \in W_{sing}\}$$

□

Proposition 3.11 *The locally compact space $\mathscr{B}_{\mathscr{F}}$ admits an oriented C^∞-manifold with boundary structure of dimension $N+1$.*

Proof : We will cover $\mathscr{B}_{\mathscr{F}}$ with three explicit charts by using the three open subsets of $\mathscr{B}_{\mathscr{F}}$, $U^0_{\mathscr{F}}$, $U^1_{\mathscr{F}}$ and $U^2_{\mathscr{F}}$ of the lemma above.

Chart $(U^0_{\mathscr{F}}, \Psi^0_{\mathscr{F}})$: We let $W^0_{\mathscr{F}} = \mathbb{R}^N \times (0,1]$ and $\Psi^0_{\mathscr{F}}$ to be the identity:

$$W^0_{\mathscr{F}} \overset{\Psi^0_{\mathscr{F}} = id}{\longrightarrow} U^0_{\mathscr{F}}.$$

Chart $(U^1_{\mathscr{F}}, \Psi^1_{\mathscr{F}})$: We let $W^1_{\mathscr{F}} = \mathbb{R}^{N-1} \times [0,1)_t \times (0,1)_s$ and $\Psi^1_{\mathscr{F}}$,

$$W^1_{\mathscr{F}} \overset{\Psi^1_{\mathscr{F}}}{\longrightarrow} U^1_{\mathscr{F}}$$

defined by

$$\begin{cases} (X_{N-1}, 0, s) \mapsto (X_{N-1}, s) \in \mathbb{R}^{N-1} \times (0,1) \subset \mathscr{N}_{sing} \\ (X_{N-1}, t, s) \mapsto (X_{N-1}, s, t) \in (\mathbb{R}^{N-1} \times (0,1)) \times (0,1) \subset \mathbb{R}^N \times (0,1]. \end{cases} \tag{3.24}$$

Chart $(U^2_{\mathscr{F}}, \Psi^2_{\mathscr{F}})$**:** We let $W^2_{\mathscr{F}} = W_{sing} \times [0,1)_t$ and $\Psi^2_{\mathscr{F}}$,

$$W^2_{\mathscr{F}} \xrightarrow{\Psi^2_{\mathscr{F}}} U^2_{\mathscr{F}}$$

defined by

$$\begin{cases} (w,0) \in W \mapsto (f^{-1}(w),0) \in N(X) \times \{0\} \subset \mathscr{N}_{sing} \\ (w_\partial, s, 0) \mapsto (f_\partial(s \cdot f_\partial^{-1}(w_\partial)), s) \in W_\partial \times (0,1) \subset \mathscr{N}_{sing}. \end{cases} \tag{3.25}$$

if $t = 0$, and by

$$\begin{cases} (w,t) \mapsto (f(t \cdot f^{-1}(w)), t) \in W \times (0,1) \subset \mathbb{R}^N \times (0,1] \\ (w_\partial, s, t) \mapsto (f_\partial((s+t) \cdot f_\partial^{-1}(w_\partial)), s, t) \in (W_\partial \times [0,1)) \times (0,1) \subset \mathbb{R}^N \times (0,1]. \end{cases} \tag{3.26}$$

for $t \neq 0$.

We will check now the compatibility of the charts, together with the fact that the changes of coordinates have positive sign:

- $((\Psi^0_{\mathscr{F}})^{-1} \circ \Psi^1_{\mathscr{F}})$: This is the easiest case, indeed we have that

  $$(\Psi^1_{\mathscr{F}})^{-1}(U^0_{\mathscr{F}} \bigcap U^1_{\mathscr{F}}) = (\mathbb{R}^{N-1} \times (0,1)) \times (0,1),$$

  $$(\Psi^0_{\mathscr{F}})^{-1}(U^0_{\mathscr{F}} \bigcap U^1_{\mathscr{F}}) = (\mathbb{R}^{N-1} \times (0,1)) \times (0,1)$$

  and

  $$(\Psi^1_{\mathscr{F}})^{-1}(U^0_{\mathscr{F}} \bigcap U^1_{\mathscr{F}}) \xrightarrow{(\Psi^0_{\mathscr{F}})^{-1} \circ \Psi^1_{\mathscr{F}}} (\Psi^0_{\mathscr{F}})^{-1}(U^0_{\mathscr{F}} \bigcap U^1_{\mathscr{F}})$$

  is the identity.

- $((\Psi^0_{\mathscr{F}})^{-1} \circ \Psi^2_{\mathscr{F}})$:

  $$(\Psi^2_{\mathscr{F}})^{-1}(U^0_{\mathscr{F}} \bigcap U^2_{\mathscr{F}}) = W_{sing} \times (0,1),$$

  $$(\Psi^0_{\mathscr{F}})^{-1}(U^0_{\mathscr{F}} \bigcap U^2_{\mathscr{F}}) = W_{sing} \times (0,1)$$

  and

  $$(\Psi^2_{\mathscr{F}})^{-1}(U^0_{\mathscr{F}} \bigcap U^2_{\mathscr{F}}) \xrightarrow{(\Psi^0_{\mathscr{F}})^{-1} \circ \Psi^2_{\mathscr{F}}} (\Psi^0_{\mathscr{F}})^{-1}(U^0_{\mathscr{F}} \bigcap U^2_{\mathscr{F}})$$

  is given by

  $$\begin{cases} (w,t) \mapsto (f(t \cdot f^{-1}(w)), t) \\ (w_\partial, s, t) \mapsto (f_\partial((s+t) \cdot f_\partial^{-1}(w_\partial)), s, t) \end{cases}$$

  which is evidently a diffeomorphism with positive determinant[4].

- $((\Psi^1_{\mathscr{F}})^{-1} \circ \Psi^2_{\mathscr{F}})$:

  $$(\Psi^2_{\mathscr{F}})^{-1}(U^1_{\mathscr{F}} \bigcap U^2_{\mathscr{F}}) = W_\partial \times (0,1) \times [0,1),$$

  $$(\Psi^1_{\mathscr{F}})^{-1}(U^1_{\mathscr{F}} \bigcap U^2_{\mathscr{F}}) = W_\partial \times [0,1) \times (0,1)$$

  and

  $$(\Psi^2_{\mathscr{F}})^{-1}(U^1_{\mathscr{F}} \bigcap U^2_{\mathscr{F}}) \xrightarrow{(\Psi^1_{\mathscr{F}})^{-1} \circ \Psi^2_{\mathscr{F}}} (\Psi^1_{\mathscr{F}})^{-1}(U^1_{\mathscr{F}} \bigcap U^2_{\mathscr{F}})$$

  is given by

  $$(w_\partial, s, t) \mapsto (f_\partial((s+t) \cdot f_\partial^{-1}(w_\partial)), t, s)$$

  which is evidently a diffeomorphism with positive determinant[5].

[4] It is a diffeomorphism whose linear representation is of the form $A \cdot \lambda \cdot A^{-1}$ with positive λ.

[5] same as previous footnote.

□

In particular, we have obtained an oriented cobordism $\mathscr{B}_{\mathscr{F}}$ from $\mathscr{N}_{sing}(X)$ to $\mathbb{R}^N$. From now on, we orient $\mathscr{B}_{\mathscr{F}}$ such that the induced orientation on the boundary is

$$\partial\mathscr{B}_{\mathscr{F}} = -\mathscr{N}_{sing}(X) \bigcup \mathbb{R}^N.$$

We can hence apply a Stoke's theorem argument to obtain the following proposition:

Proposition 3.12 *The following diagram is commutative*

$$\begin{array}{ccccc} K^0(\mathscr{N}_{sing}(X)) & \xleftarrow[\approx]{(e_0)_*} & K^0(\mathscr{B}_{\mathscr{F}}) & \xrightarrow{(e_1)_*} & K^0(\mathbb{R}^N) \\ & \searrow^{\int_{\mathscr{N}_{sing}(X)} ch(\cdot)} & & \swarrow_{\int_{\mathbb{R}^N} ch(\cdot)} & \\ & & \mathbb{R} & & \end{array}$$

Proof : By definition, the algebra morphisms $e_0 : C_0(\mathscr{B}_{\mathscr{F}}) \to C_0(\mathscr{N}_{sing}(X))$ and $e_1 : C_0(\mathscr{B}_{\mathscr{F}}) \to C_0(\mathbb{R}^N)$ are induced by the closed embeddings $i_0 : \mathscr{N}_{sing}(X) \hookrightarrow \mathscr{B}_{\mathscr{F}}$ and $i_1 : \mathbb{R}^N \hookrightarrow \mathscr{B}_{\mathscr{F}}$ respectively. The Chern character being natural we have that the following diagram is commutative

$$\begin{array}{ccccc} K^0(\mathscr{N}_{sing}(X)) & \xleftarrow[\approx]{(e_0)_*} & K^0(\mathscr{B}_{\mathscr{F}}) & \xrightarrow{(e_1)_*} & K^0(\mathbb{R}^N) \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} \\ H^{ev}_{dR}(\mathscr{N}_{sing}(X)) & \xleftarrow[(i_0)_*]{} & H^{ev}_{dR}(\mathscr{B}_{\mathscr{F}}) & \xrightarrow[(i_1)_*]{} & H^{ev}_{dR}(\mathbb{R}^N) \end{array} \tag{3.27}$$

The result now follows from Stoke's theorem, indeed, for any ω N-closed differential form on $\mathscr{B}_{\mathscr{F}}$ with compact support, we have by Stoke's that

$$\int_{\partial\mathscr{B}_{\mathscr{F}}} \omega = O$$

and hence $\int_{\mathscr{N}_{sing}(X)} \omega = \int_{\mathbb{R}^N} \omega$.

□

4 Atiyah-Patodi-Singer theorem in K-theory

4.1 The Fredholm index morphism

Deformation groupoids induce index morphisms. The groupoid $\Gamma(X)_{\mathcal{F}}$ is parametrized by the closed interval $[0,1]$. Its algebra comes equipped with evaluations to the algebra of $\mathcal{T}_{nc}X$ (at t=0) and to the algebra of $\overset{\circ}{X} \times \overset{\circ}{X}$ (for $t \neq 0$). We have a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1]) \longrightarrow C^*(\Gamma(X)_{\mathcal{F}}) \xrightarrow{e_0} C^*(\mathcal{T}_{nc}X) \longrightarrow 0 \tag{4.1}$$

where the algebra $C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1])$ is contractible. Hence applying the K-theory functor to this sequence we obtain an index morphism

$$ind_{\mathcal{F}} = (e_1)_* \circ (e_0)_*^{-1} : K^0(\mathcal{T}_{nc}X) \longrightarrow K^0(\overset{\circ}{X} \times \overset{\circ}{X}) \approx \mathbb{Z}.$$

Proposition 4.1 *[16, 18, 10]. For any fully elliptic operator D on X, there is a "non commutative symbol" $[\sigma_D] \in K^0(\mathcal{T}_{nc}X)$ and*

$$ind_{\mathcal{F}}(\sigma_D) = \mathrm{Index}_{APS}(D) \tag{4.2}$$

Proof : For the sake of completeness, we briefly explain the proof. Let $D \in \Psi^0(\Gamma(X); E, F)$ be a zero order fully elliptic b-operator. Here E, F are hermitian bundles on X, pulled-back to $\Gamma(X)$ with the target map. Let $Q \in \Psi^0(\Gamma(X); F, E)$ be a full parametrix of P. This means:

$$PQ - 1 \in \Psi^{-1}(\Gamma(X); F, F), \quad QP - 1 \in \Psi^{-1}(\Gamma(X); E, E) \tag{4.3}$$

and that, moreover,

$$P_\partial Q_\partial = 1, \quad Q_\partial P_\partial = 1 \tag{4.4}$$

where we have denoted by A_∂ the $\partial X \times \partial X \times \mathbb{R}$-pseudodifferential operator obtained by restriction of any $\Gamma(X)$-operator A. The equations (4.3) reflect the interior ellipticity while (4.3) reflect the boundary ellipticity. It is well known that $P : L^2_b(X, E) \longrightarrow L^2_b(X, F)$ is bounded and Fredholm [20, 23] where the hermitian structure of E, F as bundles over X is used together the measure on X associated with a b-metric, and

$$\mathrm{Index}_{APS}(P) = \mathrm{Index}_{\mathrm{Fred}}(P) = \dim\ker P - \mathrm{dimcoker} P.$$

This can also be recovered as the analytical index of a K-homology class of a compact space. Let $X^c = X/\partial X$ be the conical space associated with X and we note π the canonical projection map. We represent $C(X^c)$ into $L^2_b(X, G)$, where G is any hermitian bundle over X, as follows:

$$f \in C(X^c), \quad \xi \in L^2_b(X, G); \quad \forall y \in X, \quad m(f)(\xi)(y) = f(\pi(y))\xi(y).$$

It is immediate to check that

$$(P) = \left(L^2_b(X, E \oplus F); m; \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \right) \tag{4.5}$$

is a Kasparov $(C(X^c), \mathbb{C})$-module and that the resulting K-homology class $[P] \in K_0(X^c)$ does not depend on the choices of the parametrix Q nor the particular b-metric. Then $\mathrm{Index}_{\mathrm{APS}}(P) = \mathrm{Index}_{\mathrm{ana}}(P) = p_*([P])$ where $p : X^c \to \{\mathrm{point}\}$.

We define $\sigma_P \in K^0(\mathcal{T}_{nc}X)$ to be the Poincaré dual class of P. Let us describe this element more explicitly.

Let $\widetilde{P}$, $\widetilde{Q}$ be any elliptic operators on $\Gamma(X)^{tan}$ such that $\widetilde{P}|_{t=1} = P$, $\widetilde{Q}|_{t=1} = Q$ and:

$$\widetilde{P}\widetilde{Q} - 1, \widetilde{Q}\widetilde{P} - 1 \in \Psi^{-1}(\Gamma(X)^{tan}).$$

We have by construction:

$$(\widetilde{P}_{t=1})_\partial(\widetilde{Q}_{t=1})_\partial - 1 = 0 = (\widetilde{Q}_{t=1})_\partial(\widetilde{P}_{t=1})_\partial - 1,$$

Hence:

$$(\widetilde{P}) = \left(C^*(\Gamma(X)_{\mathcal{F}}, E \oplus F), 1, \begin{pmatrix} 0 & \widetilde{Q} \\ \widetilde{P} & 0 \end{pmatrix} \right) \tag{4.6}$$

is a Kasparov $(\mathbb{C}, C^*(\Gamma(X)_{\mathcal{F}}))$-module, and the restriction:

$$\sigma_{nc}(\widetilde{P}) := \widetilde{P}|_{\mathcal{T}_{nc}X} \tag{4.7}$$

provides a Kasparov $(\mathbb{C}, C^*(\mathcal{T}_{nc}X))$-module:

$$(\sigma_{nc}(\widetilde{P})) := \left(C^*(\mathcal{T}_{nc}X, E \oplus F), 1, \begin{pmatrix} 0 & \sigma_{nc}(\widetilde{Q}) \\ \sigma_{nc}(\widetilde{P}) & 0 \end{pmatrix} \right). \tag{4.8}$$

Denoting by e_0 the $*$-homomorphism $C^*(\Gamma(X)_{\mathcal{F}}) \to C^*(\mathcal{T}_{nc}X)$, the previous class satisfies:

$$(e_0)_*[\widetilde{P}] = [\sigma_{nc}(\widetilde{P})] \in K^0(\mathcal{T}_{nc}X) \tag{4.9}$$

By construction the Poincaré duality isomorphism [11, 18] sends $[P]$ to $[\sigma_{nc}(\widetilde{P})]$, and we thus set:

$$\sigma_P = [\sigma_{nc}(\widetilde{P})]$$

Now, μ_0 denoting the Morita equivalence $\overset{\circ}{X} \times \overset{\circ}{X} \sim$ point, we compute:

$$\begin{aligned} ind_{\mathcal{F}}(\sigma(P)) &= \mu_0 \circ (e_1)_* \circ (e_0)_*^{-1}([\sigma_{nc}(\widetilde{P})] && (4.10)\\ &= \mu_0 \circ (e_1)_*(\widetilde{P}) && (4.11)\\ &= \mu_0 \left(C^*(\overset{\circ}{X} \times \overset{\circ}{X}, E \oplus F), 1, \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \right) && (4.12)\\ &= \left(L^2_b(X, E \oplus F), 1, \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \right) && (4.13)\\ &= \mathrm{Index}_{\mathrm{ana}}(P) = p_*([P]) = \mathrm{Index}_{\mathrm{APS}}(P) && (4.14) \end{aligned}$$

□

It is also interesting to manage geometric operators (for instance, Dirac type operators on X equipped with an exact b-metric g_b) instead of abstract 0-order pseudodifferential operators. Under appropriate assumptions, they also give rise to K-homology classes of X^c and thus one may look for a *geometric* representative of their Poincaré dual class in $K^0(\mathcal{T}_{nc}X)$.

Rather than dealing exaclty with Dirac operators on (X, g_b), we shall consider the following class of differential operators on X containing them. Let E be a smooth hermitian vector bundle over X endowed with an orthogonal decomposition $E = E_0 \oplus E_1$ and an isomorphism $U : E|_{\partial X \times (0,\varepsilon)} \to (p_1)^* E_\partial \oplus (p_1)^* E_\partial$ where p_1 is the first projection subordonated to a collar identification near the boundary and $E_\partial = E_0|_{\partial X}$. Then we consider first order elliptic differential operators D, which have, after conjugation by U, the following expression near the boundary:

$$UDU^{-1} = \begin{pmatrix} 0 & -x\frac{\partial}{\partial x} + S \\ x\frac{\partial}{\partial x} + S & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \tag{4.15}$$

where $S \in \mathrm{Diff}^1(\partial X, E_\partial)$. We require S to be elliptic, symmetric, and to simplify independant of $x < \varepsilon$. It follows from these assumptions that $D^2 + 1$ is invertible, as a linear map, on $C^\infty(X, E)$ [20] and since $\Psi(\Gamma(X))$ is spectrally invariant (it is understood that the appropriate Schwartz algebra is added in the calculus [17]), we have $(D^2+1)^{-1/2} \in \Psi^{-1}(\Gamma(X))$. Now, we moreover require the invertibility of S, which is equivalent here to the full ellipticity of D. Observe that this assumption is not sufficient to let D into an unbounded $(C(X^c), \mathbb{C})$-Kasparov module in the sense of [6], but it nevertheless implies that $W(D) := D(D^2+1)^{-1/2} \in \Psi^0(\Gamma(X), E)$ is fully elliptic since the indicial family map $I : \Psi(\Gamma(X)) \to \Psi(\partial X, \mathbb{R})$ is a homomorphism of algebras:

$$I(W(D), \tau) = \begin{pmatrix} 0 & \frac{S - i\tau}{(S^2+\tau^2+1)^{1/2}} \\ \frac{S+i\tau}{(S^2+\tau^2+1)^{1/2}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & W(D)_- \\ W(D)_+ & 0 \end{pmatrix}.$$

Thus one can already associate to D the (bounded) K-homology class $[W(D)_+]$, with Poincaré dual given by the (bounded) K-theory class $\sigma_{W(D)_+}$ as above. Alternatively, we may look for a more geometric representative of these classes.

For that purpose, we define a lift $\widetilde{D} \in \Psi(\Gamma(X)^{tan})$ of D as follows. Let s be the complete symbol of S with respect to the exponential map of the metric on the boundary (see [14]), and d the complete symbol of D with respect to the exponential map of the metric on X. We rescale s and d as follows:

$$\forall 0 < \varepsilon \leq 1, \forall (x,\xi) \in T^*\partial X, \quad s_{\mathrm{ad}}(y,\xi) = s(y,\varepsilon\xi) \text{ and } \forall (z,\zeta) \in T^*X, \quad d_{\mathrm{ad}}(z,\zeta) = d(z,\varepsilon\zeta)$$

Setting $S_{\mathrm{ad}} = s_{\mathrm{ad}}(y, D_y)$, $D_{\mathrm{ad}} = d_{\mathrm{ad}}(z, D_z)$ and using positive functions $\varphi, \psi \in C^\infty(X)$ such that $\varphi + \psi = 1$ and $\varphi = 1$ if $x < \varepsilon$, $\varphi = 0$ if $x \geq 2\varepsilon$, we let

$$\widetilde{D}|_{x\varepsilon>0} = \varphi U^{-1} \begin{pmatrix} 0 & -\varepsilon x\frac{\partial}{\partial x} + S_{\mathrm{ad}} \\ \varepsilon x\frac{\partial}{\partial x} + S_{\mathrm{ad}} & 0 \end{pmatrix} U + \psi D_{\mathrm{ad}},$$

$$\widetilde{D}|_{x=0,\varepsilon>0} = U^{-1} \begin{pmatrix} 0 & -\frac{\partial}{\partial \lambda} + S_{\mathrm{ad}} \\ \frac{\partial}{\partial \lambda} + S_{\mathrm{ad}} & 0 \end{pmatrix} U,$$

and

$$\widetilde{D}|_{\varepsilon=0} = \varphi U^{-1} \begin{pmatrix} 0 & -\frac{\partial}{\partial \lambda} + s(y, D_Y) \\ \frac{\partial}{\partial \lambda} + s(y, D_Y) & 0 \end{pmatrix} U + \psi d(z, D_Z),$$

where D_Y, D_Z stand for the differentiation in the fibers coordinates of $T\partial X$ and TX respectively.

On the other hand, we can also rescale $\widetilde{D}|_{x=0,\varepsilon>0}$ into:

$$\sigma_{\partial-\mathrm{ubd}}(D) = U^{-1} \begin{pmatrix} 0 & -\frac{\partial}{\partial \lambda} + \frac{1}{1-\varepsilon} S_{\mathrm{ad}} \\ \frac{\partial}{\partial \lambda} + \frac{1}{1-\varepsilon} S_{\mathrm{ad}} & 0 \end{pmatrix} U$$

and define $\sigma_{\mathrm{ubd}}(D) \in \mathrm{Diff}^1(\mathcal{T}_{nc}X)$ by

$$\sigma_{\mathrm{ubd}}(D) = \sigma_{\partial-\mathrm{ubd}}(D) \text{ if } x = 0 \text{ and } \sigma_{\mathrm{ubd}}(D) = \widetilde{D}|_{\varepsilon=0} \text{ if } x > 0$$

We also note $\widetilde{E}$ the pull back of E for the map $X \times [0,1] \xrightarrow{p_1} X$ and then E_{ad} its restriction to $\{x\varepsilon = 0\}$. By construction, we get an elliptic symmetric element $\sigma_{\mathrm{ubd}}(D) \in \mathrm{Diff}^1(\mathcal{T}_{nc}X, r^*E_{\mathrm{ad}})$, whose closure $\overline{\sigma_{\mathrm{ubd}}(D)}$ as an unbounded operator on the Hilbert module $\mathcal{E} := C^*(\mathcal{T}_{nc}X, r^*E_{\mathrm{ad}})$ with domain $C^{\infty,0}(\mathcal{T}_{nc}X, r^*E_{\mathrm{ad}})$ is a regular selfadjoint operator [6]. To prove that assertion, we can not directly apply proposition 3.6.2 and lemma 3.6.3 in [32] since the unit space of $\mathcal{T}_{nc}X$ is not compact. Nevertheless, we can pick up a suitable parametrix $q \in \Psi^{-1}(\overline{\mathcal{T}_{nc}X}, r^*E_{\mathrm{ad}})$ of $\sigma_{\mathrm{ubd}}(D)$, where $\overline{\mathcal{T}_{nc}X} := \Gamma(X)^{tan}|_{x\varepsilon=0}$, in such a way that the proofs given in [32] apply verbatim : we define q by combining, using cut-off functions, a parametrix given by the inversion of the principal symbol of $\sigma_{\mathrm{ubd}}(D)$ with the true inverse of $\sigma_{\partial-\mathrm{ubd}}(D)|_\varepsilon$ when $1 - \alpha < \varepsilon < 1$ and extended by 0 at $\varepsilon = 1$. Then, we have by construction:

$$\sigma_{\mathrm{ubd}}(D)q = 1 + k_1,\ q\sigma_{\mathrm{ubd}}(D) = 1 + k_2 \text{ with } q, k_i \in \Psi^{-1}(\overline{\mathcal{T}_{nc}X}, r^*E_{\mathrm{ad}}),\ q|_{\varepsilon=1} = 0,\ k_i|_{\varepsilon=1} = 0.$$

The first of the two conditions on the operators q, k_1, k_2 implies that they extend into compact morphisms on $C^*(\overline{\mathcal{T}_{nc}X}, r^*E_{\mathrm{ad}})$ and the second that they actually are compact on $C^*(\mathcal{T}_{nc}X, r^*E_{\mathrm{ad}})$.

It then follows that $(\sigma_{ubd}(D)^2 + 1)^{-1}$ is a compact morphism on $C^*(\mathcal{T}_{nc}X, r^*E_{\mathrm{ad}})$. Thus, we get an unbounded $(\mathbb{C}, C^*(\mathcal{T}_{nc}X))$-Kasparov class in the sense of [6]:

$$\sigma_{\mathrm{ubd},D} := \left(C^*(\mathcal{T}_{nc}X, r^*E_{\mathrm{ad}}), \sigma_{ubd}(D)\right) \in \mathcal{E}_{\mathrm{ubd}}(\mathbb{C}, C^*(\mathcal{T}_{nc}X)).$$

In the equality above, $\mathcal{E}_{\mathrm{ubd}}(A,B)$ denotes the family of unbounded Kasparov A-B-bimodules as defined in [6].

To check that the latter is an unbounded representative of the Poincaré dual of $[W(D)_+]$ (and thus can be used for the computation of the index of D_+), we have to prove the equality :

$$[\sigma_{W(D_+)}] = [W(\sigma_{ubd}(D))] \in KK(\mathbb{C}, C^*(\mathcal{T}_{nc}X)),$$

which can be achieved by comparing the operator part of these KK-classes. When ε, they coincide. When $\varepsilon > 0$, the operator part in $[\sigma_{W(D_+)}]$ can be represented by:

$$\begin{pmatrix} 0 & \frac{S_{\mathrm{ad}} - i\partial_\lambda}{(S_{\mathrm{ad}}^2 + \partial_\lambda^2 + 1)^{1/2}} \\ \frac{S_{\mathrm{ad}} + i\partial_\lambda}{(S_{\mathrm{ad}}^2 + \partial_\lambda^2 + 1)^{1/2}} & 0 \end{pmatrix},$$

and for $[W(\sigma_{ubd}(D))]$ we have:

$$\begin{pmatrix} 0 & \frac{S_{\mathrm{ad}} - i(1-\varepsilon)\partial_\lambda}{(S_{\mathrm{ad}}^2 + (1-\varepsilon)^2(\partial_\lambda^2 + 1))^{1/2}} \\ \frac{S_{\mathrm{ad}} + i(1-\varepsilon)\partial_\lambda}{(S_{\mathrm{ad}}^2 + (1-\varepsilon)^2(\partial_\lambda^2 + 1))^{1/2}} & 0 \end{pmatrix}.$$

Homotoping the numerical factor $(1-\varepsilon)$ with 1 provides an operator homotopy between both, and this proves the assertion.

Observe also that:

$$W(\sigma_{ubd}(D))|_{\varepsilon=1} = \begin{pmatrix} 0 & \frac{S}{|S|} \\ \frac{S}{|S|} & 0 \end{pmatrix},$$

and thus that, playing again with homotopies, this value at $\varepsilon = 1$ can be conserved for $\varepsilon \in]\alpha, 1]$ for arbitrary $\alpha > 0$.

4.2 Atiyah-Patodi-Singer index theorem in K-theory

Theorem 4.2 *[K-theoretic APS] Let X be a manifold with boundary, consider an embedding of X in $\mathbb{R}^N$ as in the previous section. Hence, the Fredholm index morphism $ind_{\mathcal{F}}$ can be decomposed as the composition of the following three morphisms*

1. *The Connes-Thom isomorphism $\mathscr{C}\mathscr{T}_0$ followed by the Morita equivalence $\mathscr{M}_0$:*

$$K^0(\mathcal{T}_{nc}X) \xrightarrow{\mathscr{C}\mathscr{T}_0} K^0((\mathcal{T}_{nc}X)_{h_0}) \xrightarrow{\mathscr{M}_0} K^0(\mathcal{N}_{sing}(X))$$

2. *The index morphism of the deformation space $\mathscr{B}_{\mathcal{F}}$:*

$$K^0(\mathcal{N}_{sing}(X)) \xleftarrow[\approx]{(e_0)_*} K^0(\mathscr{B}_{\mathcal{F}}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)$$

3. *The usual Bott periodicity morphism: $K^0(\mathbb{R}^N) \xrightarrow{Bott} \mathbb{Z}$.*

Proof : The morphism $h : \Gamma(X)_{\mathcal{F}} \to \mathbb{R}^N$ is by definition also parametrized by $[0,1]$, *i.e.*, we have morphisms $h_0 : \mathcal{T}_{nc}X \to \mathbb{R}^N$ and $h_1 : \overset{\circ}{X} \times \overset{\circ}{X} \to \mathbb{R}^N$, for $t = 1$. We can consider the associated groupoids, which are free and proper.

The following diagram, in which the morphisms $\mathscr{C}\mathscr{T}$ and $\mathscr{M}$ are the Connes-Thom and Morita isomorphisms respectively, is trivially commutative by naturality of the Connes-Thom isomorphism:

$$\begin{array}{ccccc}
K^0(\mathcal{T}_{nc}X) & \xleftarrow[\approx]{e_0} & K^0(\Gamma(X)_{\mathcal{F}}) & \xrightarrow{e_1} & K^0(\overset{\circ}{X}\times\overset{\circ}{X}) \\
\downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx & & \downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx & & \downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx \\
K^0((\mathcal{T}_{nc}X)_{h_0}) & \xleftarrow[\approx]{e_0} & K^0((\Gamma(X)_{\mathcal{F}})_h) & \xrightarrow{e_1} & K^0((\overset{\circ}{X}\times\overset{\circ}{X}))_{h_1}) \\
\downarrow{\scriptstyle \mathscr{M}}\approx & & \downarrow{\scriptstyle \mathscr{M}}\approx & & \downarrow{\scriptstyle \mathscr{M}}\approx \\
K^0(\mathcal{N}_{sing}(X)) & \xleftarrow[\approx]{e_0} & K^0(\mathscr{B}_{\mathscr{F}}) & \xrightarrow{e_1} & K^0(\mathbb{R}^N).
\end{array} \tag{4.16}$$

The left vertical line gives the first part of the topological index map. The bottom line is the morphism induced by the deformation space $\mathscr{B}$. And the right vertical line is precisely the inverse of the Bott isomorphism $\mathbb{Z} = K^0(\{pt\}) \approx K^0(\overset{\circ}{X}\times\overset{\circ}{X}) \to K^0(\mathbb{R}^N)$. Since the top line gives $ind_{\mathcal{F}}$, we obtain the result.

□

Definition 4.3 *[Atiyah-Patodi-Singer topological index morphism for a manifold with boundary] Let X be a manifold with boundary. The topological index morphism of X is the morphism*

$$ind_t^X : K^0(\mathcal{T}_{nc}X) \longrightarrow \mathbb{Z}$$

defined (using an embedding as above) as the composition of the following three morphisms

1. *The Connes-Thom isomorphism $\mathscr{C}\mathscr{T}_0$ followed by the Morita equivalence $\mathscr{M}_0$:*

$$K^0(\mathcal{T}_{nc}X) \xrightarrow{\mathscr{C}\mathscr{T}_0} K^0((\mathcal{T}_{nc}X)_{h_0}) \xrightarrow{\mathscr{M}_0} K^0(\mathcal{N}_{sing}(X))$$

2. *The index morphism of the deformation space $\mathscr{B}$:*

$$K^0(\mathcal{N}_{sing}(X)) \xleftarrow[\approx]{(e_0)_*} K^0(\mathscr{B}_{\mathscr{F}}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)$$

   and

3. *the usual Bott periodicity morphism: $K^0(\mathbb{R}^N) \xrightarrow{Bott} \mathbb{Z}$.*

Because of the theorem it does not depend on the embedding.

Remark 4.4 *The topological index defined above is a natural generalisation of the topological index theorem defined by Atiyah-Singer. Indeed, in the smooth case, they coincide.*

5 The cohomological APS formula

The theorem 4.2 (see also diagram (4.16)) tells us that the computation of the index can be performed (modulo Connes-Thom and Morita) as the computation of the index of a deformation space :

$$K^0(\mathcal{N}_{sing}(X)) \xleftarrow[\approx]{(e_0)_*} K^0(\mathscr{B}_{\mathscr{F}}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)$$

Now, consider the following diagram

$$\begin{array}{ccc}
K^0(\mathbb{R}^N) & \xrightarrow{Ch} & H^*_{dR}(\mathbb{R}^N) \\
\downarrow{\scriptstyle Bott^{-1}} & & \downarrow{\scriptstyle \int_{\mathbb{R}^N}\cdot} \\
\mathbb{Z} & \longrightarrow & \mathbb{C},
\end{array} \tag{5.1}$$

where $\int_{\mathbb{R}^N}$ is the integration with respect to the fundamental class of $\mathbb{R}^N$. It is well known that this diagram is commutative.

We can summarize the previous statements in the following result, which is then an immediate consequence of theorem 4.2 (see again diagram (4.16)):

Corollary 5.1 *Let $(X, \partial X)$ be a manifold with boundary. Given $i : X \hookrightarrow \mathbb{R}^N$ an embedding as in 3.3, we have, using the notations of last sections, that the index morphism $ind_{\mathcal{F}}$ fits in the following commutative diagram*

$$\begin{array}{ccccc} K^0(\mathcal{T}_{nc}(X)) & \xrightarrow{\mathscr{C}\mathscr{T}_{h_0}} & K^0((\mathcal{T}_{nc}(X))_{h_0}) & \xrightarrow{Morita} & K^0(\mathcal{N}_{sing}(X)) \\ & \searrow^{ind_{\mathcal{F}}} & & & \downarrow Ch \\ & & & & H^*(\mathcal{N}_{sing}(X)) \\ & & & & \downarrow \int_{\mathcal{N}_{sing}(X)} \\ & & & & \mathbb{C} \end{array} \tag{5.2}$$

For keeping short notations we will denote by $\mathscr{C}\mathscr{T}$ the composition of $\mathscr{C}\mathscr{T}_h$ followed by the Morita equivalence induced isomorphism $\mathscr{M}$.

In particular, for any fully elliptic operator D on X with "non commutative symbol" $[\sigma_D] \in K^0(\mathcal{T}_{nc}X)$ we have the following cohomological formula for the APS index:

$$Index_{APS}(D) = \int_{\mathcal{N}_{sing}(X)} Ch((\mathscr{C}\mathscr{T}([\sigma_D]))) \tag{5.3}$$

Remember that the space $\mathcal{N}_{sing}(X)$ already splits in two, exhibiting in this way the contributions from the interior and from the boundary. The interior contribution looks classic but an explicit comparison between the Thom isomorphism and the Connes-Thom isomorphism is needed. This will be detailed elsewhere.

In particular, picking up a differential form ω_D on $\mathcal{N}_{sing}(X)$ representing $Ch(\mathscr{C}\mathscr{T}([\sigma_D])$, we obtain:

$$Index_{APS}(D) = \int_{\mathcal{N}(X)} \omega_D + \int_{D_\partial} \omega_D. \tag{5.4}$$

The first integral above involves the restriction of ω_D to $\mathcal{N}(X)$, which is related to the ordinary principal symbol of D. More precisely, the principal symbol $\sigma_{pr}(D)$ of D provides a K-theory class of $C^*(A^*(\Gamma(X)))$, that is a compactly supported K-theory class of the dual of the Lie algebroid of $\Gamma(X)$ or in other words of the b-cotangent bundle ${}^bT^*X$, and by functoriality of both the Chern character and Thom-Connes maps, we have

$$[(\omega_D)|_{\mathcal{N}(X)}] = Ch(\mathscr{C}\mathscr{T}([\sigma_{pr}(D)]).$$

The second integral can thus be viewed as a correction term, which contains the eta invariant appearing in APS formula and which also depends on the choice of the representative $\omega_D \in Ch(\mathscr{C}\mathscr{T}([\sigma_D]))$.

5.1 A more explicit description of (5.3)

Let $\sigma_{\mathrm{ubd},D}$ be an unbounded noncommutative symbol as defined in paragraph 4.1. As explained above, we want to emphasize the computatibility of

$$\mathscr{C}\mathscr{T}([\sigma_{\mathrm{ubd},D}]) \in K^0(\mathcal{N}_{sing}(X)),$$

where $\mathscr{C}\mathscr{T} := \mathscr{M} \circ \mathscr{C}\mathscr{T}_{h_0}$ Now, this class is given by definition as follows (see section 2.3):

$$\mathscr{M} \circ (e_{1,t})_* \circ (e_{0,t})_*^{-1} \circ B(\sigma_{\mathrm{ubd},D}).$$

We assume that $N = 2M$ is even, we identify $\mathbb{R}^N$ with $\mathbb{C}^M$, we denote by $\Lambda^*(\mathbb{C}^M)$ the exterior algebra of $\mathbb{C}^M$, by $c(v) = v \wedge \cdot - \overline{v}\llcorner \cdot$ the Clifford multiplication by v, and we consider the following unbounded representative of the M^{th} power of Bott class $\beta \in KK(\mathbb{C}, C_0(\mathbb{R}^2))$:

$$\beta^M = \big(C_0(\mathbb{R}^N, \Lambda^*(\mathbb{C}^M)), 1, c\big) \in \mathcal{E}_{\text{ubd}}(\mathbb{C}, C_0(\mathbb{R}^N)) \tag{5.5}$$

with the grading given by even/odd forms.

Then $B(\sigma_{\text{ubd},D})$ is represented by

$$\Sigma_{\text{ubd},D} := \big(C^*(\mathcal{T}_{nc}X \times \mathbb{R}^N, r^*E_{\text{ad}} \otimes \Lambda^*(\mathbb{C}^M)), 1, \Sigma_{\text{ubd}}(D)\big) \in \mathcal{E}_{\text{ubd}}((\mathbb{C}, C^*(\mathcal{T}_{nc}X \times \mathbb{R}^N)) \tag{5.6}$$

where we have set $\Sigma_{\text{ubd}}(D) := \sigma_{\text{ubd}}(D)\hat{\otimes}1 + 1\hat{\otimes}c$. Next, we look for a representative of $(e_{1,t})_* \circ (e_{0,t})_*^{-1} \circ \Sigma_{\text{ubd},D}$. For that purpose, we introduce the following $C^{\infty,0}$-diffeomorphism between $\mathcal{T}_{nc}X \times \mathbb{R}^N$ and $\mathcal{T}_{nc}X \ltimes \mathbb{R}^N$:

$$\begin{aligned} \phi : \mathcal{T}_{nc}X \times \mathbb{R}^N &\longrightarrow \mathcal{T}_{nc}X \ltimes \mathbb{R}^N \\ (\gamma, X) &\longmapsto (\gamma, X - h(\gamma)) \end{aligned} \tag{5.7}$$

which commutes with source maps of both groupoids, that is : $s = s \circ \phi$. Using ϕ, we see that any given right Haar system $d\lambda$ on $\mathcal{T}_{nc}X$, we get a right Haar system $d\lambda^h$ on $\mathcal{T}_{nc}X \ltimes \mathbb{R}^N$ by the formula:

$$\int_{(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N)_{(x,X)}} f d\lambda^h_{(x,X)} = \int_{(\mathcal{T}_{nc}X)_x} f \circ \phi d\lambda_x \tag{5.8}$$

This leads to the following formulas for the reduced C^*-norms:

$$\|f\|_{C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N)} = \|f \circ \phi\|_{C^*(\mathcal{T}_{nc}X \times \mathbb{R}^N)} = \sup_{X \in \mathbb{R}^N} \|f \circ \phi(\cdot, X)\|_{C^*(\mathcal{T}_{nc}X)} \tag{5.9}$$

We also observe that a straight computation proves that if a is any $\mathcal{T}_{nc}X \times \mathbb{R}^N$-operator, then $a_h := (\phi^{-1})^* \circ a \circ (\phi)^*$ is a $\mathcal{T}_{nc}X \ltimes \mathbb{R}^N$-operator. Thus, we let:

$$\Sigma_{\text{ubd}\,-h}(D) := (\Sigma_{\text{ubd}}(D))_h \tag{5.10}$$

that is, for all $f \in C_c^{\infty,0}(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N, r^*E_{\text{ad}} \times \Lambda^*(\mathbb{C}^M))$ and $\forall (\gamma, X) \in \mathcal{T}_{nc}X \ltimes \mathbb{R}^N$:

$$\Sigma_{\text{ubd}\,-h}(D)(f)(\gamma, X) := (\sigma_{\text{ubd}}(D)\hat{\otimes}1)(f \circ \phi)(\gamma, X + h(\gamma)) + (1\hat{\otimes}c(X + h(\gamma)))(f)(\gamma, X). \tag{5.11}$$

We prove that:

Proposition 5.2 *The operator $\Sigma_{\text{ubd}\,-h}(D)$ belongs to $\text{Diff}^1(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N, r^*E_{\text{ad}} \times \Lambda^*(\mathbb{C}^M))$, it is symmetric, elliptic, and the following assertion holds:*

$$\Sigma_h = \big(C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N, r^*E_{\text{ad}} \otimes \Lambda^*(\mathbb{C}^M)), 1, \Sigma_{\text{ubd}\,-h}(D)\big) \in \mathcal{E}_{\text{ubd}}((\mathbb{C}, C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N). \tag{5.12}$$

and we have $[\Sigma_h] = (e_{1,t})_* \circ (e_{0,t})_*^{-1} \circ B(\sigma_{\text{ubd},D}) \in KK(\mathbb{C}, C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N))$.

Proof : Let us consider $p := \Sigma_{\text{ubd}}(D)^2 + 1 = \sigma_{\text{ubd}}(D)^2 + |X|^2 + 1 \in \text{Diff}^2(\mathcal{T}_{nc}X \times \mathbb{R}^N, r^*E_{\text{ad}} \times \Lambda^*(\mathbb{C}^M))$, where we have taken into account the identity $c^2(f)(\gamma, X) = |X|^2.f(\gamma, X)$. It is self-adjoint and for any (x, X) belonging to the unit space $(\mathcal{T}_{nc}X \times \mathbb{R}^N)^{(0)} = X_\partial \times \mathbb{R}^N$, the operator $\Sigma_{\text{ubd}}(D)^2_{(x,X)} + 1$ is invertible by classical arguments. Thus, p itself is invertible, which means that p^{-1} exists and belongs to $\Psi^{-2}(\mathcal{T}_{nc}X \times \mathbb{R}^N, r^*E_{\text{ad}} \times \Lambda^*(\mathbb{C}^M))$. As a direct consequence, $\Sigma_{\text{ubd}}(D)$ is regular. Moreover, let us consider the groupoid $\overline{\mathcal{T}_{nc}X} \times B^N \rightrightarrows \overline{X_\partial} \times B^N$ where B^N is the compactification of $\mathbb{R}^N$ by a

sphere at infinity and we recall that $\overline{\mathcal{T}_{nc}X} = \mathcal{T}_{nc}X \cup \partial X \times \partial X \times \mathbb{R} \times \{\varepsilon = 1\} \rightrightarrows X \times [0,1]_\varepsilon$. The groupoid structure of $\overline{\mathcal{T}_{nc}X} \times B^N$ is the obvious one and it inherits a natural $C^{\infty,0}$-structure in such a way that it contains $\mathcal{T}_{nc}X \times \mathbb{R}^N$ as an open saturated $C^{\infty,0}$-subgroupoid. We also extend the vector bundles r^*E_{ad} and $\Lambda^*(\mathbb{C}^M)$ onto $\overline{X_\partial}$ and B^N without changing the notations. Thanks to the decay of p^{-1} when $\varepsilon \to 1$ or $|X| \to \infty$, we see that the extension of p^{-1} by 0 gives an element of $\Psi^{-2}(\overline{\mathcal{T}_{nc}X} \times B^N, r^*E_{\mathrm{ad}} \times \Lambda^*(\mathbb{C}^M))$. Since $\overline{\mathcal{T}_{nc}X} \times B^N$ has a compact unit space, we know by [24, 32] that p^{-1} is a compact morphism of the $C^*(\overline{\mathcal{T}_{nc}X} \times B^N)$-Hilbert module $\overline{\mathcal{E}} := C^*(\overline{\mathcal{T}_{nc}X} \times B^N, r^*E_{\mathrm{ad}} \times \Lambda^*(\mathbb{C}^M))$. Considering the closed saturated $C^{\infty,0}$-subgroupoid $\partial(\overline{\mathcal{T}_{nc}X} \times B^N)$ given by the equation $z := (1-\varepsilon)(1/(|X|+1)) = 0$ and the corresponding Hilbert submodule $\partial\overline{\mathcal{E}}$, we get an exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{E}) \longrightarrow \mathcal{K}(\overline{\mathcal{E}}) \stackrel{\mathrm{rest}|_{z=0}}{\longrightarrow} \mathcal{K}(\partial\overline{\mathcal{E}}) \longrightarrow 0. \tag{5.13}$$

Since $\mathrm{rest}|_{z=0}(p^{-1}) = 0$, we conclude that p^{-1} belongs to $\mathcal{K}(\mathcal{E})$.

Replacing h by $H = (th)_{t\in[0,1]}$ and extending the previous construction to the groupoid $\mathcal{T}_{nc}X \rtimes_H \mathbb{R}^N \rightrightarrows X_\partial \times \mathbb{R}^N \times [0,1]$, we get and unbounded Kasparov class $\Sigma_H := (C^*(\mathcal{T}_{nc}X \rtimes_H \mathbb{R}^N, r^*E_{\mathrm{ad}} \otimes \Lambda^*(\mathbb{C}^M)), 1, \Sigma_{\mathrm{ubd}\,-H}(D)) \in \Psi^1(\mathbb{C}, C^*(\mathcal{T}_{nc}X \rtimes_H \mathbb{R}^N)$ such that $(e_{0,t})_*([\Sigma_H]) = B(\sigma_{\mathrm{ubd},D})$ and $(e_{1,t})_*([\Sigma_H]) = [\Sigma_h]$.

□

Following [19, 30], we now implement the Morita isomorphism $\cdot \otimes \mathscr{M} : K_0(C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N)) \to K_0(C_0(\mathscr{N}_{sing}(X)))$ by the Kasparov module:

$$\mathscr{M} = (\mathcal{E}_{\mathscr{M}}, \pi, 0) \in KK(C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N), C_0(\mathscr{N}_{sing}(X))). \tag{5.14}$$

Let us recall its content. We note G instead of $\mathcal{T}_{nc}X \ltimes \mathbb{R}^N$, O instead of $G^{(0)}/G$ and λ the Haar system previously introduced to lighten the reading. The space $C_0(G^{(0)})$ inherits the following $C_0(O)$-pre hilbertian module structure:

$$< \xi, \eta > (o) \underset{\forall z \ s.t \ [z]=o}{=} \int_{G_z} \overline{\xi(r(\gamma))}\eta(r(\gamma))d\lambda_z(\gamma), \tag{5.15}$$

and the right $C_0(O)$-module structure of $C_0(G^{(0)})$ is given by

$$\xi.a(z) = \xi(z)a([z]). \tag{5.16}$$

Then we take $\mathcal{E}_{\mathscr{M}}$ as the Hilbert module completion of this pre-hilbertian module. Finally, the following formula:

$$\pi(b).\xi(z) = \int_{G_z} b(\gamma^{-1})\xi(r(\gamma))d\lambda_z(\gamma). \tag{5.17}$$

defines the appropriate $*$-homomorphism $\pi : C^*(G) \to \mathcal{K}(\mathcal{E}_{\mathscr{M}})$.

The Kasparov product $\Sigma_h \underset{C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N)}{\otimes} \mathscr{M}$ is given by:

$$\left(C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N, r^*E_{\mathrm{ad}} \otimes \Lambda^*(\mathbb{C}^M)) \underset{\pi}{\otimes} \mathcal{E}_{\mathscr{M}}, 1, \Sigma_{\mathrm{ubd}\,-h}(D)\widehat{\otimes}1\right) \in KK(\mathbb{C}, C_0(\mathscr{N}_{sing}(X))) \tag{5.18}$$

We use again the shorcut notations G, O for $\mathcal{T}_{nc}X \ltimes \mathbb{R}^N$, $\mathscr{N}_{sing}(X)$ and we moreover set $E = E_{\mathrm{ad}} \otimes \Lambda^*(\mathbb{C}^M)$. We extend to $C_c(G^{(0)}, E)$ the previous $C_0(O)$-prehilbertian module structure:

$$< \xi, \eta > ([z]) = \int_{G_z} < \xi(r(\gamma)), \eta(r(\gamma)) >_{E_{r(\gamma)}} d\lambda_z(\gamma). \tag{5.19}$$

and we denote by $\mathcal{E}_E$ its Hilbert module completion. We also observe that (5.17) makes sense for $b \in C_c(G, r^*E)$ since in the integral $b(\gamma^{-1}) \in E_{r(\gamma^{-1})} = E_z$, yielding a map again denoted by π:

$$\pi : C_c(G, r^*E) \longrightarrow \mathcal{L}(C_c(G^{(0)}), C_c(G^{(0)}, E))$$

We then introduce:

$$U : C_c(G, r^*E) \otimes C_c(G^{(0)}) \longrightarrow C_c(G^{(0)}, E) \tag{5.20}$$

$$a \otimes f \longrightarrow \pi(a)f \tag{5.21}$$

Proposition 5.3 *The map U defined in (5.20) extends to a unitary of $C_0(O)$-Hilbert module:*

$$U : C^*(G, r^*E) \underset{\pi}{\otimes} \mathcal{E}_{\mathscr{M}} \longrightarrow \mathcal{E}_E, \tag{5.22}$$

providing a unitary equivalence between $\Sigma_h \underset{C^*(\mathcal{T}_{nc}X \ltimes \mathbb{R}^N)}{\otimes} \mathscr{M}$ and:

$$(\mathcal{E}_E, 1, r_*(\Sigma_{\mathrm{ubd}\,-h}(D))) \tag{5.23}$$

where r denotes the target map of the groupoid $\mathcal{T}_{nc}X \ltimes \mathbb{R}^N$ and r_* is the map defined by $r_*(P)(f)(z) = P(f \circ r)(\gamma)$ for any γ such that $(r(\gamma) = z$ and and any $\mathcal{T}_{nc}X \ltimes \mathbb{R}^N$-operator P.

Proof : If $E = X \times \mathbb{C}$ is the product rank one bundle, then the fact that U is isometric directly follows from the fact that $\pi : C^*(G) \to \mathcal{L}(\mathcal{E}_{\mathscr{M}})$ is a $*$-homomorphism:

$$\begin{aligned} < U(a \otimes f) | U(b \otimes g) >_{\mathcal{E}_{\mathscr{M}}} &= < \pi(a)f | \pi(b)g >_{\mathcal{E}_{\mathscr{M}}} = < f | \pi(a)^*\pi(b)g >_{\mathcal{E}_{\mathscr{M}}} \\ &= < f | \pi(a^*b)g >_{\mathcal{E}_{\mathscr{M}}} \\ &= < f | \pi(< a | b >_{C^*(G)})g >_{\mathcal{E}_{\mathscr{M}}} =: < a \otimes f | b \otimes g >_{C^*(G) \underset{\pi}{\otimes} \mathcal{E}_{\mathscr{M}}} \end{aligned}$$

Since for any $b \in C_c(G)$, $\xi \in C_c(G^{(0)})$, we have $\pi(b)(\xi) = (b.r^*(\xi))|_{G^{(0)}}$ where . denotes the convolution product of functions on G (well defined thanks to the compactness of the support of b), we get, taking an approximate unit for $C^*(G)$, the density of the image of U and thus its surjectivity.

In the general case, we imbed $E \hookrightarrow X \times \mathbb{C}^k$ into a trivial bundle in such a way that we get an imbedding $C^*(G, r^*E) \hookrightarrow C^*(G)^k$ of $C^*(G)$-Hilbert module. Then we use the previous arguments coordinatewise.

Now, a straight computation shows that

$$U \circ (\Sigma_{\mathrm{ubd}\,-h}(D) \widehat{\otimes} 1) = r_*(\Sigma_{\mathrm{ubd}\,-h}(D)) \circ U$$

□

Remark 5.4

1. *Proposition 5.3 gives a computable representative of the integrand in the right hand side of (5.3). Indeed, this integrand is the Chern character of a K-theory class of the locally compact space $\mathscr{N}_{sing}(X)$, namely $\mathscr{M}(CT_h([\sigma_D])$, which is now represented by an explicit and rather simple differential operator on $X_\partial \times \mathbb{R}^N$ and this operator is elliptic along the fibers of the projection map : $p : X_\partial \times \mathbb{R}^N \to \mathscr{N}_{sing}(X)$. In other words, $\mathscr{M}(CT_h([\sigma_D])$ is represented by a family of elliptic order 1 differential operators parametrized by the orbit space $\mathscr{N}_{sing}(X)$ and computing its Chern character is a classical issue, achieved for instance using the technics of [7]. This will be made more precise in a forthcoming paper where we will address the case of manifolds with corners.*

2. *The computations above can be reproduced identically for 0-order pseudodifferential operators and bounded K-classes.*

A Deformation to the normal cone functor and tangent groupoids

The tangent groupoid is a particular case of a geometric construction that we describe here.

Let M be a C^∞ manifold and $X \subset M$ be a C^∞ submanifold. We denote by $\mathscr{N}_X^M$ the normal bundle to X in M. We define the following set

$$\mathscr{D}_X^M := \mathscr{N}_X^M \times 0 \bigsqcup M \times \mathbb{R}^* \tag{A.1}$$

The purpose of this section is to recall how to define a C^∞-structure in $\mathscr{D}_X^M$. This is more or less classical, for example it was extensively used in [15].

Let us first consider the case where $M = \mathbb{R}^p \times \mathbb{R}^q$ and $X = \mathbb{R}^p \times \{0\}$ (here we identify X canonically with $\mathbb{R}^p$). We denote by $q = n - p$ and by $\mathscr{D}_p^n$ for $\mathscr{D}_{\mathbb{R}^p}^{\mathbb{R}^n}$ as above. In this case we have that $\mathscr{D}_p^n = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}$ (as a set). Consider the bijection $\psi : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \to \mathscr{D}_p^n$ given by

$$\psi(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0 \\ (x, t\xi, t) & \text{if } t \neq 0 \end{cases} \tag{A.2}$$

whose inverse is given explicitly by

$$\psi^{-1}(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0 \\ (x, \frac{1}{t}\xi, t) & \text{if } t \neq 0 \end{cases}$$

We can consider the C^∞-structure on $\mathscr{D}_p^n$ induced by this bijection.

We pass now to the general case. A local chart $(\mathcal{U}, \phi)$ in M is said to be a X-slice if

1) $\phi : \mathcal{U} \to U \subset \mathbb{R}^p \times \mathbb{R}^q$ is a diffeomorphsim.

2) If $V = U \cap (\mathbb{R}^p \times \{0\})$, then $\phi^{-1}(V) = \mathcal{U} \cap X$, denoted by $\mathcal{V}$.

With this notation, $\mathscr{D}_V^U \subset \mathscr{D}_p^n$ as an open subset. We may define a function

$$\tilde{\phi} : \mathscr{D}_{\mathcal{V}}^{\mathcal{U}} \to \mathscr{D}_V^U \tag{A.3}$$

in the following way: For $x \in \mathcal{V}$ we have $\phi(x) \in \mathbb{R}^p \times \{0\}$. If we write $\phi(x) = (\phi_1(x), 0)$, then

$$\phi_1 : \mathcal{V} \to V \subset \mathbb{R}^p$$

is a diffeomorphism. We set $\tilde{\phi}(v, \xi, 0) = (\phi_1(v), d_N\phi_v(\xi), 0)$ and $\tilde{\phi}(u, t) = (\phi(u), t)$ for $t \neq 0$. Here $d_N\phi_v : N_v \to \mathbb{R}^q$ is the normal component of the derivative $d\phi_v$ for $v \in \mathcal{V}$. It is clear that $\tilde{\phi}$ is also a bijection (in particular it induces a C^∞ structure on $\mathscr{D}_{\mathcal{V}}^{\mathcal{U}}$).

Let us define, with the same notations as above, the following set

$$\Omega_V^U = \{(x, \xi, t) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} : (x, t \cdot \xi) \in U\}. \tag{A.4}$$

which is an open subset of $\mathbb{R}^p \times \mathbb{R}^q \times [0, 1]$ and thus a C_c^∞ manifold (with border). It is immediate that $\mathscr{D}_V^U$ is diffeomorphic to Ω_V^U through the restriction of Ψ, used in (A.2). Now we consider an atlas $\{(\mathcal{U}_\alpha, \phi_\alpha)\}_{\alpha \in \Delta}$ of M consisting of $X-$slices. It is clear that

$$\mathscr{D}_X^M = \cup_{\alpha \in \Delta} \mathscr{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha} \tag{A.5}$$

and if we take $\mathscr{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha} \overset{\varphi_\alpha}{\to} \Omega_{V_\alpha}^{U_\alpha}$ defined as the composition

$$\mathscr{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha} \overset{\tilde{\phi_\alpha}}{\to} \mathscr{D}_{V_\alpha}^{U_\alpha} \overset{\Psi_\alpha^{-1}}{\to} \Omega_{V_\alpha}^{U_\alpha}$$

then we have (proposition 3.1 in [8]).

Proposition A.1 *$\{(\mathscr{D}^{\mathcal{U}_\alpha}_{\mathcal{V}_\alpha}, \varphi_\alpha)\}_{\alpha\in\Delta}$ is a C^∞ atlas over $\mathscr{D}^M_X$.*

Definition A.2 (Deformation to the normal cone) *Let $X \subset M$ be as above. The set $\mathscr{D}^M_X$ equipped with the C^∞ structure induced by the atlas of X-slices is called the deformation to the normal cone associated to the embedding $X \subset M$.*

One important feature about the deformation to the normal cone is the functoriality. More explicitly, let $f : (M, X) \to (M', X')$ be a C^∞ map $f : M \to M'$ with $f(X) \subset X'$. Define $\mathscr{D}(f) : \mathscr{D}^M_X \to \mathscr{D}^{M'}_{X'}$ by the following formulas:

1) $\mathscr{D}(f)(m, t) = (f(m), t)$ for $t \neq 0$,

2) $\mathscr{D}(f)(x, \xi, 0) = (f(x), d_N f_x(\xi), 0)$, where $d_N f_x$ is by definition the map

$$(\mathscr{N}^M_X)_x \xrightarrow{d_N f_x} (\mathscr{N}^{M'}_{X'})_{f(x)}$$

induced by $T_xM \xrightarrow{df_x} T_{f(x)}M'$.

Then we have, (proposition 3.4 in [8]),

Proposition A.3 *$\mathscr{D}(f) : \mathscr{D}^M_X \to \mathscr{D}^{M'}_{X'}$ is a C^∞-map. In the language of categories, the deformation to the normal cone construction defines a functor*

$$\mathscr{D} : \mathscr{C}^\infty_2 \longrightarrow \mathscr{C}^\infty, \tag{A.6}$$

where $\mathscr{C}^\infty$ is the category of C^∞-manifolds and $\mathscr{C}^\infty_2$ is the category of pairs of C^∞-manifolds.

Definition A.4 (Tangent groupoid of a continuous families groupoid) *Let $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ be a continuous families groupoid. The tangent groupoid associated to $\mathscr{G}$ is the groupoid that has*

$$\mathscr{D}^{\mathscr{G}}_{\mathscr{G}^{(0)}} = \mathscr{N}^{\mathscr{G}}_{\mathscr{G}^{(0)}} \times \{0\} \bigsqcup \mathscr{G} \times \mathbb{R}^*$$

as the set of arrows and $\mathscr{G}^{(0)} \times \mathbb{R}$ as the units, with:

1. *$s^T(x, \eta, 0) = (x, 0)$ and $r^T(x, \eta, 0) = (x, 0)$ at $t = 0$.*
2. *$s^T(\gamma, t) = (s(\gamma), t)$ and $r^T(\gamma, t) = (r(\gamma), t)$ at $t \neq 0$.*
3. *The product is given by $m^T((x, \eta, 0), (x, \xi, 0)) = (x, \eta + \xi, 0)$ and $m^T((\gamma, t), (\beta, t)) = (m(\gamma, \beta), t)$ if $t \neq 0$ and if $r(\beta) = s(\gamma)$.*
4. *The unit map $u^T : \mathscr{G}^{(0)} \to \mathscr{G}^T$ is given by $u^T(x, 0) = (x, 0)$ and $u^T(x, t) = (u(x), t)$ for $t \neq 0$.*

We denote $\mathscr{G}^T = \mathscr{D}^{\mathscr{G}}_{\mathscr{G}^{(0)}}$ and $A\mathscr{G} = \mathscr{N}^{\mathscr{G}}_{\mathscr{G}^{(0)}}$ as a vector bundle over $\mathscr{G}^{(0)}$. Then we have a family of continuous families groupoids parametrized by $\mathbb{R}$, which itself is a continuous families groupoid

$$\mathscr{G}^T = A\mathscr{G} \times \{0\} \bigsqcup \mathscr{G} \times \mathbb{R}^* \rightrightarrows \mathscr{G}^{(0)} \times \mathbb{R}.$$

As a consequence of the functoriality of the deformation to the normal cone, one can show that the tangent groupoid is in fact a continuous families groupoid compatible with the continuous families groupoid structures of $\mathscr{G}$ and $A\mathscr{G}$ (considered as a continuous families groupoid with its vector bundle structure).

Proposition A.5 *Let $\mathscr{G}$ be a continuous families groupoid together with an injective continuous families groupoid morphism $\mathscr{G} \xrightarrow{h} \mathbb{R}^N$. Consider also the induced infinitesimal continuous families groupoid morphism $A(\mathscr{G}) \xrightarrow{A(h)} \mathbb{R}^N$. Assume that $A(h)$ is also injective and that both crossed product groupoids, $\mathscr{G}_h$ and $A(\mathscr{G})_{A(h)}$ are free and proper. Then the induced morphism $\mathscr{G}^{tan} \xrightarrow{h^T} \mathbb{R}^N$ gives a free proper crossed groupoid as well.*

Proof : We will use again properness caracterization (ii) of proposition 2.14 in [30]. In particular we have to verify only property (A) (of section 3.2 above), that is, in our case we have to check that the map

$$\mathscr{G}^{tan} \times \mathbb{R}^N \xrightarrow{(t,s)} (\mathscr{G}^{(0)} \times [0,1] \times \mathbb{R}^N) \times (\mathscr{G}^{(0)} \times [0,1] \times \mathbb{R}^N) \tag{A.7}$$

given by

$$\begin{cases} ((\gamma,\epsilon),X) \mapsto ((t(\gamma),\epsilon,X),(s(\gamma),\epsilon,X+\frac{h(\gamma)}{\epsilon})) \\ ((x,\xi),X) \mapsto ((x,0,X),(x,0,X+A(h)(x,\xi))) \end{cases} \tag{A.8}$$

is closed.

Let $(A_n)_n := (\tilde{\gamma_n}, X_n)_n$ a sequence in $\mathscr{G}^{tan} \times \mathbb{R}^N$ such that

$$lim_{n\to\infty}(t,s)(\tilde{\gamma_n}, X_n) = P \tag{A.9}$$

with P a point in $(\mathscr{G}^{(0)} \times [0,1] \times \mathbb{R}^N) \times (\mathscr{G}^{(0)} \times [0,1] \times \mathbb{R}^N)$. It is enough to justify that there is a subsequence of $(A_n)_n$ converging to an antecedent of P: The point P is of the form $((x,\epsilon_1,X),(y,\epsilon_2,Y))$. The first consequence of (A.9) is that $\epsilon_1 = \epsilon_2$, hence $P = ((x,\epsilon,X),(y,\epsilon,Y))$

We will separate the analysis in two cases:

- (a) The case $\epsilon \neq 0$: By the explicit form of (A.8), we can assume (or there is a subsequence) that $(A_n)_n \subset \mathscr{G} \times (0,1] \times \mathbb{R}^N$, i.e., that the elements os the sequence are of the form $A_n = (\gamma_n, \epsilon_n, X_n)$ with $\epsilon \neq 0$. But then we have the following convergences: $t(\gamma_n) \to x$, $\epsilon_n \to \epsilon$, $X_n \to X$, $s(\gamma_n) \to y$ and $X_n + \frac{h(\gamma_n)}{\epsilon_n}$.

  In particular we obtain that $h(\gamma_n) \to \epsilon \cdot (Y - X)$, and since $\mathscr{G}_h$ is proper we have that there is a subsequence of $(\gamma_{n_k})_k$ of $(\gamma_n)_n$ and a $\gamma \in \mathscr{G}$ such that $\gamma_{n_k} \to \gamma$ with $t(\gamma) = x$, $s(\gamma) = y$ and $h(\gamma) = \epsilon \cdot (Y - X)$. In particular, letting $A = (\gamma, \epsilon, X)$ we have that $A_{n_k} \to A$ and $(t,s)(A) = P$.

- (b) The case $\epsilon = 0$: In this case we have two subcases:

  - (b_1) There is a subsequence of $(A_n)_n$ entirely contained in $A(\mathscr{G}) \times \mathbb{R}^N$. In this case we might assume that $A_n = ((x_n,\xi_n),X_n) \in A_{x_n}(\mathscr{G}) \times \mathbb{R}^N$. Then, (A.9) implies that $x_n \to x = y$, $X_n \to X$ and $X_n + A(h)(x_n,\xi_n) \to Y$. In particular, $A(h)(x_n,\xi_n) \to Y - X$ and since $A(\mathscr{G})_{A(h)}$ is proper we have that there is a subsequence (x_{n_k},ξ_{n_k}) converging in $A(\mathscr{G})$ to an element $(x,\xi) \in A_x(\mathscr{G})$. Then letting $A = ((x,\xi),X)$ we have that $A_{n_k} \to A$ and $(t,s)(A) = P$.
  - (b_2) There is a subsequence of $(A_n)_n$ entirely contained in $\mathscr{G} \times (0,1] \times \mathbb{R}^N$. In this case we might assume that $A_n = (\gamma_n,\epsilon,X_n) \in \mathscr{G} \times (0,1] \times \mathbb{R}^N$. Then, (A.9) implies that $t(\gamma_n) \to x$, $\epsilon_n \to 0$, $X_n \to X$, $s(\gamma_n) \to y$ and $X_n + \frac{h(\gamma_n)}{\epsilon_n} \to Y$. This implies $\frac{h(\gamma_n)}{\epsilon_n} \to Y - X$ in $\mathbb{R}^N$ or in other words $(h(\gamma_n),\epsilon_n) \to (Y - X, 0)$ in $(\mathbb{R}^N)^{tan}$. In particular we have also that $h(\gamma_n) \to 0$ in $\mathbb{R}^N$ and since $\mathscr{G}_h$ is proper and h is injective, we deduce that $x = y$ and that there is a subsequence $(\gamma_{n_k})_k$ of $(\gamma_n)_n$ such that $\gamma_{n_k} \to x$. Now, from the injectivity of $A(h)$ and the fact that $\frac{h(\gamma_n)}{\epsilon_n} \to Y - X$ we deduce that there is an unique $\xi \in A_x(\mathscr{G})$ such that $A(h)(x,\xi) = Y - X$. Finally, letting $A = ((x,\xi),X)$ we have that $A_{n_k} \to A$ and $(t,s)(A) = P$.

□

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