Rational Blanchfield forms, S-equivalence, and null LP-surgeries

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Abstract

Garoufalidis and Rozansky introduced null-moves on the set of pairs (M, K), where M is an integral homology sphere and K is a knot in M. These null-moves are suitable to study the Kricker lift of the Kontsevich integral. They defined a filtration on the space generated by pairs (M, K) up to orientation-preserving homeomorphism. This filtration splits with respect to the isomorphism classes of integral Alexander modules equipped with their Blanchfield forms. Null Lagrangian-preserving surgeries are a generalization of the Garoufalidis and Rozansky null-moves in the setting of pairs (M, K) composed of a rational homology sphere M and a null-homologous knot K in M. They are defined as replacements of null-homologous rational homology handlebodies of $M \setminus K$ by other such handlebodies with identical Lagrangian. We prove that two pairs (M, K) can be obtained from one another by a finite sequence of null Lagrangian-preserving surgeries if and only if they have isomorphic rational Alexander modules and Blanchfield forms.

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Keywords: Alexander module; Blanchfield form; equivariant linking pairing; homology sphere; homology handlebody; Lagrangian-preserving surgery; Seifert matrix; S-equivalence; lift of the Kontsevich integral; null-move; Euler degree of the Kontsevich integral.

Contents

1 Introduction

1.1 Context

In [GR], Garoufalidis and Rozansky studied the rational vector space generated by the pairs (M, K) modulo orientation-preserving homeomorphism, where M is an integral homology 3-sphere ($\mathbb{Z}$HS), that is an oriented compact 3-manifold which has the same homology with integral coefficients as S^3, and K is a knot in M. They defined a filtration on this space by means of null-moves, that are surgeries on claspers whose leaves are trivial in $H_1(M \setminus K; \mathbb{Z})$. They studied this filtration with the Kricker lift of the Kontsevich integral defined in [GK]. The first step in the study of this filtration is the determination of the classes of pairs (M, K) up to null-moves. As a corollary of results of Matveev in [Ma], Naik and Stanford in [NS], and Trotter in [T], Garoufalidis and Rozansky established that two pairs can be obtained from one another by a finite sequence of null-moves if and only if they admit S-equivalent Seifert matrices, and if and only if they have isomorphic integral Alexander modules and Blanchfield forms.

In this article, we consider pairs (M, K), where M is a *rational homology 3-sphere* ($\mathbb{Q}$HS), *i.e.* an oriented compact 3-manifold which has the same homology with rational coefficients as S^3, and K is a *null-homologous knot* in M, *i.e.* a knot whose class in $H_1(M; \mathbb{Z})$ is trivial. We define the null Lagrangian-preserving surgeries, which play the role played by the null-moves in the integral case. We prove that the classes of pairs (M, K) modulo null Lagrangian-preserving surgeries are characterized by the classes of rational S-equivalence of their Seifert matrices, and by the isomorphism classes of their rational Alexander modules equipped with their Blanchfield forms. Null Lagrangian-preserving

surgeries define a filtration of the rational vector space generated by pairs (M, K) modulo orientation-preserving homeomorphism. This article is a first step in the study of this filtration, that is useful in the study of equivariant finite type knot invariants.

1.2 Statement of the results

When it does not seem to cause confusion, we use the same notation for a curve and its homology class.

We first recall the definition of the Alexander module and of the Blanchfield form. Let (M, K) be a $\mathbb{Q}SK$-*pair*, that is a pair made of a rational homology sphere M and a null-homologous knot K in M. Let $T(K)$ be a tubular neighborhood of K. The *exterior* of K is $X = M \setminus Int(T(K))$. Consider the projection $\pi : \pi_1(X) \to \frac{H_1(X;\mathbb{Z})}{torsion} \cong \mathbb{Z}$, and the covering map $p : \tilde{X} \to X$ associated with its kernel. The covering $\tilde{X}$ is the *infinite cyclic covering* of X. The automorphism group of the covering, $Aut(\tilde{X})$, is isomorphic to $\mathbb{Z}$. It acts on $H_1(\tilde{X};\mathbb{Q})$. Denoting the action of a generator τ of $Aut(\tilde{X})$ as the multiplication by t, we get a structure of $\mathbb{Q}[t^{\pm 1}]$-module on $\mathcal{A}(K) = H_1(\tilde{X};\mathbb{Q})$. This $\mathbb{Q}[t^{\pm 1}]$-module is the *Alexander module* of K. It is a torsion $\mathbb{Q}[t^{\pm 1}]$-module.

On the Alexander module $\mathcal{A}(K)$, one can define the *Blanchfield form*, or *equivariant linking pairing*, $\phi_K : \mathcal{A}(K) \times \mathcal{A}(K) \to \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]}$, as follows. First define the equivariant linking number of two knots.

Definition 1.1. Let J_1 and J_2 be two knots in $\tilde{X}$ such that $J_1 \cap \tau^k(J_2) = \emptyset$ for all $k \in \mathbb{Z}$. Let $\delta(t)$ be the annihilator of $\mathcal{A}(K)$. Then $\delta(\tau)J_1$ and $\delta(\tau)J_2$ are rationally null-homologous links. The *equivariant linking number* of J_1 and J_2 is

$$lk_e(J_1, J_2) = \frac{1}{\delta(t)\delta(t^{-1})} \sum_{k\in\mathbb{Z}} lk(\delta(\tau)J_1, \tau^k(\delta(\tau)J_2))t^k.$$

One can easily see that $lk_e(J_1, J_2) \in \frac{1}{\delta(t)}\mathbb{Q}[t^{\pm 1}]$, $lk_e(J_2, J_1)(t) = lk_e(J_1, J_2)(t^{-1})$, and $lk_e(P(\tau)J_1, Q(\tau)J_2)(t) = P(t)Q(t^{-1})lk_e(J_1, J_2)(t)$. Now, if γ (resp. η) is the homology class of J_1 (resp. J_2) in $\mathcal{A}(K)$, define $\phi_K(\gamma, \eta)$ by:

$$\phi_K(\gamma, \eta) = lk_e(J_1, J_2) \ mod \ \mathbb{Q}[t^{\pm 1}].$$

The Blanchfield form is hermitian: $\phi_K(\gamma, \eta)(t) = \phi_K(\eta, \gamma)(t^{-1})$ and $\phi_K(P(t)\gamma, Q(t)\eta)(t) = P(t)Q(t^{-1})\,\phi_K(\gamma, \eta)(t)$ for all $\gamma, \eta \in \mathcal{A}(K))$ and all $P, Q \in \mathbb{Q}[t^{\pm 1}]$. Moreover, it is non degenerate: $\phi_K(\gamma, \eta) = 0$ for all $\eta \in \mathcal{A}(K)$ implies $\gamma = 0$.

We turn to the notion of Seifert matrices. Let Σ be a *Seifert surface* of K, *i.e.* a compact connected oriented surface in M such that $\partial\Sigma = K$. Such a surface exists since

K is null-homologous. Let g be the genus of Σ. Let $(f_i)_{1\leq i\leq 2g}$ be a symplectic basis of $H_1(\Sigma;\mathbb{Z})$, *i.e.* a basis such that the matrix of the intersection form in $(f_i)_{1\leq i\leq 2g}$ is $-J$, where J is made of blocks $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ on the diagonal, and zeros elsewhere. The *Seifert matrix* of K associated with Σ and $(f_i)_{1\leq i\leq 2g}$ is the matrix $V\in\mathcal{M}_{2g}(\mathbb{Q})$ defined by $V_{ij}=lk(f_i,f_j^+)$, where f_j^+ is a push-off of f_j in the direction of the positive normal of Σ. This matrix satisfies $V-V^t=J$. Any rational matrix with this property will be called a *Seifert matrix*. In Section 4, we prove that any such matrix is indeed the Seifert matrix of a $\mathbb{Q}$SK-pair (M,K).

Given the Seifert matrix V, one can compute the Alexander module $\mathcal{A}(K)$ and the Blanchfield form ϕ_K. Let $T(\Sigma)$ be a tubular neighborhood of Σ. For $1\leq i\leq 2g$, let $e_i\subset\partial T(\Sigma)$ be a meridian of f_i. The module $\mathcal{A}(K)$ can be presented as:

$$\mathcal{A}(K)=\frac{\bigoplus_{1\leq i\leq 2g}\mathbb{Q}[t^{\pm 1}]b_i}{\bigoplus_{1\leq j\leq 2g}\mathbb{Q}[t^{\pm 1}]\partial S_j},$$

where the b_i are lifts of the e_i in $\tilde{X}$, and the S_j are lifts of the $[-1,1]\times f_j$. When the lifts are suitably chosen, then $\partial S_j=\sum_{1\leq i\leq 2g}(tV-V^t)_{ij}b_i$, hence $tV-V^t$ is a presentation matrix of $\mathcal{A}(K)$ (see [L, Chapter 6]). Moreover, the computation of the $lk_e(b_i,\partial S_j)$ shows that the form ϕ_K is given by $\phi_K(b_i,b_j)=(1-t)((tV-V^t)^{-1})_{ji}\ mod\ \mathbb{Q}[t^{\pm 1}]$ (see Kearton in [K, §8]).

Definition 1.2. A *row enlargement* of a matrix $V\in\mathcal{M}_{2g}(\mathbb{Q})$ is a matrix $W=\begin{pmatrix}0 & 0 & 0\\ 1 & x & \xi^t\\ 0 & \xi & V\end{pmatrix}$, where $x\in\mathbb{Q}$ et $\xi\in\mathbb{Q}^{2g}$. Then the matrix V is a *row reduction* of W. A *column enlargement* of V is a matrix $W=\begin{pmatrix}0 & -1 & 0\\ 0 & x & \xi^t\\ 0 & \xi & V\end{pmatrix}$, where $x\in\mathbb{Q}$ et $\xi\in\mathbb{Q}^{2g}$. Then the matrix V is a *column reduction* of W.

Note that an enlargement or a reduction of a Seifert matrix still is a Seifert matrix. An enlargement of a Seifert matrix corresponds to the addition of a tube to the Seifert surface.

Definition 1.3. Two Seifert matrices are *rationally S-equivalent* if they can be obtained from one another by a sequence of enlargements, reductions, and invertible rational congruences.

In particular, two Seifert matrices of a $\mathbb{Q}$SK-pair (M, K) are rationally S-equivalent (see [L, Theorem 8.4] for the integral case, which easily generalises). By extension, we say that two $\mathbb{Q}$SK-pairs (M, K) are *rationally S-equivalent* if they admit rationally S-equivalent Seifert matrices. In Section 2, we prove that rationally S-equivalent $\mathbb{Q}$SK-pairs have isomorphic Blanchfield forms. The converse easily follows from a result of Trotter in [T]. In Section 3, we prove the stronger statement:

Proposition 1.4. *Let (M, K) and (M', K') be two $\mathbb{Q}$SK-pairs whose Blanchfield forms are isomorphic. Let V and V' be associated Seifert matrices. Then V and V' can be related by a finite sequence of enlargements, reductions, and integral unimodular congruences.*

We now define homology handlebodies and Lagrangian-preserving surgeries.

Definition 1.5. For $g \in \mathbb{N}$, a *genus g rational (resp. integral) homology handlebody* ($\mathbb{Q}$HH, resp. $\mathbb{Z}$HH) is a 3-manifold which is compact, oriented, and which has the same homology with rational (resp. integral) coefficients as the standard genus g handlebody.

Such a $\mathbb{Q}$HH (resp. $\mathbb{Z}$HH) is connected, and its boundary is necessarily homeomorphic to the standard genus g surface.

Definition 1.6. The *Lagrangian* $\mathcal{L}_A$ of a $\mathbb{Q}$HH A is the kernel of the map

$$i_* : H_1(\partial A; \mathbb{Q}) \to H_1(A; \mathbb{Q})$$

induced by the inclusion. Two $\mathbb{Q}$HH's A and B have *LP-identified* boundaries if (A, B) is equipped with a homeomorphism $h : \partial A \to \partial B$ such that $h_*(\mathcal{L}_A) = \mathcal{L}_B$.

The Lagrangian of a $\mathbb{Q}$HH A is indeed a Lagrangian subspace of $H_1(\partial A; \mathbb{Q})$ with respect to the intersection form.

Let M be a $\mathbb{Q}$HS, let $A \subset M$ be a $\mathbb{Q}$HH, and let B be a $\mathbb{Q}$HH whose boundary is LP-identified with ∂A. Set $M(\frac{B}{A}) = (M \setminus Int(A)) \cup_{\partial A = \partial B} B$. We say that the $\mathbb{Q}$HS $M(\frac{B}{A})$ is obtained from M by *Lagrangian-preserving surgery*, or *LP-surgery*.

Given a $\mathbb{Q}$SK-pair (M, K), a *null-$\mathbb{Q}$HH* in $M \setminus K$ is a $\mathbb{Q}$HH $A \subset M \setminus K$ such that the map $i_* : H_1(A; \mathbb{Q}) \to H_1(M \setminus K; \mathbb{Q})$ induced by the inclusion has a trivial image. A *null LP-surgery* on (M, K) is an LP-surgery $(\frac{B}{A})$ such that A is null in $M \setminus K$. The $\mathbb{Q}$SK-pair obtained by surgery is denoted by $(M, K)(\frac{B}{A})$.

In Section 2, we prove that null LP-surgeries preserve the Alexander module and the Blanchfield form. In Section 4, we finally obtain:

Theorem 1.7. *Given two $\mathbb{Q}$SK-pairs (M, K) and (M', K'), the following assertions are equivalent :*

1. *(M, K) and (M', K') have isomorphic Alexander modules and Blanchfield forms,*

2. *(M, K) and (M', K') are rationally S-equivalent,*

3. *(M', K') can be obtained from (M, K) by a finite sequence of null LP-surgeries.*

We end the article by proving the following proposition in Section 5.

Proposition 1.8. *There are $\mathbb{Q}$SK-pairs (M, K) and (M', K') that can be obtained from one another by a finite sequence of null LP-surgeries, but not by a single null LP-surgery.*

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2 Conservation of the Blanchfield form

In this section, we prove that the isomorphism class of the Blanchfield form is preserved by null LP-surgery (Lemma 2.1) and by rational S-equivalence (Lemma 2.3).

Lemma 2.1. *Let (M, K) be a $\mathbb{Q}$SK-pair. Let A be a null-$\mathbb{Q}$HH in $M \setminus K$. Let B be a $\mathbb{Q}$HH whose boundary is LP-identified with ∂A. Then the $\mathbb{Q}$SK-pairs (M, K) and $(M, K)(\frac{B}{A})$ have isomorphic rational Alexander modules and Blanchfield forms.*

Proof. Throughout this proof, the homology modules are considered with rational coefficients. Let $\tilde{X}$ (resp. $\tilde{X}'$) be the infinite cyclic covering associated with (M, K) (resp. (M', K')). The preimage $\tilde{A}$ of A in $\tilde{X}$ (resp. $\tilde{B}$ of B in $\tilde{X}'$) is the disjoint union of $\mathbb{Z}$ copies A_i of A (resp. B_i of B).

Set $Y = \tilde{X} \setminus Int(\tilde{A})$. The Mayer-Vietoris sequence associated with $\tilde{X} = \tilde{A} \cup Y$ yields the exact sequence:

$$H_1(\partial\tilde{A}) \to H_1(\tilde{A}) \oplus H_1(Y) \to H_1(\tilde{X}) \to 0.$$

Since $H_1(\partial\tilde{A}) \cong H_1(\tilde{A}) \oplus (\mathbb{Q}[t, t^{-1}] \otimes \mathcal{L}_A)$, we get $H_1(\tilde{X}) \cong \dfrac{H_1(Y)}{\mathbb{Q}[t, t^{-1}] \otimes \mathcal{L}_A}$. Similarly, $H_1(\tilde{X}') \cong \dfrac{H_1(Y)}{\mathbb{Q}[t, t^{-1}] \otimes \mathcal{L}_B}$. Since $\mathcal{L}_A = \mathcal{L}_B$, the Alexander modules $H_1(\tilde{X})$ and $H_1(\tilde{X}')$ are canonically identified.

Now consider two null-homologous knots J and J' in $\tilde{X}$ that do not meet $\tilde{A}$. Consider a Seifert surface Σ of J. Assume that Σ is transverse to $\partial\tilde{A}$ and J'. Write $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 = \Sigma \cap Y$ and $\Sigma_2 = \Sigma \cap \tilde{A}$. Since J' does not meet Σ_2, the linking number

$lk_{\tilde{X}}(J, J')$ is equal to the algebraic intersection number $< J', \Sigma_1 >$. Now $\partial\Sigma_2$ is an integral linear combination of curves $\alpha_i \in \mathcal{L}_{A_i}$. In $\tilde{X}'$, each α_i lies in $\mathcal{L}_{B_i}$, so each α_i has a multiple that bounds a surface in B_i. Thus, there is a surface $\Sigma_3 \subset \tilde{B}$ such that $\partial\Sigma_3 = n\partial\Sigma_2$ for some integer n. We have $nJ = \partial(n\Sigma_1 \cup \Sigma_3)$, thus:

$$lk_{\tilde{X}'}(J, J') = \frac{1}{n} < J', n\Sigma_1 \cup \Sigma_3 >=< J', \Sigma_1 >= lk_{\tilde{X}}(J, J').$$

Since any class γ in $H_1(\tilde{X})$ has a multiple that can be represented by a knot J in Y such that $P(t).J$ is null-homologous for some $P \in \mathbb{Q}[t, t^{-1}]$, the Blanchfield form is preserved. □

Remark: We can work with integers, and define integral null LP-surgeries. Then, the previous proof still works, with $\mathbb{Q}$ replaced by $\mathbb{Z}$. Thus:

Lemma 2.2. *If two $\mathbb{Q}$SK-pairs (M, K) and (M', K') can be obtained from one another by a finite sequence of integral null LP-surgeries, then they have isomorphic integral Alexander modules and Blanchfield forms.*

Lemma 2.3. *Let (M, K) and (M', K') be $\mathbb{Q}$SK-pairs. If (M, K) and (M', K') are rationally S-equivalent, then they have isomorphic rational Alexander modules and Blanchfield forms.*

Proof. Let V be a Seifert matrix associated with (M, K). As recalled in the introduction, there is a family $(b_i)_{1\leq i\leq 2g}$ of generators of $\mathcal{A}(K)$ that yields the presentation matrix $W = tV - V^t$, and the Blanchfield form is given by $\phi_K(b_i, b_j) = (1-t)(W^{-1})_{ji}$. Set $b = \begin{pmatrix} b_1 & b_2 & \dots & b_{2g} \end{pmatrix}$, and $r = \begin{pmatrix} r_1 & r_2 & \dots & r_{2g} \end{pmatrix} = bW$. We have:

$$\mathcal{A}(K) = \frac{\bigoplus_{1\leq i\leq 2g} \mathbb{Q}[t, t^{-1}]b_i}{\bigoplus_{1\leq j\leq 2g} \mathbb{Q}[t, t^{-1}]r_j}.$$

Define the same notation with primes for the $\mathbb{Q}$SK-pair (M', K').

First assume that $V' = P^tVP$ for an invertible rational matrix P. Note that $W' = P^tWP$. Define a $\mathbb{Q}[t, t^{-1}]$-isomorphism $\tilde{\psi} : \bigoplus_{1\leq i\leq 2g} \mathbb{Q}[t, t^{-1}]b_i \to \bigoplus_{1\leq i\leq 2g} \mathbb{Q}[t, t^{-1}]b'_i$ by $\tilde{\psi}(b_i) = (b'P^t)_i$. We have $\tilde{\psi}(r_j) = (b'P^tW)_j = (r'P^{-1})_j$, thus $\tilde{\psi}(\bigoplus_{1\leq j\leq 2g} \mathbb{Q}[t, t^{-1}]r_j) =$

$\bigoplus_{1\leq j\leq 2g}\mathbb{Q}[t,t^{-1}]r'_j$. Hence $\tilde{\psi}$ induces an isomorphism $\psi:\mathcal{A}(K)\to\mathcal{A}(K')$. Now, we have:

$$\begin{aligned}\phi_{K'}(\psi(b_i),\psi(b_j)) &= \phi_{K'}((b'P^t)_i,(b'P^t)_j)\\ &= \sum_{k,l} p_{ik}p_{jl}(1-t)((W')^{-1})_{lk}\\ &= (1-t)\Big(P(P^{-1}W^{-1}(P^{-1})^t)P^t\Big)_{ji}\\ &= \phi_K(b_i,b_j).\end{aligned}$$

It remains to treat the case of an enlargement. Assume $V=\begin{pmatrix}0&0&0\\1&x&\xi^t\\0&\xi&V'\end{pmatrix}$. Then:

$$W=\begin{pmatrix}0&-1&0\\t&x(t-1)&(t-1)\xi^t\\0&(t-1)\xi&W'\end{pmatrix}.$$

Thus b_2 is trivial, and b_1 is a linear combination over $\mathbb{Q}[t,t^{-1}]$ of the b_i for $3\leq i\leq 2g$. Hence there is an isomorphism $(\mathcal{A}(K),\phi_K)\cong(\mathcal{A}(K'),\phi'_K)$ which identifies b_i with b'_{i-2} for $3\leq i\leq 2g$. Proceed similarly for a column enlargement. □

3 Relating Seifert matrices

Proposition 1.4 is a direct consequence of Lemmas 3.1, 3.2, and 3.3 below.

Lemma 3.1. *Let (M,K) and (M',K') be two $\mathbb{Q}$SK-pairs that have isomorphic Blanchfield forms. Let Σ and Σ' be Seifert surfaces of K and K', respectively, and let V and V' be two associated Seifert matrices. Then V and V' are rationally S-equivalent.*

Proof. If V is non invertible, there exists $f_1\in H_1(\Sigma;\mathbb{Z})$, such that $lk(f_1,\gamma^+)=0$ for all $\gamma\in H_1(\Sigma;\mathbb{Z})$. Since the intersection form $<.,.>_\Sigma$ on Σ is non degenerate, there is $f_2\in H_1(\Sigma;\mathbb{Z})$ such that $<f_1,f_2>_\Sigma=1$. Consider a symplectic basis $(f_i)_{3\leq i\leq 2g}$ of the orthogonal of $\mathbb{Z}f_1\oplus\mathbb{Z}f_2$ in $H_1(\Sigma;\mathbb{Z})$ with respect to the intersection form. Then $(f_i)_{1\leq i\leq 2g}$ is a symplectic basis of $H_1(\Sigma;\mathbb{Z})$, and the associated Seifert matrix V_1 is a row enlargement of a Seifert matrix V_2. Since V and V_1 are associated with the same Seifert surface, they are related by a change of basis of $H_1(\Sigma;\mathbb{Z})$, *i.e.* they are congruent. Hence V is S-equivalent to the smaller matrix V_2. Iterating this process, we see that V is S-equivalent to an invertible Seifert matrix W, where we consider that there exists an empty matrix, which is invertible. Similarly, V' is S-equivalent to an invertible Seifert

matrix W'. The matrices W and W' are invertible Seifert matrices that define isomorphic Blanchfield forms. Hence, by [T, Proposition 2.12], they are congruent. □

We now treat the case of rationally congruent Seifert matrices. Consider two Seifert matrices V and V' such that $V' = PVP^t$, with P rational and invertible. Since $V - V^t = V' - (V')^t = J$, we have $PJP^t = J$, *i.e.* P is symplectic. We shall prove that we can realise a symplectic rational congruence by a finite sequence of enlargements, reductions, and integral unimodular symplectic congruences. We first treat a particular type of congruence matrices.

Lemma 3.2. *Let V and W be two Seifert matrices such that $\Delta_n V \Delta_n = W$, where n or $\frac{1}{n}$ is a positive integer, and*

$$\Delta_n = \begin{pmatrix} n & & & & \\ & \frac{1}{n} & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then there are enlargements $\tilde{V}$ of V and $\tilde{W}$ of W that are integrally unimodularly congruent.

Proof. Assume n is a positive integer.

Set $V = \begin{pmatrix} a & b & \zeta^t \\ c & d & \xi^t \\ \zeta & \xi & U \end{pmatrix}$. Then $W = \begin{pmatrix} n^2 a & b & n\zeta^t \\ c & \frac{d}{n^2} & \frac{1}{n}\xi^t \\ n\zeta & \frac{1}{n}\xi & U \end{pmatrix}$. Note that $c = b + 1$. Set:

$$P = \left(\begin{array}{cccc|c} 0 & n & 0 & -1 & \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & n & 0 & \\ 0 & 1 & 0 & 0 & \\ \hline & & 0 & & I \end{array}\right), \quad \tilde{V} = \left(\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{d}{n^2} & \frac{c}{n} & \frac{d}{n} & \frac{1}{n}\xi^t \\ 0 & \frac{c}{n} & a & b & \zeta^t \\ 0 & \frac{d}{n} & c & d & \xi^t \\ \hline 0 & \frac{1}{n}\xi & \zeta & \xi & U \end{array}\right),$$

$$\text{and } \tilde{W} = \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & a & na & \frac{c}{n} & \zeta^t \\ 0 & na & n^2a & b & n\zeta^t \\ 0 & \frac{c}{n} & c & \frac{d}{n^2} & \frac{1}{n}\xi^t \\ \hline 0 & \zeta & n\zeta & \frac{1}{n}\xi & U \end{array}\right).$$

The matrix P is integral unimodular and symplectic, and we have $\tilde{W} = P\tilde{V}P^t$.

Since $\Delta_{\frac{1}{n}} = \Delta_n^{-1}$, the case $\frac{1}{n} \in \mathbb{N} \setminus \{0\}$ follows. □

Lemma 3.3. *Any symplectic rational matrix P can be written as a product of integral unimodular symplectic matrices and matrices Δ_n or $\Delta_{\frac{1}{n}}$ for positive integers n.*

Proof.

Step 1: There is no loss in assuming that the first column of P is $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Denote by d a common denominator for the terms of the first column of P. The matrix $P\Delta_d$ has integral coefficients in its first column. Denote by δ their gcd. The terms of the first column of $P\Delta_d\Delta_{\frac{1}{\delta}}$ are coprime integers. There is an integral unimodular symplectic matrix Q with the same first column. The matrix $Q^{-1}P\Delta_d\Delta_{\frac{1}{\delta}}$ has the required first column.

Step 2: We can assume that the first two columns of P are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$.

The matrix P^{-1} has the same first column as P. Since it is symplectic, its second

column is $\begin{pmatrix} x_1 \\ \vdots \\ x_{2g} \end{pmatrix}$, with $x_2 = 1$. Set:

$$Q = \begin{pmatrix} 1 & x_1 & -x_4 & x_3 & \dots & -x_{2g} & x_{2g-1} \\ 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & x_3 & 1 & & & & \\ \vdots & \vdots & & \ddots & & 0 & \\ \vdots & \vdots & & & \ddots & & \\ \vdots & \vdots & & 0 & & \ddots & \\ 0 & x_{2g} & & & & & 1 \end{pmatrix}.$$

Since Q has the same first two columns as P^{-1}, the matrix PQ has the required first two columns. Now, if n is a common denominator for all the x_i, the matrix $\Delta_n Q \Delta_{\frac{1}{n}}$ has integral coefficients, and is symplectic.

Step 3: Induction.

We have $P = \begin{pmatrix} I_2 & R \\ 0 & Q \end{pmatrix}$. Since P is symplectic, $R = 0$ and Q is symplectic. Thus we can conclude by induction on g. □

4 Topological realization of matrix relations

In this section, we conclude the proof of Theorem 1.7. Lemma 3.1 implies $(1 \Rightarrow 2)$. Since $(2 \Rightarrow 1)$ and $(3 \Rightarrow 1)$ were proved in Section 2, it suffices to show $(1 \Rightarrow 3)$. The proof of this implication will be based on Proposition 1.4.

Lemma 4.1. *Let M be a $\mathbb{Q}$HS (resp. $\mathbb{Z}$HS). Let Σ be a genus g closed connected surface embedded in M. Then $M \setminus \Sigma$ has exactly two connected components, whose closures are $\mathbb{Q}$HH's (resp. $\mathbb{Z}$HH's) of genus g.*

Proof. Any point of $M \setminus \Sigma$ can be joined to a point of $\Sigma \times [-1, 1]$ in $M \setminus \Sigma$. Since $(\Sigma \times [-1, 1]) \setminus \Sigma$ has two connected components, $M \setminus \Sigma$ contains at most two connected components. Let x_1 and x_2 be points of $(\Sigma \times [-1, 1]) \setminus \Sigma$, one in each connected component. If there were a path from x_1 to x_2 in $M \setminus \Sigma$, we could construct a closed curve in M which would meet Σ exactly once. Since M is a $\mathbb{Q}$HS, this is not possible. Hence $M \setminus \Sigma$

has exactly two connected components. Let A_1 and A_2 be their closures. Note that $\partial A_1 = \partial A_2 = \Sigma$ (up to orientation).

For $i = 1, 2$, we have $H_3(A_i; \mathbb{Z}) = 0$ and $H_0(A_i; \mathbb{Z}) = \mathbb{Z}$. The Mayer-Vietoris sequence associated with $M = A_1 \cup A_2$ yields the exact sequence:

$$H_3(M; \mathbb{Z}) \xrightarrow{\partial} H_2(\Sigma; \mathbb{Z}) \longrightarrow H_2(A_1; \mathbb{Z}) \oplus H_2(A_2; \mathbb{Z}) \longrightarrow 0.$$

The map ∂ is an isomorphism that identifies the fondamental classes. Thus $H_2(A_1; \mathbb{Z}) = H_2(A_2; \mathbb{Z}) = 0$. It follows that A_1 and A_2 are $\mathbb{Q}$HH's. Their genus is given by their boundary.

Assume M is a $\mathbb{Z}$HS. The Mayer-Vietoris sequence associated with $M = A_1 \cup A_2$ yields an isomorphism $H_1(\Sigma; \mathbb{Z}) \cong H_1(A_1; \mathbb{Z}) \oplus H_1(A_2; \mathbb{Z})$. Hence, for $i = 1, 2$, $H_1(A_i; \mathbb{Z})$ is torsion-free, thus A_i is a $\mathbb{Z}$HH. □

Lemma 4.2. *If two $\mathbb{Q}$SK-pairs (M, K) and (M', K') have a common Seifert matrix, then they can be obtained from one another by a null LP-surgery.*

Proof. Let Σ and Σ' be Seifert surfaces of K and K', with the same associated Seifert matrix V, with respect to symplectic bases $(f_1, \dots, f_{2g})$ of $H_1(\Sigma, \mathbb{Z})$ and $(f'_1, \dots, f'_{2g})$ of $H_1(\Sigma', \mathbb{Z})$. Let $T(\Sigma)$ be a regular neighborhood of Σ, and set $A = M \setminus Int(T(\Sigma))$. Note that $T(\Sigma)$ is homeomorphic to the standard genus $2g$ handlebody. By Lemma 4.1, A is a $\mathbb{Q}$HH, which is clearly null in $M \setminus K$. We have a similar decomposition $M' = A' \cup T(\Sigma')$.

For $1 \leq i \leq 2g$, let e_i be a meridian of f_i on $\partial T(\Sigma)$. The Lagrangian $\mathcal{L}_A$ is generated by homology classes $\alpha_i = f_i^+ - \sum_{j=1}^{2g} V_{ji} e_j \in H_1(\partial A; \mathbb{Q})$. Similarly, define curves e'_i in $\partial A'$ and homology classes α'_i in $H_1(\partial A'; \mathbb{Q})$. There is a homeomorphism from $T(\Sigma)$ to $T(\Sigma')$ that sends f_i to f'_i and e_i to e'_i. The associated isomorphism in homology sends α_i to α'_i. Thus we have $M' = A' \cup T(\Sigma') \cong M(\frac{A'}{A})$, and the replacement of A by A' is Lagrangian-preserving. □

Lemma 4.3. *Let (M, K) be a $\mathbb{Q}$SK-pair. Let Σ be a Seifert surface of K, and let V be the associated Seifert matrix with respect to a basis $(f_1, \dots, f_{2g})$ of $H_1(\Sigma, \mathbb{Z})$. Let W be an enlargement of V. Then W is a Seifert matrix of a $\mathbb{Q}$SK-pair (M', K') that can be obtained from (M, K) by a null LP-surgery.*

Proof. We have

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & \xi^t \\ 0 & \xi & V \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -1 & 0 \\ 0 & x & \xi^t \\ 0 & \xi & V \end{pmatrix}.$$

We want to add a tube to Σ, whose linking numbers with the f_i are given by ξ. This may not be possible in M, so we first modify M by null LP-surgeries.

Set $\xi = \begin{pmatrix} \frac{c_1}{d_1} \\ \vdots \\ \frac{c_{2g}}{d_{2g}} \end{pmatrix}$, where the c_i and d_i are integers, and $d_i > 0$. Consider trivial knots J_i, disjoint from Σ, such that $lk(J_i, f_j) = \delta_{ij} c_i$. For each i, consider a tubular neighborhood $T(J_i)$ of J_i. By [M2, Lemma 1.5], there are rational homology tori A_i that satisfy:

- $H_1(\partial A_i; \mathbb{Z}) = \mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i$, with $< \alpha_i, \beta_i >= 1$,
- $\beta_i = d_i \gamma_i$ in $H_1(A_i; \mathbb{Z})$, where γ_i is a curve in A_i,
- $H_1(A_i; \mathbb{Z}) = \mathbb{Z}\gamma_i \oplus \frac{\mathbb{Z}}{d_i \mathbb{Z}} \alpha_i$.

Let N be the manifold obtained from M by the null LP-surgeries $(\frac{A_i}{T(J_i)})$, where the identifications $\partial T(J_i) = \partial A_i$ identify α_i with a meridian of J_i, and β_i with a parallel of J_i that does not link J_i. We get $lk(\gamma_i, f_j) = \delta_{ij} \frac{c_i}{d_i}$.

In N, consider a ball B disjoint from Σ and all the A_i. Consider a rational homology ball B' that contains a curve γ_0 with self-linking $x \; mod \; \mathbb{Z}$. Set $M' = N(\frac{B'}{B})$, and $K' = K$.

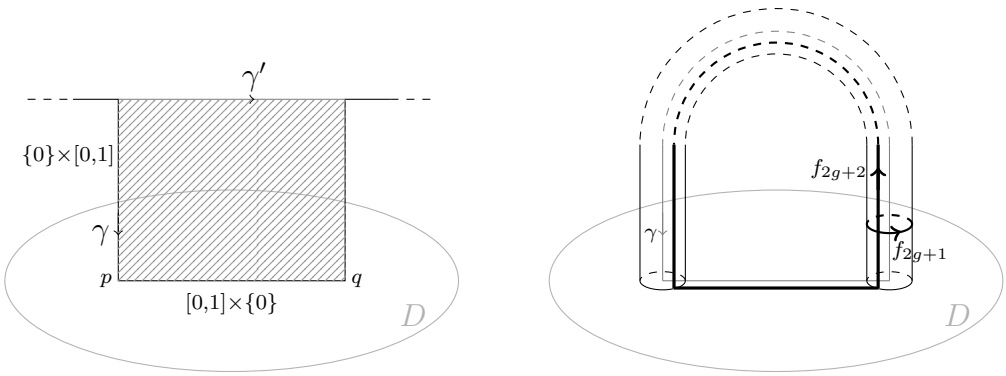

Figure 1: Adding a tube to Σ

Define a curve γ' in M' as a band sum of the γ_i for $0 \leq i \leq 2g$, with bands outside Σ. Consider a disk D in Σ, and two distinct points p and q in D. Consider an embedded band $[0,1] \times [0,1]$ in M' such that $[0,1] \times \{0\} = ([0,1] \times [0,1]) \cap \Sigma$ is a curve from p to q in D, $[0,1] \times \{1\} = ([0,1] \times [0,1]) \cap \gamma'$, and the vector tangent to $\{0\} \times [0,1]$ at $\{0\} \times \{0\}$ is the positive normal vector of Σ if W is a row enlargement of V, and the negative one if it is a column enlargement. Figure 1 represents the first case. Now set $\gamma = (\gamma' \cup \partial([0,1] \times [0,1])) \setminus (]0,1[\times\{1\})$, and construct a surface Σ' by adding a tube around $\gamma \setminus ([0,1] \times \{0\})$

to Σ. The surface Σ' is a Seifert surface for K'. On Σ', consider a meridian f_{2g+1} of the tube and a parallel f_{2g+2} of γ such that $< f_{2g+1}, f_{2g+2} >_{\Sigma'} = 1$ and $lk(f_{2g+2}, \gamma) = x$. Note that the orientation of the meridian depends on the type of enlargement. The Seifert matrix associated with Σ' with respect to the basis $(f_{2g+1}, f_{2g+2}, f_1, \ldots, f_{2g})$ is W.

Since the different $\mathbb{Q}$HH replaced by surgery are disjoint, they can be connected by tubes. Thus (M', K') can be obtained from (M, K) by one surgery on a genus $2g$ $\mathbb{Q}$HH. □

Lemma 4.4. *For any matrix $V \in \mathcal{M}_{2g}(\mathbb{Q})$, with $g > 0$, satisfying $V - V^t = J$, there exists a $\mathbb{Q}$SK-pair (M, K) that admits V as a Seifert matrix.*

Proof. Set $V = (v_{ij})_{1 \leq i,j \leq 2g}$. By [M1, Corollary 2.13], there is a $\mathbb{Q}$HS M and pairwise

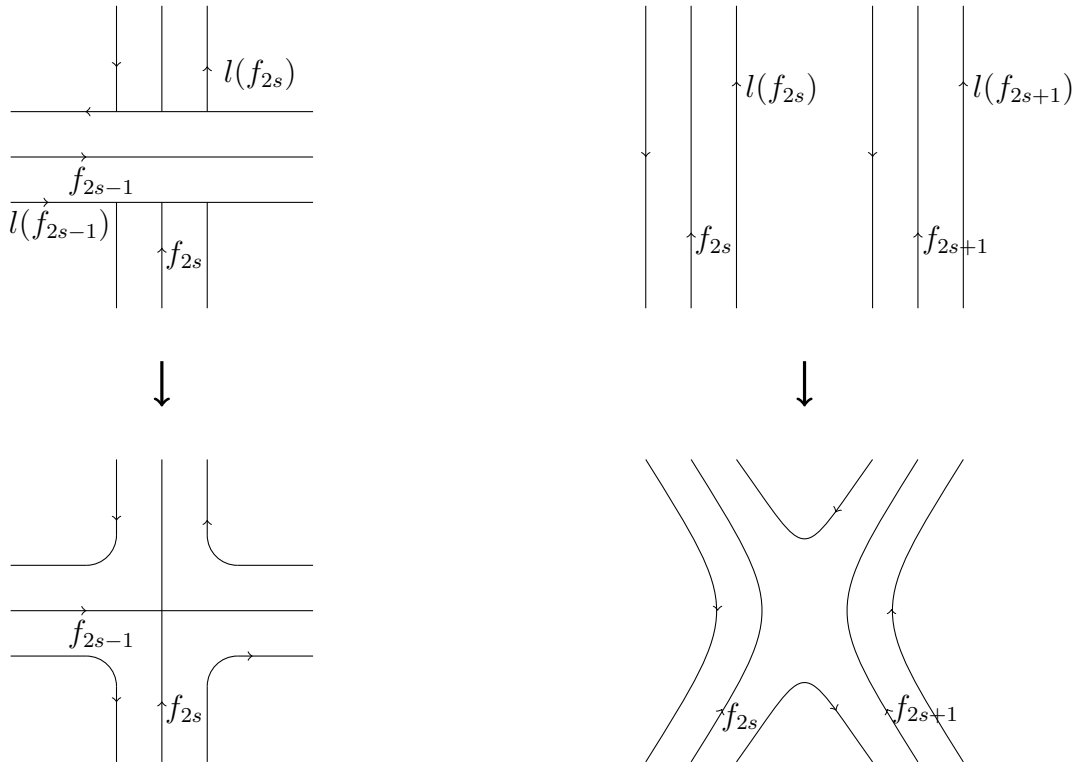

Figure 2: Gluing bands

disjoint simple closed framed curves f_i, $1 \leq i \leq 2g$, in M, such that $lk(f_i, f_j) = v_{ij}$ for $j \leq i$. Consider bands around the f_i, that are images of embeddings $h_i : [-1, 1] \times S^1 \hookrightarrow M$ such that $h_i(\{0\} \times S^1) = f_i$, and $h_i(\{1\} \times S^1) = l(f_i)$. Connecting these bands as

indicated in Figure 2, we get a surface bounded by a knot K which satisfies the required conditions. □

Proof of (1 ⇒ 3) in Theorem 1.7. By Proposition 1.4, Seifert matrices associated with ℚSK-pairs which have isomorphic Blanchfield forms can be related by a finite sequence of enlargements, reductions, and integral unimodular congruences. If two ℚSK-pairs (M, K) and (M', K') have integrally unimodularly congruent Seifert matrices V and V' respectively, then V' is a Seifert matrix for (M, K), obtained from V by a change of basis of the first homology module with integral coefficients of the Seifert surface. Hence Lemma 4.2 shows that (M, K) and (M', K') are related by a null LP-surgery. Enlargements are realised by Lemma 4.3. Now, null LP-surgeries are invertible. Since a reduced Seifert matrix always corresponds to a ℚSK-pair (M, K) by Lemma 4.4, Lemma 4.3 also realises reductions. □

5 Sequences of LP-surgeries

In this section, we prove Proposition 1.8.

Lemma 5.1. *There exist two knots in S^3 which have isomorphic rational Blanchfield forms, and different integral Alexander modules.*

Proof. In S^3, consider a knot K with Seifert matrix $\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$, and a knot K' with Seifert matrix $\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$. Their Alexander modules have presentation matrices $\begin{pmatrix} 1-t & -1 \\ t & 2t-2 \end{pmatrix}$ and $\begin{pmatrix} 3t-3 & t-2 \\ 2t-1 & 0 \end{pmatrix}$. Both have Alexander polynomial $\Delta(t) = (2t-1)(2-t)$. Since it is the product of two dual non symmetric prime polynomials, their rational Blanchfield forms are isomorphic (see [M1, Lemma 3.6]). But K has integral Alexander module $\frac{\mathbb{Z}[t,t^{-1}]}{(\Delta(t))}$, whereas the integral Alexander module of K' has a non trivial second elementary ideal (the k-th elementary ideal associated with K is the ideal of $\mathbb{Z}[t, t^{-1}]$ generated by the minors of size $2g - k + 1$ of the presentation matrix, see [L, Chapter 6]). Indeed, this ideal is generated by $(t-2)$ and $(2t-1)$ in $\mathbb{Z}[t, t^{-1}]$, so the evaluation at $t = -1$ maps it onto $3\mathbb{Z}$. □

Proof of Proposition 1.8. Consider the ℚSK-pairs (S^3, K) and (S^3, K') of Lemma 5.1. By Theorem 1.7, (S^3, K') can be obtained from (S^3, K) by a finite sequence of null LP-surgeries. Suppose we can restrict to a single surgery $(\frac{B}{A})$. Then A and B would be

$\mathbb{Q}$HH's embedded in a $\mathbb{Z}$HS. It follows from Lemma 4.1 that A and B would be $\mathbb{Z}$HH's. Thus, by Lemma 2.2, the surgery would preserve the integral Alexander module. $\square$

References

[GK] S. Garoufalidis, A. Kricker, *A rational noncommutative invariant of boundary links*, Geometry & Topology 8, p.115-204 (2004).

[GR] S. Garoufalidis, L. Rozansky, *The loop expansion of the Kontsevich integral, the null-move and S-equivalence*, Topology 43, p.1183-1210 (2004).

[K] C. Kearton, *Blanchfield duality and simple knots*, Transactions of the AMS 202, p.141-160 (1975).

[L] W.B.R. Lickorish, *An introduction to knot theory*, GTM, Springer-Verlag (1997).

[Ma] S.V. Matveev, *Generalized surgery of three-dimensional manifolds and representations of homology spheres*, Mathematical Notes of the Academy of Sciences of the USSR 42 (2), p.651-656 (1987).

[M1] D. Moussard, *On Alexander modules and Blanchfield forms of null-homologous knots in rational homology spheres*, Journal of Knot Theory and Its Ramifications Vol. 21, No. 5, 1250042 (21 p.), (2012).

[M2] D. Moussard, *Finite type invariants of rational homology 3-spheres*, arXiv : 1203.1603.

[NS] S. Naik, T. Stanford, *A move on diagrams that generates S-equivalence of knots*, Journal of Knot Theory and Its Ramifications Vol. 12, No. 5, p.717-724, (2003).

[T] H.F. Trotter, *On S-Equivalence of Seifert Matrices*, Inventiones math. 20, p.173-207, Springer-Verlag (1973).