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Patxi Ritter, Alessandro D. A. M. Spallicci, Sofiane Aoudia, Stéphane Cordier

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Patxi Ritter[1,2] **Alessandro D.A.M. Spallicci**[1]‡
Sofiane Aoudia[3] **Stéphane Cordier**[2]

[1] Université d'Orléans, Observatoire des Sciences de l'Univers en région Centre, LPC2E Campus CNRS, 3A Av. Recherche Scientifique, 45071 Orléans, France
[2] Université d'Orléans, Laboratoire de Mathématiques - Analyse, Probabilités, Modèlisation - Orléans, MAPMO, Rue de Chartres, 45067 Orléans, France
[3] Max Planck Institut für Gravitationphysik, A. Einstein, Am Mühlenberg 1, 14476 Potsdam, Deutschland

Abstract. On the basis of a recently proposed strategy of finite element integration in time domain for partial differential equations with a singular source term, we present a fourth order algorithm for non-rotating black hole perturbations in the Regge-Wheeler gauge. Herein, we address even perturbations induced by a particle plunging in. The forward time value at the upper node of the (r^*, t) grid cell is obtained by an algebraic sum of i) the preceding node values of the same cell, ii) analytic expressions, related to the jump conditions on the wave function and its derivatives, iii) the values of the wave function at adjacent cells. In this approach, the numerical integration does not deal with the source and potential terms directly, for cells crossed by the particle world line. This scheme has also been applied to circular and eccentric orbits and it will be object of a forthcoming publication.

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1. Introduction

In the scenario of the capture of compact objects by a supermassive black hole of mass M, the seized object is compared to a small mass m (henceforth the particle or the source) perturbing the background spacetime curvature and generating gravitational radiation. A comprehensive introduction to the general relativistic issues related to EMRI (Extreme Mass Ratio Inspiral) sources is contained in a topical volume [1].

Schwarzschild-Droste (SD) [2–5] (see Rothman [6] for a justification of this terminology), black hole perturbations have been largely developed in the Regge-Wheeler (RW) gauge, before in vacuum [7] and after in the presence of a particle by Zerilli [8–11]. The first finite difference scheme in time domain was proposed by Lousto and Price [12]. The initial conditions, reflecting the past motion of the particle and the initial amount of gravitational waves, were parametrised by Martel and Poisson [13].

If the gravitational radiation emitted and the mass of the captured object are to be taken into account for the determination of the motion of the latter, it is necessary to compute the derivatives of the perturbations that implies the third derivative of the wave function $\Psi(r^*, t)$, see e.g. [14]. For a given accuracy $\mathcal{O}(h)$ of the third derivative

‡ Corresponding author: spallicci@cnrs-orleans.fr

of Ψ, the error on Ψ itself should be $\mathcal{O}(h^4)$. Effectively, the reminder ought to be $\mathcal{O}(h^5)$ due to the presence in the mesh of the particle that lowers by one more degree the convergence order of the code for geometrical effects [15]. We have therefore developed a fourth order scheme.

The complexity in assessing the continuity of the perturbations at the position of the particle and the compatibility of the self-force to the harmonic (Lorenz-de Donder§) gauge [16, 17] has led researchers to convey their efforts to this gauge, as commenced by Barack and Lousto [19]. Conversely, work in harmonic gauge is made cumbersome by the presence of a system of ten coupled equations which replace the single wave equation of the RW gauge.

We have proposed [20, 21] a finite element method of integration, in RW gauge, based on the jump conditions that the wave function and its derivatives have to satisfy for the SD black hole perturbations to be continuous at the position of the particle. We first deal with the radial trajectory and the associated even parity perturbations, while in a forthcoming paper we shall present the circular and eccentric orbital cases, referring thus to both odd and even parity perturbations.

The main feature of this method consists in avoiding the direct and explicit integration of the wave equation (the potential and the source term with the associated singularities) whenever the grid cells are crossed by the particle. Indeed, the information on the wave equation is implicitly given by the jump conditions on the wave function and its derivatives. Conversely, for cells not crossed by the particle world line, the integrating method might retain the previous approach by Lousto [22] and Haas [15]. Among the efforts using jump discontinuities, although in a different context, it is worthwhile to mention those of Haas [15], Sopuerta and coworkers [23–25] getting the self-force in a scalar case. For the geodesic gravitational case, like Sopuerta and coworkers, Jung et al. [26], Chakraborty et al. [27] rely on spectral methods; Zumbusch [28], Field et al. [29] use a discontinuous Galerkin method; Hopper and Evans [30] work partially in frequency domain. Among recent results not based on jump discontinuities but concerning fourth order time domain codes, the one proposed by Thornburg [31] deals with and adaptive mesh refinement, while Nagar and coworkers replace the delta distribution with a narrow Gaussian [32, 33].

For the computation of the back-action, this method ensures a well behaved wave function at the particle position, since the approach is governed by the analytical values of the jump conditions at the particle position.

In [21] we have provided waveforms at infinity and the wave function at the position of the particle at first order. Herein, we focus instead on the improvement of the algorithm at fourth order and refer to [21] for all complementary information. The features of this method can be summarised as follows:

- Avoidance of direct and explicit integration of the wave equation (the potential and the source term with the associated singularities) for the grid cells crossed by the particle.
- Improvement of the reliability, since analytic expressions partly replace numerical ones (the replacement is total at first order [20, 21]).
- Applicability of the method to generic orbits, assuming that the even and odd wave equations are satisfied by Ψ, respectively R, being C^{-1}‖.

§ FitzGerald is considered to have also identified the harmonic gauge [18].

‖ A C^{-1} continuity class element, like a Heaviside step distribution, may be seen as an element which after integration transforms into an element belonging to the C^0 class of functions.

Geometric units ($G = c = 1$) are used, unless stated otherwise. The metric signature is $(-,+,+,+)$.

2. The wave equation

The wave function (its dimension is such that the energy is proportional to $\int_0^\infty \dot{\Psi}^2\, dt$), in the Moncrief form [34] and RW gauge [7], is defined by

$$\Psi_l(t,r) = \frac{r}{\lambda+1}\left[K^l + \frac{r-2M}{\lambda r+3M}\left(H_2^l - r\frac{\partial K^l}{\partial r}\right)\right], \tag{1}$$

where $K(t,r)$ and $H_2(t,r)$ are the perturbations, and the Zerilli [9] normalisation is used for Ψ_l. The wave equation is given by the operator $\mathcal{Z}$ acting on the wave function

$$\mathcal{Z}\Psi^l(t,r) = \partial^2_{r^*}\Psi^l(t,r) - \partial_t^2\Psi^l(t,r) - V^l(r)\Psi^l(t,r) = S^l(t,r) \quad , \tag{2}$$

where $r^* = r + 2M\ln(r/2M-1)$ is the tortoise coordinate and the potential $V^l(r)$ is

$$V^l(r) = \left(1-\frac{2M}{r}\right)\frac{2\lambda^2(\lambda+1)r^3+6\lambda^2Mr^2+18\lambda M^2r+18M^3}{r^3(\lambda r+3M)^2} \quad , \tag{3}$$

being $\lambda = 1/2(l-1)(l+2)$. The source $S^l(t,r)$ includes the derivative of the Dirac distribution (the latter appear in the process of forming the wave equation out of the ten linearised Einstein equations)

$$S^l = \frac{2(r-2M)\kappa}{r^2(\lambda+1)(\lambda r+3M)}\times$$

$$\left\{\frac{r(r-2M)}{2U^0}\delta'\left[r-r_u(t)\right] - \left[\frac{r(\lambda+1)-3M}{2U^0} - \frac{3MU^0(r-2M)^2}{r(\lambda r+3M)}\right]\delta\left[r-r_u(t)\right]\right\} \quad , \tag{4}$$

$U^0 = E/(1-2M/r_u)$ being the time component of the 4-velocity, $E = \sqrt{1-2M/r_{u0}}$ the conserved energy per unit mass, and $\kappa = 4m\sqrt{(2l+1)\pi}$. The geodesic in the unperturbed SD metric $z_u(\tau) = \{t_u(\tau), r_u(\tau), \theta_u(\tau), \phi_u(\tau)\}$ assumes different forms according to the initial conditions. For radial infall of a particle starting from rest at finite distance r_{u0}, $r_u(t)$ is the inverse function in coordinate time t of the trajectory in the background field, corresponding to the geodesic in proper time τ (u stands for unperturbed)

$$\frac{t(r_u)}{2M} = \sqrt{1-\frac{2M}{r_{u0}}}\sqrt{1-\frac{r_u}{r_{u0}}}\left(\frac{r_{u0}}{2M}\right)\left(\frac{r_u}{2M}\right)^{1/2} + 2\mathrm{arctanh}\left(\frac{\sqrt{\frac{2M}{r_u}-\frac{2M}{r_{u0}}}}{\sqrt{1-\frac{2M}{r_{u0}}}}\right) +$$

$$\sqrt{1-\frac{2M}{r_{u0}}}\left(1+\frac{4M}{r_{u0}}\right)\left(\frac{r_{u0}}{2M}\right)^{3/2}\arctan\left(\sqrt{\frac{r_{u0}}{r}-1}\right) \tag{5}$$

The expressions above correspond to those in [14], where some of the errors of previously published literature on radial fall are indicated.

3. Jump conditions

From the visual inspection of the Zerilli wave equation (2), it is evinced that the wave function Ψ is of C^{-1} continuity class (the second derivative of the wave function is proportional to the first derivative of the Dirac distribution, in itself a C^{-3} class element). Thus, the wave function and its derivatives can be written as (the l index is dropped henceforth for simplicity of notation)

$$\Psi = \Psi^+\Theta_1 + \Psi^-\Theta_2 \ , \tag{6}$$
$$\Psi_{,r} = \Psi^+_{,r}\Theta_1 + \Psi^-_{,r}\Theta_2 + \left(\Psi^+ - \Psi^-\right)\delta \ , \tag{7}$$
$$\Psi_{,t} = \Psi^+_{,t}\Theta_1 + \Psi^-_{,t}\Theta_2 - \dot{r}_u\left(\Psi^+ - \Psi^-\right)\delta \ , \tag{8}$$
$$\Psi_{,rr} = \Psi^+_{,rr}\Theta_1 + \Psi^-_{,rr}\Theta_2 + 2\left(\Psi^+_{,r} - \Psi^-_{,r}\right)\delta + \left(\Psi^+ - \Psi^-\right)\delta' \ , \tag{9}$$
$$\Psi_{,tt} = \Psi^+_{,tt}\Theta_1 + \Psi^-_{,tt}\Theta_2 - 2\dot{r}_u\left(\Psi^+_{,t} - \Psi^-_{,t}\right)\delta - \ddot{r}_u\left(\Psi^+ - \Psi^-\right)\delta + \dot{r}_u^2\left(\Psi^+ - \Psi^-\right)\delta', \tag{10}$$
$$\Psi_{,tr} = \Psi^+_{,tr}\Theta_1 + \Psi^-_{,tr}\Theta_2 + \left(\Psi^+_{,t} - \Psi^-_{,t}\right)\delta - \dot{r}_u\left(\Psi^+_{,r} - \Psi^-_{,r}\right)\delta - \dot{r}_u\left(\Psi^+ - \Psi^-\right)\delta' \ , \tag{11}$$

where in shortened notation $\Theta_1 = \Theta\left[r - r_u(t)\right]$, and $\Theta_2 = \Theta\left[r_u(t) - r\right]$ are two Heaviside step distributions, while $\delta = \delta\left[r - r_u(t)\right]$ and $\delta' = \delta'\left[r - r_u(t)\right]$ are the Dirac delta - and its derivative - distributions. The dot and the prime indicate time and space derivatives, respectively.

3.1. Jump conditions from the wave equation

For the computation of back-action effects, we need first order derivatives of the perturbations and thus third order wave function derivatives. To this end, we operate directly on the wave equation, Eq. 2. The source term is cast in the following form

$$S(t,r) = G(t,r)\delta + F(t,r)\delta' = \tilde{G}_{r_u(t)}\delta + F_{r_u(t)}\delta' \ , \tag{12}$$

where $\tilde{G}_{r_u(t)} = G_{r_u(t)} - F'_{r_u(t)}$ and one of the properties of the Dirac delta distribution, namely $\phi(r)\delta'\left[r - r_u(t)\right] = \phi_{r_u(t)}\delta'\left[r - r_u(t)\right] - \phi'_{r_u(t)}\delta\left[r - r_u(t)\right]$, has been used at the position of the particle. The determination of the jump conditions imposes the transformation of Eq. 2 into the corresponding equation in (r,t) domain (the tortoise coordinate can't be inverted). Turning to the r variable, we get ($f = 1 - 2M/r$)

$$\begin{aligned}\partial^2_{r^*}\Psi &= ff'\partial_r\Psi + f^2\partial^2_r\Psi \\ &= \left[ff'\Psi^+_{,r} + f^2\Psi^+_{,rr}\right]\Theta_1 + \left[ff'\Psi^-_{,r} + f^2\Psi^-_{,rr}\right]\Theta_2 + ff'\left(\Psi^+ - \Psi^-\right)\delta \\ &\quad + 2f^2\left(\Psi^+_{,r} - \Psi^-_{,r}\right)\delta + f^2\left(\Psi^+ - \Psi^-\right)\delta' \ ,\end{aligned} \tag{13}$$

$$\begin{aligned}\partial^2_t\Psi &= \Psi^+_{,tt}\Theta_1 + \Psi^+_{,tt}\Theta_2 - 2\dot{r}_u\partial_t\left(\Psi^+ - \Psi^-\right)\delta - \ddot{r}_u\left(\Psi^+ - \Psi^-\right)\delta \\ &\quad + \dot{r}_u^2\left(\Psi^+ - \Psi^-\right)\delta' \ ,\end{aligned} \tag{14}$$

$$V\Psi = V\Psi^+\Theta_1 + V\Psi^-\Theta_2 \ . \tag{15}$$

The notation $[\Psi]$ stands for the difference $\left(\Psi^+ - \Psi^-\right)_{r_u}$ and a likewise notation is used for the derivatives at the point r_u. Equating the coefficients of δ', and owing

to the above mentioned property of the delta derivative for which $(\Psi^+ - \Psi^-)\,\delta' = [\Psi]\,\delta' - [\Psi_{,r}]\,\delta$, we get the jump condition for Ψ

$$[\Psi] = \frac{1}{f_{r_u}^2 - \dot{r}_u^2} F_{r_u} \quad . \tag{16}$$

Equating the coefficients of δ, we get the jump condition on the space derivative

$$[\Psi_{,r}] = \frac{1}{f_{r_u}^2 - \dot{r}_u^2}\left[\tilde{G}_{r_u} + \left(f_{r_u} f'_{r_u} - \ddot{r}_u\right)[\Psi] - 2\dot{r}_u \frac{d}{dr_u}[\Psi]\right] \quad , \tag{17}$$

and therefore the jump condition on the first time derivative

$$[\Psi_{,t}] = \dot{r}_u \frac{d}{dr_u}[\Psi] - \dot{r}_u [\Psi_{,r}] \quad . \tag{18}$$

Since $\mathcal{Z}\Psi^{\pm} = 0$, the coefficients of Θ_1 and Θ_2 ought to be equal. We thus obtain

$$[\Psi_{,tt}] - f_{r_u} f'_{r_u}[\Psi_{,r}] - f_{r_u}^2[\Psi_{,rr}] + V_{r_u}[\Psi] = 0 \quad , \tag{19}$$

which is an equation with two unknowns. We circumvent the difficulty by using i) the commutativity of the derivatives, $[\Psi_{,tr}] = [\Psi_{,rt}]$, ii) the transformation $d/dt = \dot{r}_u d/dr_u$, and write

$$\begin{aligned}[\Psi_{,tt}] &= \frac{d}{dt}[\Psi_{,t}] - \dot{r}_u[\Psi_{,tr}] = \frac{d}{dt}[\Psi_{,t}] - \dot{r}_u\left\{\frac{d}{dt}[\Psi_{,r}] - \dot{r}_u[\Psi_{,rr}]\right\} = \\ &= \dot{r}_u \frac{d}{dr_u}[\Psi_{,t}] - \dot{r}_u^2 \frac{d}{dr_u}[\Psi_{,r}] + \dot{r}_u^2[\Psi_{,rr}] \quad . \end{aligned} \tag{20}$$

The jump condition on the second space derivative can now be expressed by

$$[\Psi_{,rr}] = \frac{1}{f_{r_u}^2 - \dot{r}_u^2}\left\{\dot{r}_u \frac{d}{dr_u}[\Psi_{,t}] - \dot{r}_u^2 \frac{d}{dr_u}[\Psi_{,r}] - f_{r_u} f'_{r_u}[\Psi_{,r}] + V_{r_u}[\Psi]\right\} \quad . \tag{21}$$

The other second derivatives are obtained by

$$[\Psi_{,tr}] = [\Psi_{,rt}] = \frac{d}{dt}[\Psi_{,r}] - \dot{r}_u[\Psi_{,rr}] \quad , \tag{22}$$

$$[\Psi_{,tt}] = \frac{d}{dt}[\Psi_{,t}] - \dot{r}_u[\Psi_{,tr}] \quad . \tag{23}$$

For the third order derivatives, we derive the wave equation with respect to r and obtain

$$\begin{aligned}[\Psi_{,rrr}] = \frac{1}{\dot{r}_u^2 - f_{r_u}^2}\Bigg\{ & \dot{r}_u^2 \frac{d}{dr_u}[\Psi_{,rr}] - \dot{r}_u \frac{d}{dr_u}[\Psi_{,rt}] \\ & + \left(f'^2_{r_u} + f_{r_u} f''_{r_u} - V_{r_u}\right)[\Psi_{,r}] + 3 f_{r_u} f'_{r_u}[\Psi_{,rr}] - V'_{r_u}[\Psi]\Bigg\} \quad , \end{aligned} \tag{24}$$

while deriving with respect to t, we obtain

$$[\Psi_{,ttt}] = \frac{\dot{r}_u^2}{\dot{r}_u^2 - f_{r_u}^2}\left\{f_{r_u}^2 \frac{d}{dr_u}[\Psi_{,rt}] - \dot{r}_u^{-1} f_{r_u}^2 \frac{d}{dr_u}[\Psi_{,tt}] + f_{r_u} f'_{r_u}[\Psi_{,rt}] - V_{r_u}[\Psi_{,t}]\right\} \quad , \tag{25}$$

$$[\Psi_{,ttr}] = [\Psi_{,trt}] = [\Psi_{,rtt}] = \frac{d}{dr_u}[\Psi_{,tt}] - \dot{r}_u^{-1}[\Psi_{,ttt}] \quad , \tag{26}$$

$$[\Psi_{,trr}] = [\Psi_{,rtr}] = [\Psi_{,rrt}] = \frac{d}{dr_u}[\Psi_{,tr}] - \dot{r}_u^{-1}[\Psi_{,ttr}] \quad , \tag{27}$$

$$[\Psi_{,rrr}] = \frac{d}{dr_u}[\Psi_{,rr}] - \dot{r}_u^{-1}[\Psi_{,trr}] \quad . \tag{28}$$

Finally, we similarly proceed for the fourth derivatives

$$[\Psi_{,tttt}] = \frac{\dot{r}_u^2}{\dot{r}_u^2 - f_{r_u}^2} \times$$
$$\left\{ f_{r_u}^2 \frac{d}{dr_u}[\Psi_{,ttr}] - \dot{r}_u^{-1} f_{r_u}^2 \frac{d}{dr_u}[\Psi_{,ttt}] + f_{r_u} f'_{r_u}[\Psi_{,ttr}] - V_{r_u}[\Psi_{,tt}] \right\}, \tag{29}$$

$$[\Psi_{,tttr}] = [\Psi_{,ttrt}] = [\Psi_{,trtt}] = [\Psi_{,rttt}] \frac{d}{dr_u}[\Psi_{,ttt}] - \dot{r}_u^{-1}[\Psi_{,tttt}] \quad , \tag{30}$$

$$[\Psi_{,ttrr}] = [\Psi_{,trtr}] = [\Psi_{,trrt}] = [\Psi_{,rttr}][\Psi_{,rtrt}] = [\Psi_{,rrtt}] =$$
$$= \frac{d}{dr_u}[\Psi_{,ttr}] - \dot{r}_u^{-1}[\Psi_{,tttr}] \quad , \tag{31}$$

$$[\Psi_{,trrr}] = [\Psi_{,rtrr}] = [\Psi_{,rrtr}][\Psi_{,rrrt}] \frac{d}{dr_u}[\Psi_{,trr}] - \dot{r}_u^{-1}[\Psi_{,ttrr}] \quad , \tag{32}$$

$$[\Psi_{,rrrr}] = \frac{d}{dr_u}[\Psi_{,rrr}] - \dot{r}_u^{-1}[\Psi_{,rrrt}] \quad . \tag{33}$$

3.1.1. Jump conditions in explicit form. We list hereafter the jump conditions in explicit form.

Jump conditions

$$[\Psi] = \frac{\kappa E r_u}{(\lambda+1)(3M+\lambda r_u)} \tag{34}$$

First derivative jump conditions

$$[\Psi_{,t}] = -\frac{\kappa E r_u \dot{r}_u}{(2M-r_u)(3M+\lambda r_u)} \tag{35}$$

$$[\Psi_{,r}] = \frac{\kappa E\left[6M^2 + 3M\lambda r_u + \lambda(\lambda+1)r_u^2\right]}{(\lambda+1)(2M-r_u)(3M+\lambda r_u)^2} \tag{36}$$

Second derivative jump conditions

$$[\Psi_{,rr}] = -\frac{\kappa E\left[3M^3(5\lambda-3) + 6M^2\lambda(\lambda-3)r_u + 3M\lambda^2(\lambda-1)r_u^2 - 2\lambda^2(\lambda+1)r_u^3\right]}{(\lambda+1)(2M-r_u)^2(3M+\lambda r_u)^3} \tag{37}$$

$$[\Psi_{,tr}] = \frac{\kappa E\left(3M^2 + 3M\lambda r_u - \lambda r_u^2\right)\dot{r}_u}{(2M-r_u)^2(3M+\lambda r_u)^2} \tag{38}$$

$$[\Psi_{,tt}] = -\frac{\kappa\, E\, M}{{r_u}^2\,(3\,M + r_u\,\lambda)} \tag{39}$$

Third derivative jump conditions

$$[\Psi_{,rrr}] = \frac{\kappa E}{r_u\left(\lambda+1\right)\left(2M-r_u\right)^3(3M+r_u\lambda)^4}\Big[81\left(\lambda+1\right)M^5+9r_u\left(19\lambda^2+18E^2\lambda+ 3\lambda+18E^2\right)M^4+9r_u^2\lambda\left(7\lambda^2+24E^2\lambda-14\lambda+24E^2+3\right)M^3+3r_u^3\lambda^2\left(7\lambda^2 +36E^2\lambda-11\lambda+36E^2+18\right)M^2+3r_u^4\lambda^3\left(8E^2\lambda-7\lambda+8E^2-1\right)M+ 2r_u^5\lambda^3\left(\lambda+1\right)\left(E^2\lambda+3\right)\Big] \tag{40}$$

$$[\Psi_{,trr}] = \frac{-\kappa E\dot{r}_u}{r_u(2M-r_u)^3(3M+r_u\lambda)^3}\Big[27M^4+6r_u\left(5\lambda+9E^2-3\right)M^3+3r_u^2\lambda\left(5\lambda+ 18E^2-6\right)M^2+6r_u^3\lambda^2\left(3E^2-2\right)M+2r_u^4\lambda^2\left(E^2\lambda+1\right)\Big] \tag{41}$$

$$[\Psi_{,ttr}] = \frac{\kappa E}{r_u^3\left(2M-r_u\right)\left(3M+r_u\lambda\right)^2}\Big[39M^3+9r_u\left(3\lambda+2E^2-2\right)M^2+r_u^2\lambda\left(4\lambda+ 12E^2-13\right)M+2r_u^3\lambda^2\left(E^2-1\right)\Big] \tag{42}$$

$$[\Psi_{,ttt}] = \frac{-\kappa E\dot{r}_u}{r_u^3\left(2M-r_u\right)\left(3M+r_u\lambda\right)}\Big[9M^2+2r_u\left(2\lambda+3E^2-2\right)M+2r_u^2\lambda\left(E^2-1\right)\Big] \tag{43}$$

Fourth derivative jump conditions

$$[\Psi_{,rrrr}] = \frac{-3\kappa E}{r_u^2\left(\lambda+1\right)\left(2M-r_u\right)^4(3M+r_u\lambda)^5}\Big[567\left(\lambda+1\right)M^7+162r_u\left(\lambda+1\right)\left(6\lambda +16E^2-5\right)M^6+6r_u^2\left(139\lambda^3+738E^2\lambda^2-123\lambda^2+162E^4\lambda+441E^2\lambda -171\lambda+162E^4-297E^2+27\right)M^5+12r_u^3\lambda\left(21\lambda^3+252E^2\lambda^2-85\lambda^2+ 135E^4\lambda-24\lambda+135E^4-252E^2+18\right)M^4+3r_u^4\lambda^2\left(21\lambda^3+344E^2\lambda^2- 95\lambda^2+360E^4\lambda-340E^2\lambda+100\lambda+360E^4-684E^2+24\right)M^3+2r_u^5\lambda^3\cdot \left(88E^2\lambda^2-47\lambda^2+180E^4\lambda-260E^2\lambda+25\lambda+180E^4-348E^2-24\right)M^2 +2r_u^6\lambda^4\left(6E^2\lambda^2+30E^4\lambda-53E^2\lambda+23\lambda+30E^4-59E^2+11\right)M+ 4r_u^7\lambda^4\left(\lambda+1\right)\left(E^4\lambda-2E^2\lambda-2\right)\Big] \tag{44}$$

$$[\Psi_{,trrr}] = \frac{3\kappa E\dot{r}_u}{r_u^2(2M-r_u)^4(3M+r_u\lambda)^4}\Big[135M^6+27r_u\left(7\lambda+32E^2-6\right)M^5+3r_u^2\cdot \left(35\lambda^2+396E^2\lambda-75\lambda+108E^4-144E^2+18\right)M^4+r_u^3\lambda\left(35\lambda^2+ 612E^2\lambda-120\lambda+432E^4-594E^2+72\right)M^3+r_u^4\lambda^2\left(140E^2\lambda-45\lambda+ 216E^4-306E^2+36\right)M^2+2r_u^5\lambda^3\left(6E^2\lambda+24E^4-35E^2+9\right)M+ 2r_u^6\lambda^3\left(2E^4\lambda-3E^2\lambda-1\right)\Big] \tag{45}$$

$$[\Psi_{,ttrr}] = \frac{-\kappa E}{r_u^4(2M-r_u)^2(3M+r_u\lambda)^3}\Big[1431M^5+6r_u\left(251\lambda+234E^2-210\right)M^4+ 9r_u^2\left(59\lambda^2+160E^2\lambda-148\lambda+36E^4-66E^2+30\right)M^3+6r_u^3\lambda\left(10\lambda^2+ 82E^2\lambda-79\lambda+54E^4-102E^2+48\right)M^2+2r_u^4\lambda^2\left(28E^2\lambda-27\lambda+54E^4 -105E^2+52\right)M+12r_u^5\lambda^3\left(E^2-1\right)^2\Big] \tag{46}$$

$$[\Psi_{,tttr}] = \frac{\kappa E\dot{r}_u}{r_u^4\left(2M-r_u\right)^2\left(3M+r_u\lambda\right)^2}\Big[243M^4+3r_u\left(61\lambda+132E^2-64\right)M^3+3r_u^2\cdot \left(12\lambda^2+92E^2\lambda-49\lambda+36E^4-48E^2+12\right)M^2+2r_u^3\lambda\left(24E^2\lambda-15\lambda+ 36E^4-51E^2+14\right)M+6r_u^4\lambda^2\left(E^2-1\right)\left(2E^2-1\right)\Big] \tag{47}$$

$$[\Psi_{,tttt}] = \frac{-\kappa E}{r_u^6\left(3M+r_u\lambda\right)}\Big[189M^3+2r_u\left(36\lambda+84E^2-77\right)M^2+6r_u^2\left(E^2-1\right)(10\lambda+ 6E^2-5\big)M+12r_u^3\lambda\big(E^2-1\big)^2\Big] \tag{48}$$

While heuristic arguments [35, 36] have been put forward to show that, for radial fall in the RW gauge, even metric perturbations belong to the C^0 continuity class at the position of th particle, in [20, 21] we have provided an analysis *vis à vis* the jump conditions that the wave function and its (first and second) derivatives have to satisfy for guaranteeing the continuity of the perturbations at the position of the particle. Therein, we have derived the same jump conditions (34 - 38) from the inverse relations (expressions giving the perturbations as function of the wave function and its derivatives) by fulfilment of the continuity conditions (equal coefficients for the two Heaviside distributions, and null coefficients for the Dirac distribution and its derivative).

4. The algorithm

The integration method considers cells belonging to two groups for cells never crossed by the world line, the integrating method may be drawn by previous approaches explored by Lousto [22] and Haas [15], whereas for cells crossed by a particle, we propose a new algorithm. The grid is in the r^*, t domain.

Initial conditions require knowledge of the situation prior to $t = 0$. At fourth order, the wave function may be Taylor-expanded around $t = 0$. For the boundary conditions, simplicity suggests a sufficiently large grid to avoid unwanted reflections.

4.1. Empty cells

Empty cells are those cells which are not crossed by the particle. In this case, the cell upper point is obtained by performing an integration of the wave equation over the entire surface A of the cell, identified by the nodes $\alpha, \beta, \gamma, \delta$. We briefly recall the algorithm used by Haas [15]. Therein, the sole numerical computation to be carried

out is represented by the product of the potential term and the wave function $V\Psi = g$. It is performed via a double Simpson integral, using points of the past light cone of the upper node α, Fig. 1. We set $g_q = g(r^*_q, t_q) = V(r_q)\Psi(r^*_q, t_q)$, $V_q = V(r_q)$ and $\Psi_q = \Psi(r^*_q, t_q)$, where q is one of the points shown in Fig. 1. The increment h is defined as $h = \frac{1}{2}\Delta r^* = \frac{1}{2}\Delta t$ where Δr^* is the spatial step and Δt is the time step. We have

$$\int\int_{Cell} g dA = \left(\frac{h}{3}\right)^2 \Big[g_\alpha + g_\beta + g_\gamma + g_\delta + 4\left(g_{\beta\gamma} + g_{\alpha\beta} + g_{\delta\gamma} + g_{\alpha\delta}\right) + 16 g_\sigma\Big] + \mathcal{O}(h^6) \quad , \tag{49}$$

where the sum of the intermediate terms between nodes is given by

$$\begin{aligned} g_{\beta\gamma} + g_{\alpha\beta} + g_{\delta\gamma} + g_{\alpha\delta} = {} & 2V_\sigma\Psi_\sigma\Big[1 - \frac{1}{2}\left(\frac{h}{2}\right)^2 V_\sigma\Big] + V_{\beta\gamma}\Psi_\beta\Big[1 - \frac{1}{2}\left(\frac{h}{2}\right)^2 V_{\beta\gamma}\Big] + \\ & V_{\delta\gamma}\Psi_\delta\Big[1 - \frac{1}{2}\left(\frac{h}{2}\right)^2 V_{\delta\gamma}\Big] + \frac{1}{2}\Big[V_{\beta\gamma} - 2V_\sigma + V_{\delta\gamma}\Big]\left(\Psi_\beta + \Psi_\delta\right) \\ & + \mathcal{O}(h^4) \quad . \end{aligned} \tag{50}$$

The last intermediate term g_σ in Eq. 49 is evaluated using given nodes in the past light cone of α, Fig. 1

$$g_\sigma = \frac{1}{16}\Big[8g_\beta + 8g_\gamma + 8g_\delta - 4g_{\gamma_1} - 4g_{\gamma_2} + g_{\mu_1} - g_{\mu_2} - g_{\mu_3} + g_{\mu_4}\Big] + \mathcal{O}(h^4) \quad . \tag{51}$$

For the differential operators, an exact integration simply leads to

$$\int\int_{Cell} \left(\partial^2_{r^*} - \partial^2_t\right)\Psi(r^*, t) dA = -4\Big[\Psi_\alpha - \Psi_\beta + \Psi_\gamma - \Psi_\delta\Big] \quad . \tag{52}$$

Finally, we get

$$\begin{aligned} \Psi_\alpha = {} & -\Psi_\gamma + \Psi_\beta\Big[1 - \frac{1}{4}\left(\frac{h}{2}\right)^2 \left(V_\sigma + V_\beta\right) + \frac{1}{16}\left(\frac{h}{2}\right)^4 V_\sigma\left(V_\sigma + V_\beta\right)\Big] \\ & + \Psi_\delta\Big[1 - \frac{1}{4}\left(\frac{h}{2}\right)^2 \left(V_\sigma + V_\delta\right) + \frac{1}{16}\left(\frac{h}{2}\right)^4 V_\sigma\left(V_\sigma + V_\delta\right)\Big] \\ & - \left(\frac{h}{2}\right)^2 \Big[1 - \frac{1}{4}\left(\frac{h}{2}\right)^2 V_\sigma\Big]\Big[g_{\beta\gamma} + g_{\alpha\beta} + g_{\delta\gamma} + g_{\alpha\delta} + 4g_\sigma\Big] \end{aligned} \tag{53}$$

For cells adjacent to cells crossed by the particle, the requirement of good accuracy suggests a different dealing for the computation of g_σ, since the past light cone of an adjacent cell can cross the path of the particle. In such a case, g_σ is approximated by non-centred spatial finite difference expressions [15].

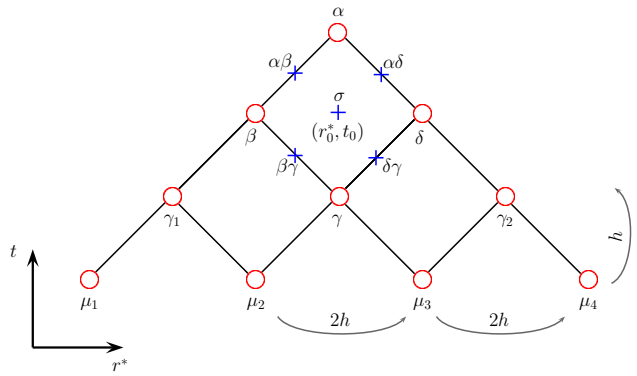

Figure 1: Set of points (circles and crosses) used for the integration of the $V\Psi = g$ term in the vacuum case. The crosses don't overlap with grid nodes; thus the field g at these points, Eqs. (50,51), is approximated by the field at the nodes on the past light cone of the grid node α.

4.2. Cells crossed by the world line

With reference to Figs. 2 - 4, we define $\alpha_0, \beta_0, \gamma_0, \delta_0$ the four corners of the cell. There are three different envisaged cases depending upon the trajectory of the particle across the cell. The fourth case, the trajectory entering through the γ_0 - δ_0 side, leaving through the β_0 - α_0 side, and passing below the cell centre, is to be dealt as the case of Fig. 2. We start considering the case shown by Fig. 2. The notation $[\Psi]_q$ stands for the difference $(\Psi^+ - \Psi^-)_{r_u = r_u(t_q)}$, where t_q is the coordinate time at the point $q = a, b$. A likewise notation is used for the derivatives.
In case 1, Fig. 2, the trajectory crosses the segment $[\alpha_0\gamma_0]$ at the point b. We define the lapse $\epsilon_b = t_{\alpha_0} - t_b$. Our aim is the determination of the value of $\Psi^+_{\alpha_0}$, knowing: i) ϵ_b, ii) the jump (analytical) conditions on Ψ and its derivatives at the point b; iii) the values of Ψ on a set of 14 points $\{\alpha_i, \beta_j, \gamma_i, \mu_j\}$ at the left hand-side of the world line. A Taylor expansion is applied at each point around b up to 4^{th} order

$$\Psi^+_{\alpha_0} = \Psi^+\left(t_b + \epsilon_b, r^*_b\right) = \sum_{n=0}^{4} \frac{\epsilon_b^n}{n!} \partial_t^n \Psi^+_b + \mathcal{O}\left(\epsilon_b^5\right) \quad , \tag{54}$$

$$\Psi^-_{\gamma_0} = \Psi^-\left(t_b - (2h - \epsilon_b), r^*_b\right) = \sum_{n=0}^{4} (-1)^n \frac{(2h - \epsilon_b)^n}{n!} \partial_t^n \Psi^-_b + \mathcal{O}\left((2h - \epsilon_b)^5\right) \quad , \tag{55}$$

$$\Psi^-_{\alpha_i} = \Psi^-\left(t_b + \epsilon_b, r^*_b - 2ih\right) = \sum_{n+m\leq 4} (-1)^n \frac{(2ih)^n}{n!} \frac{\epsilon_b^m}{m!} \frac{\partial^{n+m}}{\partial_{r^*}^n \partial_t^m} \Psi^-_b + \mathcal{O}\left(h^5\right) \quad , \tag{56}$$

$$\Psi^-_{\beta_j} = \Psi^-\left(t_b - (h - \epsilon_b), r^*_b - h(2j+1)\right) = \sum_{n+m\leq 4} (-1)^{n+m} \frac{(2jh + h)^n}{n!} \frac{(h - \epsilon_b)^m}{m!} \frac{\partial^{n+m}}{\partial_{r^*}^n \partial_t^m} \Psi^-_b + \mathcal{O}\left(h^5\right), \tag{57}$$

$$\Psi^-_{\gamma_i} = \Psi^-\left(t_b - (2h - \epsilon_b), r^*_b - 2ih\right) = \sum_{n+m\leq 4} (-1)^{n+m} \frac{(2ih)^n}{n!} \frac{(2h - \epsilon_b)^m}{m!} \frac{\partial^{n+m}}{\partial_{r^*}^n \partial_t^m} \Psi^-_b + \mathcal{O}\left(h^5\right) \quad , \tag{58}$$

$$\Psi^-_{\mu_j} = \Psi^-\left(t_b - (3h - \epsilon_b), r^*_b - h(2j+1)\right) = \sum_{n+m\leq 4} (-1)^{n+m} \frac{(2jh+h)^n}{n!} \frac{(3h-\epsilon_b)^m}{m!} \frac{\partial^{n+m}}{\partial^n_{r^*}\partial^m_t} \Psi^-_b + \mathcal{O}\left(h^5\right), \tag{59}$$

for the indexes running as $i = 1, 2, 3$, $j = 0, 1, 2, 3$ and concerning the α, γ and β, μ nodes, respectively.

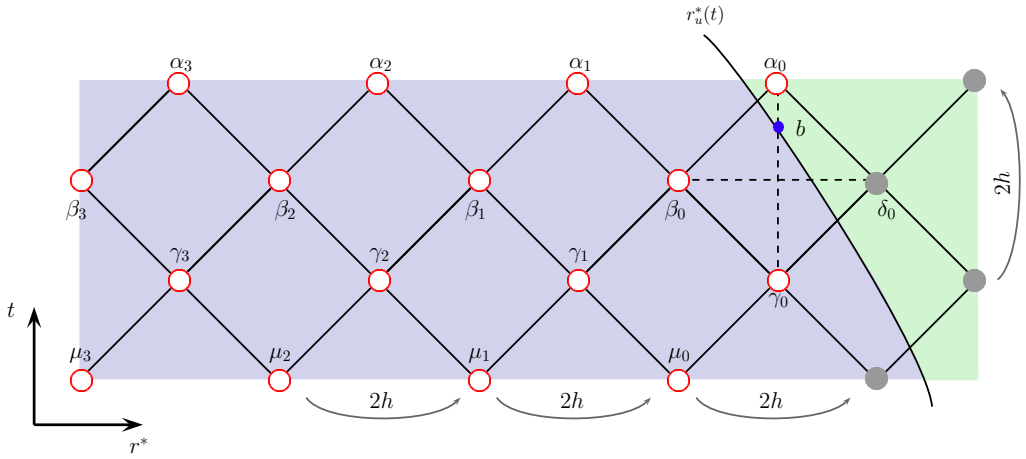

Figure 2: Cell crossed by the particle and cells used for the computation of $\Psi^+_{\alpha_0}$. Since the points α_0 and γ_0 are used to determine the jump conditions, the elimination of the derivatives of Ψ^-_b demands (60) the utilisation of the points on the left of the path $r^*_u(t)$. The points used in the algorithm are represented by circles. In this case, the particle crosses the segment $[\alpha_0\gamma_0]$ at the point b. The background distinguishes two zones: one where $\Psi(r^* < r^*_u(t), t) = \Psi^-(r^*, t)$, the other where $\Psi(r^* > r^*_u(t), t) = \Psi^+(r^*, t)$, the path $r^*_u(t)$ representing the separation between the two zones.

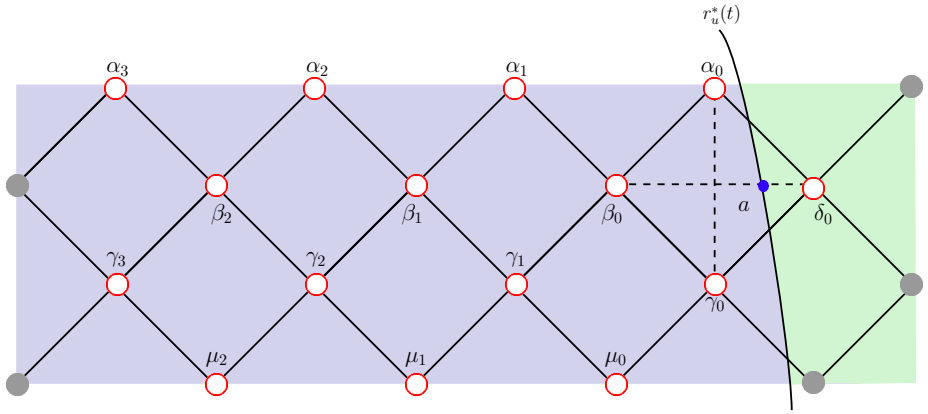

Figure 3: Cell crossed by the particle and cells used for the computation of $\Psi^-_{\alpha_0}$. Since the points α_0 and δ_0 are used to determine the jump conditions, the elimination of the derivatives of Ψ^-_a demands the utilisation of the points on the left of the path $r^*_u(t)$. The points used in the algorithm are represented by circles. In this case, the particle crosses the segment $[\beta_0\delta_0]$ at the point a. The background distinguishes two zones: one where $\Psi(r^* < r^*_u(t), t) = \Psi^-(r^*, t)$, the other where $\Psi(r^* > r^*_u(t), t) = \Psi^+(r^*, t)$, the path $r^*_u(t)$ representing the separation between the two zones.

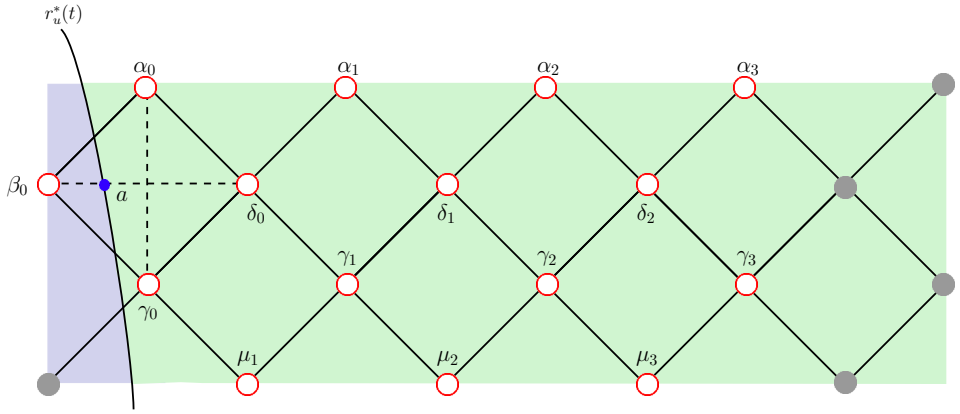

Figure 4: Cell crossed by the particle and cells used for the computation of $\Psi^+_{\alpha_0}$. Since the points α_0 and β_0 are used to determine the jump conditions, the elimination of the derivatives of Ψ^+_a demands the utilisation of the points on the right of the path $r^*_u(t)$. The points used in the algorithm are represented by circles. In this case, the particle crosses the segment $[\beta_0\delta_0]$ at the point a. The background distinguishes two zones: one where $\Psi(r^* < r^*_u(t), t) = \Psi^-(r^*, t)$, the other where $\Psi(r^* > r^*_u(t), t) = \Psi^+(r^*, t)$, the path $r^*_u(t)$ representing the separation between the two zones.

$$\Psi^+_{\alpha 0} = \Psi^-_{\gamma 0} + [\Psi]_b + \epsilon_b [\Psi_{,t}]_b + c_1 \partial_t \Psi^-_b + \frac{\epsilon_b^2}{2} [\Psi_{,tt}]_b + c_2 \partial_t^2 \Psi^-_b + \frac{\epsilon_b^3}{3!} [\Psi_{,ttt}]_b +$$

$$c_3\partial_t^3\Psi_b^- + \frac{\epsilon_b^4}{4!}\left[\Psi_{,tttt}\right]_b + c_4\partial_t^4\Psi_b^- + \mathcal{O}(h^5) \quad , \tag{60}$$

where for an accuracy of order 4, all quantities $\mathcal{O}(h^5)$ are disregarded, and the c coefficients are given by

$$c_1 = 2h \tag{61}$$
$$c_2 = 2h(\epsilon_b - h) \tag{62}$$
$$c_3 = \frac{h}{3}(4h^2 - 6h\epsilon_b + 3\epsilon_b^2) \tag{63}$$
$$c_4 = \frac{h}{3}(-2h^3 + 4h^2\epsilon_b - 3h\epsilon_b^2 + \epsilon_b^3) \tag{64}$$

The sum $c_1\partial_t\Psi_b^- + c_2\partial_t^2\Psi_b^- + c_3\partial_t^3\Psi_b^- + c_4\partial_t^4\Psi_b^-$, Eq. 60, is composed by numerical derivatives of lower order than $\mathcal{O}(h^5)$. However, we can adopt a combination of wave function values at adjacent cells $\left\{\Psi_{\alpha_i}^-, \Psi_{\beta_j}^-, \Psi_{\gamma_i}^-, \Psi_{\mu_j}^-\right\}$, Fig. 2, that removes this sum. Therefore, we look for coefficients $\{\mathcal{A}_i\}$, $\{\mathcal{B}_j\}$, $\{\mathcal{G}_i\}$, $\{\mathcal{M}_j\}$ such that

$$\begin{aligned}\Psi_{\alpha 0}^+ = \Psi_{\gamma 0}^- + \left[\Psi\right]_b + \epsilon_b\left[\Psi_{,t}\right]_b + \frac{\epsilon_b^2}{2}\left[\Psi_{,tt}\right]_b + \frac{\epsilon_b^3}{3!}\left[\Psi_{,ttt}\right]_b + \frac{\epsilon_b^4}{4!}\left[\Psi_{,tttt}\right]_b \\ + \sum_{i=1}^{3}\left(\mathcal{A}_i\Psi_{\alpha_i}^- + \mathcal{G}_i\Psi_{\gamma_i}^-\right) + \sum_{j=0}^{3}\left(\mathcal{B}_j\Psi_{\beta_j}^- + \mathcal{M}_j\Psi_{\mu_j}^-\right) \quad ,\end{aligned} \tag{65}$$

In other words, the coefficients must satisfy the following equality

$$\begin{aligned}&c_1\partial_t\Psi_b^- + c_2\partial_t^2\Psi_b^- + c_3\partial_t^3\Psi_b^- + c_4\partial_t^4\Psi_b^- = \\ &\sum_{i=1}^{3}\left(\mathcal{A}_i\Psi_{\alpha_i}^- + \mathcal{G}_i\Psi_{\gamma_i}^-\right) + \sum_{j=0}^{3}\left(\mathcal{B}_j\Psi_{\beta_j}^- + \mathcal{M}_j\Psi_{\mu_j}^-\right) \quad .\end{aligned} \tag{66}$$

By injection of Eqs. 56-59 into Eq. 66, the system can be cast in a matrix form

$$\mathbb{T}\cdot\mathbb{P} = \mathbb{C} \quad , \tag{67}$$

where $\mathbb{P}$ is the unknown vector formed by the coefficients $\{\mathcal{A}_i\}$, $\{\mathcal{B}_j\}$, $\{\mathcal{G}_i\}$, $\{\mathcal{M}_j\}$

$$\mathbb{P} = (\mathcal{A}_1, \ldots, \mathcal{A}_3, \mathcal{B}_0, \ldots, \mathcal{B}_3, \mathcal{G}_1, \ldots, \mathcal{G}_3, \mathcal{M}_0, \ldots, \mathcal{M}_3)^t \quad , \tag{68}$$

$\mathbb{C}$ is given by

$$\mathbb{C} = (0,\ c_1,\ c_2,\ c_3,\ c_4,\ 0, \cdots,\ 0)^t \quad , \tag{69}$$

while $\mathbb{T}$ is the matrix constructed from the Taylor coefficients in Eqs. 56-59 (see appendix). By inversion of $\mathbb{T}$, we get $\mathbb{P}$ and specifically

$$\mathcal{A}_1 = \mathcal{B}_2 = \mathcal{G}_2 = \mathcal{M}_1 = -3 \quad , \tag{70}$$
$$\mathcal{A}_2 = \mathcal{B}_1 = \mathcal{G}_1 = \mathcal{M}_2 = +3 \quad , \tag{71}$$
$$\mathcal{A}_3 = \mathcal{B}_0 = \mathcal{M}_3 = -1 \quad , \tag{72}$$
$$\mathcal{B}_3 = \mathcal{M}_0 = \mathcal{G}_3 = +1 \quad , \tag{73}$$

which leads to

$$\begin{aligned}\Psi^{+}_{\alpha 0} = & \left[\Psi\right]_b + \epsilon_b \left[\Psi_{,t}\right]_b + \frac{\epsilon_b^2}{2}\left[\Psi_{,tt}\right]_b + \frac{\epsilon_b^3}{3!}\left[\Psi_{,ttt}\right]_b + \frac{\epsilon_b^4}{4!}\left[\Psi_{,tttt}\right]_b \\ & +3\left(\Psi^{-}_{\alpha_2} + \Psi^{-}_{\beta_1} + \Psi^{-}_{\gamma_1} + \Psi^{-}_{\mu_2} - \Psi^{-}_{\alpha_1} - \Psi^{-}_{\beta_2} - \Psi^{-}_{\gamma_2} - \Psi^{-}_{\mu_1}\right) \\ & +\left(\Psi^{-}_{\beta_3} + \Psi^{-}_{\mu_0} + \Psi^{-}_{\gamma_3} - \Psi^{-}_{\alpha_3} - \Psi^{-}_{\beta_0} - \Psi^{-}_{\mu_3}\right) + \Psi^{-}_{\gamma_0} \quad . \end{aligned} \tag{74}$$

We thus have obtained, without direct integration of the singular source and potential term, the value of the upper node. Equation 74 shows three types of terms: the preceding node values of the same cell, the jump conditions which are fully analytical quantities, and the wave function values at adjacent cells. Incidentally, at first order [21], the latter type of terms disappears and a simpler expression is obtained.

Similar relations are found for the other two remaining cases. For case 2, Fig. 3, we obtain (having defined the shift $\epsilon_a = t_{\delta_0} - r_a$)

$$\begin{aligned}\Psi^{-}_{\alpha 0} = & -\left[\Psi\right]_a - \epsilon_a \left[\Psi_{,r^*}\right]_a - \frac{\epsilon_a^2}{2}\left[\Psi_{,r^*r^*}\right]_a - \frac{\epsilon_a^3}{3!}\left[\Psi_{,r^*r^*r^*}\right]_a - \frac{\epsilon_a^4}{4!}\left[\Psi_{,r^*r^*r^*r^*}\right]_a \\ & +2\left(\Psi^{-}_{\gamma_0} + \Psi^{-}_{\mu_1} - \Psi^{-}_{\alpha_1} - \Psi^{-}_{\beta_2}\right) + \left(\Psi^{-}_{\alpha_2} + \Psi^{-}_{\gamma_3} - \Psi^{-}_{\mu_0} - \Psi^{-}_{\mu_2}\right) \\ & +3\left(\Psi^{-}_{\beta_1} - \Psi^{-}_{\gamma_1}\right) + \Psi^{+}_{\delta_0} \quad , \end{aligned} \tag{75}$$

while for case 3, Fig. 4

$$\begin{aligned}\Psi^{+}_{\alpha 0} = & \left[\Psi\right]_a - \epsilon_a \left[\Psi_{,r^*}\right]_a + \frac{\epsilon_a^2}{2}\left[\Psi_{,r^*r^*}\right]_a - \frac{\epsilon_a^3}{3!}\left[\Psi_{,r^*r^*r^*}\right]_a + \frac{\epsilon_a^4}{4!}\left[\Psi_{,r^*r^*r^*r^*}\right]_a \\ & +2\left(\Psi^{+}_{\gamma_0} + \Psi^{+}_{\mu_1} - \Psi^{+}_{\alpha_1} - \Psi^{+}_{\delta_2}\right) + \left(\Psi^{+}_{\alpha_2} + \Psi^{+}_{\gamma_3} - \Psi^{+}_{\mu_0} - \Psi^{+}_{\mu_2}\right) \\ & +3\left(\Psi^{+}_{\delta_1} - \Psi^{+}_{\gamma_1}\right) + \Psi^{-}_{\beta_0} \quad , \end{aligned} \tag{76}$$

where the jump conditions in the tortoise r^* relate to those previously computed in the r variable

$$\left[\Psi_{,r^*}\right] = f_{r_u}\left[\Psi_{,r}\right] \quad , \tag{77}$$

$$\left[\Psi_{,r^*r^*}\right] = f_{r_u} f'_{r_u}\left[\Psi_{,r}\right] + f^2_{r_u}\left[\Psi_{,rr}\right] \quad , \tag{78}$$

$$\left[\Psi_{,r^*r^*r^*}\right] = f_{r_u}\left(f'^2 + ff''\right)_{r_u}\left[\Psi_{,r}\right] + 3f^2_{r_u} f'_{r_u}\left[\Psi_{,rr}\right] + f^3_{r_u}\left[\Psi_{,rrr}\right] \quad , \tag{79}$$

$$\begin{aligned}\left[\Psi_{,r^*r^*r^*r^*}\right] = & f_{r_u}\left(f'^3 + 4ff'f'' + f^2 f'''\right)_{r_u}\left[\Psi_{,r}\right] + f^2_{r_u}\left(7f'^2 + 4ff''\right)_{r_u}\left[\Psi_{,rr}\right] \\ & + 6f^3_{r_u} f'_{r_u}\left[\Psi_{,rrr}\right] + f^4_{r_u}\left[\Psi_{,rrrr}\right] \quad . \end{aligned} \tag{80}$$

5. Numerical implementation

Waveforms at infinity and at the particle position at first order are to be found in [21], as well as comparisons with other methods. Herein we are concerned on the numerical improvement. To tis end, we have considered a distant observer, located at $r^* = 400(2M)$. The observer is reached by a pulse produced by a Gaussian, time-symmetric perturbation

$$\Psi(r^*, t)_{t=0} = \exp\left[-(r^* - r^*_0)^2\right] \quad , \tag{81}$$

$$\partial_t \Psi(r^*, t)_{t=0} = 0 \quad . \tag{82}$$

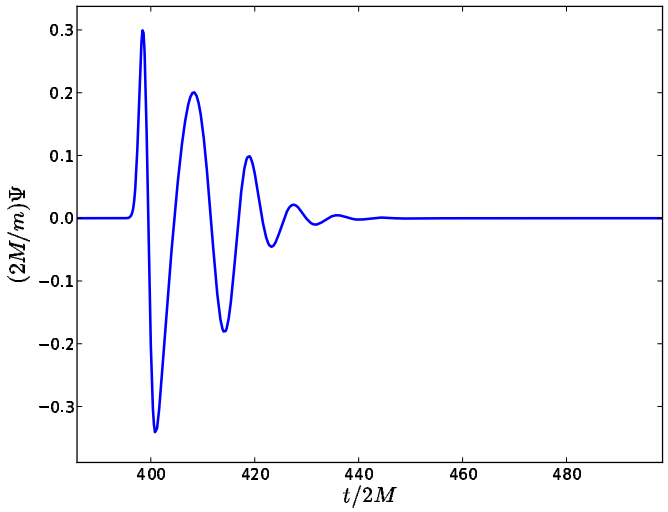

Figure 5: The waveform, $r_{u0} = 5(2M)$, of a Gaussian, time-symmetric initial pulse. The observer is located at $r^* = 400(2M)$.

Fig. 5, obtained for $r_{u0} = 5(2M)$, shows the waveform produced in the homogeneous case. The convergence rate is computed as ($\epsilon^{(n)}(\xi)$ is the unknown error function of order ≈ 1)

$$n = \log\left|\frac{\psi(4h) - \psi(2h)}{\psi(2h) - \psi(h)}\right| / \log(2) + \log\left|\epsilon^{(n)}(\xi)\right| / \log(2) \quad . \tag{83}$$

Fig. 6, obtained for $r_{u0} = 5(2M)$, shows the fourth and second order convergence rates (we remind that the first order code [21] includes empty cells dealt at second order).

6. Conclusions

We have presented a fourth order novel integration method in time domain for the Zerilli wave equation. We have focused our attention to the even perturbations produced by a particle plunging in a non-rotating black hole. For cells crossed by the particle world line, the forward time wave function value at the upper node of the (t, r^*) grid cell is obtained by the combination of the preceding node values of the same cell, analytic expressions related to the jump conditions, and the values of the wave function at adjacent cells. In this manner, the numerical integration does not deal directly nor with the source term and the associated singularities, nor with the potential term. In short, the direct integration of the wave equation is avoided. For other cells, we refer instead to already published approaches [15].

The scheme has also been applied to circular and eccentric orbits and it will be object of a forthcoming publication.

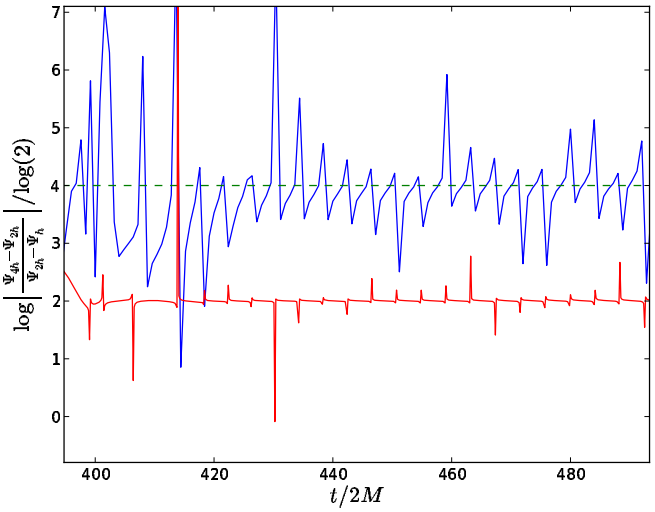

Figure 6: Convergence rates of the fourth and second order algorithms, $r_{u0} = 5(2M)$.

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Appendix

Through Eq. 60, we have determined the value of Ψ at the upper node of the cell as function of the analytic jump conditions and of the time derivatives of the wave function up to fourth order. The derivatives are evaluated at the point b and weighted by four coefficients c_1, c_2, c_3 and c_4. Afterwards, the derivatives are converted into a linear combination of the wave function values taken on the left side of the trajectory. Indeed, Eq. 66 represents such a system of linear equations. By injection of Eqs. 56-59 into Eq. 66, we get

$$\begin{array}{c} \mathcal{A}_1 T_{\alpha_1}^{(0,0)}\Psi_b + \mathcal{A}_1 T_{\alpha_1}^{(0,1)}\partial_t\Psi_b + \mathcal{A}_1 T_{\alpha_1}^{(0,2)}\partial_t^2\Psi_b + \cdots + \mathcal{A}_1 T_{\alpha_1}^{(1,3)}\partial_{r*}\partial_t^3\Psi_b \\ + \\ \mathcal{A}_2 T_{\alpha_2}^{(0,0)}\Psi_b + \mathcal{A}_2 T_{\alpha_2}^{(0,1)}\partial_t\Psi_b + \mathcal{A}_2 T_{\alpha_2}^{(0,2)}\partial_t^2\Psi_b + \cdots + \mathcal{A}_2 T_{\alpha_2}^{(1,3)}\partial_{r*}\partial_t^3\Psi_b \\ + \\ \vdots \\ + \\ \mathcal{M}_3 T_{\mu_3}^{(0,0)}\Psi_b + \mathcal{M}_3 T_{\mu_3}^{(0,1)}\partial_t\Psi_b + \mathcal{M}_3 T_{\mu_3}^{(0,2)}\partial_t^2\Psi_b + \cdots + \mathcal{M}_3 T_{\mu_3}^{(1,3)}\partial_{r*}\partial_t^3\Psi_b \end{array} = \begin{array}{c} c_1\partial_t\Psi_b \\ + \\ c_2\partial_t^2\Psi_b \\ + \\ c_3\partial_t^3\Psi_b \\ + \\ c_4\partial_t^4\Psi_b \end{array} , \quad (84)$$

where $T_p^{(n,m)}$ represent the Taylor series coefficients at p in the neighbourhood of b and the indexes correspond to n^{th} space and time m^{th} derivatives. The wave function

at p is thus given by

$$\Psi_p = \sum_{n+m\leq 4} T_p^{(n,m)} \frac{\partial^{n+m}}{\partial_{r*}^n \partial_t^m} \Psi_b + \mathcal{O}\left(h^5\right) \quad . \tag{85}$$

An example shows the procedure applicable to all cases. We pick the node α_1, Eq. 56, where $T_{\alpha_1}^{(n,m)} = (-1)^n \frac{(2h)^n}{n!} \frac{\epsilon_b^m}{m!}$ and remind that $T_p^{(0,0)} = 1 \ \forall \ p$. By grouping the derivatives, we get

$$\begin{array}{l} \left(\mathcal{A}_1 T_{\alpha_1}^{(0,0)} + \mathcal{A}_2 T_{\alpha_2}^{(0,0)} + \cdots + \mathcal{M}_3 T_{\mu_3}^{(0,0)}\right) \Psi_b \\ \quad + \\ \left(\mathcal{A}_1 T_{\alpha_1}^{(0,1)} + \mathcal{A}_2 T_{\alpha_2}^{(0,1)} + \cdots + \mathcal{M}_3 T_{\mu_3}^{(0,1)}\right) \partial_t \Psi_b \\ \quad + \\ \left(\mathcal{A}_1 T_{\alpha_1}^{(0,2)} + \mathcal{A}_2 T_{\alpha_2}^{(0,2)} + \cdots + \mathcal{M}_3 T_{\mu_3}^{(0,2)}\right) \partial_t^2 \Psi_b \\ \quad + \\ \quad \vdots \\ \quad + \\ \left(\mathcal{A}_1 T_{\alpha_1}^{(1,3)} + \mathcal{A}_2 T_{\alpha_2}^{(1,3)} + \cdots + \mathcal{M}_3 T_{\mu_3}^{(1,3)}\right) \partial_{r*} \partial_t^3 \Psi_b \end{array} \quad = \quad \begin{array}{c} c_1 \partial_t \Psi_b \\ + \\ c_2 \partial_t^2 \Psi_b \\ + \\ c_3 \partial_t^3 \Psi_b \\ + \\ c_4 \partial_t^4 \Psi_b \end{array} \quad . \tag{86}$$

By identification, we obtain a linear system, that is cast in the form

$$\underbrace{\begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ T_{\alpha_1}^{(0,1)} & \cdots & T_{\beta_0}^{(0,1)} & \cdots & T_{\gamma_1}^{(0,1)} & \cdots & T_{\mu_3}^{(0,1)} \\ T_{\alpha_1}^{(0,2)} & \cdots & T_{\beta_0}^{(0,2)} & \cdots & T_{\gamma_1}^{(0,2)} & \cdots & T_{\mu_3}^{(0,2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ T_{\alpha_1}^{(1,0)} & \cdots & T_{\beta_0}^{(1,0)} & \cdots & T_{\gamma_1}^{(1,0)} & \cdots & T_{\mu_3}^{(1,0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ T_{\alpha_1}^{(1,3)} & \cdots & T_{\beta_0}^{(1,3)} & \cdots & T_{\gamma_1}^{(1,3)} & \cdots & T_{\mu_3}^{(1,3)} \end{pmatrix}}_{\mathbb{T}} \underbrace{\begin{pmatrix} \mathcal{A}_1 \\ \vdots \\ \mathcal{B}_0 \\ \vdots \\ \mathcal{G}_1 \\ \vdots \\ \mathcal{M}_3 \end{pmatrix}}_{\mathbb{P}} = \underbrace{\begin{pmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbb{C}} \quad , \tag{87}$$

where the upper indexes (n, m) cover all combinations such that $n+m \leq 4$. Finally, by inversion of the $\mathbb{T}$ matrix, the unknown terms of the $\mathbb{P}$ vector are identified.

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