

# Sharp ultimate bounds of solutions to a class of second order linear evolution equations with bounded forcing term

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## **Abstract**

We establish a precise estimate of the ultimate bound of solutions to some second order evolution equations with possibly unbounded linear damping and bounded forcing term.

## Introduction

Let  $H$  be a real Hilbert space. In the sequel we denote by  $(u, v)$  the inner product of two vectors  $u, v$  in  $H$  and by  $|u|$  the  $H$ -norm of  $u$ . Given  $f \in L^\infty(\mathbb{R}, H)$ , we consider the second order evolution equation with possibly unbounded and time-dependent damping operator  $B$ :

$$u'' + Au + Bu' = f(t) \quad (0.1)$$

where  $A$  is a fixed linear, self-adjoint and positive operator in  $H$ . We assume that the domain of  $A$  is dense in  $H$  and  $A$  is coercive, in other terms:

$$\exists \lambda > 0, \quad \forall u \in D(A), \quad (Au, u) \geq \lambda |u|^2. \quad (0.2)$$

Obviously the set of  $\lambda$  satisfying (0.2) is closed. For our purpose the best possible is the largest one, ie.

$$\lambda = \inf_{u \in D(A), |u|=1} (Au, u) =: \lambda_1(A).$$

We introduce  $V = D(A^{\frac{1}{2}})$  endowed with the norm given by

$$\forall u \in V, \quad \|u\| = |A^{\frac{1}{2}}u|.$$

This norm defined on  $V$  is equivalent to the graph norm of  $A^{\frac{1}{2}}$  as a result of the coerciveness hypothesis on  $A$ .

In the sequel,  $B : V \rightarrow V'$  may be a time-dependent continuous operator. When  $B$  is linear and time-independent, we write (0.1) in the following form:

$$U' + LU = F(t) \quad (0.3)$$

with  $U = (u, u')$ ,  $L = \begin{pmatrix} 0 & -I \\ A & B \end{pmatrix}$  and  $F = (0, f)$ . If  $B \in L(V, V')$  satisfies

$$\langle Bv, v \rangle \geq 0 \quad \forall v \in V$$

then it is not difficult to check (cf.e.g. [1, 3, 4]) that  $L$  is a maximal monotone operator with dense domain  $D(L) = \{(u, v) \in V \times V, Au + Bv \in H\}$  in  $V \times H$ . Then, by Hille -Yosida's Theorem (cf.e.g. [3, 13]),  $L$  generates a  $C^0$  contraction semi-group  $S(t)$  that insures the existence and uniqueness of a mild solution  $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$  to (0.1) on  $\mathbb{R}^+$  for any pair of initial data  $u_0 = u(0) \in V; u_1 = u'(0) \in H$ . Moreover, the two following properties are equivalent cf [9]:

1)  $S(t)$  is exponentially damped on  $V \times H$  which means that for some constants  $M \geq 1, \delta > 0$

$$\forall t \geq 0, \quad \|S(t)\|_{L(H)} \leq M \exp(-\delta t)$$

2)  $\forall F \in L^\infty(\mathbb{R}^+, H)$ , any solution of (0.3) is bounded in  $V \times H$  for  $t \geq 0$ .

In addition in this case we have

$$\overline{\lim}_{t \rightarrow \infty} \|U(t)\| \leq \frac{M}{\delta} \overline{\lim}_{t \rightarrow \infty} \|F(t)\|_H$$

In applications to infinite or even finite dimensional second order equations, this method does not give the best possible estimate because it is not easy to optimize on  $M$  and  $\delta$ . This was already observed in [11] and [12] where precise estimates of  $\overline{\lim}_{t \rightarrow \infty} \|U(t)\|$  were given in the case of (0.1) with  $B = cI$  or  $B = cA^{\frac{1}{2}}$ .

The main objective of this paper is to generalize the results of [11, 12] for  $B$  time independent and improve some of the results in the specific cases  $B = cI$  and  $B = cA^{\frac{1}{2}}$ . We shall consider also the case  $B = cA$  which was not studied before.

The plan of the paper is the following: section 1 contains an improvement of the main result from [11] in the general case  $B = \beta(t)$ . Section 2 is devoted to the case where  $B = B(t)$  is linear and self-adjoint. Section 3 gives the precise statements when  $B = cA^\alpha$  with a special treatment in the case  $B = B_0 = cA^{\frac{1}{2}}$  and Section 4 is devoted to the main concrete applications of Theorem 2.1. Finally Section 5 is devoted to slightly different examples and some additional remarks.

## 1 An ultimate bound valid for general time-dependent damping terms

We consider the equation:

$$u'' + \beta(t)u' + Au = f(t) \tag{1.1}$$

where  $t \in \mathbb{R}^+$ . For this equation, we improve some general estimates obtained in [11] when  $\beta(t) : \mathbb{R}^+ \rightarrow C(V, V')$  is a measurable family of possibly nonlinear continuous operators which satisfies the two hypotheses:

$$\exists c > 0, \quad \forall t \in \mathbb{R}^+, \quad \forall v \in V, \quad \langle \beta(t)v, v \rangle \geq c|v|^2. \tag{1.2}$$

$$\exists C > 0, \quad \forall t \in \mathbb{R}^+, \quad \forall v \in V, \quad \|\beta(t)v\|_*^2 \leq C\langle \beta(t)v, v \rangle. \tag{1.3}$$

It is immediate (cf. e.g. [11]) that  $c \leq C\lambda_1$  where  $\lambda_1 = \lambda_1(A)$ . Our main result is the following

**Theorem 1.1.** For any solution  $u \in W_{loc}^{1,\infty}(\mathbb{R}^+, V) \cap W_{loc}^{2,\infty}(\mathbb{R}^+, H)$  of (1.1) we have the estimate :

$$\max(\overline{\lim}_{t \rightarrow \infty} \|u(t)\|, \overline{\lim}_{t \rightarrow \infty} |u'(t)|) \leq \max(\sqrt{12}\sqrt{\frac{C}{c}}, \frac{3}{c}) \overline{\lim}_{t \rightarrow \infty} |f(t)| \quad (1.4)$$

**Proof.** For simplicity of the formulas, we drop the variable  $t$  whenever possible and we denote by  $z'$  the time derivative of a (scalar or vector) time-dependent function  $z$ . We consider for some  $\alpha > 0$  to be chosen later the following modified energy functional:

$$\Phi = |u'|^2 + \|u\|^2 + \alpha(u, u') - \frac{\alpha^2}{4}|u|^2.$$

Then

$$\begin{aligned} \Phi' &= -2\langle \beta u', u' \rangle + \alpha|u'|^2 - \alpha\|u\|^2 - \alpha\langle \beta u', u \rangle + \langle f, 2u' + \alpha u \rangle - \frac{\alpha^2}{2}(u, u') \\ &= -\frac{\alpha}{2}(|u'|^2 + \|u\|^2 + \alpha(u, u')) - 2\langle \beta u', u' \rangle + \frac{3\alpha}{2}|u'|^2 - \frac{\alpha}{2}\|u\|^2 - \alpha\langle \beta u', u \rangle + \langle f, 2u' + \alpha u \rangle \end{aligned}$$

we set  $\Psi = |u'|^2 + \|u\|^2 + \alpha(u, u') \geq \Phi$ .

Then, by using (1.2), we have:

$$\Phi' \leq -\frac{\alpha}{2}\Phi - \frac{1}{2}\langle \beta u', u' \rangle - \left(\frac{3c}{2} - \frac{3\alpha}{2}\right)|u'|^2 - \frac{\alpha}{2}\|u\|^2 - \alpha\langle \beta u', u \rangle + \langle f, 2u' + \alpha u \rangle$$

we have, from (1.3):

$$|\langle \beta u', u \rangle| \leq \sqrt{C}\langle \beta u', u' \rangle^{\frac{1}{2}}\|u\|$$

By using Young's inequality we deduce :

$$|\alpha\langle \beta u', u \rangle| \leq \alpha C\langle \beta u', u' \rangle + \alpha\frac{\|u\|^2}{4}$$

Assuming  $\alpha C \leq \frac{1}{2}$ , then:

$$\Phi' + \frac{\alpha}{2}\Phi \leq -\frac{3}{2}(c - \alpha)|u'|^2 + 2\langle f, u' \rangle - \frac{\alpha}{4}\|u\|^2 + \alpha\langle f, u \rangle$$

Assuming  $\frac{3}{2}(c - \alpha) \geq \frac{1}{2}c$ , then  $\alpha \leq \frac{2}{3}c$ .

We have, by using Young's inequality:

$$\begin{aligned} -\frac{3}{2}(c - \alpha)|u'|^2 + 2\langle f, u' \rangle &\leq -\frac{c}{2}|u'|^2 + 2\langle f, u' \rangle \\ &\leq \frac{2}{c}|f|^2 \end{aligned}$$

Moreover

$$\alpha \langle f, u \rangle \leq \frac{\alpha}{\sqrt{\lambda_1}} |f| \|u\|$$

Therefore, by Young's inequality:

$$\begin{aligned} -\frac{\alpha}{4} \|u\|^2 + \alpha \langle f, u \rangle &\leq \alpha \left( -\frac{\|u\|^2}{4} + \frac{1}{\sqrt{\lambda_1}} |f| \|u\| \right) \\ &\leq \frac{\alpha}{\lambda_1} |f|^2 \\ &\leq \frac{\alpha C}{c} |f|^2 \\ &\leq \frac{1}{2c} |f|^2 \end{aligned}$$

Then

$$\Phi' + \frac{\alpha}{2} \Phi \leq \frac{5}{2c} |f|^2$$

Then, we find that  $\Phi$  is bounded with:

$$\overline{\lim}_{t \rightarrow \infty} \Phi(t) \leq \frac{5}{c\alpha} \overline{\lim}_{t \rightarrow \infty} |f(t)|^2.$$

Moreover, we have:

$$-\alpha(u, u') \leq |u'|^2 + \frac{\alpha^2}{4} |u|^2$$

We set  $F = \overline{\lim}_{t \rightarrow \infty} |f(t)|^2$ .

In particular for any  $\epsilon > 0$  we have for  $t$  large enough

$$\left(1 - \frac{\alpha^2}{2\lambda_1}\right) \|u(t)\|^2 \leq \|u(t)\|^2 - \frac{\alpha^2}{2} |u(t)|^2 \leq \Phi(t) \leq \frac{5}{c\alpha} F + \frac{\epsilon}{2}.$$

Now since  $\alpha \leq \frac{2}{3}c$  and  $\alpha \leq \frac{1}{2C}$ , we have

$$\frac{\alpha^2}{2\lambda_1} \leq \frac{c}{6\lambda_1 C} \leq \frac{1}{6}$$

Then we find

$$\overline{\lim}_{t \rightarrow \infty} \|u(t)\|^2 \leq \frac{6}{c\alpha} F + 2\epsilon$$

Finally, by choosing  $\alpha = \inf(\frac{2}{3}c, \frac{1}{2C})$ , we obtain by letting  $\epsilon \rightarrow 0$ :

$$\overline{\lim}_{t \rightarrow \infty} \|u(t)\| \leq \max\left(\sqrt{\frac{12C}{c}}, \frac{3}{c}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)|.$$

In order to estimate  $u'$ , observe that for  $t$  large enough:

$$|u'(t)|^2 + \lambda_1 |u(t)|^2 + \alpha(u, u') - \frac{\alpha^2}{4} |u(t)|^2 \leq \frac{5}{c\alpha} F + \frac{\epsilon}{2}$$

Since  $\alpha \leq \frac{2}{3}c \leq c$  and  $\alpha \leq \frac{1}{2C} \leq \frac{\lambda_1}{2c}$ , then  $\alpha^2 \leq \alpha c \leq \frac{\lambda_1}{2}$ .

Consequently for  $t$  large enough

$$\frac{5}{6} |u'(t)|^2 + 2\alpha^2 |u(t)|^2 + \frac{1}{6} |u'(t)|^2 + \alpha(u, u') - \frac{\alpha^2}{4} |u(t)|^2 \leq \frac{5}{c\alpha} F + \frac{\epsilon}{2}$$

In other terms

$$\frac{5}{6} |u'(t)|^2 + \frac{\alpha^2}{4} |u(t)|^2 + \left| \frac{1}{\sqrt{6}} u' + \frac{\sqrt{3}}{\sqrt{2}} \alpha u \right|^2 \leq \frac{5}{c\alpha} F + \frac{\epsilon}{2}$$

Then:

$$\overline{\lim}_{t \rightarrow \infty} |u'(t)|^2 \leq \frac{6}{c\alpha} F + 2\epsilon$$

Also assuming  $\alpha = \inf(\frac{1}{2C}, \frac{2}{3}c)$  and letting  $\epsilon \rightarrow 0$ , we have:

$$\overline{\lim}_{t \rightarrow \infty} |u'(t)| \leq \max\left(\sqrt{\frac{12C}{c}}, \frac{3}{c}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)|.$$

□

**Remark 1.2.** If  $\beta(t) = B_0 \in L(V, V')$ , it is well known that the conditions  $(u_0, u_1) \in D(A) \times V$  and  $f \in C^1(\mathbb{R}^+, V)$  imply  $u \in C^1(\mathbb{R}^+, V) \cap C^2(\mathbb{R}^+, H)$ . By density on  $(u_0, u_1, f)$  we obtain easily the following

**Corollary 1.3.** *Let  $\beta(t) = B_0 \in L(V, V')$ . In this case any mild solution  $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$  of (1.1) satisfies (1.4).*

**Remark 1.4.** In [11], the following estimate was established

$$\sup\{\overline{\lim}_{t \rightarrow \infty} \|u(t)\|, \overline{\lim}_{t \rightarrow \infty} |u'(t)|\} \leq \sqrt{3}\left(C + \frac{4}{c}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)| \quad (1.5)$$

Since

$$\sqrt{\frac{12C}{c}} \leq \frac{\sqrt{12}}{4}\left(C + \frac{4}{c}\right) = \frac{\sqrt{3}}{2}\left(C + \frac{4}{c}\right)$$

and

$$\frac{3}{c} \leq \frac{3}{4}\left(C + \frac{4}{c}\right) \leq \frac{\sqrt{3}}{2}\left(C + \frac{4}{c}\right),$$

we can see that Theorem 1.1 improves the estimate (1.5) by a factor 2 for all values of  $c$  and  $C$ . Moreover if  $C \rightarrow \infty$  with  $\frac{C}{c}$  bounded,  $\max(\sqrt{\frac{12C}{c}}, \frac{3}{c})$  remains bounded and  $(C + \frac{4}{c})$  tends to infinity, therefore (1.4) improves (1.5) by an arbitrarily large amount. A typical case is :  $\beta = cB_0$  with  $c \rightarrow \infty$  since then  $\frac{C}{c}$  is fixed and  $C \rightarrow \infty$ .

## 2 The case of a linear self-adjoint damping operator

In this section, we study the equation (0.1) where  $B : \mathbb{R}^+ \longrightarrow L(V, V')$  is a self-adjoint and possibly unbounded operator and satisfies the following hypotheses:

$$\exists c > 0, \quad \forall t \in \mathbb{R}^+, \quad \forall v \in V, \quad \langle B(t)v, v \rangle \geq c|v|^2 \quad (2.1)$$

$$\exists C > 0, \quad \forall t \in \mathbb{R}^+, \quad \forall v \in V, \quad \langle B(t)v, v \rangle \leq C\langle Av, v \rangle \quad (2.2)$$

The following result, will give close to optimal estimates even when  $B$  is independent of time.

**Theorem 2.1.** *Any solution  $u \in W_{loc}^{1,\infty}(\mathbb{R}^+, V) \cap W_{loc}^{2,\infty}(\mathbb{R}^+, H)$  of (0.1) satisfies the following estimate:*

$$\max(\overline{\lim}_{t \rightarrow \infty} \|u(t)\|, \overline{\lim}_{t \rightarrow \infty} |u'(t)|) \leq \max\left(\sqrt{\frac{3C}{c}}, \frac{3}{\sqrt{2c}}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)| \quad (2.3)$$

**Proof.** Considering again the energy functional  $\Phi = |u'|^2 + \|u\|^2 + \alpha(u, u') - \frac{\alpha^2}{4}|u|^2$  we find:

$$\begin{aligned} \Phi' &= -2|B^{\frac{1}{2}}u'|^2 + \alpha|u'|^2 - \alpha\|u\|^2 - \alpha(Bu', u) + (f, 2u' + \alpha u) - \frac{\alpha^2}{2}(u, u') \\ &= -\frac{\alpha}{2}\Psi - \left(2 - \frac{3\alpha}{2c}\right)|B^{\frac{1}{2}}u'|^2 - \frac{\alpha}{2}\|u\|^2 - \alpha(Bu', u) + (f, 2u' + \alpha u) \\ &\leq -\frac{\alpha}{2}\Phi - \left(2 - \frac{3\alpha}{2c}\right)|B^{\frac{1}{2}}u'|^2 - \frac{\alpha}{2}\|u\|^2 - \alpha(Bu', u) + (f, u' + \alpha u) + (f, u') \end{aligned}$$

where  $\Psi = |u'|^2 + \|u\|^2 + \alpha(u, u') \geq \Phi$ . By (2.1) and Young's inequality, we have

$$\begin{aligned} (f, u') &\leq \frac{1}{2c}|f|^2 + \frac{c}{2}|u'|^2 \\ &\leq \frac{1}{2c}|f|^2 + \frac{1}{2}|B^{\frac{1}{2}}u'|^2 \end{aligned}$$

Therefore by using (2.2), we obtain

$$\Phi' \leq -\frac{\alpha}{2}\Phi - \left(\frac{3}{2} - \frac{3\alpha}{2c}\right)|B^{\frac{1}{2}}u'|^2 - \frac{\alpha}{2C}|B^{\frac{1}{2}}u|^2 - \alpha(Bu', u) + \frac{1}{2c}|f|^2 + (f, u' + \alpha u)$$

Assuming

$$\frac{3}{2} - \frac{3\alpha}{2c} \geq \frac{1}{2} \quad \text{and} \quad \alpha^2 \leq \frac{\alpha}{C}$$

which means

$$\alpha \leq \frac{2}{3}c \quad \text{and} \quad \alpha \leq \frac{1}{C}$$

we deduce

$$\begin{aligned}\Phi' &\leq -\frac{\alpha}{2}\Phi - \frac{1}{2}|B^{\frac{1}{2}}u'|^2 - \frac{\alpha^2}{2}|B^{\frac{1}{2}}u|^2 - \alpha(Bu', u) + \frac{1}{2c}|f|^2 + (f, u' + \alpha u) \\ &\leq -\frac{\alpha}{2}\Phi - \frac{1}{2}|B^{\frac{1}{2}}(u' + \alpha u)|^2 + \frac{1}{2c}|f|^2 + (f, u' + \alpha u)\end{aligned}$$

By using (2.1), we find

$$\Phi' \leq -\frac{\alpha}{2}\Phi - \frac{c}{2}|u' + \alpha u|^2 + \frac{1}{2c}|f|^2 + (f, u' + \alpha u)$$

By using Young's inequality in the last term, we have

$$(f, u' + \alpha u) \leq \frac{1}{2c}|f|^2 + \frac{c}{2}|u' + \alpha u|^2$$

Then

$$\Phi' \leq -\frac{\alpha}{2}\Phi + \frac{1}{c}|f|^2$$

Then we find that  $\Phi$  is bounded with

$$\overline{\lim}_{t \rightarrow \infty} \Phi(t) \leq \frac{2}{c\alpha} \overline{\lim}_{t \rightarrow \infty} |f(t)|^2$$

By setting  $F = \overline{\lim}_{t \rightarrow \infty} |f(t)|^2$  we see that for  $t$  large enough and any  $\epsilon > 0$

$$|u'(t)|^2 + \|u(t)\|^2 + \alpha(u(t), u'(t)) - \frac{\alpha^2}{4}|u(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

In other terms

$$\|u(t)\|^2 + |u'(t) + \frac{\alpha}{2}u(t)|^2 - \frac{\alpha^2}{2}|u(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

By using  $\alpha \leq \frac{2}{3}c$  and (2.1), we obtain for  $t$  large enough:

$$\|u(t)\|^2 - \frac{\alpha}{3}|B^{\frac{1}{2}}u(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

now using  $\alpha \leq \frac{1}{C}$  and (2.2), for  $t$  large enough we obtain :

$$\|u(t)\|^2 \leq \frac{3}{\alpha c}F + 2\epsilon$$

Finally by selecting  $\alpha = \inf(\frac{2}{3}c, \frac{1}{C})$  and letting  $\epsilon \rightarrow 0$  we find :

$$\overline{\lim}_{t \rightarrow \infty} \|u(t)\| \leq \max\left(\sqrt{\frac{3C}{c}}, \frac{3}{\sqrt{2c}}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$



In order to estimate  $u'$ , for  $t$  large enough by using (0.2)

$$\frac{2}{3}|u'(t)|^2 + \lambda_1|u(t)|^2 + \alpha(u(t), u'(t)) + \frac{1}{3}|u'(t)|^2 - \frac{\alpha^2}{4}|u(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

Since  $\alpha \leq \frac{2}{3}c \leq c$  and  $\alpha \leq \frac{1}{c} \leq \frac{\lambda_1}{c}$ , we have  $\alpha^2 \leq \alpha c \leq \lambda_1$ .

Therefore, for  $t$  large enough:

$$\frac{2}{3}|u'(t)|^2 + \alpha^2|u(t)|^2 + \alpha(u(t), u'(t)) + \frac{1}{3}|u'(t)|^2 - \frac{\alpha^2}{4}|u(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

Then, for  $t$  large enough

$$\frac{2}{3}|u'(t)|^2 + \frac{3\alpha^2}{4}|u(t)|^2 + \alpha(u(t), u'(t)) + \frac{1}{3}|u'(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

In other terms

$$\frac{2}{3}|u'(t)|^2 + \left| \frac{\sqrt{3}}{2}\alpha u(t) + \frac{1}{\sqrt{3}}u'(t) \right|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

Hence, for  $t$  large enough

$$|u'(t)|^2 \leq \frac{3}{\alpha c}F + 2\epsilon$$

Finally by letting  $\epsilon \rightarrow 0$

$$\overline{\lim}_{t \rightarrow \infty} |u'(t)| \leq \max\left(\sqrt{\frac{3C}{c}}, \frac{3}{\sqrt{2c}}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

□

By using Remark 1.2 we obtain

**Corollary 2.2.** *Let  $\beta(t) = B_0 \in L(V, V')$ . In this case any mild solution  $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$  of (0.1) satisfies (2.3).*

**Remark 2.3.** When  $B$  is linear and self-adjoint, Theorem 2.1 improves the result (1.4) with  $\beta(t) = B(t)$  by a factor  $\in [\sqrt{2}, 2]$  depending on the values of  $C$  and  $c$ . Indeed in this case (but not in general) the two inequalities (1.3) and (2.2) are equivalent, see Section 5 below.

### 3 Applications when $B = \gamma A^\alpha$ , $0 \leq \alpha \leq 1$

In this section we consider the case of a time independent self-adjoint  $B$  proportional to some positive power of  $A$ . In order to guarantee exponential damping of the associated semi-group the power will be taken  $\leq 1$ .

### 3.1 The ODE case

We consider the equation:

$$u'' + \gamma u' + \omega^2 u = f(t) \quad (3.1)$$

We apply theorem 2.1 to (3.1) with  $c = \gamma$  and  $C = \frac{\gamma}{\omega^2}$ , we find

$$\forall t \in \mathbb{R}, |u(t)| \leq \max\left(\frac{\sqrt{3}}{\omega^2}, \frac{3}{\sqrt{2}\gamma\omega}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)| \quad (3.2)$$

By comparison with the estimates in [10], we find that the result of theorem 2.1 is optimal up to a factor  $K(\omega, \gamma) = \frac{3\pi}{4\sqrt{2}}$ , if  $\gamma < 2\omega$  and  $\sqrt{3}$  if  $\gamma \geq 2\omega$ . More precisely, in [10] the exact minimum global bound for solutions bounded on the whole line is given, and the minimum turns out to be achieved on some periodic solutions (corresponding to a periodic source term) for which the ultimate bound of course coincides with the global bound on  $\mathbb{R}$ .

### 3.2 The case $B = \gamma A^\alpha$ , $0 \leq \alpha \leq 1$

We consider the equation

$$u'' + \gamma A^\alpha u' + Au = f(t) \quad (3.3)$$

In this case (cf. Proposition 5.4) we have  $c = \gamma \lambda_1^\alpha$  and  $C = \frac{\gamma}{\lambda_1^{1-\alpha}}$ , then, by Theorem 2.1, we have the following estimates

$$\max(\overline{\lim}_{t \rightarrow \infty} \|u(t)\|, \overline{\lim}_{t \rightarrow \infty} |u'(t)|) \leq \max\left(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma\lambda_1^\alpha}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)| \quad (3.4)$$

Considering the special case  $H = \mathbb{R}$ ,  $A = \omega^2 I$  we conclude that this result is always sharp up to a factor  $\sqrt{3}$ .

### 3.3 The case $B = \gamma I$ :

we consider the equation:

$$u'' + \gamma u' + Au = f(t) \quad (3.5)$$

Applying Theorem 2.1 to (3.5) with  $C = \frac{\gamma}{\lambda_1}$  and  $c = \gamma$  we find :

$$\max(\overline{\lim}_{t \rightarrow \infty} \|u(t)\|, \overline{\lim}_{t \rightarrow \infty} |u'(t)|) \leq \max\left(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)| \quad (3.6)$$

**Remark 3.1.** Let us compare our result on (3.6) with the estimates from [8]. In [8] it was shown that

$$\overline{\lim}_{t \rightarrow \infty} \|u(t)\| \leq \sqrt{\frac{4}{\gamma^2} + \frac{1}{\lambda_1}} \overline{\lim}_{t \rightarrow \infty} |f(t)|. \quad (3.7)$$

If  $\gamma$  is fixed and  $\lambda_1 \rightarrow \infty$  we have:

$$\max\left(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma}\right) = \frac{3}{\sqrt{2}\gamma}$$

and

$$\sqrt{\frac{4}{\gamma^2} + \frac{1}{\lambda_1}} \simeq \frac{2}{\gamma}$$

therefore we find that (3.6) is worse than (3.7), hence Theorem 2.1 is weaker than the result of [8] in this case.

If  $\lambda_1$  is fixed and  $\gamma \rightarrow \infty$  we have:

$$\max\left(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma}\right) = \sqrt{\frac{3}{\lambda_1}}$$

and

$$\sqrt{\frac{4}{\gamma^2} + \frac{1}{\lambda_1}} \simeq \sqrt{\frac{1}{\lambda_1}}$$

therefore in this case Theorem 2.1 is also weaker than [8].

Let us determine the values of  $\gamma$  and  $\lambda_1$  for which condition (3.6) is better than (3.7). To this end we can study the condition:

$$\frac{\sqrt{\frac{4}{\gamma^2} + \frac{1}{\lambda_1}}}{\max\left(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma}\right)} > 1$$

Therefore, we introduce:

$$g(\gamma, \lambda_1) = \frac{\sqrt{4 + \frac{\gamma^2}{\lambda_1}}}{\max\left(\sqrt{\frac{3\gamma^2}{\lambda_1}}, \frac{3}{\sqrt{2}}\right)}$$

By setting  $r = \frac{\gamma}{\sqrt{\lambda_1}}$ , we obtain:

$$g(\gamma, \lambda_1) = p(r) = \frac{\sqrt{4 + r^2}}{\max\left(\sqrt{3r^2}, \frac{3}{\sqrt{2}}\right)}$$

Introducing  $\tau = r^2$ , we have:

$$P(\tau) = \frac{4 + \tau}{\max(\frac{9}{2}, 3\tau)}$$

A simple calculation shows that

$$P(\tau) > 1 \iff \tau \in ]\frac{1}{2}, 2[ \iff r^2 \in ]\frac{1}{2}, 2[ \iff r \in ]\frac{1}{\sqrt{2}}, \sqrt{2}[.$$

Finally, we obtain that if  $\gamma \in ]\sqrt{\frac{\lambda_1}{2}}, \sqrt{2\lambda_1}[$ , Theorem 2.1 improves the result of [8].

### 3.4 The case $B = \gamma A$ :

Let us consider the equation:

$$u'' + \gamma Au' + Au = f(t) \tag{3.8}$$

with  $\gamma > 0$ .

When we apply Theorem 2.1 to the equation (3.8) with  $C = \gamma$  and  $c = \gamma\lambda_1$ , we obtain immediately:

**Corollary 3.2.** *Any solution of (3.8) satisfies the following hypotheses:*

$$\max(\overline{\lim}_{t \rightarrow \infty} \|u(t)\|, \overline{\lim}_{t \rightarrow \infty} |u'(t)|) \leq \max\left(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma\lambda_1}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)| \tag{3.9}$$

**Remark 3.3.** This result is new and was not obtained in [11].

### 3.5 The case $B = \gamma A^{\frac{1}{2}}$

In this subsection we consider the so-called structural damping (cf [5, 6, 7] for the terminology and main properties). Therefore we consider as in [12] the equation:

$$u'' + \gamma A^{\frac{1}{2}}u' + Au = f(t) \tag{3.10}$$

with  $\gamma > 0$ .

If we apply theorem (2.1) with  $c = \gamma\sqrt{\lambda_1}$  and  $C = \frac{\gamma}{\sqrt{\lambda_1}}$ , we obtain

$$\max(\overline{\lim}_{t \rightarrow \infty} \|u(t)\|, \overline{\lim}_{t \rightarrow \infty} |u'(t)|) \leq \max\left(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma\sqrt{\lambda_1}}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)| \tag{3.11}$$

By comparison with [12], we remark that (2.1) gives a weaker result. We shall now recover the estimate on  $u$  from [12] in the case of large damping by a method introduced by C. Fitouri (cf. [8]) which is less complicated than the method of [12].

We recall the main result from [12].

**Theorem 3.4.** *The bounded solution of (3.10) satisfies the estimate*

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \leq \frac{1}{\sqrt{\lambda_1}} \max(1, \frac{2}{\gamma}) \|f(t)\|_{L^\infty(\mathbb{R}, H)}.$$

**Proof.** In the case of a small damping we refer to [2]. We now prove (3.4) when

$$\gamma \geq 2 \tag{3.12}$$

We choose the energy functional

$$\Phi = |A^{\frac{1}{4}}u'|^2 + |A^{\frac{3}{4}}u|^2 + \alpha(A^{\frac{1}{2}}u', A^{\frac{1}{2}}u)$$

Then, we have:

$$\begin{aligned} \Phi' &= (2A^{\frac{1}{2}}u', u'' + Au) + \alpha|A^{\frac{1}{2}}u'|^2 + \alpha(Au, u'') \\ &= -2\gamma|A^{\frac{1}{2}}u'|^2 + \alpha|A^{\frac{1}{2}}u'|^2 - \gamma\alpha(Au, A^{\frac{1}{2}}u') - \alpha|Au|^2 + (f, 2A^{\frac{1}{2}}u' + \alpha Au) \\ &= -\frac{\alpha}{2}(|A^{\frac{1}{2}}u'|^2 + \alpha(Au, A^{\frac{1}{2}}u') + |Au|^2) + (\frac{3\alpha}{2} - 2\gamma)|A^{\frac{1}{2}}u'|^2 + (\frac{\alpha^2}{2} - \gamma\alpha)(Au, A^{\frac{1}{2}}u') \\ &\quad - \frac{\alpha}{2}|Au|^2 + (f, 2A^{\frac{1}{2}}u' + \alpha Au) \end{aligned}$$

we set

$$\Psi = |A^{\frac{1}{2}}u'|^2 + \alpha(Au, A^{\frac{1}{2}}u') + |Au|^2$$

Then:

$$\Phi' = -\frac{\alpha}{2}\Psi + (\frac{3\alpha}{2} - 2\gamma)|A^{\frac{1}{2}}u'|^2 + (\frac{\alpha^2}{2} - \gamma\alpha)(Au, A^{\frac{1}{2}}u') - \frac{\alpha}{2}|Au|^2 + (f, 2A^{\frac{1}{2}}u' + \alpha Au)$$

by using Young's inequality, we obtain:

$$(f, 2A^{\frac{1}{2}}u' + \alpha Au) \leq \frac{\alpha}{2}|f|^2 + \frac{1}{2\alpha}(4|A^{\frac{1}{2}}u'|^2 + 4\alpha(Au, A^{\frac{1}{2}}u') + \alpha^2|Au|^2)$$

Therefore

$$\Phi' \leq -\frac{\alpha}{2}\Psi + (\frac{3\alpha}{2} + \frac{2}{\alpha} - 2\gamma)|A^{\frac{1}{2}}u'|^2 + (\frac{\alpha^2}{2} - \gamma\alpha + 2)(Au, A^{\frac{1}{2}}u') + \frac{\alpha}{2}|f|^2$$

we remark that  $\alpha = \gamma - \sqrt{\gamma^2 - 4}$  is a solution of the equation:

$x^2 - 2\gamma x + 4 = 0$ , then:

$$\frac{\alpha^2}{2} - \gamma\alpha + 2 = 0$$

we have also

$$2\gamma - \frac{3\alpha}{2} - 2\gamma + \frac{2}{\alpha} = \alpha - \gamma < 0$$

then

$$\Phi' \leq -\frac{\alpha}{2}\Psi + \frac{\alpha}{2}|f|^2$$

We have:

$$\alpha = \gamma - \sqrt{\gamma^2 - 4} = \frac{4}{\gamma + \sqrt{\gamma^2 - 4}} \leq \frac{4}{\gamma}$$

then, from (3.12)

$$0 < \frac{\alpha^2}{4} \leq \frac{4}{\gamma^2} < 1$$

We have

$$\begin{aligned} \Psi &= |A^{\frac{1}{2}}u'|^2 + \alpha(Au, A^{\frac{1}{2}}u') + |Au|^2 \\ &= |A^{\frac{1}{4}}(A^{\frac{1}{4}}u' + \frac{\alpha}{2}A^{\frac{3}{4}}u)|^2 + (1 - \frac{\alpha^2}{4})|Au|^2 \\ &\geq \sqrt{\lambda_1}|A^{\frac{1}{4}}u' + \frac{\alpha}{2}A^{\frac{3}{4}}u|^2 + (1 - \frac{\alpha^2}{4})|A^{\frac{3}{4}}u|^2 \\ &= \sqrt{\lambda_1}\Phi \end{aligned}$$

Hence

$$\Phi' \leq -\frac{\alpha\sqrt{\lambda_1}}{2}\Phi + \frac{\alpha}{2}|f|^2$$

since  $\Phi$  is bounded, we have

$$\forall t \in \mathbb{R}, \quad \Phi(t) \leq \frac{1}{\sqrt{\lambda_1}}\|f(t)\|_\infty^2$$

which means

$$\forall t \in \mathbb{R}, \quad |A^{\frac{1}{4}}u'(t)|^2 + |A^{\frac{3}{4}}u(t)|^2 + \alpha(A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}u'(t)) \leq \frac{1}{\sqrt{\lambda_1}}\|f(t)\|_\infty^2$$

Then

$$\forall t \in \mathbb{R}, \quad \sqrt{\lambda_1}|A^{\frac{1}{2}}u(t)|^2 + \frac{\alpha}{2}\frac{d}{dt}|A^{\frac{1}{2}}u(t)|^2 \leq \frac{1}{\sqrt{\lambda_1}}\|f(t)\|_\infty^2$$

Finally, since  $u$  is bounded in  $V$  on  $\mathbb{R}$ , we obtain

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \leq \frac{1}{\sqrt{\lambda_1}}\|f(t)\|_\infty \tag{3.13}$$

□

**Remark 3.5.** By this method, we do not recover the estimate of  $u'$  from [12] in the strongly damped case  $\gamma > 2$ .

## 4 Main examples

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  and  $\gamma > 0$ .

**Example 4.1.** We consider the following equation

$$\begin{cases} u_{tt} - \Delta u + \gamma u_t = f \\ u_{/\partial\Omega} = 0 \end{cases} \quad (4.1)$$

Then, as a consequence of (3.5) we have the following result valid for all mild solutions

$$\overline{\lim}_{t \rightarrow \infty} \left\{ \int_{\Omega} \|\nabla u\|^2 dx \right\}^{\frac{1}{2}} \leq \max \left( \sqrt{\frac{3}{\lambda_1(\Omega)}}, \frac{3}{\sqrt{2}\gamma} \right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

This result improves on [11] when  $\sqrt{\frac{\lambda_1(\Omega)}{2}} < \gamma < \sqrt{2\lambda_1(\Omega)}$ .

**Example 4.2.** We consider the equation

$$\begin{cases} u_{tt} - \Delta u - \gamma \Delta u_t = f \\ u_{/\partial\Omega} = 0 \end{cases} \quad (4.2)$$

We have the following result valid for all mild solutions

$$\overline{\lim}_{t \rightarrow \infty} \left\{ \int_{\Omega} \|\nabla u\|^2 dx \right\}^{\frac{1}{2}} \leq \max \left( \sqrt{\frac{3}{\lambda_1(\Omega)}}, \frac{3}{\sqrt{2}\gamma\lambda_1(\Omega)} \right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

**Example 4.3.** We consider the equation

$$\begin{cases} u_{tt} + \Delta^2 u - \gamma \Delta u_t = f \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{cases} \quad (4.3)$$

Then, we have for all mild solutions

$$\overline{\lim}_{t \rightarrow \infty} \left\{ \int_{\Omega} |\Delta u|^2 dx \right\}^{\frac{1}{2}} \leq \frac{1}{\lambda_1(\Omega)} \max \left( 1, \frac{2}{\gamma} \right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

This follows from Theorem 3.4 since here  $\lambda_1(A) = \lambda_1(\Omega)^2$

**Example 4.4.** We consider the equation

$$\begin{cases} u_{tt} + \Delta^2 u - \gamma \Delta u_t = f \\ u = |\nabla u| = 0 \quad \text{on } \partial\Omega \end{cases} \quad (4.4)$$

Then, we shall establish

$$\overline{\lim}_{t \rightarrow \infty} \left\{ \int_{\Omega} |\Delta u|^2 dx \right\}^{\frac{1}{2}} \leq \max \left( \sqrt{\frac{3}{\lambda_1(\Omega)\lambda_1(A)}}, \frac{3}{\sqrt{2}\gamma\lambda_1(\Omega)} \right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

Indeed, in this example, we have

$$B = -\gamma\Delta : H_0^1 \rightarrow H^{-1}; \quad A = \Delta^2$$

with domain

$$D(A) = \{u \in H^2(\Omega) \mid u = |\nabla u| = 0 \text{ on } \partial\Omega\}$$

and

$$cI \leq B \leq CA,$$

with

$$c = \gamma\lambda_1(\Omega)$$

To get an estimate for  $C$  we observe that

$$\begin{aligned} (Bv, v) &= \gamma \int_{\Omega} \|\nabla u\|^2 dx \\ &= -\gamma \int_{\Omega} \Delta v \cdot v dx \\ &= \gamma \left( \int_{\Omega} |\Delta v|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |v|^2 ds \right)^{\frac{1}{2}} \\ &= \frac{\gamma}{\lambda_1(A)} \left( \int_{\Omega} |\Delta v|^2 dx \right) \\ &= \frac{\gamma}{\lambda_1(A)} (Av, v) \end{aligned}$$

Therefore, we can take  $C \leq \frac{\gamma}{\lambda_1(A)}$  and this shows the claim.

**Remark 4.5.** Actually, since we used a Cauchy-Schwarz inequality for two linearly independent functions it is clear that the optimal value of  $C$  is strictly less than  $\frac{\gamma}{\lambda_1(A)}$ . More precisely to obtain the optimum we need to evaluate

$$\mu = \inf \left\{ \frac{\int_{\Omega} |\Delta v|^2 dx}{\int_{\Omega} |\nabla v|^2 dx}, v \in H_0^2(\Omega), v \neq 0 \right\} = \inf \left\{ \int_{\Omega} |\Delta v|^2 dx, v \in H_0^2(\Omega), \int_{\Omega} |\nabla v|^2 = 1 \right\}$$

By the Lagrange multiplier theory, there is  $v \neq 0$  such that

$$\begin{cases} \Delta^2 v = -\mu \Delta v \\ v \in H_0^2(\Omega) \end{cases}$$



with

$$\Delta v \in L^2(\Omega)$$

and

$$-(\Delta + \mu)(\Delta v) = 0$$

Then we have  $C = \frac{1}{\mu}$ . To illustrate this we consider the one dimensional case.

**Proposition 4.6.** *If  $N = 1$ ,  $\Omega = (0, \pi)$  then  $C = \frac{1}{4}$ .*

*Proof.* In order to compute  $C$  we need to find the minimal value of  $\mu$  when

$$u^{(4)} = -\mu u'', \quad u \in H_0^2(0, \pi)$$

Then, setting  $\lambda = \sqrt{\mu}$ , we have

$$u = c_1 x + c_2 + c_3 \cos(\lambda x) + c_4 \sin(\lambda x)$$

$$u' = c_1 - \lambda c_3 \sin(\lambda x) + \lambda c_4 \cos(\lambda x)$$

$$0 = c_2 + c_3$$

$$0 = c_1 \pi + c_2 + c_3 \cos(\pi \lambda) + c_4 \sin(\pi \lambda)$$

$$0 = c_1 + \lambda c_4$$

$$0 = c_1 - \lambda c_3 \sin(\lambda \pi) + \lambda c_4 \cos(\lambda \pi)$$

$$c_4 = -c_3 \sin(\pi \lambda) + c_4 \cos(\pi \lambda)$$

$$c_4(1 - \cos(\pi \lambda)) = -c_3 \sin(\pi \lambda)$$

We distinguish 3 possibilities.

**case 1:** If  $\sin(\pi \lambda) = 0$  and  $\cos(\pi \lambda) \neq 1$  ( $= -1$ ) then

$$c_4 = 0 \implies c_1 = 0, \quad c_2 = -c_3 \cos(\pi \lambda) = c_3 \implies c_2 = c_3 = 0$$

then  $u \equiv 0$  and this case is excluded.

**case 2:** If  $\sin(\pi \lambda) = 0$  and  $\cos(\pi \lambda) = 1 \implies \lambda = 2k$ ,  $k \in \mathbb{N}$

then

$$0 = c_1 \pi + c_2 + c_3 = c_2 + c_3 \implies c_1 = 0$$

and

$$c_4 = -\frac{1}{\lambda} c_1 = 0.$$

Therefore  $u = c_2(1 - \cos(2kx)) = 2c_2 \sin^2 kx$ . In this case  $\mu = 4k^2$  and therefore  $\mu \geq 4$ .

**case 3:** If  $\sin(\pi\lambda) \neq 0$ , then

$$2c_4 \sin^2\left(\frac{\pi\lambda}{2}\right) = -2c_3 \sin\left(\frac{\pi\lambda}{2}\right) \cos\left(\frac{\pi\lambda}{2}\right)$$

hence

$$c_3 = -c_4 \tan\left(\frac{\pi\lambda}{2}\right) \quad c_2 = -c_3, \quad c_1 = -\lambda c_4.$$

and

$$-\lambda\pi c_4 + c_4 \tan\left(\frac{\pi\lambda}{2}\right) - c_4 \cos(\pi\lambda) \tan\left(\frac{\pi\lambda}{2}\right) + c_4 \sin(\pi\lambda) = 0.$$

If  $c_4 = 0$ , then  $u = 0$ .

If  $c_4 \neq 0$ , we can reduce to  $c_4 = 1$ , then we find

$$\tan\left(\frac{\pi\lambda}{2}\right)(1 - \cos(\pi\lambda) + 2\cos^2\left(\frac{\pi\lambda}{2}\right)) = \lambda\pi \iff 2\tan\left(\frac{\pi\lambda}{2}\right) = \lambda\pi \iff \tan\left(\frac{\pi\lambda}{2}\right) = \frac{\pi\lambda}{2}$$

Therefore

$$\frac{\pi\lambda}{2} > \pi \implies \lambda > 2$$

and

$$\mu = \lambda^2 > 4.$$

Summarizing the 3 cases we conclude that the minimal possible value of  $\mu$  is 4. □

**Corollary 4.7.** *Any mild solution  $u$  of*

$$\begin{cases} u_{tt} + u_{xxxx} - \gamma u_{xxt} = f \\ u(t, 0) = u(t, \pi) = u_x(t, 0) = u_x(t, \pi) = 0 \end{cases} \quad (4.5)$$

*satisfies the asymptotic bound:*

$$\overline{\lim}_{t \rightarrow \infty} \left\{ \int_{\Omega} |u_{xx}|^2 dx \right\}^{\frac{1}{2}} \leq \max\left(\frac{\sqrt{3}}{2}, \frac{3}{\sqrt{2}\gamma}\right) \overline{\lim}_{t \rightarrow \infty} |f(t)| \quad (4.6)$$

## 5 Additional results

### 5.1 The first eigenvalue of a square root.

At several places in this paper we used implicitly the property

$$\lambda_1(A^{\frac{1}{2}}) = (\lambda_1(A))^{\frac{1}{2}}$$

where  $A$  is a self-adjoint coercive operator. This property is obvious when  $A$  has compact inverse, but it is natural to ask what happens in general. In the next subsection we shall derive a similar property for any positive power of  $A$ , but in the case of square roots an easier proof can be given. The result is as follows

**Proposition 5.1.** *Let  $A$  be as the introduction. Then  $A^{\frac{1}{2}}$  is also coercive and  $\lambda_1(A^{\frac{1}{2}}) = (\lambda_1(A))^{\frac{1}{2}}$ .*

The proof of this proposition relies on 2 simple lemmas :

**Lemma 5.2.** *Let  $B \in L(H)$  be symmetric and nonnegative. Then we have*

$$\begin{aligned} \|B^2\| &= \|B\|^2 \\ \forall v \in H, \quad |Bv|^2 &\leq \|B\|(Bv, v) \end{aligned}$$

**Proof.** First we have  $B^2 \in L(H)$  and  $\|B^2\| \leq \|B\|^2$ . The reverse inequality is also immediate since

$$|Bu|^2 = (B^2u, u) \leq \|B^2\|\|u\|^2$$

Finally we have for any  $v \in H$

$$|Bv|^2 = |B^{\frac{1}{2}}(B^{\frac{1}{2}}v)|^2 \leq \|B^{\frac{1}{2}}\|^2 |B^{\frac{1}{2}}v|^2 = \|B\|(Bv, v)$$

□

**Lemma 5.3.** *Let  $A$  be a self-adjoint, positive, coercive operator. Then*

$$\lambda_1(A) = \frac{1}{\|A^{-1}\|}$$

**Proof.** By definition it is clear that

$$\lambda_1(A) = \frac{1}{\|A^{-\frac{1}{2}}\|^2}$$

Then the result follows from the previous Lemma.

□

**Proof of Proposition 5.1.** We first show that  $A^{\frac{1}{2}}$  is coercive. Actually  $A^{\frac{1}{2}} \in L(V, H)$  is clearly injective. Moreover for any  $h \in H$ , there is  $u \in D(A)$  with  $Au = h$ . But then  $v = A^{\frac{1}{2}}u \in V$  and  $A^{\frac{1}{2}}v = h$ . Hence  $A^{\frac{1}{2}} \in L(V, H)$  is onto and by Banach Theorem,  $A^{-\frac{1}{2}} \in L(H, V)$ . By Lemma 5.2 we find that that  $A^{\frac{1}{2}}$  is coercive. Then  $\lambda_1(A^{\frac{1}{2}}) = \frac{1}{\|A^{-\frac{1}{2}}\|}$  and the result follows from a last application of Lemma 5.2  $\square$

## 5.2 The first eigenvalue of a fractional power.

Let  $A$  be a self-adjoint coercive operator. The fractional power  $A^\alpha$  with  $\alpha \in (0, 1)$  is defined as the inverse of the operator

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{-\alpha} (tI + A)^{-1} dt$$

with domain equal to the range of  $A^{-\alpha}$ .  $D(A^\alpha)$  is also the closure of  $D(A)$  under the seminorm  $p_\alpha(u) := |A^\alpha(u)|$  (cf. e.g. [2, 13]).

We now generalize Proposition 5.1 to any positive power by relying on the above formula.

**Proposition 5.4.** *For any  $\alpha \in (0, 1)$ ,  $A^\alpha$  is also coercive and  $\lambda_1(A^\alpha) = (\lambda_1(A))^\alpha$ .*

**Proof.** By homogeneity it is clearly sufficient to establish the result when  $\lambda_1(A) = 1$ . Then applying the result to  $A_1 = \lambda_1(A)^{-1}A$  gives the general case. First we show that

$$\lambda_1(A) = 1 \implies \lambda_1(A^\alpha) \geq 1$$

Indeed we have

$$\|A^{-\alpha}\| \leq \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{-\alpha} \|(tI + A)^{-1}\| dt \leq \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{-\alpha} (t+1)^{-1} dt = 1$$

and then Lemma 5.3 gives the result. Now we have for any  $u \in D(A)$

$$(Au, u) = (A^\alpha A^{1-\alpha} u, u) = (A^\alpha A^\beta u, A^\beta u)$$

with  $\beta = \frac{1-\alpha}{2}$ . Hence

$$(Au, u) \geq \lambda_1(A^\alpha) |A^\beta u, A^\beta u|^2 = \lambda_1(A^\alpha) (A^{1-\alpha} u, u) \geq \lambda_1(A^\alpha) \lambda_1(A^{1-\alpha}) |u|^2$$

Then

$$1 = \lambda_1(A) \geq \lambda_1(A^\alpha) \lambda_1(A^{1-\alpha})$$

Finally

$$\lambda_1(A^\alpha) = \lambda_1(A^{1-\alpha}) = 1$$

$\square$

### 5.3 The relationship between the two main results.

In Remark 2.3 we said that when  $B$  is linear and self-adjoint, the two inequalities (1.3) and (2.2) are equivalent. This is a consequence of the following

**Proposition 5.5.** *Let  $A$  be as the introduction and  $\in L(V, V')$  be symmetric and nonnegative. Then the 3 following conditions are equivalent*

$$\|B\|_{L(V, V')} \leq C \quad (5.1)$$

$$B \leq CA \quad (5.2)$$

$$\forall u \in V, \quad \|Bu\|_*^2 \leq C\langle Bu, u \rangle \quad (5.3)$$

**Proof.** We proceed in 3 steps

1) Proof of (5.1)  $\implies$  (5.2). Assuming (5.1) we have

$$\forall u \in V, \quad \langle Bu, u \rangle \leq \|B\|_{L(V, V')} \|u\| \leq C \|u\|^2 = C \langle Au, u \rangle$$

Hence  $B \leq CA$ .

2) Proof of (5.3)  $\implies$  (5.1). Assuming (5.3) we have

$$\forall u \in V, \quad \|Bu\|_*^2 \leq C \langle Bu, u \rangle \leq C \|Bu\|_* \|u\|$$

Hence, either  $Bu = 0$  or  $\|Bu\|_* \leq C \|u\|$  and we have (5.1).

3) Proof of (5.2)  $\implies$  (5.3). Since  $B \geq 0$  we have

$$\forall (u, v) \in V \times V, \quad \langle Bu, v \rangle^2 \leq \langle Bu, u \rangle \langle Bv, v \rangle$$

In this formula we choose  $v = A^{-1}(Bu)$  Then

$$\langle Bu, v \rangle = \langle Bu, A^{-1}(Bu) \rangle = \|A^{-1}Bu\|^2 = \|Bu\|_*^2$$

so that we find

$$\|Bu\|_*^4 \leq \langle Bu, u \rangle \langle BA^{-1}(Bu), A^{-1}(Bu) \rangle \leq C \langle Bu, u \rangle \langle AA^{-1}(Bu), A^{-1}(Bu) \rangle$$

by using (5.2). Now

$$\langle Bu, u \rangle \langle AA^{-1}(Bu), A^{-1}(Bu) \rangle = \langle Bu, u \rangle \|A^{-1}(Bu)\|^2 = \|Bu\|_*^2 \langle Bu, u \rangle$$

and if  $Bu \neq 0$  we obtain (5.1) on dividing through by  $\|Bu\|_*^2$ .  $\square$

**Remark 5.6.** For a general positive operator the conditions are not equivalent . For instance take  $V = H = \mathbb{C}$  and for some  $\alpha > 0, \beta \in \mathbb{R}$

$$\forall v \in V, \quad Bv = (\alpha + i\beta)v$$

In this case we have

$$\|B\| = (\alpha^2 + \beta^2)^{\frac{1}{2}}$$

$$\forall v \in V, \quad (Bv, v) = \alpha|v|^2$$

so that the optimal value of  $C$  in (5.2) is  $\alpha$ . The optimal value of  $C$  in (5.3) is  $\frac{\alpha^2 + \beta^2}{\alpha}$ . As soon as  $\beta \neq 0$  we have

$$\alpha < (\alpha^2 + \beta^2)^{\frac{1}{2}} < \frac{\alpha^2 + \beta^2}{\alpha}$$

and therefore the three constants are all different.

## 5.4 Some more examples.

Sometimes Theorem 2.1 can be applied to equations in unbounded domains. For brevity we give only 2 typical examples

**Example 5.7.** Let  $\Omega$  be a possibly unbounded domain in  $\mathbb{R}^N$  and  $m > 0, \gamma > 0$ . We consider the following equation

$$\begin{cases} u_{tt} - \Delta u + mu + \gamma u_t = f \\ u_{/\partial\Omega} = 0 \end{cases} \quad (5.4)$$

Then, as a consequence of (3.5) we have the following result valid for all mild solutions

$$\overline{\lim}_{t \rightarrow \infty} \left\{ \int_{\Omega} m|u|^2 + \|\nabla u\|^2 dx \right\}^{\frac{1}{2}} \leq \max \left( \sqrt{\frac{3}{m + \lambda_1(\Omega)}}, \frac{3}{\sqrt{2\gamma}} \right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

**Example 5.8.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  and  $\gamma > 0$ . We consider the cylinder  $\mathcal{C} = \Omega \times \mathbb{R}$  and the following equation in  $\mathbb{R}^+ \times \mathcal{C}$

$$\begin{cases} u_{tt} - \Delta u + \gamma u_t = f \\ u_{/\partial\mathcal{C}} = 0 \end{cases} \quad (5.5)$$

Then, as a consequence of (3.5), since  $A = -\Delta$  is coercive in  $\mathcal{C}$  with  $\lambda_1(\mathcal{C}) = \lambda_1(\Omega)$  we have the following result valid for all mild solutions

$$\overline{\lim}_{t \rightarrow \infty} \left\{ \int_{\Omega} \|\nabla u\|^2 dx \right\}^{\frac{1}{2}} \leq \max \left( \sqrt{\frac{3}{\lambda_1(\Omega)}}, \frac{3}{\sqrt{2\gamma}} \right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

We conclude this section by giving 2 examples of application for Theorem 1.1 and Theorem 2.1: a case where  $B$  is not selfadjoint and a case where  $B$  is non linear.

**Example 5.9.** Let  $V = H = \mathbb{C}$ . Then any solution  $u$  of the ODE

$$u'' + u + (\alpha + i\beta)u' = f \tag{5.6}$$

satisfies

$$\max(\overline{\lim}_{t \rightarrow \infty} |u(t)|, \overline{\lim}_{t \rightarrow \infty} |u'(t)|) \leq \max(\sqrt{12} \sqrt{1 + \frac{\beta^2}{\alpha^2}}, \frac{3}{\alpha}) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

We did not investigate how close from optimality this estimate is.

**Example 5.10.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  and  $\gamma > 0$ . We consider the following equation

$$\begin{cases} u_{tt} - \Delta u + \alpha(t, x)u_t^+ - \beta(t, x)u_t^- = f \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5.7}$$

where  $\alpha, \beta \in C^1(\mathbb{R}^+, C^0(\overline{\Omega}))$  are nonnegative functions with

$$0 < a \leq \min(\alpha(t, x), \beta(t, x)) \leq \max(\alpha(t, x), \beta(t, x)) \leq A.$$

It is tempting to apply Theorem 1.1 in this situation. However it is better to use Theorem 2.1 as follows. First we can approach the solutions by strong solutions with  $f$  replaced by a smooth function with a smaller or equal  $L^2(\Omega)$ -ultimate bound. For such a solution we can write

$$\alpha(t, x)u_t^+ - \beta(t, x)u_t^- = B(t, x)u_t$$

where

$$B(t, x) = \alpha(t, x)\chi(u_t > 0) - \beta(t, x)\chi(u_t \leq 0)$$

is a multiplication operator. Then, as a consequence of Theorem 2.1 we find

$$\overline{\lim}_{t \rightarrow \infty} \left\{ \int_{\Omega} \|\nabla u\|^2 dx \right\}^{\frac{1}{2}} \leq \max \left( \sqrt{\frac{3A}{a\lambda_1(\Omega)}}, \frac{3}{a\sqrt{2}} \right) \overline{\lim}_{t \rightarrow \infty} |f(t)|$$

We skip the details.

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