Package CAPUSHE pour le logiciel R

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Introduction

Cadre : Sélection de modèle par minimisation d'un critère pénalisé.

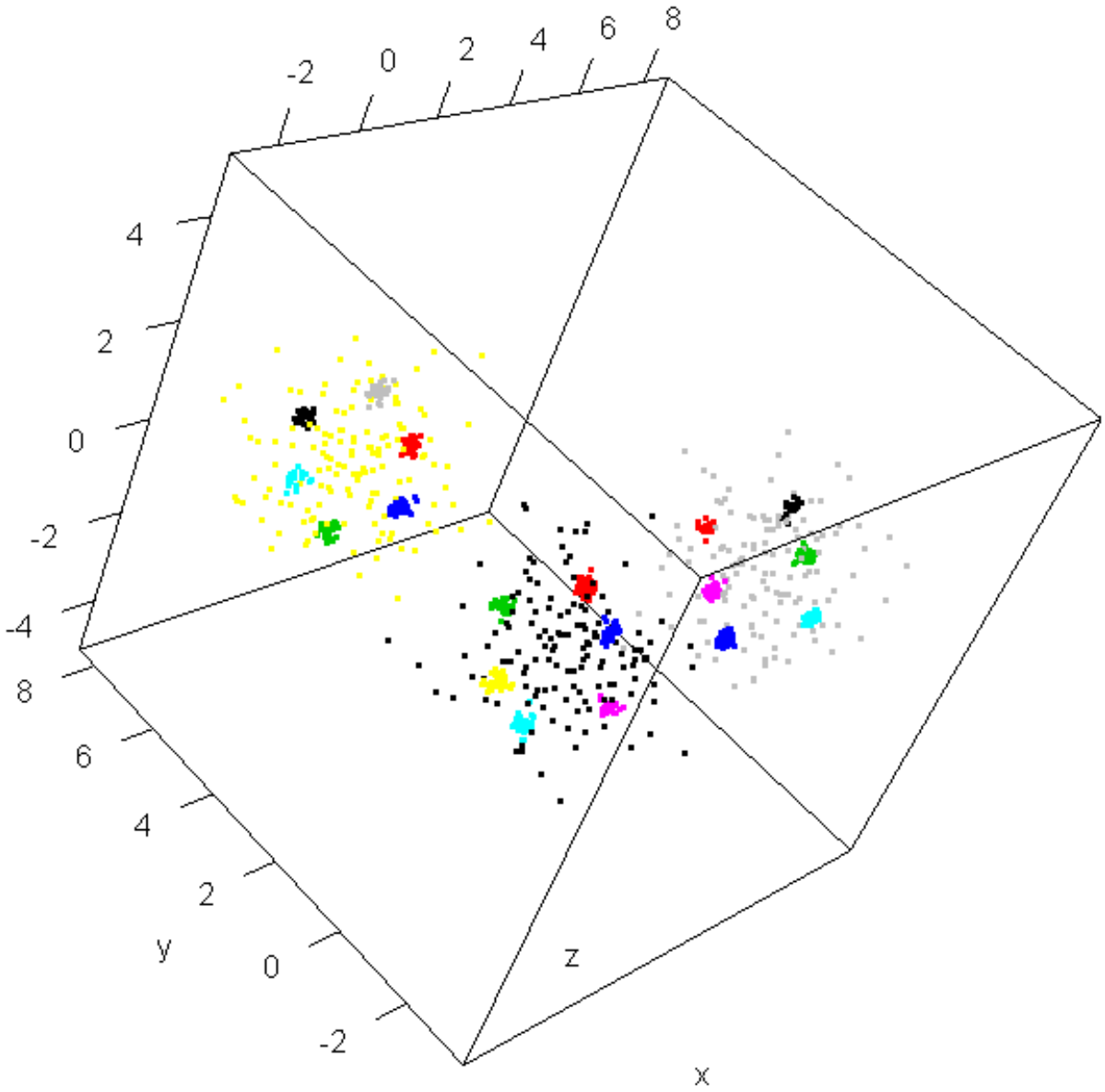
Hypothèse : La pénalité est connue à une constante près.

Objectif : Capushe permet de calibrer cette dernière.

Plan

1. Théorie
2. Heuristique de pente
3. DDSE
4. Djump
5. Capushe

Introduction



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n-échantillon $X_1, \ldots, X_n$ de $\mathcal{X}$

Paramètre d'intérêt $s \in \mathscr{S}$

Fonction de contraste $\gamma : \mathscr{S} \times \mathcal{X} \to \mathbb{R}$ telle que

$$s \in \operatorname*{argmin}_{t \in \mathscr{S}} \mathbb{E}_X \left[\gamma(t, X)\right]$$

<u>Exemple</u> :

$n = 1000$ et $\mathcal{X} = \mathbb{R}^3$

s la densité de l'échantillon

$\gamma(s, X) = -\log s(X)$

Contraste empirique $\forall t \in \mathcal{S}, \gamma_n(t) = \frac{1}{n} \sum_{i=1}^{n} \gamma(t, X_i)$

Exemple :

$$\begin{aligned} \gamma_n(s_m) &= -\log\left(\prod_{i=1}^{n} s(X_i;\theta)\right) \\ &= -\sum_{i=1}^{n} \log s(X_i;\theta) \end{aligned}$$

Famille de modèles $(\mathcal{S}_m)_{m\in\mathcal{M}}$ de complexité respective C_m.
Estimateur $\widehat{s}_m$ de s dans $\mathcal{S}_m$:

$$\widehat{s}_m \in \underset{t\in\mathcal{S}_m}{\operatorname{argmin}}\ \gamma_n(t).$$

Exemple :

$\mathcal{S}_m$ ensemble des mélanges sphériques avec m composantes.

$$X \sim \sum_{i=1}^{m} p_i \mathcal{N}\left(\mu_i, \sigma_i^2 I_3\right)$$

$\Rightarrow C_m = 4m + (m-1)$ nombre de paramètres libres de tout mélange de $\mathcal{S}_m$

$$\gamma_n(s_m) = -\frac{1}{n}\sum_{i=1}^{n} \log\left[\sum_{i=1}^{m} p_i (2\pi)^{-3/2}\sigma_i^{-3} \exp\left(-\frac{1}{2\sigma_i^2}||X_i-\mu_i||_2^2\right)\right]$$

Modèle oracle :

$$m^\star \in \underset{m \in \mathcal{M}}{\operatorname{argmin}}\, \mathbb{E}_X \left[\ell \left(s, \widehat{s}_m \right) \right]$$

où $\ell\left(s,t\right) = \mathbb{E}_X\left[\gamma(t,X)\right] - \mathbb{E}_X\left[\gamma(s,X)\right]$

ℓ est la divergence de Kullback-Leibler

Modèle oracle :

$$m^\star \in \underset{m \in \mathcal{M}}{\operatorname{argmin}}\, \mathbb{E}_X \left[\ell\left(s, \widehat{s}_m\right)\right]$$

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ℓ est la divergence de Kullback-Leibler

$$\mathrm{crit}(m) = \gamma_n(\widehat{s}_m)$$

Modèle oracle :

$$m^\star \in \operatorname*{argmin}_{m\in\mathcal{M}} \mathbb{E}_X \left[\ell\left(s, \widehat{s}_m\right)\right]$$

où $\ell\left(s,t\right) = \mathbb{E}_X\left[\gamma(t,X)\right] - \mathbb{E}_X\left[\gamma(s,X)\right]$

ℓ est la divergence de Kullback-Leibler

Fonction de pénalité pen : $\mathcal{M} \to \mathbb{R}^+$ telle que le modèle $\widehat{m}$ minimisant le critère pénalisé associé

$$\text{crit}(m) = \gamma_n(\widehat{s}_m) + \text{pen}(m)$$

ait un risque $\mathbb{E}_X\left[\ell(s, \widehat{s}_{\widehat{m}})\right]$ proche de celui de l'oracle.

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Pénalité optimale

$$\underset{m\in\mathcal{M}}{\operatorname{argmin}}\ \mathrm{crit}_{\mathrm{opt}}(m) = \underset{m\in\mathcal{M}}{\operatorname{argmin}}\ \ell(s,\hat{s}_m)$$

La pénalité idéale serait :

$$\begin{aligned}\mathrm{crit}_{\mathrm{opt}}(m) = \ell(s,\hat{s}_m) \quad &\Leftrightarrow \quad \gamma_n(\hat{s}_m) + \mathrm{pen}_{\mathrm{opt}}(m) = \ell(s,\hat{s}_m)\\ &\Leftrightarrow \quad \mathrm{pen}_{\mathrm{opt}}(m) = \ell(s,\hat{s}_m) - \gamma_n(\hat{s}_m)\end{aligned}$$

Pénalité optimale

$$
\begin{aligned}
\mathrm{pen}_{\mathrm{opt}}(m) &= \ell(s, \hat{s}_m) - \gamma_n(\hat{s}_m) \\
&= \underbrace{\left(\mathbb{E}_X\left[\gamma(\hat{s}_m, X)\right] - \mathbb{E}_X\left[\gamma(s_m, X)\right]\right)}_{=v_m} \\
&\quad + \left(\mathbb{E}_X\left[\gamma(s_m, X)\right] - \mathbb{E}_X\left[\gamma(s, X)\right]\right) \\
&\quad + \underbrace{\left(\gamma_n(s_m) - \gamma\left(\hat{s}_m\right)\right)}_{=\widehat{v}_m} \\
&\quad - \left(\gamma_n(s_m) - \gamma_n(s)\right) \\
&\quad - \gamma_n(s) \\
&= v_m + \widehat{v}_m + \Delta_n(s_m) - \gamma_n(s)
\end{aligned}
$$

Heuristique de pente [SH1]

$$\text{pen}_\kappa = \kappa \widehat{v}_m$$

$$\begin{aligned}
\text{crit}_\kappa(m) &= \gamma_n(\widehat{s}_m) + \kappa \widehat{v}_m \\
&= \gamma_n(\widehat{s}_m) + \kappa \left(\gamma_n(s_m) - \gamma\left(\widehat{s}_m\right)\right) \\
&= (1-\kappa)\gamma_n(\widehat{s}_m) + \kappa \gamma_n(s_m)
\end{aligned}$$

Birgé et Massart (2006) :

$$\text{pen}_{\min}(m) \approx \widehat{v}_m$$

Heuristique de pente [SH2]

$$v_m \approx \widehat{v}_m$$

$$\Delta_n(s_m) \approx 0$$

Nous obtenons :

$$\begin{aligned} \mathrm{pen}_{\mathrm{opt}}(m) &= v_m + \widehat{v}_m + \Delta_n(s_m) \\ &\approx \widehat{v}_m + \widehat{v}_m \\ &\approx 2 \times \mathrm{pen}_{\mathrm{min}}(m) \end{aligned}$$

Forme de la pénalité $\text{pen}_{\text{shape}}$

La pénalité à calibrer est de la forme

$$\begin{aligned} \text{pen}_{\text{shape}}(m) &= \kappa \hat{v}_m \\ \text{pen}_{\text{min}}(m) &= \kappa_{\text{min}}\, \text{pen}_{\text{shape}}(m) \end{aligned}$$

Pour les mélanges gaussiens :

$$\text{pen}_{\text{shape}}(m) = C_m$$

Données

```
> datacapushe
   model   pen complexity contrast
1    K=1 0.004          4  7.02975
2    K=2 0.009          9  6.16194
3    K=3 0.014         14  5.22071
4    K=4 0.019         19  5.17469
5    K=5 0.024         24  5.13399
6    K=6 0.029         29  5.08707
7    K=7 0.034         34  5.05738
8    K=8 0.039         39  5.02772
9    K=9 0.044         44  4.99665
10  K=10 0.049         49  4.95389
```

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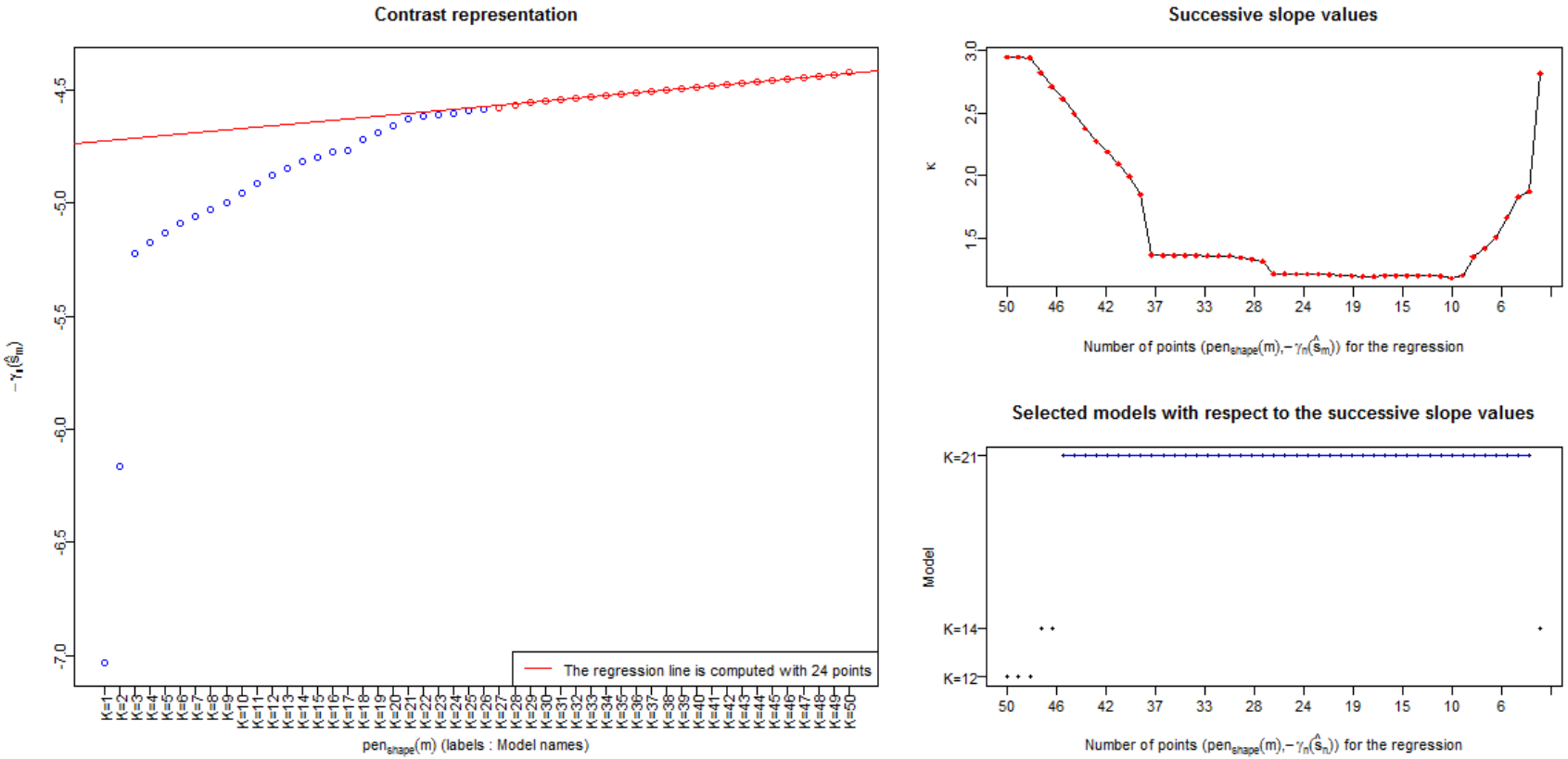
DDSE

$$
\begin{aligned}
\mathrm{pen}_{\min}(m) &\approx \hat{v}_m \\
&\approx \gamma_n(s_m) - \gamma_n(\hat{s}_m) \\
&\approx \gamma_n(s_m) - \gamma_n(s) + \gamma_n(s) - \gamma_n(\hat{s}_m) \\
&\approx \mathbb{E}_X\left[\ell(s, s_m)\right] + \gamma_n(s) - \gamma_n(\hat{s}_m)
\end{aligned}
$$

Plot

$$\gamma_n(\widehat{s}_m) - \gamma_n(s) \approx \mathbb{E}_X\left[\ell(s, s_m)\right] - \kappa_{\min}\mathrm{pen}_{\mathrm{shape}}(m)$$



Commande

```
> ResDDSE=DDSE(datacapushe,pct=0.15,point=0,psi.rlm=psi.bisquare,scoef=2)
```

Commande

```
> ResDDSE=DDSE(datacapushe,pct=0.15,point=0,psi.rlm=psi.bisquare,scoef=2)
> ResDDSE
[1] "K=21"
```

Commande

```
> ResDDSE=DDSE(datacapushe,pct=0.15,point=0,psi.rlm=psi.bisquare,scoef=2)
> ResDDSE
[1] "K=21"
> summary(ResDDSE)

Selected model: K=21

Corresponding Slope interval: [1.18;2.62](88% of points)

Selected model complexity: 104
```

Validation

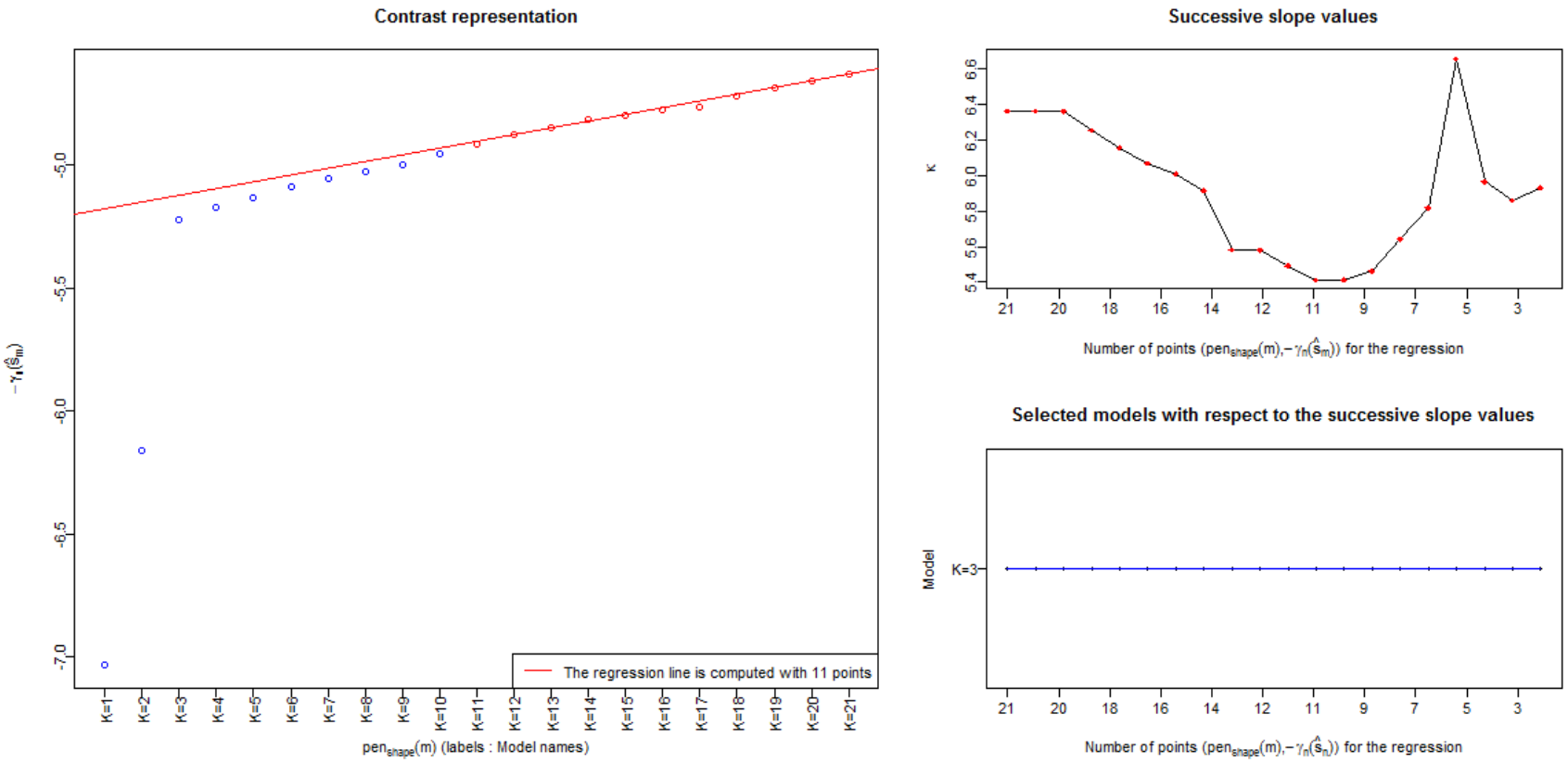
```
> ResDDSEpartial=DDSE(datapartialcapushe)
```

Validation

```
> ResDDSEpartial=DDSE(datapartialcapushe)
> plot(ResDDSEpartial)
```

Validation

```
> ResDDSEpartial=DDSE(datapartialcapushe)
> plot(ResDDSEpartial)
```

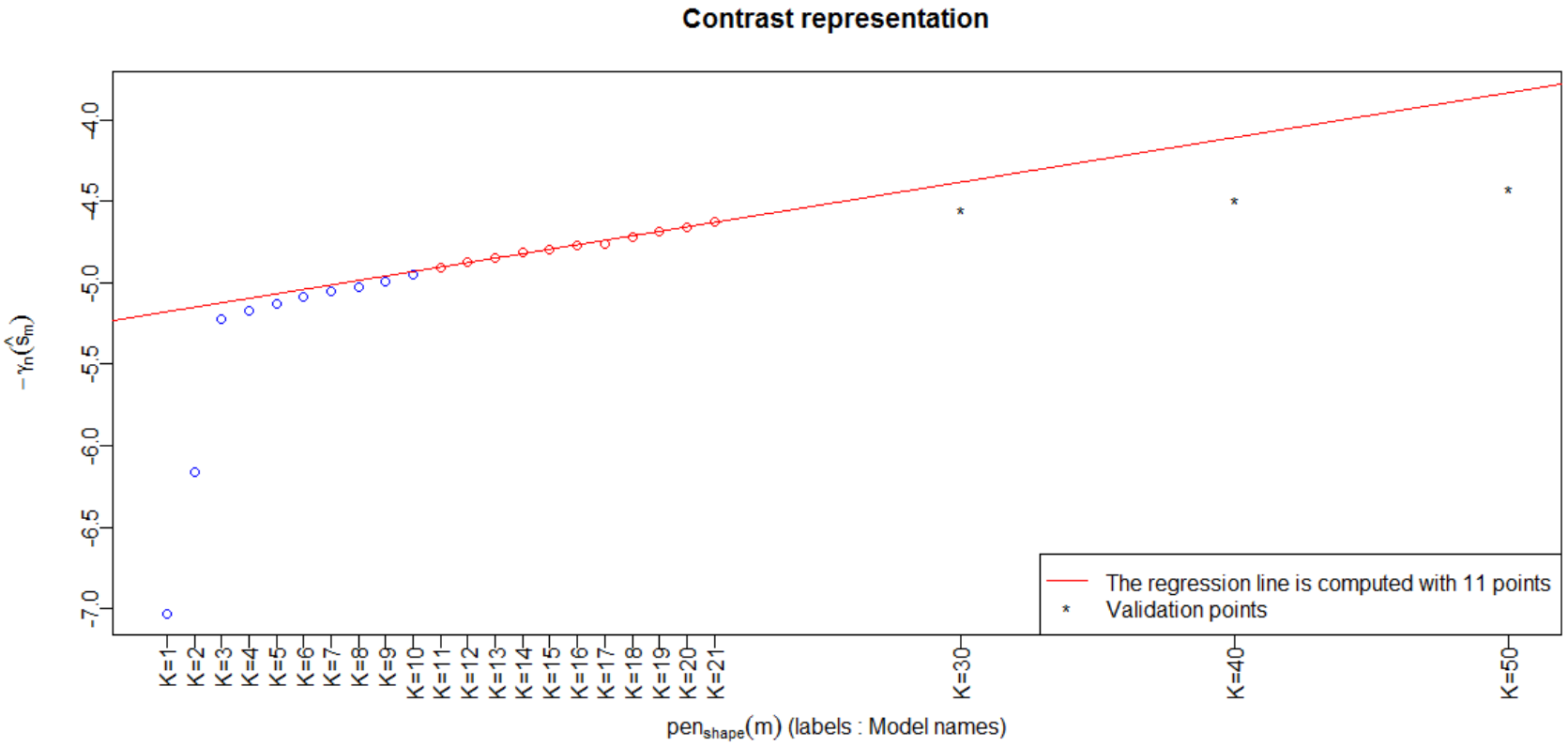


Validation

```
> ResDDSEpartial=DDSE(datapartialcapushe)
> plot(ResDDSEpartial)
> validation(ResDDSEpartial,datavalidcapushe)
```

Validation

```
> ResDDSEpartial=DDSE(datapartialcapushe)
> plot(ResDDSEpartial)
> validation(ResDDSEpartial,datavalidcapushe)
```



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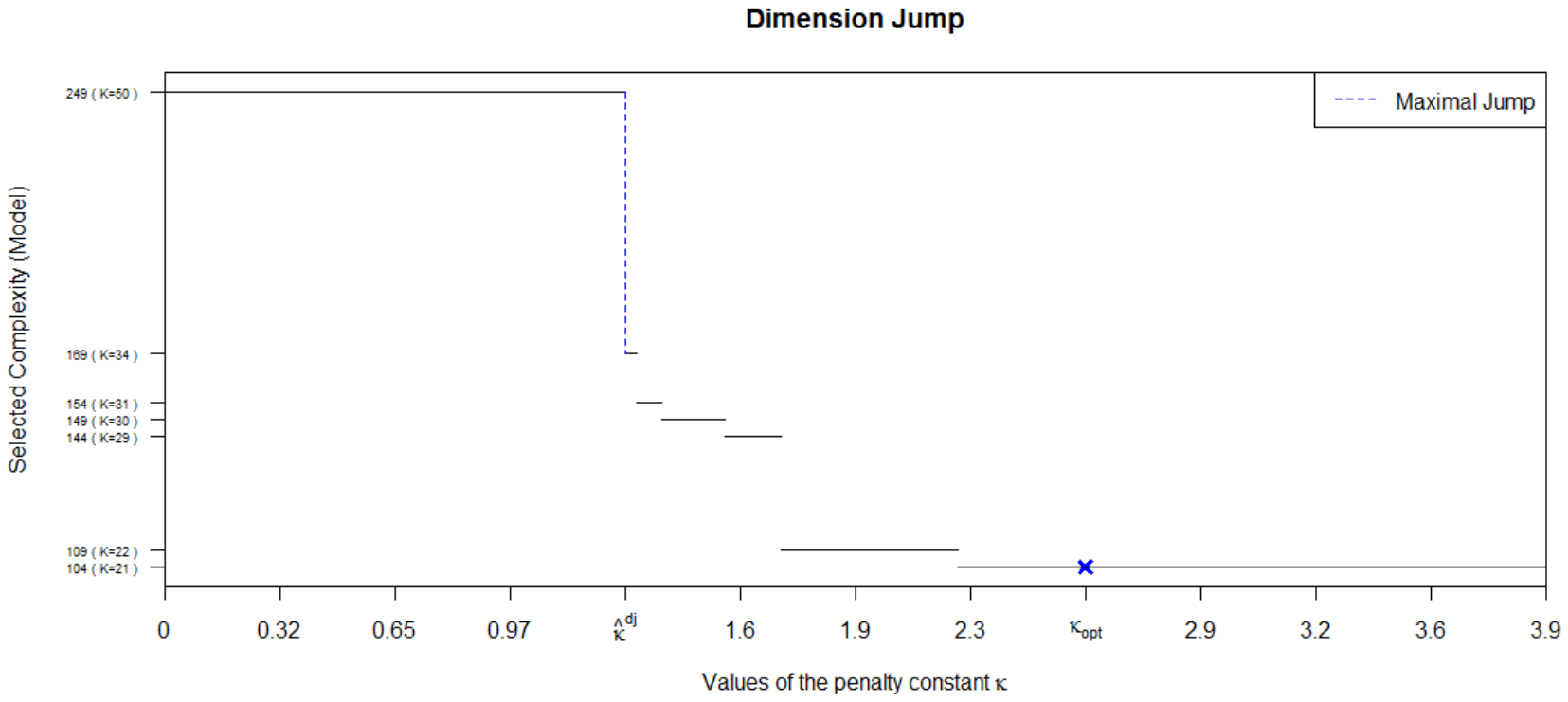
Djump

$C_{m(\kappa)}$ la complexité du modèle minimisant le critère $\text{crit}_\kappa(\cdot)$
$\kappa \mapsto C_{m(\kappa)}$ doit faire un saut au voisinage du κ_{min}

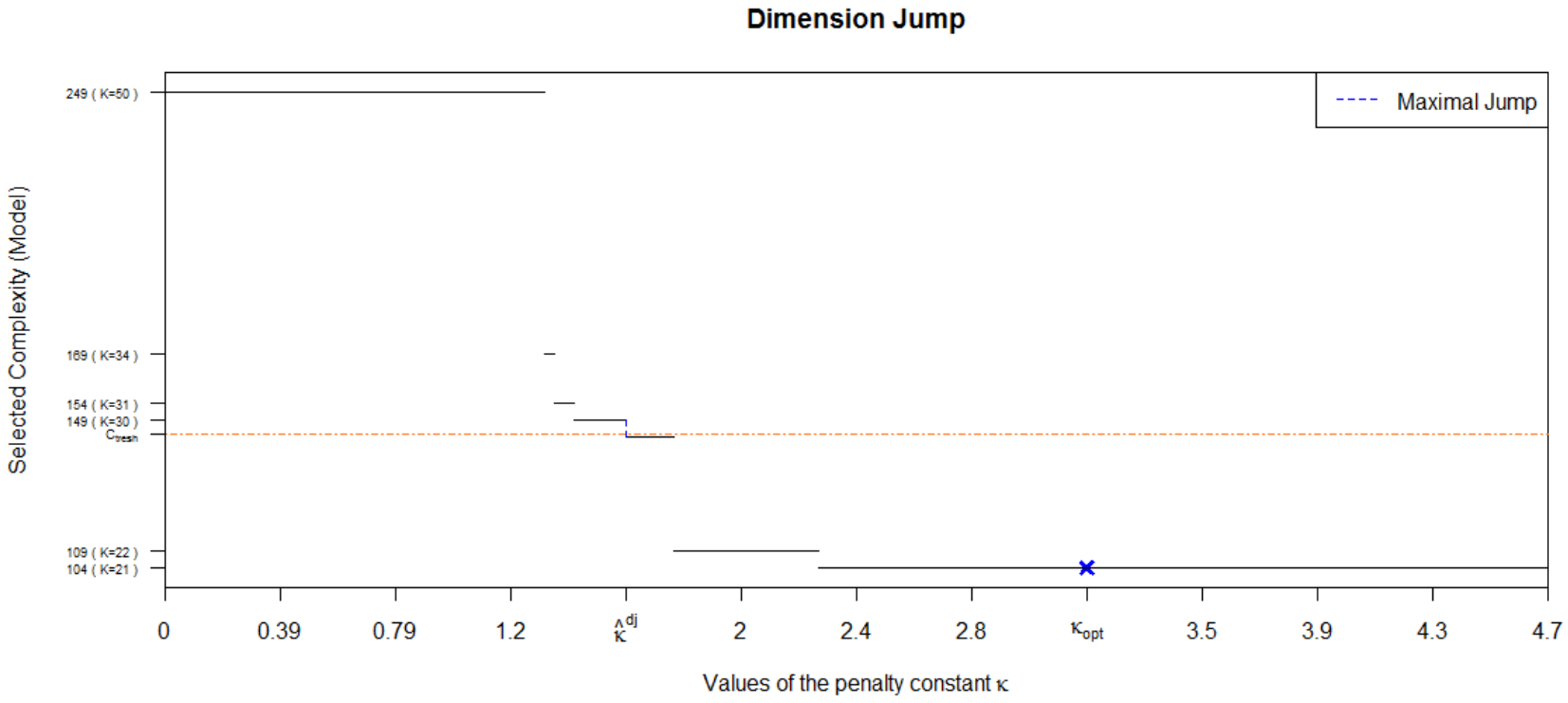


C_{tresh}

```
> plot(Djump(datacapushe,Ctresh=1000/log(1000)))
```

C_{tresh}

```
> plot(Djump(datacapushe,Ctresh=1000/log(1000)))
```

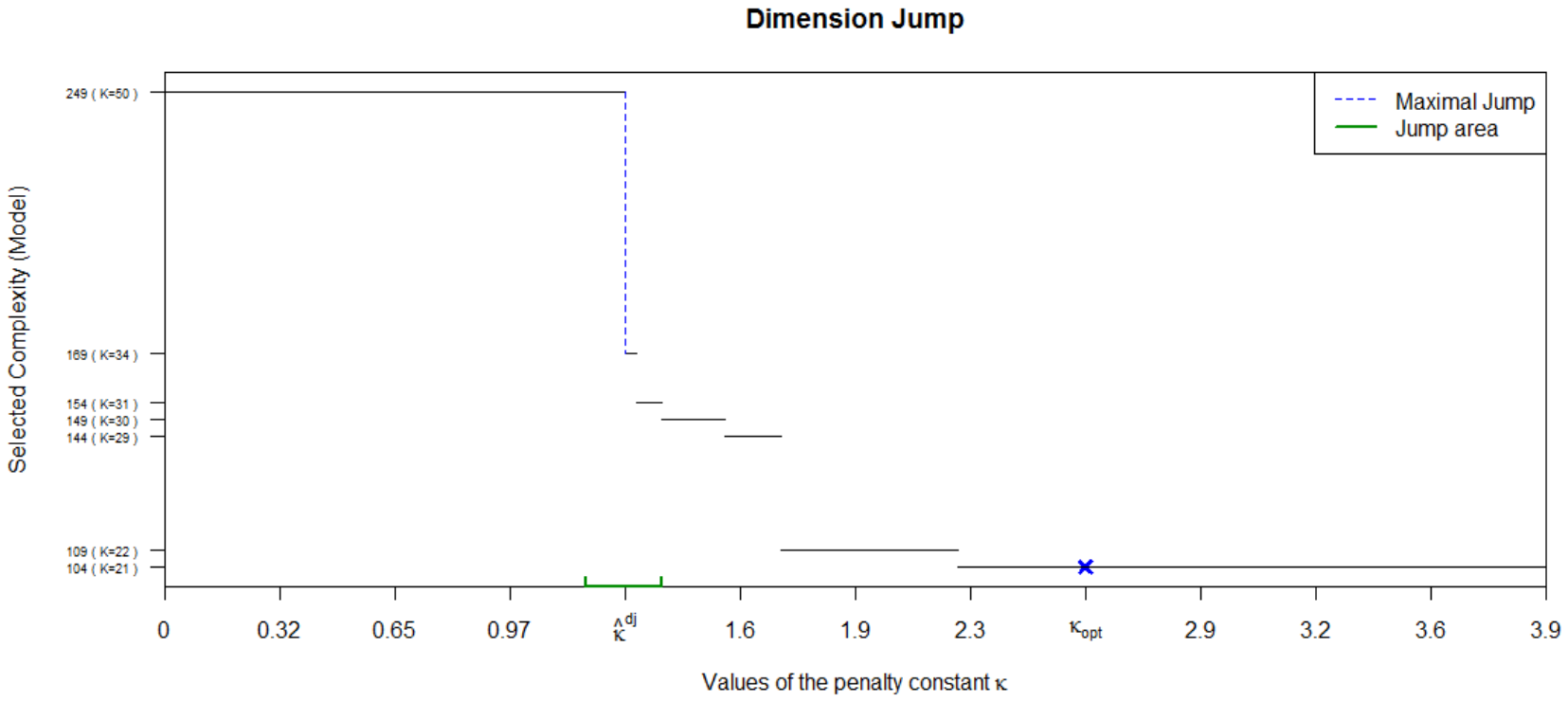


$C_{AreaJump}$

```
> plot(Djump(datacapushe,Careajump=sqrt(log(1000)/1000)))
```

$C_{AreaJump}$

```
> plot(Djump(datacapushe,Careajump=sqrt(log(1000)/1000)))
```



Commande

```
> ResDjump=Djump(datacapushe,scoef=2,Careajump=sqrt(log(1000)/1000),Ctresh=0)
```

Commande

```
> ResDjump=Djump(datacapushe,scoef=2,Careajump=sqrt(log(1000)/1000),Ctresh=0)
> ResDjump
[1] "K=21"
```

Commande

```
> ResDjump=Djump(datacapushe,scoef=2,Careajump=sqrt(log(1000)/1000),Ctresh=0)
> ResDjump
[1] "K=21"
> summary(ResDjump)

Selected model: K=21

Estimated penalty constant: 1.29
Selected model complexity: 104

Jump area: [1.18;1.4]
Model selected by jump area: K=21
```

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Commande

```
> ResCapushe=capushe(datacapushe,n=1000)
```

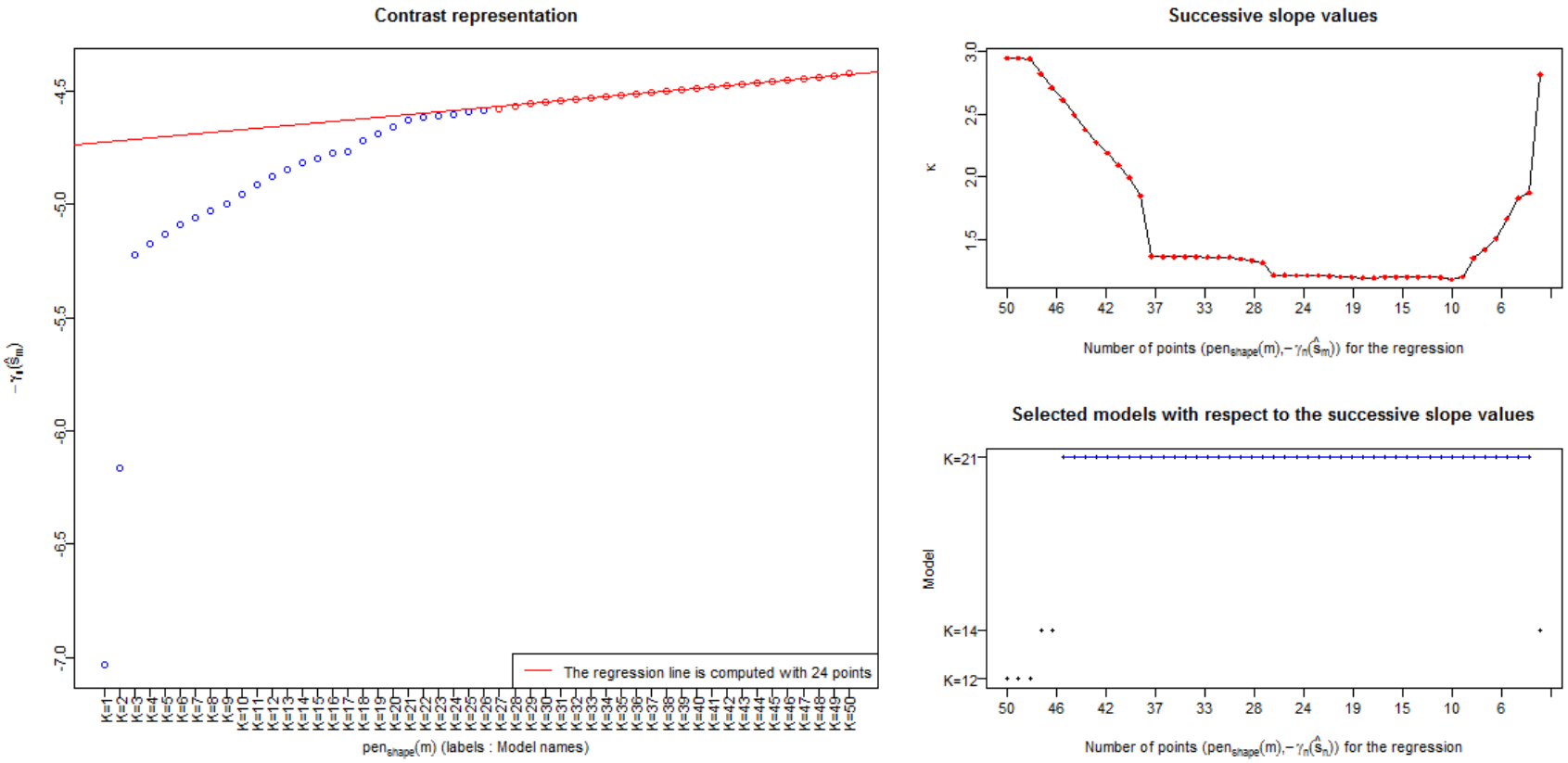
Commande

```
> ResCapushe=capushe(datacapushe,n=1000)
> summary(ResCapushe)

Model selected by DDSE: K=21

Corresponding Slope interval: [1.18;2.62](88% of points)
Selected model complexity: 104


Model selected by Djump: K=21

Estimated penalty constant: 1.29
Selected model complexity: 104

Jump area: [1.18;1.4]
Model selected by jump area: K=21


Model selected by AIC: K=50

Model selected by BIC: K=21
```

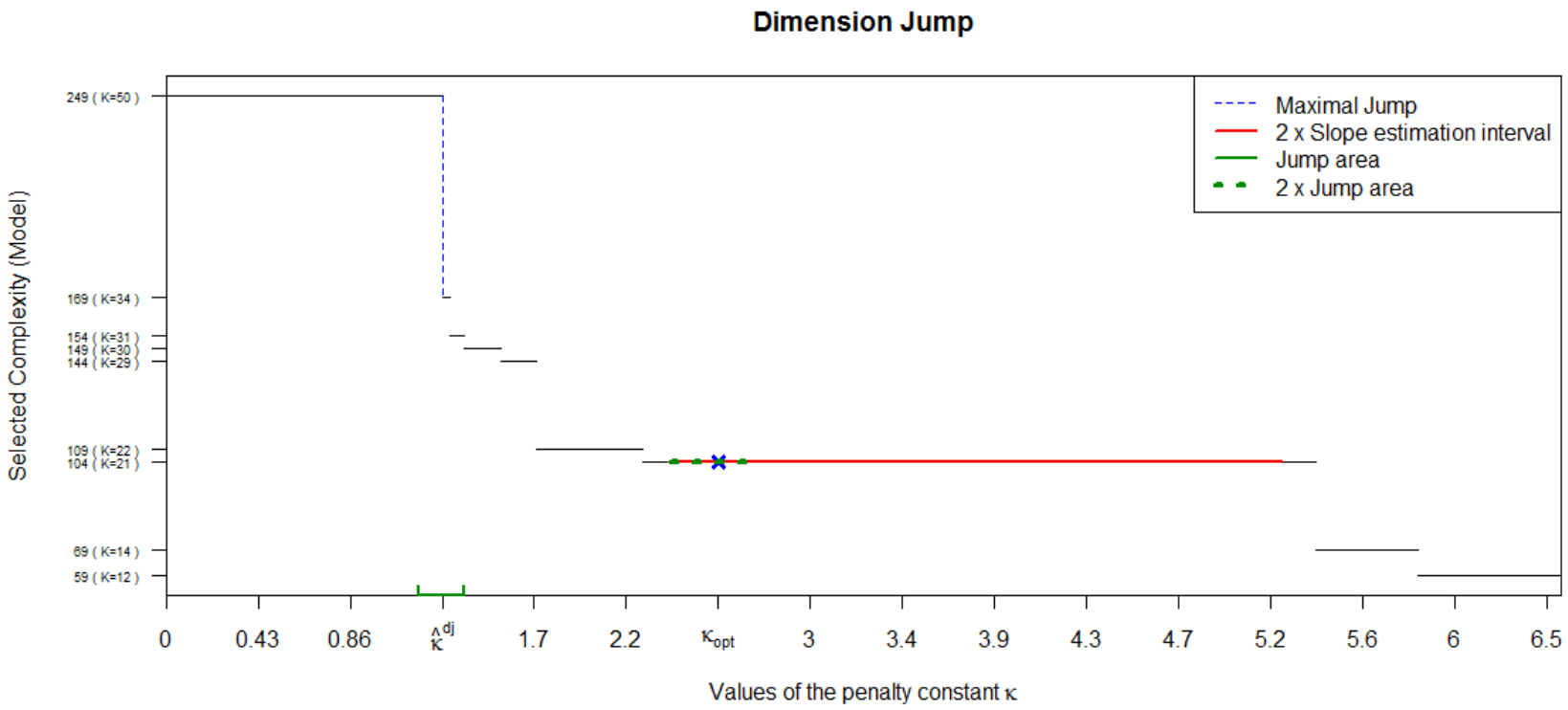
Commande

```
> ResCapushe=capushe(datacapushe,n=1000)
> summary(ResCapushe)

Model selected by DDSE: K=21

Corresponding Slope interval: [1.18;2.62](88% of points)
Selected model complexity: 104


Model selected by Djump: K=21

Estimated penalty constant: 1.29
Selected model complexity: 104

Jump area: [1.18;1.4]
Model selected by jump area: K=21


Model selected by AIC: K=50

Model selected by BIC: K=21
> plot(ResCapushe)
```

Plot


Contrast representation
The regression line is computed with 24 points
pen_shape(m) (labels : Model names)
Successive slope values
κ
Number of points (pen_shape(m), −γ_n(ŝ_m)) for the regression
Selected models with respect to the successive slope values
Model
K=21
K=14
K=12
Number of points (pen_shape(m), −γ_n(ŝ_m)) for the regression

Plot


Dimension Jump
Maximal Jump
2 x Slope estimation interval
Jump area
2 x Jump area
Selected Complexity (Model)
249 (K=50)
169 (K=34)
154 (K=31)
149 (K=30)
144 (K=29)
109 (K=22)
104 (K=21)
69 (K=14)
59 (K=12)
0
0.43
0.86
$\hat{\kappa}^{dj}$
1.7
2.2
κ_{opt}
3
3.4
3.9
4.3
4.7
5.2
5.6
6
6.5
Values of the penalty constant κ

Question ?

Birgé, L. et Massart, P. (2001). Gaussian model selection. *Journal of the European Mathematical Society*, 3(3) :203-268.

Birgé, L. et Massart, P. (2006). Minimal penalties for Gaussian model selection. *Probability Theory and Related Fields*, 138(1-2) :33-73.

Lebarbier, E. (2005). Detecting multiple change-points in the mean of Gaussian process by model selection. *Signal Processing*, 85(4) :717-736.

Massart, P. (2007). *Concentration Inequalities and Model Selection. École d'été de Probabilités de Saint-Flour 2003. Lecture Notes in Mathematics*. Springer.

Maugis, C. et Michel, B. (2011). A non asymptotic penalized criterion for Gaussian mixture model selection. *ESAIM* : P & S, 15, p 41-68.

Baudry, J.-P., Maugis, C. and Michel, B. (2011) Slope Heuristics : Overview and Implementation. *Statistics and Computing*, 22(2), 455-470.