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On closed subgroups of the group of homeomorphisms of a manifold

Frédéric Le Roux

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Abstract

Let $M$ be a triangulable compact manifold. We prove that, among closed subgroups of $\text{Homeo}_0(M)$ (the identity component of the group of homeomorphisms of $M$), the subgroup consisting of volume preserving elements is maximal.

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1 Introduction

The theory of groups acting on the circle is very rich (see in particular the monographs [Ghy01, Nav07]). The theory is far less developed in higher dimension, where it seems difficult to discover more than some isolated islands in a sea of chaos. In this note, we are interested in the closed subgroups of the group $\text{Homeo}_0(M)$, the identity component of the group of homeomorphisms of some compact topological $n$-dimensional manifold $M$. We will show that, when $n \geq 2$, for any good (nonatomic and with total support) probability measure $\mu$, the subgroup of elements that preserve $\mu$ is maximal among closed subgroups.

Let us recall some related results in the case when $M$ is the circle. De La Harpe conjectured that $\text{PSL}(2, \mathbb{R})$ is a maximal closed subgroup ([Bes]). Ghys proposed a list of closed groups acting transitively, asking whether, up to conjugacy, the list was complete ([Ghy01]); the list consists in the whole group, $SO(2)$, $\text{PSL}(2, \mathbb{R})$, the group $\text{Homeo}_{k,0}(S^1)$ of elements that commutes with some rotation of order $k$, and the group $\text{PSL}_k(2, \mathbb{R})$ which is defined analogously. The first conjecture was solved by Giblin and Markovic in [GM06]. These authors also answered Ghys’s question affirmatively, under the additional hypothesis that the group contains some non trivial arcwise connected component. Thinking of the two-sphere with these results in mind, one is naturally led to the following questions.

Question 1. Let $G$ be a proper closed subgroup of $\text{Homeo}_0(S^2)$ acting transitively. Assume that $G$ is not a (finite dimensional) Lie group. Is $G$ conjugate to one of the two subgroups: (1) the centralizer of the antipodal map $x \mapsto -x$, (2) the subgroup of area-preserving elements?

Note that the centralizer of the antipodal map is the group of lifts of homeomorphisms of the projective plane; it is the spherical analog of the groups $\text{Homeo}_{k,0}(S^1)$. 

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**Question 2.** Is $\text{PSL}(2, \mathbb{C})$ maximal among closed subgroups of $\text{Homeo}_0(S^2)$?

On the circle the group of measure-preserving elements coincides with $\text{SO}(2)$. It is not a maximal closed subgroup since it is included in $\text{PSL}(2, \mathbb{R})$. In contrast, we propose to prove that the closed subgroup of area-preserving homeomorphisms of the two-sphere is maximal. To put this into a general context, let $M$ be a compact topological manifold whose dimension is greater or equal to 2. We assume that $M$ is triangulable and (for simplicity) without boundary. Let us equip $M$ with a probability measure $\mu$ which is assumed to be good: this means that every finite set has measure zero, and every non-empty open set has positive measure. We consider the group $\text{Homeo}_0(M)$ of homeomorphisms of $M$ that are isotopic to the identity, and the subgroup $\text{Homeo}_0(M, \mu)$ of elements that preserve the measure $\mu$. According to the famous Oxtoby-Ulam theorem ([OU41], see also [Fat80]), if $\mu'$ is another good probability measure on $M$ then it is homeomorphic to $\mu$, meaning that there exists an element $h \in \text{Homeo}_0(M)$ such that $h_*\mu = \mu'$. In particular the subgroup $\text{Homeo}_0(M, \mu')$ is isomorphic to $\text{Homeo}_0(M, \mu)$. We equip these transformation groups with the topology of uniform convergence, which turns them into topological groups. The subgroup $\text{Homeo}_0(M, \mu)$ is easily seen to be closed in $\text{Homeo}_0(M)$. Note that according to Fathi’s theorem (first theorem in [Fat80]), $\text{Homeo}_0(M, \mu)$ coincides with the identity component in the group of measure preserving homeomorphisms. The aim of the present note is to prove the following.

**Theorem.** The group $\text{Homeo}_0(M, \mu)$ is maximal among closed subgroups of the group $\text{Homeo}_0(M)$.

In what follows we consider some element $f \in \text{Homeo}_0(M)$ that does not preserves the measure $\mu$, and we denote by $G_f$ the subgroup of $\text{Homeo}_0(M)$ generated by

$$\{f\} \cup \text{Homeo}_0(M, \mu).$$

Our aim is to show that the group $G_f$ is dense in $\text{Homeo}_0(M)$.

## 2 Localization

In this section we show how to find some element in $G_f$ that has small support and contracts the volume of some given ball.

**Good balls** A ball is any subset of $M$ which is homeomorphic to a euclidean ball in $\mathbb{R}^n$, where $n$ is the dimension of $M$. We will need to consider balls which are locally flat and whose boundary has measure zero. More precisely, let us denote by $B_r(0)$ the euclidean ball with radius $r$ and center $0$ in $\mathbb{R}^n$. A ball $B$ will be called good if $\mu(\partial B) = 0$ and if there exists a topological embedding (continuous one-to-one map) $\gamma : B_2(0) \to M$ such that $\gamma(B_1(0)) = B$. Note that, due to countable additivity, if $\gamma : B_1(0) \to M$ is any topological embedding, then for almost every $r \in (0, 1)$ the ball $\gamma(B_r(0))$ is good.
Oxtoby-Ulam theorem  We will need the following consequence of the Oxtoby-Ulam theorem. Let \( B_1, B_2 \) be two good balls in the interior of some manifold \( M' \), with or without boundary (what we have in mind is either \( M' = M \) or \( M' \) is a euclidean ball). Let \( \mu' \) be a good probability measure on \( M' \) which assigns measure zero to the boundary \( \partial M' \). Denote by \( \text{Homeo}_0(M', \mu') \) the identity component of the group of homeomorphisms of \( M' \) which are supported in the interior of \( M' \) and preserve \( \mu' \). Assume \( \mu'(B_1) = \mu'(B_2) \). Then there exists \( \phi \in \text{Homeo}_0(M', \mu') \) such that \( \phi(B_1) = B_2 \). To construct \( \phi \), we first choose a good ball \( B \) in the interior of \( M' \) that contains \( B_1, B_2 \) in its interior. According to the annulus theorem ([Kir69, Qui82]), we may find a homeomorphism \( \phi' \) supported in the ball \( B \) that sends \( B_1 \) onto \( B_2 \). A first use of the Oxtoby-Ulam theorem provides a homeomorphism \( \phi_1 \) supported in \( B_2 \) and sending the measure \( (\phi'_1, \mu')|_{B_2} \) to the measure \( \mu'|_{B_2} \). A second use of the same theorem gives a homeomorphism \( \phi_2 \) supported in \( B \setminus B_2 \) and sending the measure \( (\phi'_2, \mu')|_{B \setminus B_2} \) to the measure \( \mu'|_{B \setminus B_2} \). Then \( \phi \) is obtained as \( \phi_2 \phi_1 \phi' \). Note that, since \( \phi \) is supported in the ball \( B \), Alexander’s trick ([Ale23]) provides an isotopy from the identity to \( \phi \) within the homeomorphisms of \( B \) that preserves the measure \( \mu' \), which shows that \( \phi \) belongs to \( \text{Homeo}_0(M', \mu') \).

### Triangulations

We will also need triangulations which have good properties with respect to the measure \( \mu \). We begin with any triangulation \( \mathcal{T} \) of \( M \). We would like the \((n - 1)\)-skeleton of \( \mathcal{T} \) to have measure zero, but some \((n - 1)\)-dimensional simplices may have positive measure. We fix this as follows. Each \( n \)-dimensional simplex \( s \) of \( \mathcal{T} \) is homeomorphic to the standard \( n \)-dimensional simplex; let \( \mu_s \) be a probability measure on \( s \) which is the homeomorphic image of the Lebesgue measure on the standard simplex. The measure

\[
\mu' = \frac{1}{N} \sum \mu_s
\]

(where \( N \) denotes the number of \( n \)-dimensional simplices of \( \mathcal{T} \)) is a good probability measure on \( M \) for which the \((n - 1)\)-dimensional simplices have measure zero. We apply the Oxtoby-Ulam theorem to get a homeomorphism \( h \) of \( M \) sending \( \mu' \) to \( \mu \). Then we consider the image triangulation \( \mathcal{T}_0 = h_*(\mathcal{T}) \), whose \((n - 1)\)-skeleton has measure zero. In addition to this, all the simplices of \( \mathcal{T}_0 \) have the same mass. Using successive barycentric subdivisions we get a sequence \( (\mathcal{T}_p)_{p \geq 0} \) of nested triangulations with both properties: the \((n - 1)\)-skeleton have no mass and all the simplices have the same mass. Denote by \( m_p \) the common mass of the simplices of \( \mathcal{T}_p \), and by \( d_p \) the supremum of the diameters of the simplices of \( \mathcal{T}_p \) (for some metric which is compatible with the topology on \( M \)). Then the sequences \( (m_p) \) and \( (d_p) \) tends to zero.

Here is a useful consequence. Let \( O \) be any open subset of \( M \). We define inductively \( \mathcal{O}_p \) as the set of all the \( n \)-dimensional open simplices of \( \mathcal{T}_p \) that are included in \( O \) but not in some \( s \in \mathcal{O}_{p-1} \). The elements of \( \mathcal{O} := \cup \mathcal{O}_p \) are pairwise

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\(^1\)One may probably avoid the use of the annulus theorem here, since the ball \( B \) may be constructed explicitly by gluing the two good balls \( B_1 \) and \( B_2 \) to a piecewise linear tube connecting them.
disjoint and their closures cover $O$. Since the $(n - 1)$-skeleton of our triangulations have no mass, we have the equality

$$\mu(O) = \sum_{U \in O} \mu(U) \quad (1).$$

We call a (closed) simplex of some $T_p$ good if it is a good ball in $M$. We notice that for every $p > 0$, all the $n$-dimensional simplices that are disjoint from the $(n - 1)$-skeleton of $T_0$ are good. Thus equality (1) still holds if, in the definition of the $O_p$’s, we replace the simplices by the simplices whose closure is good. As a consequence, if two probability measures $\mu, \mu'$ give the same mass to all the good simplices of $T_p$ for every $p$, then they are equal.

In the first Lemma we look for elements of the group $G_f$ that do not preserve the measure and have small support.

**Lemma 2.1.** For every positive $\varepsilon$ there exists a good ball $B$ of measure less than $\varepsilon$ and an element $g \in G_f$ which is supported in $B$ and does not preserve the measure $\mu$.

*Proof.* By hypothesis the probability measures $\mu$ and $f_*\mu$ are not equal. According to the discussion preceding the Lemma, there exists some $p > 0$ and some simplex of the triangulation $T_p$ whose closure $B_1$ is a good ball, and such that $\mu(B_1) \neq \mu(f^{-1}(B_1))$. To fix ideas let us assume that

$$\mu(f^{-1}(B_1)) > \mu(B_1).$$

This implies the same inequality for at least one of the simplices of $T_{p+1}$ that are included in $B_1$; thus, by induction, we see that we may choose $p$ to be arbitrarily large. Note that we have $\mu(f^{-1}(M \setminus B_1)) < \mu(M \setminus B_1)$. Thus the same reasoning, applied to $M \setminus B_1$, provides a (closed) simplex $B_2$ of some $T_{p'}$, disjoint from $B_1$, such that

$$\mu(f^{-1}(B_2)) < \mu(B_2).$$

Again, by induction, we may assume that $p' = p$ and this is an arbitrarily large integer. In particular $B_1$ and $B_2$ are good balls with the same mass. Let $B'$ be a ball whose interior contains $B_1$ and $B_2$. Since $B_1$ and $B_2$ have the same measure, by the above mentioned version of the Oxtoby-Ulam theorem there exists $\phi \in \text{Homeo}_0(M, \mu)$ supported in $B'$ and sending $B_1$ onto $B_2$. Now we consider the element

$$g = f^{-1}\phi f$$

of the group $G_f$. It has support in the ball $B = f^{-1}(B')$. It sends the ball $f^{-1}(B_1)$ to the ball $f^{-1}(B_2)$, and we have

$$\mu(f^{-1}(B_1)) > \mu(B_1) = \mu(B_2) > \mu(f^{-1}(B_2))$$

so that $g$ does not preserve the measure $\mu$, as required by the Lemma.

It remains to see that in the above construction we may have chosen $B$ to be a good ball of arbitrarily small measure. Since $\mu$ has no atom, for every $\varepsilon > 0$
there exists some \( \eta > 0 \) such that every subset of \( M \) of diameter less than \( \eta \) has measure less than \( \varepsilon \). Thus by choosing \( p = p' \) large enough we may require that

\[
\mu(f^{-1}(B_1)) + \mu(f^{-1}(B_2)) < \varepsilon.
\]

Then we choose \( B \) as a ball whose interior contains \( f^{-1}(B_1) \) and \( f^{-1}(B_2) \) and which still has measure less than \( \varepsilon \). Finally we shrink \( B \) a little bit to turn it into a good ball. This completes the proof of the Lemma. \( \square \)

We subdivide the euclidean unit ball \( B_1(0) \) of \( \mathbb{R}^n \) into the half-balls \( B_1^{-} = B_1(0) \cap \{ x \leq 0 \} \) and \( B_1^{+} = B_1(0) \cap \{ x \geq 0 \} \). Let \( \Sigma \) be the disk \( B_1^{-} \cap B_1^{+} \) that separates the half-balls. We consider a given ball \( B \) and some homeomorphism \( g \) supported in \( B \). For every homeomorphism \( \gamma : B_1(0) \to B \) we let \( \gamma^\pm = \gamma(B_1^{\pm}) \); we say that \( \gamma \) is thin if \( \gamma(\Sigma) \) has measure zero. We now consider the set \( \mathcal{I}(\gamma, g) \) of all the numbers of the type

\[
\mu(g(\gamma^+)) - \mu(\gamma^+)
\]

where \( \gamma \) is thin.

**Lemma 2.2.** If \( g \) does not preserve the measure \( \mu \) then \( \mathcal{I}(\gamma, g) \) contains an interval \([a^-, a^+]\) with \( a^- < 0 < a^+ \).

**Proof.** First we want to prove that there exists some \( \gamma : B_1(0) \to B \) which is thin and such that \( \mu(g(\gamma^+)) \neq \mu(\gamma^+) \). Since \( g \) does not preserve the measure \( \mu \), we may find some good ball \( b \) in the interior of \( B \) such that \( \mu(b) \neq \mu(f^{-1}(b)) \). To fix ideas we assume that \( \mu(b) < \mu(f^{-1}(b)) \). Thanks to the Oxtoby-Ulam theorem we may identify \( B \) with a euclidean ball in \( \mathbb{R}^n \), \( b \) with another euclidean ball inside \( B \), and \( \mu \) with the restriction of the Lebesgue measure on \( \mathbb{R}^n \). All our balls are centered at the origin. Let \( b' \) be a ball slightly greater than \( b \), and \( T \) be a thin tube in \( B \setminus b' \) connecting the boundary of \( B \) and that of \( b' \). There exists a homeomorphism \( \gamma : B_1(0) \to B \) such that \( \gamma^+ = T \cup b' \). The construction may be done so that the (Lebesgue) measure of \( \gamma^+ \) is arbitrarily close to that of \( b \), and then we have \( \mu(\gamma^+) < \mu(g^{-1}(\gamma^+)) \), as wanted.

We can find a continuous family \( \{ R_t \}_{t \in [0, 1]} \) of rotations of \( B_1(0) \) such that \( R_0 \) is the identity and \( R_1 \) is a rotation that exchanges \( B_1^{-} \) and \( B_1^{+} \). Setting \( \gamma_t := \gamma \circ R_t \), we have \( \gamma_1^{-} = \gamma_0^{-} = \gamma^{-} \). Note that it may happen that \( \gamma_t(\Sigma) \) has positive measure for some \( t \). To remedy for this we consider \( \gamma' = \phi \circ \gamma \), where \( \phi : B \to B \) is a homeomorphism that fixes \( \gamma(\Sigma) \), such that the image under \( \gamma' \) of the Lebesgue measure on \( B_1(0) \) is equivalent to the restriction of \( \mu \) to the ball \( B \), in the sense that both measures share the same measure zero sets; such a \( \phi \) is provided by the Oxtoby-Ulam theorem. This ensures that \( \gamma^\prime_t := \gamma' \circ R_t \) is thin for every \( t \). Note that \( \gamma^\prime_0 = \gamma^\prime_0^\pm = \gamma^\pm_1 \). We have

\[
\mu(g(\gamma^+_1)) - \mu(\gamma^+_1) = \mu(g(\gamma^+_0)) - \mu(\gamma^+_0) = (1 - \mu(g(\gamma^+_0))) - (1 - \mu(\gamma^+_0)) = -\mu(g(\gamma^+_0)) + \mu(\gamma^+_0) \neq 0.
\]

Thus the set \( \mathcal{I}(\gamma, g) \) contains the interval

\[
\{ \mu(g(\gamma^+_t)) - \mu(\gamma^+_t), t \in [0, 1] \}
\]
Corollary 2.3. Let $\gamma_0 : B_1(0) \to M$ be a topological embedding in $M$ with $\mu(\gamma_0(\Sigma)) = 0$, let $B_0 = \gamma_0(B_1(0))$, and let $\varepsilon > 0$ be less than the measure of $\gamma_0^+$. Then there exists some element $g \in G_f$, supported in $B_0$, such that

$$\mu(g(\gamma_0^+)) = \mu(\gamma_0^+) - \varepsilon.$$  

In the situation of the corollary we will say that $g$ transfers a mass $\varepsilon$ from $\gamma_0^+$ to $\gamma_0^-$. 

Proof. Lemma 2.1 provides some element $g' \in G_f$ that does not preserve the measure $\mu$, and which is supported on a good ball $B$ whose measure is less than the minimum of $\mu(\gamma_0^+) - \varepsilon$ and $\mu(\gamma_0^-)$. Then Lemma 2.2 provides some homeomorphism $\gamma : B_1(0) \to B$ which is thin and such that $g'$ transfers some mass $a$ from $\gamma^+$ to $\gamma^-$:

$$\mu(g'(\gamma^+)) - \mu(\gamma^+)) = a.$$  

Since such a number $a$ may be chosen freely in an open interval containing 0, we may assume that $a = \frac{\varepsilon}{N}$ for some positive integer $N$. Choose some homeomorphism $\Phi_1 \in \mathrm{Homeo}_0(M, \mu)$ that sends $B$ inside $B_0$, $\gamma^+$ inside $\gamma_0^+$ and $\gamma^-$ inside $\gamma_0^-$. Such a $\Phi_1$ is provided by Oxtoby-Ulam theorem, thanks to the fact that we have chosen the measure of $B$ to be small enough and that $\mu(\gamma(\Sigma)) = \mu(\gamma_0(\Sigma)) = 0$. Now the conjugate $g_1 = \Phi_1 g' \Phi_1^{-1}$ transfers a mass $a$ from $\gamma_0^+$ to $\gamma_0^-$: 

$$\mu(g_1(\gamma_0^+)) = \mu(\gamma_0^+) - a.$$  

We repeat the process with $\gamma_1 = g_1 \circ \gamma_0$ instead of $\gamma_0$, getting an element $g_2 \in G_f$ that transfers a mass $a$ from $\gamma_1^+$ to $\gamma_1^-$: 

$$\mu(g_2 g_1(\gamma_0^+)) = \mu(g_2(\gamma_0^+)) = \mu(g_1(\gamma_0^+)) - a = \mu(\gamma_0^+) - 2a.$$  

We repeat the process $N$ times, and get the final homeomorphism $g$ as a composition of the $N$ homeomorphisms $g_N, \ldots, g_1$. 

3 Proof of the theorem 

We consider as before some element $f \in \mathrm{Homeo}_0(M) \setminus \mathrm{Homeo}_0(M, \mu)$. Let $g$ be some other element in $\mathrm{Homeo}_0(M)$. In order to prove the theorem we want to approximate $g$ with some element in the group $G_f$ generated by $f$ and $\mathrm{Homeo}_0(M, \mu)$. We fix a triangulation $T_0$ for which the $(n-1)$-skeleton has zero measure. The first step of the proof consists in finding an element $g' \in G_f$ satisfying the following property: for every simplex $s$ of $T_0$, the measure of $g'(s)$ coincides with the measure of $g^{-1}(s)$. To achieve this, the (very natural) idea
is to use corollary \ref{s:2} to progressively transfer some mass from the simplices \( s \) whose mass is larger than the mass of their image under \( g^{-1} \), to those for which the opposite holds.

Here are some details. Given a triangulation \( T \) for which the \((n-1)\)-skeleton has zero measure, we choose two \( n \)-dimensional simplices \( s, s' \) of \( T \), and some positive \( \varepsilon \) less than \( \mu(s) \); let us explain how to transfer a mass \( \varepsilon \) from \( s \) to \( s' \). First assume that \( s \) and \( s' \) are adjacent. Then we may choose an embedding \( \gamma : B_1(0) \to s \cup s' \) with \( \gamma(\Sigma) \subset s \cap s' \), \( \gamma^+ \subset s \) and \( \gamma^- \subset s' \), and we apply corollary \ref{s:2}. Thus we get an element \( h \in G_f \), supported in \( s \cup s' \), such that \( \mu(h(s)) = \mu(s) - \varepsilon \), and consequently \( \mu(h(s')) = \mu(s') + \varepsilon \). Now consider the general case, when \( s \) and \( s' \) are not adjacent. Since \( M \) is connected, there exists a sequence \( s_0 = s, \ldots, s_\ell = s' \) of simplices of \( T \) in which two successive elements are adjacent. As described before we may transfer mass \( \varepsilon \) from \( s_0 \) to \( s_1 \), then from \( s_1 \) to \( s_2 \), and so on. Thus by successive adjacent transfers of mass we get some element in \( h \in G_f \) that transfers mass \( \varepsilon \) from \( s \) to \( s' \). Note that the masses of all the other elements do not change, that is, \( \mu(h(\sigma)) = \mu(\sigma) \) for every simplex \( \sigma \) of \( T \) different from \( s \) and \( s' \).

Now we go back to our triangulation \( T_0 \), and we construct \( g' \) the following way. If each simplex \( s \) has the same measure as its inverse image \( g^{-1}(s) \) then there is nothing to do. In the opposite case there exists some simplex \( s \) of \( T_0 \) such that \( \mu(s) > \mu(g^{-1}(s)) \). We also select some other simplex \( s' \) such that \( \mu(s') \neq \mu(g^{-1}(s')) \), and we use the previously described construction of a homeomorphism \( g_1 \in G_f \) that transfers the mass \( \mu(s) - \mu(g^{-1}(s)) \) from the simplex \( s \) to the simplex \( s' \). After doing so the number of simplices \( g_1(s) \in g_1T_0 \) whose mass differs from the mass of \( g^{-1}(s) \) has decreased by at least one compared to \( T_0 \). We proceed recursively until we get an element \( g' \in G_f \) such that \( \mu(g'(s)) = \mu(g^{-1}(s)) \) for every simplex \( s \) in \( T_0 \), as wanted for this first step.

For the second and last step we consider the triangulations \((g^{-1})_*(T_0)\) and \(g'_*(T_0)\). The homeomorphism \( g'g \) sends the first one to the second one, and each simplex \( g^{-1}(s) \in (g^{-1})_*(T_0) \) has the same measure as its image \( g'(s) \in g'_*(T_0) \). We apply Oxtoby-Ulam theorem independently on each \( g'(s) \) to get a homeomorphism \( \Phi_s : g'(s) \to g'(s) \), which is the identity on \( \partial g'(s) \), and which sends the measure \( (g'g)_*(\mu|g^{-1}(s)) \) to the measure \( \mu|g'(s) \). The homeomorphism

\[
\Phi := \left( \prod_s \Phi_s \right) g'g
\]

preserves the measure \( \mu \). Furthermore by Alexander’s trick each \( \Phi_s \) is isotopic to the identity, thus \( \Phi \) is isotopic to the identity, and belongs to the group \( \text{Homeo}_0(M, \mu) \). Now the homeomorphism \( g'' = g'^{-1} \Phi \) belongs to the group \( G_f \) and for each simplex \( s \) of the triangulation \( T_0 \) we have \( g''^{-1}(s) = g^{-1}(s) \). We may have chosen the triangulation \( T_0 \) so that each simplex has diameter less than some given \( \varepsilon \). Every point \( x \) in \( M \) belongs to some \( n \)-dimensional closed simplex \( g^{-1}(s) \) of the triangulation \((g^{-1})_*(T_0)\), and since both \( g(x) \) and \( g''(x) \) belong to \( s \) they are a distance less than \( \varepsilon \) apart. In other words the uniform distance from \( g \) to \( g'' \) is less than \( \varepsilon \). This proves that \( g \) belongs to the closure of \( G_f \), and completes the proof of the theorem.
References


