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► **To cite this version:**

Thomas Dedieu, Fabio Perroni. The fundamental group of quotients of a product of curves. *Journal of Group Theory*, 2012, 15 (3), pp.439-453. hal-00715735

HAL Id: hal-00715735

<https://hal.science/hal-00715735>

Submitted on 9 Jul 2012

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THE FUNDAMENTAL GROUP OF QUOTIENTS OF A PRODUCT OF CURVES

THOMAS DEDIEU AND FABIO PERRONI

This article is dedicated to the memory of Fritz Grunewald

ABSTRACT. We prove a structure theorem for the fundamental group of the quotient X of a product of curves by the action of a finite group G , hence for that of any resolution of the singularities of X .

1. INTRODUCTION

The study of varieties isogenous to a product of curves was initiated by Catanese in [Cat00], inspired by a construction of Beauville. These varieties are quotients of a product of smooth projective curves $C_1 \times \cdots \times C_n$ by the free action of a finite group G .

Much of the work in this area has been focused in the $n = 2$ case. Surfaces isogenous to a product of curves provide a wide class of surfaces quite manageable to work with, since they are determined by discrete combinatorial data. They were used successfully to address various questions (see e.g. the survey paper [BCP06]), and in particular to obtain substantial information about various moduli spaces of surfaces of general type (see e.g. [BC04, BCG08, BCGP09]).

In the case of a variety isogenous to a product, the action of G is free, and $X := (C_1 \times \cdots \times C_n)/G$ is smooth. Furthermore, we have the following natural description of the fundamental group of X .

Proposition 1.1. [Cat00] *If $X := (C_1 \times \cdots \times C_n)/G$ is the quotient of a product of curves by the free action of a finite group, then the fundamental group of X sits in an exact sequence*

$$1 \rightarrow \Pi_{g_1} \times \cdots \times \Pi_{g_n} \rightarrow \pi_1(X) \rightarrow G \rightarrow 1, \quad (1.1)$$

where each Π_{g_i} is the fundamental group of C_i . This extension, in the unmixed case where each Π_{g_i} is a normal subgroup, is determined by the associated maps $G \rightarrow \text{Out}(\Pi_{g_i})$ to the respective Teichmüller modular groups.

In the recent paper [BCGP09], Bauer, Catanese, Grunewald and Pignatelli prove that a similar statement still holds under weaker assumptions.

Theorem 1.2. [BCGP09, Thm. 0.10 and Thm. 4.1] *Assume that G acts faithfully on each curve C_i as a group of automorphisms, and let $X := (C_1 \times \cdots \times C_n)/G$ be the (possibly singular) quotient by the diagonal action of G . Then the fundamental group $\pi_1(X)$ has a normal subgroup of finite index isomorphic to the product of n surface groups. We call G' the quotient group.*

Date: April 1, 2010.

2000 *Mathematics Subject Classification.* 32Q55, 14H30, 20F34.

Here, by a surface group we mean a group isomorphic to the fundamental group of a compact Riemann surface. Note that, unlike in Proposition 1.1, the surface groups in Theorem 1.2 above are not necessarily isomorphic to the fundamental groups of the curves C_1, \dots, C_n , and furthermore that the corresponding quotient G' of $\pi_1(X)$ is not necessarily isomorphic to G .

The first step of the proof of Theorem 1.2 consists in showing that $\pi_1(X)$ is isomorphic to the quotient of the fibre product $\mathbf{T} := \mathbf{T}_1 \times_G \cdots \times_G \mathbf{T}_n$ of n orbifold surface groups (see Subsection 2.1) by its torsion subgroup $\text{Tors}(\mathbf{T})$. Whereas this first part rests upon geometrical considerations, the rest of the proof relies on an abstract group theoretic argument showing that this quotient necessarily contains a normal subgroup as described in Theorem 1.2. In particular, the relation occurring between the groups G and G' is not well understood.

Using a suitable resolution of the singularities of X , Bauer, Catanese, Grunewald and Pignatelli show in addition that the fundamental group of any resolution Y of X is isomorphic to the fundamental group of X , so that the same description holds for $\pi_1(Y)$.

In [BCGP09], as an important application of Theorem 1.2, many new families of algebraic surfaces S of general type with $p_g(S) = 0$ are constructed, and several new examples of groups are realized as the fundamental group of an algebraic surface S of general type with $p_g(S) = 0$. This increases notably our knowledge on algebraic surfaces. In fact the authors consider and classify all the surfaces whose canonical models arise as quotients $X := (C_1 \times C_2)/G$ of the product of two curves of genera $g(C_1), g(C_2) \geq 2$ by the action of a finite group G such that $p_g(X) = q(X) = 0$.

In the present paper, we drop the assumption that the actions of G on C_1, \dots, C_n are faithful. We obtain the following expected strengthening of Theorem 1.2.

Theorem 1.3. *Let C_1, \dots, C_n be smooth projective curves, and let G be a finite group acting on each C_i as a group of automorphisms. Then the fundamental group of the quotient $X := C_1 \times \cdots \times C_n/G$ by the diagonal action of G has a normal subgroup of finite index that is isomorphic to the product of n surface groups.*

This result should allow in the future the realization of interesting groups as fundamental groups of higher dimensional algebraic varieties, following the method developed in [BCGP09] for surfaces. Notice that, in the case where the G -actions are faithful, X can only have isolated cyclic-quotient singularities, while if the actions are not faithful, then the singular locus of X can have components of positive dimension, and the singularities are abelian-quotient singularities.

Again, one shows that any desingularization of the quotient X has a fundamental group isomorphic to that of X . This time however, we have to rely on a strong result of Kollár [K93].

The proof follows then closely the one of Theorem 1.2 of [BCGP09]. The main new difficulty one has to overcome is to find a natural counterpart to the fibered product $\mathbf{T}_1 \times_G \cdots \times_G \mathbf{T}_n$, acting discontinuously on the product $\tilde{C}_1 \times \cdots \times \tilde{C}_n$ of the universal covers of C_1, \dots, C_n . After that similar group theoretic arguments work with some slight modifications.

It has already been observed in [Cat00] that Theorem 1.3 follows directly from Theorem 1.2 when $n = 2$, by performing the quotient $(C_1 \times C_2)/G$ in successive steps. For $n > 2$ however, this procedure does not apply.

The paper is organized as follows. In Section 2, we fix notations and collect some basic facts about group actions on compact Riemann surfaces. Section 3 is devoted to the proof of Theorem 1.3: the proof itself is given in Subsection 3.1, using intermediate results proven in Subsections 3.2 and 3.3.

Acknowledgements. We are indebted to Prof. Ingrid Bauer for suggesting this problem to us and for many useful conversations. We express our thanks to Profs. Fabrizio Catanese and Keiji Ogusio for their interest and warm encouragement. In particular, we thank Prof. Ogusio for bringing the paper [K93] to our attention. We also wish to thank Wenfei Liu and Matteo Penegini for worthy conversations on the subject.

This work started as the first-named author held a post-doctoral position at the Universität Bayreuth, supported by the DFG-Forschergruppe 790 “Classification of Algebraic Surfaces and Compact Complex Manifolds”. He is grateful to this institution for providing excellent working conditions.

2. NOTATIONS AND BASIC RESULTS

2.1. Notations. We work over the field of complex numbers \mathbf{C} .

Let G be a group, and consider a subset $H \subset G$. We write $H \leq G$ when H is a subgroup of G , and $H \trianglelefteq G$ when H is a normal subgroup of G . If $A \subset G$ is any subset, then $\langle\langle A \rangle\rangle_G$ denotes the normal subgroup of G generated by A .

Let g be a non negative integer. We call Π_g the *surface group* of genus g , defined as

$$\Pi_g := \left\langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \right\rangle.$$

This is the fundamental group of compact Riemann surfaces of genus g . On the other hand, letting in addition m_1, \dots, m_r be positive integers, we denote by $\mathbf{T}(g; m_1, \dots, m_r)$ the *orbifold surface group* of signature $(g; m_1, \dots, m_r)$, defined as

$$\mathbf{T}(g; m_1, \dots, m_r) := \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r \mid \begin{aligned} &c_1^{m_1} = \dots = c_r^{m_r} = \prod_{i=1}^g [a_i, b_i] \cdot c_1 \cdot \dots \cdot c_r = 1 \end{aligned} \right\rangle.$$

It is obtained from the fundamental group of the complement of a set of r distinct points in a compact Riemann surface of genus g , by quotienting by the normal subgroup generated by $\gamma_1^{m_1}, \dots, \gamma_r^{m_r}$, where each γ_i is a simple geometric counter-clockwise loop around the i -th removed point.

Let G be a finite group. An *appropriate orbifold homomorphism* is a surjective homomorphism $\varphi : \mathbf{T}(g; m_1, \dots, m_r) \rightarrow G$ such that $\varphi(c_i)$ has order m_i for $i = 1, \dots, r$.

The action of a group G as a group of homeomorphisms on a topological space X is said to be *discontinuous* if the following two conditions are satisfied: (i) the stabilizer of each point of X is finite; (ii) each point $x \in X$ has a neighbourhood U such that $g(U) \cap U = \emptyset$, for each $g \in G$ such that $gx \neq x$.

2.2. Basic results. The following result is essentially a reformulation of Riemann's existence theorem (see [BCGP09, Thm. 2.1]).

Theorem 2.1. *A finite group G acts faithfully as a group of automorphisms on a compact Riemann surface of genus g if and only if there are natural numbers g', m_1, \dots, m_r and an appropriate orbifold homomorphism*

$$\varphi : \mathbf{T}(g'; m_1, \dots, m_r) \rightarrow G$$

such that the Riemann-Hurwitz relation holds:

$$2g - 2 = |G| \left(2g' - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right).$$

Remark 2.2. As already remarked in [BCGP09], under the above hypotheses, g' is the geometric genus of $C' := C/G$, and m_1, \dots, m_r are the branching indices at the branching points of the G -cover $p: C \rightarrow C'$. The appropriate orbifold homomorphism φ is induced by the monodromy of the Galois étale G -covering $p^\circ: C^\circ \rightarrow C'^\circ$ induced by p , where C'° is the Riemann surface obtained from C' by removing the branch points of p , and $C^\circ := p^{-1}(C'^\circ)$. In particular, $\varphi(c_i)$ generates the stabilizer of the corresponding ramification point.

Furthermore, the kernel of φ is isomorphic to the fundamental group $\pi_1(C)$, and the action of $\pi_1(C)$ on the universal cover \tilde{C} of C extends to a discontinuous action of $\mathbf{T} := \mathbf{T}(g'; m_1, \dots, m_r)$. Let $u: \tilde{C} \rightarrow C$ be the covering map. It is φ -equivariant, and $C/G \cong \tilde{C}/\mathbf{T}$.

We now give two elementary facts that will be used in the following.

Lemma 2.3. (i) *Let $x \in \tilde{C}$. Then the restriction of φ to the stabilizer St_x of x (with respect to the action of \mathbf{T} on \tilde{C}) is injective.*
(ii) *Let $t \in \text{St}_x$. Then t is conjugated to c_i^m , for some $i \in \{1, \dots, r\}$ and $m \in \mathbf{N}$.*

Proof. The $\pi_1(C)$ -action on \tilde{C} is free, so $\pi_1(C) \cap \text{St}_x = \{1\}$. This yields (i), because $\pi_1(C)$ is the kernel of φ .

To prove (ii), let $y = u(x)$. If $t = 1$, then the result is clear. Else, there exists an integer $i \in \{1, \dots, m\}$ and a point $x' \in u^{-1}(y)$ that is fixed by c_i . It then follows from (i) that $\text{St}_{x'} = \langle c_i \rangle$, hence that t is conjugated to a power of c_i . □

3. MAIN THEOREM

The main result of the paper is the following

Theorem 3.1. *Let C_1, \dots, C_n be compact Riemann surfaces, and let G be a finite group that acts as a group of automorphisms on each C_i . We consider the quotient of the product $C_1 \times \dots \times C_n$ by the diagonal action of G . Then there is a normal subgroup of finite index Π in the fundamental group*

$$\pi_1 \left(\frac{C_1 \times \dots \times C_n}{G} \right),$$

such that Π is isomorphic to the product of n surface groups.

Notice that, according to notations 2.1, a surface group is a group isomorphic to the fundamental group of a compact Riemann surface of genus a non negative integer g , in particular we admit also the “degenerate cases” where $g = 0, 1$.

The proof of this theorem follows closely that of [BCGP09, Thm. 4.1], and is given in the next subsections. Before we move on to this proof, let us give the following important consequence of Theorem 3.1.

Corollary 3.2. *Let C_1, \dots, C_n and G be as in the statement of Theorem 3.1, and let Y be a resolution of the singularities of $X := (C_1 \times \dots \times C_n)/G$. Then, the fundamental group of Y is isomorphic to the fundamental group of X , and moreover it has a normal subgroup of finite index isomorphic to the product of n surface groups.*

Proof. The natural morphism

$$f_* : \pi_1(Y) \longrightarrow \pi_1(X)$$

induced by the resolution $f : Y \rightarrow X$ is an isomorphism. This follows directly from [K93, Sec. 7]: since X is normal and only has quotient singularities, Y is locally simply connected by [K93, Thm. 7.2], hence f_* is an isomorphism by [K93, Lem. 7.2]. The second claim is now a direct consequence of Theorem 3.1. \square

3.1. Proof of the main theorem. For $i = 1, \dots, n$, we let

$$K_i = \ker(G \rightarrow \text{Aut}(C_i)) \quad \text{and} \quad H_i = G/K_i,$$

where $G \rightarrow \text{Aut}(C_i)$ is the morphism associated to the action of G on C_i . We call p_i the projection $G \rightarrow H_i$. Now H_i acts faithfully on C_i , so we have (see Remark 2.2) a short exact sequence

$$1 \rightarrow \pi_1(C_i) \rightarrow \mathbf{T}_i \xrightarrow{\varphi_i} H_i \rightarrow 1, \quad (3.1)$$

where \mathbf{T}_i is an orbifold surface group, and φ_i is an appropriate orbifold homomorphism. Let $\Sigma_i := G \times_{H_i} \mathbf{T}_i$ be the fibered product corresponding to the Cartesian diagram

$$\begin{array}{ccc} \Sigma_i & \longrightarrow & \mathbf{T}_i \\ \psi_i \downarrow & \square & \downarrow \varphi_i \\ G & \xrightarrow{p_i} & H_i. \end{array}$$

We call $\psi_i : \Sigma_i \rightarrow G$ the projection on the first factor. Pulling-back (3.1) by $p_i : G \rightarrow H_i$, we obtain a short exact sequence

$$1 \rightarrow \pi_1(C_i) \rightarrow \Sigma_i \xrightarrow{\psi_i} G \rightarrow 1, \quad (3.2)$$

where the left-hand side map is $\gamma \in \pi_1(C_i) \mapsto (1, \gamma) \in \Sigma_i$.

Next, we define $\tilde{G} := \Sigma_1 \times_G \dots \times_G \Sigma_n$ as the fibered product corresponding to the Cartesian diagram below.

$$\begin{array}{ccccc} & & \tilde{G} & & \\ & \swarrow & & \searrow & \\ \Sigma_1 & & & & \Sigma_n \\ & \swarrow & & \searrow & \\ & & G & & \end{array}$$

Let $\Delta : G \rightarrow G \times \dots \times G$ be the diagonal morphism. Then \tilde{G} can also be seen as the fibered product $G \times_{(G \times \dots \times G)} (\Sigma_1 \times \dots \times \Sigma_n) \rightarrow G$ with respect to the two

morphisms Δ and (ψ_1, \dots, ψ_n) . Therefore, the pull-back by Δ of the product of the n exact sequences (3.2) for $i = 1, \dots, n$ is a short exact sequence

$$1 \rightarrow \prod_{i=1}^n \pi_1(C_i) \rightarrow \tilde{G} \xrightarrow{\Psi} G \rightarrow 1, \quad (3.3)$$

where Ψ is the first projection $G \times_{(G \times \dots \times G)} (\Sigma_1 \times \dots \times \Sigma_n) \rightarrow G$.

Now we have the following, coming from the fact that \tilde{G} acts discontinuously on the universal cover of $C_1 \times \dots \times C_n$.

Proposition 3.3. *Let $\tilde{G}' \trianglelefteq \tilde{G}$ be the normal subgroup of \tilde{G} generated by those elements which have non-empty fixed-point set. Then*

$$\pi_1 \left(\frac{C_1 \times \dots \times C_n}{G} \right) \cong \frac{\tilde{G}}{\tilde{G}'}$$

Proof. For $i = 1, \dots, n$, the action of \mathbf{T}_i on the universal covering \tilde{C}_i of C_i (see Remark 2.2) induces an action of Σ_i on \tilde{C}_i via the projection of Σ_i on its second factor \mathbf{T}_i . We obtain in this way an action of \tilde{G} on the product $\tilde{C}_1 \times \dots \times \tilde{C}_n$.

This action is discontinuous: let St_x be the stabilizer of a point $x \in \tilde{C}_1 \times \dots \times \tilde{C}_n$ with respect to the action of \tilde{G} . Then the same argument as that in the proof of Lemma 2.3 shows that $\Psi|_{\text{St}_x}$ is injective, from which it follows that St_x is finite because G is finite. On the other hand, condition (ii) in the definition of a discontinuous action is a consequence of the fact that the \mathbf{T}_i -actions are themselves discontinuous.

Then, the main theorem in [Arm68] applies to our situation, and gives a group isomorphism

$$\pi_1 \left(\frac{\tilde{C}_1 \times \dots \times \tilde{C}_n}{\tilde{G}} \right) \cong \frac{\tilde{G}}{\tilde{G}'}$$

Eventually, since the universal covering $\mathcal{U}: \tilde{C}_1 \times \dots \times \tilde{C}_n \rightarrow C_1 \times \dots \times C_n$ is Ψ -equivariant, we have an isomorphism

$$\frac{C_1 \times \dots \times C_n}{G} \cong \frac{\tilde{C}_1 \times \dots \times \tilde{C}_n}{\tilde{G}},$$

and the proposition follows. □

Remark 3.4. The elements of \tilde{G} which have fixed-points are precisely those elements of finite order. Therefore \tilde{G}' is the torsion subgroup of \tilde{G} .

Now the proof of Theorem 3.1 relies on the following result, the proof of which we postpone to Subsection 3.3.

Proposition 3.5. *The quotient \tilde{G}/\tilde{G}' is an extension*

$$1 \rightarrow E \rightarrow \tilde{G}/\tilde{G}' \xrightarrow{\theta} T \rightarrow 1$$

of a finite group E by a group T that is a finite-index subgroup of a product of n orbifold surface groups.

Using the results of [GJZ08], the latter fact enables one to show that there is a finite index normal subgroup $\Gamma \trianglelefteq \tilde{G}/\tilde{G}'$ that injects in T :

Lemma 3.6. *Let S be a group sitting in an exact sequence*

$$1 \rightarrow E \rightarrow S \rightarrow T \rightarrow 1,$$

where E is a finite group, and T is a finite index subgroup of a product of n orbifold surface groups. Then S is residually finite. In particular, there exists a finite index normal subgroup $\Gamma \trianglelefteq S$ such that $\Gamma \cap E = \{1\}$.

Proof. By [GJZ08, Prop. 6.1], an extension of a finite group by a group that is residually finite and good in the sense of [Ser94] is residually finite. It therefore suffices to show that T enjoys the two aforementioned properties.

An orbifold surface group is residually finite. Therefore T is itself residually finite, being a finite index subgroup of a product of orbifold surface groups.

By [GJZ08, Lem. 3.2], it is enough to show that a product of orbifold surface groups is good to prove that T is good. But [GJZ08, Prop. 3.7] tells us that an orbifold surface group is good, and [GJZ08, Prop. 3.4] that a product of good groups is good. □

We are now in a position to complete the proof of our main theorem:

Proof of Theorem 3.1. Let $\mathbf{T}_1 \times \cdots \times \mathbf{T}_n$ be a product of n orbifold surface groups containing T as a finite index subgroup, and let us consider $\Gamma \trianglelefteq \tilde{G}/\tilde{G}'$ a normal subgroup of finite index such that $E \cap \Gamma = \{1\}$. Then $\theta(\Gamma) \leq \mathbf{T}_1 \times \cdots \times \mathbf{T}_n$ has finite index.

Now every orbifold surface group contains a surface group as a finite index subgroup (see e.g. [Bea95]), so let Π_i be a finite index surface group in \mathbf{T}_i for each $i = 1, \dots, n$.

For each i , we consider

$$\theta(\Gamma)_i := \theta(\Gamma) \cap (\{1\} \times \cdots \times \mathbf{T}_i \times \cdots \times \{1\})$$

as a subgroup $\theta(\Gamma)_i \leq \mathbf{T}_i$, and set

$$\Pi'_i := \bigcap_{g \in \mathbf{T}_i} g (\theta(\Gamma)_i \cap \Pi_i) g^{-1},$$

the biggest normal subgroup of \mathbf{T}_i contained in $\theta(\Gamma)_i \cap \Pi_i$. Then Π'_i has finite index in Π_i , and thus is itself a surface group. Eventually, $\Pi := \Pi'_1 \times \cdots \times \Pi'_n$ is a subgroup of $\theta(\Gamma)$, which is normal and of finite index in T . Therefore, $\theta^{-1}(\Pi) \cap \Gamma$ is a normal subgroup of \tilde{G}/\tilde{G}' , with finite index, and isomorphic to Π . □

3.2. Results in group theory. In this subsection, we prove some technical results that are needed for the proof of Proposition 3.5.

Let Σ be any group, $R \trianglelefteq \Sigma$ be a normal subgroup, and $L \subset \Sigma$ be a subset. We define

$$N(R, L) := \langle \langle \{hkh^{-1}k^{-1} \mid h \in L, k \in R\} \rangle \rangle_{\Sigma} \quad (3.4)$$

and

$$\hat{\Sigma} := \hat{\Sigma}(R, L) := \Sigma/N(R, L). \quad (3.5)$$

We call \hat{R} and \hat{L} the images of R and L respectively by the projection $\Sigma \rightarrow \hat{\Sigma}$. There is an isomorphism: $\hat{\Sigma}/\langle \langle \hat{L} \rangle \rangle_{\hat{\Sigma}} \cong \Sigma/\langle \langle L \rangle \rangle_{\Sigma}$. Notice also that $N(R, L) \trianglelefteq R$ and $N(R, L) \trianglelefteq \langle \langle L \rangle \rangle_{\Sigma}$, which implies that \hat{R} is a normal subgroup of $\hat{\Sigma}$.

Lemma 3.7. *If $R \trianglelefteq \Sigma$ has finite index, and if $L \subset \Sigma$ is a finite subset consisting of elements of finite order, then $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ is finite.*

Proof. The subgroup $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ is the image of $\langle\langle L \rangle\rangle_{\Sigma}$ under the projection $\Sigma \rightarrow \hat{\Sigma}$. Since R has finite index in Σ , and L is finite, it follows that $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ is generated by finitely many elements which are conjugated to those of \hat{L} . Since the elements of L have finite order, these generators have finite order as well.

The center $Z(\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}})$ of $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ contains $\hat{R} \cap \langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$, and hence has finite index in $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$. Now, by [BCGP09, Lem. 4.6], if a group S is generated by finitely many elements of finite order, and if its centre has finite index in S , then S is finite. From this we conclude that $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ is finite. \square

We now consider the particular case when Σ is a group constructed as in Subsection 3.1: $\Sigma = G \times_H \mathbf{T}$, where G is a finite group, H is a quotient of G , and \mathbf{T} is any group coming with a surjective morphism $\varphi : \mathbf{T} \rightarrow H$.

Lemma 3.8. *The projection on the second factor $q : \Sigma \rightarrow \mathbf{T}$ induces a morphism*

$$\bar{q} : \frac{\Sigma}{\langle\langle L \rangle\rangle_{\Sigma}} \rightarrow \frac{\mathbf{T}}{\langle\langle q(L) \rangle\rangle_{\mathbf{T}}} \quad (3.6)$$

in a natural way. It is surjective, and has finite kernel.

Proof. We have $q(\langle\langle L \rangle\rangle_{\Sigma}) = \langle\langle q(L) \rangle\rangle_{\mathbf{T}}$. The map \bar{q} is therefore induced by the composition $\Sigma \xrightarrow{q} \mathbf{T} \rightarrow \mathbf{T}/\langle\langle q(L) \rangle\rangle_{\mathbf{T}}$, which is clearly surjective. To prove the finiteness of its kernel, notice that for any $(g, t) \in q^{-1}(\langle\langle q(L) \rangle\rangle_{\mathbf{T}})$, there exists $h \in G$ with $(h, t) \in \langle\langle L \rangle\rangle_{\Sigma}$, hence $gh^{-1} \in K := \ker(G \rightarrow H)$. It follows that $q^{-1}(\langle\langle q(L) \rangle\rangle_{\mathbf{T}}) = K\langle\langle L \rangle\rangle_{\Sigma}$, where K is seen as a subgroup in Σ via the injection $k \in K \mapsto (k, 1) \in \Sigma$. Eventually, $\ker(\bar{q}) \cong K \subset G$, which is finite. \square

3.3. Realization of the fundamental group as a suitable extension. In this subsection, we give a full proof of Proposition 3.5. We use the basic results in group theory established in Subsection 3.2 above.

For $i = 1, \dots, n$, we fix the following presentation for the orbifold groups \mathbf{T}_i in (3.1):

$$\mathbf{T}_i = \left\langle a_{i1}, b_{i1}, \dots, a_{ig_i}, b_{ig_i}, c_{i1}, \dots, c_{ir_i} \mid c_{i1}^{m_{i1}} = \dots = c_{ir_i}^{m_{ir_i}} = \prod_{j=1}^{g_i} [a_{ij}, b_{ij}] \cdot c_{i1} \cdot \dots \cdot c_{ir_i} = 1 \right\rangle,$$

and set $R_i = \pi_1(C_i)$. We write the elements of \tilde{G} as (g, z_1, \dots, z_n) , with $(g, z_i) \in \Sigma_i$ for $i = 1, \dots, n$. Then we have:

Lemma 3.9. *For each $i = 1, \dots, n$, there exists a finite subset $\mathcal{N}_i \subset \tilde{G}$, such that*

$$\langle\langle \mathcal{N}_i \rangle\rangle_{\tilde{G}} = \tilde{G}',$$

and whose elements are of the form

$$(g, z_1 d_1^{\ell_1} z_1^{-1}, \dots, d_i^{\ell_i}, \dots, z_n d_n^{\ell_n} z_n^{-1})$$

for some $g \in G$, some $d_j \in \{c_{j1}, \dots, c_{jr_j}\}$ and $\ell_j \in \mathbf{N}$ for $j = 1, \dots, n$, and some $z_j \in \mathbf{T}_j$ for $j \neq i$.

Remark 3.10. As a direct consequence of Lemma 3.9, if

$$(g, z_1 d_1^{\ell_1} z_1^{-1}, \dots, d_i^{\ell_i}, \dots, z_n d_n^{\ell_n} z_n^{-1}) \in \mathcal{N}_i$$

for some i , then for any $j \neq i$, there exists

$$(h, y_1, \dots, \widehat{y}_j, \dots, y_n) \in \Sigma_1 \times_G \cdots \times_G \widehat{\Sigma}_j \times_G \cdots \Sigma_n$$

(where a hat means that the corresponding factor is omitted), such that

$$(hgh^{-1}, y_1 z_1 d_1^{\ell_1} z_1^{-1} y_1^{-1}, \dots, y_i d_i^{\ell_i} y_i^{-1}, \dots, d_j^{\ell_j}, \dots, y_n z_n d_n^{\ell_n} z_n^{-1} y_n^{-1}) \in \mathcal{N}_j.$$

Proof of Lemma 3.9. Let $s \in \tilde{G}$ be an element with non empty fixed-point set, and let us fix $i \in \{1, \dots, n\}$. By Lemma 2.3 (ii), s writes

$$s = (g, z_1 d_1^{\ell_1} z_1^{-1}, \dots, z_n d_n^{\ell_n} z_n^{-1}),$$

with notations as in the statement of the Lemma. Obviously, one can find $h \in G$, and $\zeta_j \in \mathbf{T}_j$ for each $j \neq i$, such that $(h, \zeta_1, \dots, z_i^{-1}, \dots, \zeta_n) \in \tilde{G}$, and therefore s is conjugated in \tilde{G} to an element of type

$$(g', y_1 d_1^{\ell_1} y_1^{-1}, \dots, d_i^{\ell_i}, \dots, y_n d_n^{\ell_n} y_n^{-1}). \quad (3.7)$$

Now we claim that there exists finite sets $A_j \subset \mathbf{T}_j$, $j = 1, \dots, \widehat{i}, \dots, n$, such that each element of \tilde{G} as in (3.7) is conjugated in \tilde{G} to some

$$(g'', x_1 d_1^{\ell_1} x_1^{-1}, \dots, d_i^{\ell_i}, \dots, x_n d_n^{\ell_n} x_n^{-1}),$$

with $x_j \in A_j$ for each $j \neq i$. Then it is clear that one can build \mathcal{N}_i as required.

To prove our claim, first note that if $(g, z_1 d_1^{\ell_1} z_1^{-1}, \dots, d_i^{\ell_i}, \dots, z_n d_n^{\ell_n} z_n^{-1}) \in \tilde{G}$, then an $(n+1)$ -uple $(g, \zeta_1 d_1^{\ell_1} \zeta_1^{-1}, \dots, d_i^{\ell_i}, \dots, \zeta_n d_n^{\ell_n} \zeta_n^{-1})$ corresponds to an element of the fibered product \tilde{G} if and only if for each $j \neq i$, $\varphi_j(z_j^{-1} \zeta_j)$ belongs to the centralizer $C_{H_j}(\varphi_j(d_j^{\ell_j}))$ of $\varphi_j(d_j^{\ell_j})$ in H_j .

Second, note that if $k_j \in R_j$ for some $j \neq i$, then $(1, 1, \dots, k_j, \dots, 1) \in \tilde{G}$, and therefore any element $(g, \dots, z_j d_j^{\ell_j} z_j^{-1}, \dots) \in \tilde{G}$ is conjugated to

$$(g, \dots, (k_j z_j) d_j^{\ell_j} (k_j z_j)^{-1}, \dots) \in \tilde{G}.$$

Then our claim follows from the fact that for each $j \neq i$, $R_j \trianglelefteq \varphi_j^{-1}(C_{H_j}(\varphi_j(d_j^{\ell_j})))$ has finite index. □

From now on, we let $\mathcal{N}_1, \dots, \mathcal{N}_n$ be as in Lemma 3.9.

Lemma 3.11. For $i = 1, \dots, n$, if

$$(g, z_1 d_1^{\ell_1} z_1^{-1}, \dots, d_i^{\ell_i}, \dots, z_n d_n^{\ell_n} z_n^{-1}) \in \mathcal{N}_i,$$

then for all $k_i \in R_i$, we have

$$(1, 1, \dots, d_i^{\ell_i} k_i d_i^{-\ell_i} k_i^{-1}, \dots, 1) \in \tilde{G}'.$$

Proof. Let $k_i \in R_i$. Then $\tilde{k}_i := (1, 1, \dots, k_i, \dots, 1) \in \tilde{G}$, and our result follows from the equality

$$(1, 1, \dots, d_i^{\ell_i} k_i d_i^{-\ell_i} k_i^{-1}, \dots, 1) = (g, z_1 d_1^{\ell_1} z_1^{-1}, \dots, d_i^{\ell_i}, \dots, z_n d_n^{\ell_n} z_n^{-1})^{-1} \tilde{k}_i^{-1},$$

and the fact that $\langle\langle \mathcal{N}_i \rangle\rangle_{\tilde{G}} = \tilde{G}'$. □

For $i = 1, \dots, n$, we let $L_i \subset \Sigma_i$ be the image of \mathcal{N}_i by the projection $\tilde{G} \rightarrow \Sigma_i$. The first projection $\psi_i : \Sigma_i \rightarrow G$ then induces an epimorphism $\hat{\psi}_i : \hat{\Sigma}_i := \hat{\Sigma}_i(R_i, L_i) \rightarrow G$ (see Subsection 3.2 for a definition of $\hat{\Sigma}_i$).

Eventually, we let \hat{G} be the fibered product

$$\hat{\Sigma}_1 \times_G \cdots \times_G \hat{\Sigma}_n \cong G \times_{(G \times \cdots \times G)} (\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n), \quad (3.8)$$

and we define a map $\Phi : \hat{G} \rightarrow \tilde{G}/\tilde{G}'$ by the formula

$$\Phi([s_1], \dots, [s_n]) = [(s_1, \dots, s_n)], \quad (3.9)$$

where $s_i \in \Sigma_i$ for each i (here we see \hat{G} as contained in $\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n$, using its description by the left-hand side of (3.8) rather than by its right-hand side, and similarly we see \tilde{G} as contained in $\Sigma_1 \times \cdots \times \Sigma_n$). It is a consequence of Lemma 3.11 that Φ is well-defined by (3.9). Now we have:

Lemma 3.12. *The morphism Φ is surjective, and has finite kernel.*

Proof. The surjectivity follows at once from (3.9). On the other hand, an element $([s_1], \dots, [s_n]) \in \hat{G}$ lies in $\ker \Phi$ if and only if $(s_1, \dots, s_n) \in \tilde{G}'$. This implies for $i = 1, \dots, n$ that $s_i \in \langle\langle L_i \rangle\rangle_{\Sigma_i}$, because $\tilde{G}' = \langle\langle \mathcal{N}_i \rangle\rangle_{\tilde{G}}$. We thus see that $\ker \Phi$, seen as contained in $\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n$, is contained in $\langle\langle \hat{L}_1 \rangle\rangle_{\hat{\Sigma}_1} \times \cdots \times \langle\langle \hat{L}_n \rangle\rangle_{\hat{\Sigma}_n}$, which is finite by Lemma 3.7. □

Next, we define a morphism

$$\Theta : \prod_{i=1}^n \hat{\Sigma}_i \rightarrow \prod_{i=1}^n \frac{\Sigma_i}{\langle\langle L_i \rangle\rangle} \xrightarrow{(\bar{q}_1, \dots, \bar{q}_n)} \prod_{i=1}^n \frac{\mathbf{T}_i}{\langle\langle q_i(L_i) \rangle\rangle} \quad (3.10)$$

as in Subsection 3.2: the left-hand side map in (3.10) is the product of the projections

$$\hat{\Sigma}_i \rightarrow \hat{\Sigma}_i / \langle\langle \hat{L}_i \rangle\rangle_{\hat{\Sigma}_i} \cong \Sigma_i / \langle\langle L_i \rangle\rangle_{\Sigma_i},$$

and the \bar{q}_i 's are induced by the second projections $q_i : \Sigma_i \rightarrow \mathbf{T}_i$ as in Lemma 3.8. We have

$$\ker \Phi \subset \langle\langle \hat{L}_1 \rangle\rangle_{\hat{\Sigma}_1} \times \cdots \times \langle\langle \hat{L}_n \rangle\rangle_{\hat{\Sigma}_n} \subset \ker \Theta, \quad (3.11)$$

and $\ker \Theta$ is finite by both Lemmas 3.7 and 3.8.

Let us set

$$T := \Theta(\hat{G}).$$

Notice that \hat{G} has finite index in $\prod_{i=1}^n \hat{\Sigma}_i$ because G is finite, and therefore that T has finite index in $\prod_{i=1}^n \mathbf{T}_i / \langle\langle q_i(L_i) \rangle\rangle$. We have a short exact sequence

$$1 \rightarrow E_1 \rightarrow \hat{G} \xrightarrow{\Theta|_{\hat{G}}} T \rightarrow 1. \quad (3.12)$$

Clearly, $\ker \Phi \subset E_1$. Therefore, setting $E := E_1/\ker \Phi$, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & (3.13) \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \ker \Phi & \xrightarrow{=} & \ker \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & E_1 & \longrightarrow & \hat{G} & \xrightarrow{\Theta|_{\hat{G}}} & T \longrightarrow 1 \\
 & & \downarrow & & \downarrow \Phi & & \downarrow = \\
 1 & \longrightarrow & E & \longrightarrow & \tilde{G}/\tilde{G}' & \xrightarrow{\theta} & T \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

where θ is the morphism induced by $\Theta|_{\hat{G}}$ which makes the diagram commutative.

We then claim that the lower row of the diagram (3.13) is the short exact sequence we are looking for: exactness follows from an easy diagram chase; the finiteness of E follows from that of E_1 ; eventually, each $\mathbf{T}_i/\langle\langle q_i(L_i) \rangle\rangle$ is an orbifold surface group, because $q_i(L_i)$ consists of finite order elements (see e.g. [BCGP09, Lem. 4.7]), so that T is a finite index subgroup in a product of orbifold surface groups. This concludes the proof of Proposition 3.5.

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