Gaussian multiplicative Chaos for symmetric isotropic matrices

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Abstract

Motivated by isotropic fully developed turbulence, we define a theory of symmetric matrix valued isotropic Gaussian multiplicative chaos. Our construction extends the scalar theory developed by J.P. Kahane in 1985.

1. Introduction

In the pioneering work [15], J.-P. Kahane introduced the theory of Gaussian multiplicative chaos. Given a metric space and a reference measure, Gaussian multiplicative chaos gives a mathematically rigorous definition to random measures defined as limits of measures with a lognormal density. The main application of this theory was to define the Kolmogorov-Obhukov model of energy dissipation in a turbulent flow (see [17], [19]): in this context, the metric space is the euclidean space $\mathbb{R}^3$ equipped with the Lebesgue measure and the log density has logarithmic correlations. Since this seminal work, the theory of Gaussian multiplicative chaos has found many applications in a broad number of fields among which finance ([2], [10]) and 2-d quantum gravity (see [9], [18] for the physics literature and [3], [11], [22] for the mathematics literature).

The main motivation of the Kolmogorov K41 theory ([16]) and it's extensions ([17]) is to define a realistic statistical theory of an incompressible, homogeneous, isotropic and fully developed turbulent flow (see for example [12, 13]). This ambitious program consists in defining a probabilistic model for the velocity field which satisfies the main statistical

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signatures observed experimentally, such as the mean energy transfer towards the small scales and the intermittency (or multifractal) phenomenon ([16]). Ideally, one looks for a field as close as possible to an invariant measure of the equations of motion. In [6], the authors propose a probabilistic construction of such a velocity field. Their construction, which requires a limiting procedure, is mathematically non rigorous and is based on the short time dynamics of the Euler flow, as well as further multifractal considerations. One of the key step of this construction is the introduction of the exponential of an isotropic trace-free matrix whose entries are Gaussian variables with logarithmic correlations. Unfortunately, as far as we know, there is no matrix valued theory of Gaussian multiplicative chaos. The purpose of this work is thus to define such a theory for a class of Gaussian symmetric and isotropic matrices.

In the next section, we present the framework and the main results. Section 3 is devoted to the proofs of our main results. In the appendix, we gather general formulas which are useful in our proofs.

Notations: we denote by $M(\mathbb{R}^d)$ the set of measures on $\mathbb{R}^d$ and by $M_s(\mathbb{R}^d)$ the set of signed measures on $\mathbb{R}^d$. We denote by $\mathcal{S}(M_s(\mathbb{R}^d))$ the set of symmetric matrices whose components belong to $M_s(\mathbb{R}^d)$.

2. Framework and main results

We consider an integer $N \geqslant 2$ and $c \in]-1, \frac{1}{N-1}]$. We introduce a probability space $(\Omega, \mathcal{F}, P)$ and denote expectation by E. On this space, we consider a symmetric random matrix-valued Gaussian process $X^\epsilon(x) = (X^\epsilon_{i,j}(x))_{1 \leqslant i,j \leqslant N}$ indexed by $x \in \mathbb{R}^d$ where the covariance structure is given for $\epsilon > 0$ by:

$$E[X^\epsilon_{i,i}(x)^2] = \gamma^2(\ln \frac{L}{\epsilon} + 1), \quad E[X^\epsilon_{i,i}(x)X^\epsilon_{j,j}(x)] = -c\gamma^2(\ln \frac{L}{\epsilon} + 1),\ i \neq j$$

and for $|y - x| > \epsilon$:

$$\begin{aligned} E[X^\epsilon_{i,i}(x)X^\epsilon_{i,i}(y)] =& K(x-y), \\ E[X^\epsilon_{i,i}(x)X^\epsilon_{j,j}(y)] =& -cK(x-y),\ i \neq j \end{aligned}$$

for some constant $L > 0$ and some kernel of positive type $K(x) = \gamma^2 \ln_+ \frac{L}{|x|} + g(x)$ where g is some continuous bounded function. In the sequel, we set $g(0) = m$.

We also set $\sigma^2_\epsilon = \gamma^2(\ln \frac{L}{\epsilon} + 1)$, $\sigma^2_{|y-x|} = K(y-x)$. We assume that the off diagonal terms are independent of the diagonal terms and mutually independent of variance $\bar{\sigma}^2_\epsilon = \frac{\sigma^2_\epsilon(1+c)}{2}$ and covariance for $|y - x| > \epsilon$:

$$E[X^\epsilon_{i,j}(x)X^\epsilon_{i,j}(y)] = \frac{1+c}{2}K(x-y),\ i < j.$$

In fact, the above structure is the most general situation that ensures that, for a given $x \in \mathbb{R}^d$, the Gaussian random matrix $X^\epsilon(x)$ is isotropic (see Lemma 4 in the Appendix).

Remark. Notice that the diagonal terms are independent if and only if $c = 0$. In this case, the above structure coincides with the usual Gaussian Orthogonal Ensemble (GOE) [1, 20]. Notice also that the boundary case $c = \frac{1}{N-1}$ corresponds to trace-free matrices.

Remark. The canonical example of such a kernel K is when it coincides with $\gamma^2 \ln \frac{L}{|x|}$ for x small enough. In dimension 1 and 2 we can even choose $K(x) = \gamma^2 \ln_+ \frac{L}{|x|}$. In dimension greater than 3, we can use the constructions developed in [15, 23]: for examples of such kernels, see Appendix A.1. Another approach is to use the convolution techniques developed in [24]. This does not exactly fall into the framework set out above because the convoluted kernel depends on ϵ at all scales, i.e. for $|x-y| > \epsilon$. Nevertheless, this has a non significant influence on the forthcoming computations so that we also claim that our results remain valid for such regularization procedures.

We want to study the convergence of the following random variable which lives in $\mathcal{S}(M_s(\mathbb{R}^d))$:

$$M^\epsilon(A) = \frac{1}{c_\epsilon} \int_A e^{X^\epsilon(x)} dx, \quad A \subset \mathbb{R}^d, \tag{1}$$

where c_ϵ is chosen such that $E[M^\epsilon(A)] = |A| Id$ where $|A|$ is the Lebesgue measure of A. We will prove that the normalization constant c_ϵ has the following explicit form:

$$c_\epsilon = \frac{1}{N} \frac{\Gamma(1/2)}{\Gamma(N/2)} (1+c)^{(N-1)/2} \sigma_\epsilon^{N-1} e^{\frac{\sigma_\epsilon^2}{2}}.$$

Theorem 1. *Let $0 < \gamma^2 < d$. Then there exists a random matrix measure M which lives in $\mathcal{S}(M_s(\mathbb{R}^d))$ and such that for all bounded $A \subset \mathbb{R}^d$:*

$$E[tr(M^\epsilon(A) - M(A))^2] \underset{\epsilon \to 0}{\to} 0.$$

We also have the following asymptotic structure:

$$E[trM(B(0,\ell))^2] \underset{\ell \to 0}{\sim} N^2 V_N \frac{\Gamma(N/2) e^{\gamma^2 \ln L + m}}{(1+c)^{(N-1)/2} \Gamma(1/2)} \frac{\ell^{2d-\gamma^2}}{(\gamma^2 \ln \frac{1}{\ell})^{(N-1)/2}} \tag{2}$$

with $V_N = \int_{|v|,|u| \leqslant 1} \frac{dudv}{|v-u|^{\gamma^2}}$. Furthermore, we get the following equivalent for $k \geqslant 2$:

$$\frac{\ln E[trM(B(0,\ell))^k]}{\ln \ell} \underset{\ell \to 0}{\to} \zeta(k) \tag{3}$$

where $\zeta(k) = dk - \gamma^2 \frac{k(k-1)}{2}$.

Notice that it would be interesting to prove that this matrix-valued Gaussian multiplicative chaos admits a phase transition as in the scalar case, which is likely to occur at $\gamma^2 = 2d$.

Remark. Application in turbulence. In the paper [6], the authors consider the following boundary case as a building block of their random velocity fields:

$$\gamma^2 = \frac{8}{3}\lambda^2, \; N = 3, \; c = \frac{1}{N-1} = \frac{1}{2}$$

where λ^2 is found to fit experimental data for $\lambda^2 \approx 0.025$ [6, 7]. Here, the zero trace property is reminiscent of the incompressibility condition imposed on velocity fields.

Conjecture 2. *The power law spectrum of M is given by the following expression: for all $q < \frac{2d}{\gamma^2}$, $\forall K \subset B(0,L), \forall \ell \in (0,1]$,*

$$E[trM(B(0,\ell))^q] \simeq C_q \ell^{\zeta(q)} (-\ln \ell)^{\frac{1-N}{2}},$$

where $C_q > 0$ is a constant and the structure exponent is given by

$$\zeta(q) = \big(d + \frac{\gamma^2}{2}\big)q - \frac{\gamma^2}{2}q^2.$$

If this conjecture is true, this would show that non-commutativity yields an extra log factor in the power-law spectrum of M.

Remark. Notice that one can define a notion of "metric" (actually a measure) through the quantity

$$A \in \mathcal{B}(\mathbb{R}^d) \mapsto \mathrm{tr}M(A).$$

Therefore we can define the notion of Hausdorff dimension asociated to this "metric" (see [11, 22]). It would be interesting to prove a corresponding KPZ formula and relate it with a KPZ framework .

3. Proofs of the N-dimensional case

Let us first mention that several results abouts isotropic matrices and related computations are gathered in the appendix and will be used throughout this section.

3.1 Joint law of the eigenvalues of Gaussian isotropic matrices

We consider a symmetrical random matrix $X = (X_{i,j})_{1 \leqslant i,j \leqslant N}$ made up of centered Gaussian variables with the following covariance structure: the off-diagonal terms $(X_{i,j})_{i<j}$ are i.i.d. with variance σ^2. The diagonal term $(X_{1,1}, \cdots, X_{N,N})$ is independent from the off-diagonal and it has the following covariance structure:

$$K_N = (E[X_{i,i}X_{j,j}])_{1 \leqslant i,j \leqslant N} = (1+c)\sigma_d^2 I_N - c\sigma_d^2 P_N$$

where I_N is the identity matrix, $P_N = (1)_{i,j}$ and $c \in]-1, \frac{1}{N-1}[$. By noting that $P_N^2 = NP_N$, we get the following inverse for K if $c \neq \frac{1}{N-1}$:

$$K_N^{-1} = \frac{1}{\sigma_d^2(1+c)} I_N + \frac{c}{\sigma_d^2(1+c)} \frac{1}{(1+c(1-N))} P_N$$

The density of the random matrix, with respect to the Lebesgue measure $(dx_{i,j})_{i \leqslant j}$, is therefore given by:

$$f\big((x_{i,j})_{i \leqslant j}\big) = \frac{1}{Z_N} e^{-\frac{1}{2\sigma_d^2(1+c)} \sum_{i=1}^N x_{i,i}^2 - \frac{c}{2\sigma_d^2(1+c)} \frac{1}{(1+c(1-N))} (\sum_{i=1}^N x_{i,i})^2 - \frac{1}{2\sigma^2} \sum_{i<j} x_{i,j}^2}$$

where

$$Z_N = (2\pi)^{N(N+1)/4} \sigma_d^N \sigma^{N(N-1)/2} (1+c)^{(N-1)/2} \sqrt{1-(N-1)c}$$

is a normalization constant.

Therefore if we have the following condition:

$$\sigma_d^2(1+c) = 2\sigma^2, \tag{4}$$

as we have required in section 2, we can rewrite the above density in the following matrix form:

$$f((x_{i,j})_{i \leqslant j}) = \frac{1}{Z_N} e^{-\frac{c}{2\sigma_d^2(1+c)}\frac{1}{(1+c(1-N))}(\mathrm{tr}X)^2 - \frac{1}{2\sigma_d^2(1+c)}\mathrm{tr}X^2} \tag{5}$$

with $Z_N = 2^{N/2}\pi^{N(N+1)/4}\sigma_d^{N(N+1)/2}(1+c)^{(N-1)(N+2)/4}\sqrt{1+c(1-N)}$. This shows that the matrix is isotropic, namely that for any real orthogonal matrix O, the matrices X and OX^tO have the same probability law. Therefore by applying [1, Proposition 4.1.1, page 188], we get the density of the unordered eigenvalues:

$$f((\lambda_i)_{1 \leqslant i \leqslant N}) = \frac{1}{\bar{Z}_N} e^{-\alpha(\sum_{i=1}^N \lambda_i)^2 - \frac{1}{2\sigma_d^2(1+c)}\sum_{i=1}^N \lambda_i^2} \Pi_{i<j}|\lambda_j - \lambda_i|, \tag{6}$$

where $\alpha = \frac{c}{2\sigma_d^2(1+c)}\frac{1}{(1+c(1-N))}$ and $\bar{Z}_N = 2^{N(N-1)/4}\frac{\rho(U_1(\mathbb{R}))^N N!}{\rho(U_N(\mathbb{R}))}Z_N$ (notations of [1]). We remind that $\rho(U_N(\mathbb{R})) = 2^{N/2}(2\pi)^{N(N+1)/4}\prod_{k=1}^N \frac{1}{\Gamma(k/2)}$ (see [1, page 198]) and thus:

$$\bar{Z}_N = N!(2\pi)^{N/2}(\prod_{k=1}^N \frac{\Gamma(k/2)}{\Gamma(1/2)})\sigma_d^{N(N+1)/2}(1+c)^{(N-1)(N+2)/4}\sqrt{1+c(1-N)}. \tag{7}$$

The isotropic condition (Eq. 4) ensures also that the collection of eigenvectors $(v_i)_{1 \leqslant i \leqslant N}$ is independent of the eigenvalues $(\lambda_i)_{1 \leqslant i \leqslant N}$, and they are distributed uniformly on the unit sphere according to the Haar measure [1, Corollary 2.5.4, page 53].

3.2 Computations of the renormalization

We consider here isotropic symmetric matrices $X^\epsilon(x) = (X^\epsilon_{i,j}(x))_{1 \leqslant i,j \leqslant N}$ as defined in section 2 and compute the renormalization of order 1, i.e. the constant c_ϵ such that:

$$E[e^{X^\epsilon(x)}] = c_\epsilon I_N = \frac{E[\mathrm{tr}\ e^{X^\epsilon(x)}]}{N} I_N.$$

The isotropic nature of the matrices ensures the proportionality of the former expectation to the identity matrix I_N . We want more precisely an equivalent of c_ϵ as $\epsilon \to 0$. We have:

$$c_\epsilon = \frac{1}{\bar{Z}_N}\int_{\mathbb{R}^N} e^{\lambda_1} e^{-\alpha_\epsilon(\sum_{i=1}^N \lambda_i)^2 - \frac{1}{2\sigma_\epsilon^2(1+c)}\sum_{i=1}^N \lambda_i^2}\Pi_{i<j}|\lambda_j - \lambda_i| d\lambda_1 \cdots d\lambda_N,$$

where $\alpha_\epsilon = \frac{c}{2\sigma_\epsilon^2(1+c)}\frac{1}{(1+c(1-N))}$ and the normalization constant $\bar{Z}_N$ given by Eq. 7 with $\sigma_d^2 = \sigma_\epsilon^2 = \gamma^2(\ln\frac{L}{\epsilon} + 1)$.

We set $u_i = \frac{\lambda_i}{\sigma_\epsilon}$ and therefore we get:

$$c_\epsilon = \frac{\sigma_\epsilon^{N(N+1)/2}}{\bar{Z}_N}\int_{\mathbb{R}^N} e^{\sigma_\epsilon u_1} e^{-\alpha(\sum_{i=1}^N u_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N u_i^2}\Pi_{i<j}|u_j - u_i| du_1 \cdots du_N,$$

where $\alpha = \frac{c}{2(1+c)}\frac{1}{(1+c(1-N))}$. We thus introduce

$$\varphi(u_1, \cdots, u_N) = \sigma_\epsilon u_1 - \alpha(\sum_{i=1}^N u_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N u_i^2$$

The function φ is maximal for $u_1 = S_\epsilon(1+2\alpha(1+c)(N-1)),\ i \geqslant 2 : u_i = -2\alpha S_\epsilon(1+c)$ with $S_\epsilon = \frac{\sigma_\epsilon}{\frac{1}{1+c}+2\alpha N}$. We thus set $u_1 = v_1 + S_\epsilon(1+2\alpha(1+c)(N-1)),\ i \geqslant 2 : u_i = v_i - 2\alpha S_\epsilon(1+c)$ to get:

$$c_\epsilon = \frac{\sigma_\epsilon^{N(N+1)/2} e^{\frac{\sigma_\epsilon^2}{2}}}{\bar{Z}_N} \int_{\mathbb{R}^N} e^{-\alpha(\sum_{i=1}^N v_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N v_i^2} \Pi_{2 \leqslant i} |v_1 - v_i + (1+c)\sigma_\epsilon| \\ \times \Pi_{2 \leqslant i<j} |v_j - v_i| dv_1 \cdots dv_N.$$

Therefore, we get the following equivalent by using the Laplace method:

$$c_\epsilon \underset{\epsilon \to 0}{\sim} \frac{\sigma_\epsilon^{N(N+1)/2}(1+c)^{N-1}\sigma_\epsilon^{N-1} e^{\frac{\sigma_\epsilon^2}{2}}}{\bar{Z}_N} \int_{\mathbb{R}^N} e^{-\alpha(\sum_{i=1}^N v_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N v_i^2} \prod_{2 \leqslant i<j} |v_j - v_i| dv_1 \cdots dv_N$$

By using equation (25) in the appendix, this leads finally to the following equivalent as $\epsilon \to 0$:

$$c_\epsilon \underset{\epsilon \to 0}{\sim} \frac{1}{N}\frac{\Gamma(1/2)}{\Gamma(N/2)}(1+c)^{(N-1)/2}\sigma_\epsilon^{N-1} e^{\frac{\sigma_\epsilon^2}{2}}. \tag{8}$$

3.3 Computation of the moment of order 2

In order to study the convergence, for $\epsilon \to 0$, of the Gaussian chaos $M^\epsilon(A)$ (Eq. 1), we need to consider first the second-order moment:

$$E(M^\epsilon(A)^2) = \frac{1}{c_\epsilon^2}\int_{A\times A} E(e^{X^\epsilon(x)} e^{X^\epsilon(y)}) dxdy,$$

that involves the following quantity:

$$E(e^{X^\epsilon(x)} e^{X^\epsilon(y)}) = \frac{1}{N} E\left[\mathrm{tr}(e^{X^\epsilon(x)} e^{X^\epsilon(y)})\right] I_N. \tag{9}$$

We will show that $E(M^\epsilon(A)^2)$ converges to a limit as $\epsilon \to 0$. From this convergence, one can easily deduce that the sequence $(M^\epsilon(A))_{\epsilon>0}$ is a L^2 Cauchy sequence. Again, the proportionality to the identity matrix in (9) comes from the isotropic character of matrices and we will see moreover that, because the so-defined field of matrices is homogeneous, the former quantity will depend only on $|x-y|$. The purpose of this section is to compute this quantity. We will restrict to the case $|y-x| > \epsilon$ as the case $|y-x| \leqslant \epsilon$, once integrated, leads to vanishing terms in the limit $\epsilon \to 0$. It requires first the derivation of the joint density of the two matrices $X^\epsilon(x)$ and $X^\epsilon(y)$. We will see indeed that the quantity will depend only on $|x-y|$. We will also notice that, contrary to the one-point density (Eq. 5) from which it can be shown that eigenvectors and eigenvalues are independent, eigenvalues at point x are not only correlated to eigenvalues at point y, but also with eigenvectors at

point y. This intricate correlation structure is reminiscent of the non-commutative nature of this field of matrices and is encoded in the so-called Harish-Chandra-Itzykson-Zuber integral over the orthogonal group, or angular-matrix integral, and its related moments. This is an active field of research in random matrix theory and up to now, no explicit formula are known in dimension $N \geqslant 3$ (see for instance [5, 4, 8] and references therein). Nonetheless, we will succeed to get an explicit result in the asymptotic limit $\epsilon \to 0$.

3.3.1 Joint density of two isotropic matrices

We consider here two isotropic symmetric matrices $X^\epsilon(x) = (X^\epsilon_{i,j}(x))_{1 \leqslant i,j \leqslant N}$ and $X^\epsilon(y) = (X^\epsilon_{i,j}(y))_{1 \leqslant i,j \leqslant N}$ as defined in section 2. We recall that matrix components are logarithmically correlated over space. We note $x_{i,j} = X^\epsilon_{i,j}(x)$ and $y_{i,j} = X^\epsilon_{i,j}(y)$, and in matrix form $X = X^\epsilon(x)$ and $Y = X^\epsilon(y)$.

Let us first consider the diagonal terms

$$(x_{1,1}, \cdots, x_{N,N}, y_{1,1}, \cdots, y_{N,N}).$$

The covariance structure K_{2N} of these elements is given by:

$$K_{2N} = \begin{pmatrix} \sigma^2_\epsilon A_N & \sigma^2_{|y-x|} A_N \\ \sigma^2_{|y-x|} A_N & \sigma^2_\epsilon A_N \end{pmatrix},$$

where $A_N = (1+c)I_N - cP_N$ and we recall that $\sigma^2_\epsilon = \gamma^2(\ln \frac{L}{\epsilon} + 1)$ and $\sigma^2_{|x-y|} = \gamma^2 \ln \frac{L}{|x-y|}$. We know that the inverse of K_{2N} is given by:

$$K^{-1}_{2N} = \frac{1}{\sigma^4_\epsilon - \sigma^4_{|y-x|}} \begin{pmatrix} \sigma^2_\epsilon A^{-1}_N & -\sigma^2_{|y-x|} A^{-1}_N \\ -\sigma^2_{|y-x|} A^{-1}_N & \sigma^2_\epsilon A^{-1}_N \end{pmatrix},$$

where $A^{-1}_N = \frac{1}{(1+c)} I_N + 2\alpha P_N$ with $\alpha = \frac{c}{2(1+c)} \frac{1}{(1+c(1-N))}$ which leads to the following density:

$$f((x_{i,i})_{1 \leqslant i \leqslant N}; (y_{j,j})_{1 \leqslant j \leqslant N}) =$$
$$c_N e^{-\frac{\sigma^2_\epsilon/(1+c)\sum_i x^2_{i,i} + 2\alpha\sigma^2_\epsilon(\sum_i x_{i,i})^2 + \sigma^2_\epsilon/(1+c)\sum_i y^2_{i,i} + 2\alpha\sigma^2_\epsilon(\sum_i y_{i,i})^2 - 2\sigma^2_{|y-x|}/(1+c)\sum_i x_{i,i}y_{i,i} - 4\sigma^2_{|y-x|}\alpha(\sum_i x_{i,i})(\sum_i y_{i,i})}{2(\sigma^4_\epsilon - \sigma^4_{|y-x|})}}$$

where $c_N = \frac{1}{(2\pi)^N \sqrt{\det(K_{2N})}}$. Now, $\det(K_{2N}) = (\sigma^4_\epsilon - \sigma^4_{|y-x|})^N (1+c)^{2(N-1)}(1+c(1-N))^2$ and therefore $c_N = \frac{1}{(2\pi)^N (\sigma^4_\epsilon - \sigma^4_{|y-x|})^{N/2}(1+c)^{(N-1)}(1+c(1-N))}$. A similar procedure can be performed for the remaining $N(N-1)$ off-diagonal terms of the two matrices. The density of the couple $(X = X^\epsilon(x), Y = X^\epsilon(y))$ is thus given by, in matrix form:

$$f(X,Y) =$$
$$\bar{c}_N e^{-\frac{\sigma^2_\epsilon}{2(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}(\mathrm{tr}X^2 + \mathrm{tr}Y^2) - \frac{\alpha\sigma^2_\epsilon}{(\sigma^4_\epsilon - \sigma^4_{|y-x|})}((\mathrm{tr}X)^2 + (\mathrm{tr}Y)^2) + \frac{\sigma^2_{|y-x|}}{(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}\mathrm{tr}XY + \frac{2\alpha\sigma^2_{|y-x|}}{\sigma^4_\epsilon - \sigma^4_{|y-x|}}\mathrm{tr}X\mathrm{tr}Y} \tag{10}$$

where $\bar{c}_N = c_N \frac{1}{\pi^{N(N-1)/2}(1+c)^{N(N-1)/2}(\sigma^4_\epsilon - \sigma^4_{|y-x|})^{N(N-1)/4}}$. We can see in the expression of the joint density of the two matrices X and Y (Eq. 10) two different contributions. The

first one, where is entering terms of the form $\mathrm{tr}X^2 + \mathrm{tr}Y^2$ and $(\mathrm{tr}X)^2 + (\mathrm{tr}Y)^2$, relates the density of two symmetric isotropic matrices as if they were independent. The second contribution relates an interaction term coming from the logarithmic correlation of the components. Indeed, the former vanishes if the matrices are independent, i.e. $\sigma^2_{|y-x|} = 0$.

At this stage, it is convenient to introduce two i.i.d. random matrices $M = (M_{i,j})$ and $M' = (M'_{i,j})$. These random matrices are taken to be living in the Gaussian Orthogonal Ensemble (GOE), namely they are symmetrical and isotropic with independent components with the following distribution: the components $(M_{i,j})_{i \leqslant j}$ are independent centered Gaussian variables with the following variances:

$$E[M^2_{i,j}] = \frac{(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}{2\sigma^2_\epsilon},\ i<j; \quad E[M^2_{i,i}] = \frac{(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}{\sigma^2_\epsilon}.$$

With this, we get the following expression for $E[F(X(x), X(y))]$, where F is any functional of the two matrices $X(x)$ and $X(y)$:

$$E[F(X(x),X(y))]$$
$$= \frac{1}{Z} E\left[F(M,M') e^{-\frac{\alpha\sigma^2_\epsilon}{(\sigma^4_\epsilon - \sigma^4_{|y-x|})}((\mathrm{tr}M)^2 + (\mathrm{tr}M')^2) + \frac{\sigma^2_{|y-x|}}{(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}\mathrm{tr}MM' + \frac{2\alpha\sigma^2_{|y-x|}}{\sigma^4_\epsilon - \sigma^4_{|y-x|}}\mathrm{tr}M\mathrm{tr}M'} \right],$$

where

$$Z = E\left[e^{-\frac{\alpha\sigma^2_\epsilon}{(\sigma^4_\epsilon - \sigma^4_{|y-x|})}((\mathrm{tr}M)^2 + (\mathrm{tr}M')^2) + \frac{\sigma^2_{|y-x|}}{(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}\mathrm{tr}MM' + \frac{2\alpha\sigma^2_{|y-x|}}{\sigma^4_\epsilon - \sigma^4_{|y-x|}}\mathrm{tr}M\mathrm{tr}M'} \right]. \tag{11}$$

By using classical theorems about isotropic matrices, we know that $M = OD(\lambda)^t O$, $M' = O'D(\lambda')^t O'$ where O (resp. O') is uniformly distributed on the orthogonal group of $\mathbb{R}^N$ and is independent of the diagonal matrix $D(\lambda)$ (resp. $D(\lambda')$) the diagonal entries of which are the eigenvalues of M (resp. M').

3.3.2 Joint density of eigenvalues of two correlated isotropic matrices

We are interested here in computing the renormalization constant Z (Eq. 11). To do so, we diagonalize the matrices M and M', and perform an integration over the remaining degrees of freedom left by the eigenvectors (see [4] for instance). We define the eigenvalues of M as $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^d$ and we note the Vandermonde determinant as $\triangle(\lambda) = \prod_{1 \leqslant i < j \leqslant N} |\lambda_i - \lambda_j|$. We get:

$$Z = \frac{1}{R^\epsilon_N} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Delta(\lambda)||\Delta(\lambda')| e^{-\frac{\sigma^2_\epsilon}{2(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}\sum_{i=1}^N \lambda_i^2 - \frac{\sigma^2_\epsilon}{2(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}\sum_{i=1}^N \lambda_i'^2}$$
$$\times\ e^{-\frac{\alpha\sigma^2_\epsilon}{(\sigma^4_\epsilon - \sigma^4_{|y-x|})}((\sum_{i=1}^N \lambda_i)^2 + (\sum_{i=1}^N \lambda'_i)^2) + \frac{2\alpha\sigma^2_{|y-x|}}{\sigma^4_\epsilon - \sigma^4_{|y-x|}}(\sum_{i=1}^N \lambda_i)(\sum_{i=1}^N \lambda'_i)} J(D(\lambda), D(\lambda'))d\lambda d\lambda',$$

where R^ϵ_N is a renormalization constant such that

$$\frac{1}{R^\epsilon_N} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Delta(\lambda)||\Delta(\lambda')| e^{-\frac{\sigma^2_\epsilon}{2(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}\sum_{i=1}^N \lambda_i^2 - \frac{\sigma^2_\epsilon}{2(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})}\sum_{i=1}^N \lambda_i'^2} d\lambda d\lambda' = 1,$$

and J is the following Harish-Chandra-Itzykson-Zuber integral [5, 4, 8], also called matrix angular integral (dO stands for the Haar measure on $O_N(\mathbb{R})$)

$$J(D(\lambda), D(\lambda')) = \int_{O_N(\mathbb{R})} e^{\frac{\sigma^2_{|y-x|}}{(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})} \mathrm{tr}\, D(\lambda) O D(\lambda') O^{-1}} dO,$$

obtained while integrating over the eigenvectors that enter in the term $\mathrm{tr} MM'$ of Eq. 11. We make the following change of variables $u_i = \frac{\sigma_\epsilon}{\sqrt{\sigma^4_\epsilon - \sigma^4_{|y-x|}}} \lambda_i$, $u'_i = \frac{\sigma_\epsilon}{\sqrt{\sigma^4_\epsilon - \sigma^4_{|y-x|}}} \lambda'_i$ (set $\gamma_\epsilon = \frac{\sqrt{\sigma^4_\epsilon - \sigma^4_{|y-x|}}}{\sigma_\epsilon}$) and get to:

$$\begin{aligned} Z = & \frac{\gamma_\epsilon^{N(N+1)}}{R_N^\epsilon} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Delta(u)||\Delta(u')| e^{-\frac{1}{2(1+c)} \sum_{i=1}^N u_i^2 - \frac{1}{2(1+c)} \sum_{i=1}^N u_i'^2} \\ & \times e^{-\alpha((\sum_{i=1}^N u_i)^2 + (\sum_{i=1}^N u'_i)^2) + \frac{2\alpha\sigma^2_{|y-x|}}{\sigma^2_\epsilon} (\sum_{i=1}^N u_i)(\sum_{i=1}^N u'_i)} \overline{J}(D(u), D(u')) du du', \end{aligned}$$

where we have set:

$$\overline{J}(D(u), D(u')) = \int_{O_N(\mathbb{R})} e^{\frac{1}{1+c} \frac{\sigma^2_{|y-x|}}{\sigma^2_\epsilon} \sum_{i,j=1}^N u_i u'_j |O_{i,j}|^2} dO.$$

Therefore, since $\overline{J}(D(u), D(u'))$ converges pointwise towards 0 as $\epsilon \to 0$, we can use the Lebesgue theorem to get the following equivalent as $\epsilon \to 0$:

$$\begin{aligned} Z \underset{\epsilon \to 0}{\sim} & \frac{\gamma_\epsilon^{N(N+1)}}{R_N^\epsilon} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Delta(u)||\Delta(u')| e^{-\frac{1}{2(1+c)} \sum_{i=1}^N u_i^2 - \frac{1}{2(1+c)} \sum_{i=1}^N u_i'^2} \\ & \times e^{-\alpha((\sum_{i=1}^N u_i)^2 + (\sum_{i=1}^N u'_i)^2)} du du', \end{aligned}$$

that is straightforward to compute (see the appendix).

3.3.3 Two-points correlation structure of the matrix chaos

We want to get an equivalent as $\epsilon \to 0$ of the quantity given in Eq. 9. To do so, we consider the following quantity:

$$\bar{Z} = E[\mathrm{tr}(e^M e^{M'}) e^{-\frac{\alpha\sigma^2_\epsilon}{(\sigma^4_\epsilon - \sigma^4_{|y-x|})}((\mathrm{tr}M)^2 + (\mathrm{tr}M')^2) + \frac{\sigma^2_{|y-x|}}{(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})} \mathrm{tr}MM' + \frac{2\alpha\sigma^2_{|y-x|}}{\sigma^4_\epsilon - \sigma^4_{|y-x|}} \mathrm{tr}M \mathrm{tr}M'}].$$

In the same spirit as formerly, we diagonalize the matrices M and M' and perform the integration over the eigenvectors. We get:

$$\begin{aligned} \bar{Z} = & \frac{1}{R_N^\epsilon} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Delta(\lambda)||\Delta(\lambda')| e^{-\frac{\sigma^2_\epsilon}{2(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})} \sum_{i=1}^N \lambda_i^2 - \frac{\sigma^2_\epsilon}{2(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})} \sum_{i=1}^N \lambda_i'^2} \\ & \times e^{-\frac{\alpha\sigma^2_\epsilon}{(\sigma^4_\epsilon - \sigma^4_{|y-x|})}((\sum_{i=1}^N \lambda_i)^2 + (\sum_{i=1}^N \lambda'_i)^2) + \frac{2\alpha\sigma^2_{|y-x|}}{\sigma^4_\epsilon - \sigma^4_{|y-x|}} (\sum_{i=1}^N \lambda_i)(\sum_{i=1}^N \lambda'_i)} I(D(\lambda), D(\lambda')) d\lambda d\lambda', \end{aligned}$$

where I is the following moment of the angular integral:

$$I(D(\lambda), D(\lambda')) = \int_{O_N(\mathbb{R})} \mathrm{tr}(e^{D(\lambda)} O e^{D(\lambda')} O^{-1}) e^{\frac{\sigma^2_{|y-x|}}{(1+c)(\sigma^4_\epsilon - \sigma^4_{|y-x|})} \mathrm{tr}\, D(\lambda) O D(\lambda') O^{-1}} dO.$$

We make the following change of variables $u_i = \frac{\sigma_\epsilon}{\sqrt{\sigma^4_\epsilon - \sigma^4_{|y-x|}}} \lambda_i$, $u'_i = \frac{\sigma_\epsilon}{\sqrt{\sigma^4_\epsilon - \sigma^4_{|y-x|}}} \lambda'_i$ (set $\gamma_\epsilon = \frac{\sqrt{\sigma^4_\epsilon - \sigma^4_{|y-x|}}}{\sigma_\epsilon}$):

$$\bar{Z} = \sum_{i,j=1}^N \frac{\gamma_\epsilon^{N(N+1)}}{R_N^\epsilon} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Delta(u)||\Delta(u')| e^{-\frac{1}{2(1+c)} \sum_{k=1}^N u_k^2 - \frac{1}{2(1+c)} \sum_{k=1}^N u_k'^2}$$

$$\times\, e^{-\alpha((\sum_{k=1}^N u_k)^2 + (\sum_{k=1}^N u'_k)^2) + \frac{2\alpha\sigma^2_{|y-x|}}{\sigma^2_\epsilon} (\sum_{k=1}^N u_k)(\sum_{k=1}^N u'_k)} e^{\gamma_\epsilon(u_i + u'_j)} \overline{I}_{i,j}(D(u), D(u')) du du',$$

where we have set:

$$\overline{I}_{i,j}(D(u), D(u')) = \int_{O_N(\mathbb{R})} |O_{i,j}|^2 e^{\frac{1}{1+c} \frac{\sigma^2_{|y-x|}}{\sigma^2_\epsilon} \sum_{k,k'=1}^N u_k u'_{k'} |O_{k,k'}|^2} dO,$$

known as the Morozov moment [4]. We make the following change of variables in the above integral: $u_i = v_i + \gamma_\epsilon$, $u_k = v_k - c\gamma_\epsilon$ for $k \neq i$ and $u'_j = v'_j + \gamma_\epsilon$, $u'_k = v'_k - c\gamma_\epsilon$ for $k \neq j$. We obtain the following equivalent:

$$\bar{Z} \underset{\epsilon \to 0}{\sim} \sum_{i,j=1}^N \frac{\gamma_\epsilon^{N(N+1)} e^{\sigma^2_\epsilon} (1+c)^{2(N-1)} \sigma_\epsilon^{2(N-1)}}{c_N^\epsilon} I_{i,j} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Delta_i(v)||\Delta_j(v')|$$

$$\times\, e^{-\frac{1}{2(1+c)} \sum_{k=1}^N v_k^2 - \frac{1}{2(1+c)} \sum_{k=1}^N v_k'^2 - \alpha((\sum_{k=1}^N v_k)^2 + (\sum_{k=1}^N v'_k)^2) + 2\alpha\sigma^2_{|y-x|}(1+c(1-N))^2} dv dv',$$

where $|\Delta_i(v)| = \prod_{l<l', l, l' \neq i} |v_l - v_{l'}|$ and:

$$\begin{aligned} I_{i,j} &= \int_{O_n(\mathbb{R})} |O_{i,j}|^2 e^{\frac{1}{1+c} \sigma^2_{|y-x|} \sum_{k,k'=1}^N (-c + (1+c)1_{k=i})(-c+(1+c)1_{k'=j}) |O_{k,k'}|^2} dO \\ &= e^{\sigma^2_{|y-x|}(\frac{c^2 N}{1+c} - 2c)} \int_{O_N(\mathbb{R})} |O_{1,1}|^2 e^{\sigma^2_{|y-x|}(1+c)|O_{1,1}|^2} dO, \end{aligned}$$

which is independent of i, j. Therefore, we get:

$$\bar{Z} \underset{\epsilon \to 0}{\sim} N^2 \frac{\gamma_\epsilon^{N(N+1)} e^{\sigma^2_\epsilon} (1+c)^{2(N-1)} \sigma_\epsilon^{2(N-1)}}{c_N^\epsilon} I_{1,1} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Delta_1(v)||\Delta_1(v')|$$

$$e^{-\frac{1}{2(1+c)} \sum_{k=1}^N v_k^2 - \frac{1}{2(1+c)} \sum_{k=1}^N v_k'^2 - \alpha((\sum_{k=1}^N v_k)^2 + (\sum_{k=1}^N v'_k)^2) + 2\alpha\sigma^2_{|y-x|}(1+c(1-N))^2} dv dv'.$$

In conclusion, we get:

$$\bar{Z}/Z \underset{\epsilon \to 0}{\sim} (1+c)^{N-1} (\frac{\Gamma(1/2)}{\Gamma(N/2)})^2 e^{\sigma^2_\epsilon} \sigma_\epsilon^{2(N-1)} e^{-c\sigma^2_{|y-x|}} \int_{O_N(\mathbb{R})} |O_{1,1}|^2 e^{\sigma^2_{|y-x|}(1+c)|O_{1,1}|^2} dO.$$

Including furthermore the normalization constant c_ϵ (Eq. 8), we get:

$$\bar{Z}/(Z c_\epsilon^2) \underset{\epsilon \to 0}{\sim} N^2 e^{-c\sigma^2_{|y-x|}} \int_{O_N(\mathbb{R})} |O_{1,1}|^2 e^{\sigma^2_{|y-x|}(1+c)|O_{1,1}|^2} dO.$$

3.3.4 Computation of the moment of order 2

From the above subsections, we deduce that:

$$E(trM^{\epsilon}(A)^2) \underset{\epsilon \to 0}{\to} N^2 \int_{A\times A} e^{-c\sigma^2_{|y-x|}} \int_{O_N(\mathbb{R})} |O_{1,1}|^2 e^{\sigma^2_{|y-x|}(1+c)|O_{1,1}|^2} dO\, dxdy$$

We remind that the law of $|O_{1,1}|^2$ is the one of the square of one component of a vector uniformely distributed on the unit sphere, and has thus a density given by (see Lemma 3)

$$f(v) = \frac{\Gamma(N/2)}{\Gamma(1/2)\Gamma((N-1)/2)} v^{-1/2}(1-v)^{(N-3)/2}.$$

We get the following equivalent as $|y-x| \to 0$:

$$N^2 e^{-c\sigma^2_{|y-x|}} \int_{O_N(\mathbb{R})} |O_{1,1}|^2 e^{\sigma^2_{|y-x|}(1+c)|O_{1,1}|^2} dO \underset{|y-x|\to 0}{\sim} N^2 \frac{\Gamma(N/2)}{\Gamma(1/2)} \frac{e^{\sigma^2_{|y-x|}}}{(1+c)^{(N-1)/2}\sigma^{N-1}_{|y-x|}},$$

which entails (2).

3.4 Computation of the moment of order k

We are interested here in studying the convergence, when $\epsilon \to 0$, of the Gaussian chaos $M^{\epsilon}(A)$ (Eq. 1) for higher order moments such as, $k \in \mathbb{N}$,

$$E(M^{\epsilon}(A))^k = \frac{1}{c^k_{\epsilon}} \int_{A^k} E\left(\prod_{1 \leqslant i \leqslant k} e^{X^{\epsilon}(x_i)} \right) dx_1 \cdots dx_k,$$

that involves the following quantity:

$$E\left(\prod_{1 \leqslant i \leqslant k} e^{X^{\epsilon}(x_i)} \right) = \frac{1}{N} E\left[\mathrm{tr} \prod_{1 \leqslant i \leqslant k} e^{X^{\epsilon}(x_i)} \right] I_N. \tag{12}$$

To generalize former calculations in the case $k = 2$, we will first derive the joint density of k-matrices $(X^{\epsilon}(x_i))_{1 \leqslant i \leqslant k}$. A generalized version to k-points of the Harish-Chandra-Itzykson-Zuber integral enters the expression of the density. An exact evaluation of these integrals remains an open issue. As far as we know, only their behavior in the asymptotic limit of large matrices ($N \to +\infty$) has been considered in the literature [8]. Nonetheless, a logarithmic equivalent of the quantity of interest (Eq. 12) can be obtained and allows to show the multifractal behavior of the multiplicative chaos (i.e. $\zeta(k)$ is a non linear function of the order k, see theorem 1).

3.4.1 Joint density of k isotropic Gaussian matrices

We consider here k isotropic Gaussian matrices $(X^{\epsilon}(x_i))_{1 \leqslant i \leqslant k}$. The ensemble made of the kN diagonal terms, i.e.

$$(X^{\epsilon}_{1,1}(x_1), \cdots, X^{\epsilon}_{N,N}(x_1), \cdots, X^{\epsilon}_{1,1}(x_k), \cdots, X^{\epsilon}_{N,N}(x_k)),$$

has covariance structure K_{kN}:

$$K_{kN} = \begin{pmatrix} \sigma_\epsilon^2 A_N & \sigma^2_{|x_1-x_2|}A_N & \cdots & \cdots & \sigma^2_{|x_1-x_k|}A_N \\ \sigma^2_{|x_2-x_1|}A_N & \sigma_\epsilon^2 A_N & \cdots & \cdots & \sigma^2_{|x_2-x_k|}A_N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma^2_{|x_{k-1}-x_1|}A_N & \sigma^2_{|x_{k-1}-x_2|}A_N & \cdots & \sigma_\epsilon^2 A_N & \sigma^2_{|x_{k-1}-x_k|}A_N \\ \sigma^2_{|x_k-x_1|}A_N & \sigma^2_{|x_k-x_2|}A_N & \cdots & \sigma^2_{|x_k-x_{k-1}|}A_N & \sigma_\epsilon^2 A_N \end{pmatrix},$$

where again, $A_N = (1+c)I_N - cP_N$. We know that the inverse of K_{kN} is approximately given by ($\epsilon \to 0$):

$$K_{kN}^{-1} = \frac{1}{\sigma_\epsilon^4}\begin{pmatrix} \sigma_\epsilon^2 A_N^{-1} & -\sigma^2_{|x_1-x_2|}A_N^{-1} & \cdots & \cdots & -\sigma^2_{|x_1-x_k|}A_N^{-1} \\ -\sigma^2_{|x_2-x_1|}A_N^{-1} & \sigma_\epsilon^2 A_N^{-1} & \cdots & \cdots & -\sigma^2_{|x_2-x_k|}A_N^{-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\sigma^2_{|x_{k-1}-x_1|}A_N^{-1} & -\sigma^2_{|x_{k-1}-x_2|}A_N^{-1} & \cdots & \sigma_\epsilon^2 A_N^{-1} & -\sigma^2_{|x_{k-1}-x_k|}A_N^{-1} \\ -\sigma^2_{|x_k-x_1|}A_N^{-1} & -\sigma^2_{|x_k-x_2|}A_N^{-1} & \cdots & -\sigma^2_{|x_k-x_{k-1}|}A_N^{-1} & \sigma_\epsilon^2 A_N^{-1} \end{pmatrix},$$

where $A_N^{-1} = \frac{1}{(1+c)}I_N + 2\alpha P_N$, with $\alpha = \frac{c}{2(1+c)}\frac{1}{(1+c(1-N))}$. The density of diagonal components, considering the N-dimensional vector $X^{(l)} = (X^\epsilon_{1,1}(x_l), \cdots, X^\epsilon_{N,N}(x_l))$, is thus given by:

$$f(X^{(1)}, \cdots, X^{(k)}) = c_N e^{-\frac{1}{2\sigma_\epsilon^4}\sum_{i,j=1}^k (\delta_{i,j}\sigma_\epsilon^2 - (1-\delta_{i,j})\sigma^2_{|x_i-x_j|})^t X^{(i)}(\frac{1}{(1+c)}I_N + 2\alpha P_N)X^{(j)}}$$

where $c_N = \frac{1}{(2\pi)^{kN/2}\sqrt{\det(K_{kN})}}$. For the off diagonal terms, the situation is simpler. If $i<j$, the covariance matrix of the vector $(X^\epsilon_{i,j}(x_1), \cdots, X^\epsilon_{i,j}(x_k))$, which is independent on all the remaining diagonal and off-diagonal components, is:

$$\frac{1+c}{2}\begin{pmatrix} \sigma_\epsilon^2 & \sigma^2_{|x_1-x_2|} & \cdots & \cdots & \sigma^2_{|x_1-x_k|} \\ \sigma^2_{|x_2-x_1|} & \sigma_\epsilon^2 & \cdots & \cdots & \sigma^2_{|x_2-x_k|} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma^2_{|x_{k-1}-x_1|} & \sigma^2_{|x_{k-1}-x_2|} & \cdots & \sigma_\epsilon^2 & \sigma^2_{|x_{k-1}-x_k|} \\ \sigma^2_{|x_k-x_1|} & \sigma^2_{|x_k-x_2|} & \cdots & \sigma^2_{|x_k-x_{k-1}|} & \sigma_\epsilon^2 \end{pmatrix},$$

whose inverse is approximately given by ($\epsilon \to 0$):

$$\frac{2}{(1+c)\sigma_\epsilon^4}\begin{pmatrix} \sigma_\epsilon^2 & -\sigma^2_{|x_1-x_2|} & \cdots & \cdots & -\sigma^2_{|x_1-x_k|} \\ -\sigma^2_{|x_2-x_1|} & \sigma_\epsilon^2 & \cdots & \cdots & -\sigma^2_{|x_2-x_k|} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\sigma^2_{|x_{k-1}-x_1|} & -\sigma^2_{|x_{k-1}-x_2|} & \cdots & \sigma_\epsilon^2 & -\sigma^2_{|x_{k-1}-x_k|} \\ -\sigma^2_{|x_k-x_1|} & -\sigma^2_{|x_k-x_2|} & \cdots & -\sigma^2_{|x_k-x_{k-1}|} & \sigma_\epsilon^2 \end{pmatrix}.$$

This leads to the following density, using the notations $x^{(r)}_{i,j} = X^\epsilon_{i,j}(x_r)$:

$$f(x^{(1)}_{i,j}, \cdots, x^{(k)}_{i,j}) = k_N e^{-\frac{1}{(1+c)\sigma_\epsilon^4}\sum_{r,l=1}^k (\delta_{r,l}\sigma_\epsilon^2 - (1-\delta_{r,l})\sigma^2_{|x_r-x_l|})x^{(r)}_{i,j}x^{(l)}_{i,j}}.$$

Therefore, we get the following density for the k matrices (we omit superscript ϵ for the sake of clarity):

$$f(X(x_1),\cdots,X(x_k)) = \bar{c}_N e^{-\frac{1}{2\sigma_\epsilon^4}\sum_{r,l=1}^k(\delta_{r,l}\sigma_\epsilon^2-(1-\delta_{r,l})\sigma^2_{|x_r-x_l|})^t X^{(r)}(\frac{1}{(1+c)}I_N+2\alpha P)X^{(l)}} \\ \times e^{-\sum_{i<j}\frac{1}{(1+c)\sigma_\epsilon^4}\sum_{r,l=1}^k(\delta_{r,l}\sigma_\epsilon^2-(1-\delta_{r,l})\sigma^2_{|x_r-x_l|})x_{i,j}^{(r)}x_{i,j}^{(l)}},$$

which we rewrite under matrix notation:

$$f(X(x_1),\cdots,X(x_k)) = \bar{c}_N e^{-\frac{1}{2(1+c)\sigma_\epsilon^2}\sum_{r=1}^k tr(X(x_r)^2)-\frac{\alpha}{\sigma_\epsilon^2}\sum_{r=1}^k(trX(x_r))^2} \\ \times e^{\frac{1}{(1+c)\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}trX(x_r)X(x_l)+\frac{2\alpha}{\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}trX(x_r)trX(x_l)}.$$

We introduce k i.i.d. random matrices $M^{(l)} = (M_{i,j}^{(l)})$ pertaining to the GOE ensemble. These random matrices are symmetrical and isotropic with independent components with the following distribution: the components $(M_{i,j}^{(l)})_{i\leqslant j}$ are independent centered Gaussian variables with the following variances:

$$E[(M_{i,j}^{(l)})^2] = \frac{1+c}{2}\sigma_\epsilon^2,\ i<j; \quad E[(M_{i,i}^{(l)})^2] = (1+c)\sigma_\epsilon^2.$$

With this, we get the following expression for the expectation of any functional $F(X^\epsilon(x_1),\cdots,X^\epsilon(x_k))$ of the k matrices $X^\epsilon(x_1),\cdots,X^\epsilon(x_k)$:

$$E[F(X^\epsilon(x_1),\cdots,X^\epsilon(x_k))] \\ = \frac{E[F(M^{(1)},\cdots,M^{(k)})e^{-\frac{\alpha}{\sigma_\epsilon^2}\sum_{r=1}^k(\mathrm{tr}M^{(r)})^2+\frac{1}{(1+c)\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}trM^{(r)}M^{(l)}+\frac{2\alpha}{\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}trM^{(r)}trM^{(l)}}]}{Z},$$

where:

$$Z = E[e^{-\frac{\alpha}{\sigma_\epsilon^2}\sum_{r=1}^k(\mathrm{tr}M^{(r)})^2+\frac{1}{(1+c)\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}trM^{(r)}M^{(l)}+\frac{2\alpha}{\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}trM^{(r)}trM^{(l)}}].$$

By using classical theorems about isotropic matrices, we know that, for each r, $M^{(r)} = O^{(r)}D(\lambda^{(r)})^tO^{(r)}$ where $O^{(r)}$ is uniformly distributed on the orthogonal group of $\mathbb{R}^N$ and is independent of the diagonal matrix $D(\lambda^{(r)})$ the diagonal entries of which are the eigenvalues of $M^{(r)}$.

3.4.2 Joint density of eigenvalues of k isotropic Gaussian matrices and computation of the renormalization

We start by computing Z. For $\lambda = (\lambda_1,\ldots,\lambda_N)\in\mathbb{R}^d$, we note the Vandermonde determinant as $\triangle(\lambda) = \prod_{1\leqslant i<j\leqslant N}|\lambda_i-\lambda_j|$. We get:

$$Z = \frac{1}{R_N^\epsilon}\int_{\mathbb{R}^{kN}}\Pi_{r=1}^k|\Delta(\lambda^{(r)})|e^{-\frac{1}{2(1+c)\sigma_\epsilon^2}\sum_{r=1}^k\sum_{i=1}^N(\lambda_i^{(r)})^2-\frac{\alpha}{\sigma_\epsilon^2}\sum_{r=1}^k(\sum_{i=1}^N\lambda_i^{(r)})^2} \\ \times e^{\frac{2\alpha}{\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}(\sum_{i=1}^N\lambda_i^{(r)})(\sum_{i=1}^N\lambda_i^{(l)})}J(D(\lambda^{(1)}),\cdots,D(\lambda^{(k)}))d\lambda^{(1)}\cdots d\lambda^{(k)},$$

where R_N^ϵ is a renormalization constant such that

$$\frac{1}{R_N^\epsilon}\int_{\mathbb{R}^{kN}}\Pi_{r=1}^k|\Delta(\lambda^{(r)})|e^{-\frac{1}{2(1+c)\sigma_\epsilon^2}\sum_{r=1}^k\sum_{i=1}^N(\lambda_i^{(r)})^2}=1,$$

and J is the following angular integral: ($dO^{(r)}$ stands for the Haar measure on $O_N(\mathbb{R})$)

$$J(D(\lambda^{(1)}),\cdots,D(\lambda^{(k)}))=$$
$$\int_{O_N(\mathbb{R})^k}e^{\frac{1}{(1+c)\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}\mathrm{tr}\,O^{(r)}D(\lambda^{(r)})^tO^{(r)}O^{(l)}D(\lambda^{(l)})^tO^{(l)}}dO^{(1)}\cdots dO^{(k)}.$$

We make the following change of variables $u_i^{(r)}=\frac{\lambda_i^{(r)}}{\sigma_\epsilon}$:

$$Z=\frac{\sigma_\epsilon^{N(N+1)k/2}}{R_N^\epsilon}\int_{\mathbb{R}^{kN}}\Pi_{r=1}^k|\Delta(u^{(r)})|e^{-\frac{1}{2(1+c)}\sum_{r=1}^k\sum_{i=1}^N(u_i^{(r)})^2-\alpha\sum_{r=1}^k(\sum_{i=1}^N u_i^{(r)})^2}$$
$$\times e^{\frac{2\alpha}{\sigma_\epsilon^2}\sum_{r<l}\sigma^2_{|x_r-x_l|}(\sum_{i=1}^N u_i^{(r)})(\sum_{i=1}^N u_i^{(l)})}\overline{J}(D(u^{(1)}),\cdots,D(u^{(l)}))du^{(1)}\cdots du^{(l)},$$

where we have set:

$$\overline{J}(D(u^{(1)}),\cdots,D(u^{(k)}))=$$
$$\int_{O_N(\mathbb{R})^k}e^{\frac{1}{(1+c)\sigma_\epsilon^2}\sum_{r<l}\sigma^2_{|x_r-x_l|}\mathrm{tr}\,O^{(r)}D(u^{(r)})^tO^{(r)}O^{(l)}D(u^{(l)})^tO^{(l)}}dO^{(1)}\cdots dO^{(k)}.$$

Therefore, since $\overline{J}(D(u^{(1)}),\cdots,D(u^{(k)}))$ converges pointwise towards 1 as $\epsilon\to 0$, we can use the Lebesgue theorem to get the following equivalent as $\epsilon\to 0$:

$$Z\underset{\epsilon\to 0}{\sim}\frac{\sigma_\epsilon^{N(N+1)k/2}}{R_N^\epsilon}\int_{\mathbb{R}^{kN}}\Pi_{r=1}^k|\Delta(u^{(r)})|e^{-\frac{1}{2(1+c)}\sum_{r=1}^k\sum_{i=1}^N(u_i^{(r)})^2}$$
$$\times e^{-\alpha\sum_{r=1}^k(\sum_{i=1}^N u_i^{(r)})^2}du^{(1)}\cdots du^{(k)}.$$

3.4.3 k-points correlation structure of the multiplicative chaos

For $i\leqslant j$, we want to get an equivalent as $\epsilon\to 0$ of the following quantity:

$$\bar{Z}=E[(\Pi_{r=1}^k e^{M^{(r)}})_{i,j}e^{-\frac{\alpha}{\sigma_\epsilon^2}\sum_{r=1}^k(\mathrm{tr}M^{(r)})^2+\frac{1}{(1+c)\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}trM^{(r)}M^{(l)}+\frac{2\alpha}{\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}trM^{(r)}trM^{(l)}}].$$

We get:

$$\bar{Z}=\frac{1}{R_N^\epsilon}\int_{\mathbb{R}^{kN}}\Pi_{r=1}^k|\Delta(\lambda^{(r)})|e^{-\frac{1}{2(1+c)\sigma_\epsilon^2}\sum_{r=1}^k\sum_{i=1}^N(\lambda_i^{(r)})^2-\frac{\alpha}{\sigma_\epsilon^2}\sum_{r=1}^k(\sum_{i=1}^N\lambda_i^{(r)})^2}$$
$$\times e^{\frac{2\alpha}{\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}(\sum_{i=1}^N\lambda_i^{(r)})(\sum_{i=1}^N\lambda_i^{(l)})}I(D(\lambda^{(1)}),\cdots,D(\lambda^{(k)}))d\lambda^{(1)}\cdots d\lambda^{(k)},$$

where I is the following angular integral:

$$I(D(\lambda^{(1)}),\cdots,D(\lambda^{(k)}))=\int_{O_N(\mathbb{R})^k}(\Pi_{r=1}^k O^{(r)}e^{D(\lambda^{(r)})t}O^{(r)})_{i,j}$$
$$\times e^{\frac{1}{(1+c)\sigma_\epsilon^4}\sum_{r<l}\sigma^2_{|x_r-x_l|}\mathrm{tr}\,O^{(r)}D(\lambda^{(r)})^tO^{(r)}O^{(l)}D(\lambda^{(l)})^tO^{(l)}}dO^{(1)}\cdots dO^{(k)}.$$

We make the following change of variables $u_i^{(r)} = \frac{\lambda_i^{(r)}}{\sigma_\epsilon}$:

$$\bar{Z} = \sum_{j_1,\cdots,j_k=1}^{N} \frac{\sigma_\epsilon^{N(N+1)k/2}}{R_N^\epsilon} \int_{\mathbb{R}^{kN}} \Pi_{r=1}^k |\Delta(u^{(r)})| e^{-\frac{1}{2(1+c)}\sum_{r=1}^k \sum_{i=1}^N (u_i^{(r)})^2}$$
$$\times e^{-\alpha \sum_{r=1}^k (\sum_{i=1}^N u_i^{(r)})^2 + \frac{2\alpha}{\sigma_\epsilon^2}\sum_{r<l}\sigma^2_{|x_r-x_l|}(\sum_{i=1}^N u_i^{(r)})(\sum_{i=1}^N u_i^{(l)})} e^{\sigma_\epsilon(u_{j_1}^{(1)}+\cdots+u_{j_k}^{(k)})}$$
$$\times \sum_{\substack{l_1,\cdots,l_{k-1}=1 \\ l_0=i;l_k=j}}^{N} I^{j_1,\cdots,j_k}_{l_0,l_1,\cdots,l_{k-1},l_k}(D(u^{(1)}),\cdots,D(u^{(k)}))du^{(1)}\cdots du^{(k)},$$

where we have set:

$$I^{j_1,\cdots,j_k}_{l_0,l_1,\cdots,l_{k-1},l_k}(D(u^{(1)}),\cdots,D(u^{(k)})) = \int_{O_N(\mathbb{R})^k} (\Pi_{r=1}^k O^{(r)}_{l_{r-1},j_r} O^{(r)}_{l_r,j_r})$$
$$\times e^{\frac{1}{(1+c)\sigma_\epsilon^2}\sum_{r<l}\sigma^2_{|x_r-x_l|}\mathrm{tr}\, O^{(r)}D(u^{(r)})^t O^{(r)} O^{(l)} D(u^{(l)})^t O^{(l)}} dO^{(1)}\cdots dO^{(k)}.$$

We make the following change of variables in the above integral for $1 \leqslant r \leqslant k$: $u_{j_r}^{(r)} = v_{j_r}^{(r)} + \sigma_\epsilon$, $u_k^{(r)} = v_k^{(r)} - c\sigma_\epsilon$ $k \neq j_r$. We obtain the following equivalent:

$$\bar{Z} \underset{\epsilon\to 0}{\sim} \sum_{j_1,\cdots,j_k=1}^{N} \frac{\sigma_\epsilon^{N(N+1)k/2} e^{k\sigma_\epsilon^2/2}(1+c)^{(N-1)k}\sigma_\epsilon^{(N-1)k}}{R_N^\epsilon} \int_{\mathbb{R}^{kN}} \Pi_{r=1}^k |\Delta_{j_r}(v^{(r)})|$$
$$\times e^{-\frac{1}{2(1+c)}\sum_{r=1}^k\sum_{i=1}^N (v_i^{(r)})^2 - \alpha\sum_{r=1}^k(\sum_{i=1}^N v_i^{(r)})^2 + 2\alpha(1+c(1-N))^2\sum_{r<l}\sigma^2_{|x_r-x_l|}}$$
$$\times \sum_{\substack{l_1,\cdots,l_{k-1}=1 \\ l_0=i;l_k=j}}^{N} \bar{I}^{j_1,\cdots,j_k}_{l_0,l_1,\cdots,l_{k-1},l_k} dv^{(1)}\cdots dv^{(k)},$$

where we have set:

$$\bar{I}^{j_1,\cdots,j_k}_{l_0,l_1,\cdots,l_{k-1},l_k} = \int_{O_N(\mathbb{R})^k} (\Pi_{r=1}^k O^{(r)}_{l_{r-1},j_r} O^{(r)}_{l_r,j_r})$$
$$\times e^{\frac{1}{(1+c)}\sum_{r<l}\sigma^2_{|x_r-x_l|}\sum_{m_1,m_2=1}^N \beta^{j_r,j_l}_{m_1,m_2}\sum_{n_1,n_2=1}^N O^{(r)}_{n_1,m_1}O^{(r)}_{n_2,m_1}O^{(l)}_{n_1,m_2}O^{(l)}_{n_2,m_2}} dO^{(1)}\cdots dO^{(k)}$$

with $\beta^{j_r,j_l}_{m_1,m_2} = (-c+(1+c)1_{m_1=j_r})(-c+(1+c)1_{m_2=j_l})$. This leads to

$$\bar{I}^{j_1,\cdots,j_k}_{l_0,l_1,\cdots,l_{k-1},l_k} = \int_{O_N(\mathbb{R})^k} (\Pi_{r=1}^k O^{(r)}_{l_{r-1},j_r} O^{(r)}_{l_r,j_r})$$
$$\times e^{\frac{1}{(1+c)}\sum_{r<l}\sigma^2_{|x_r-x_l|}(c^2 tr({}^tO^{(r)}O^{(r)}{}^tO^{(l)}O^{(l)})+(1+c)^2({}^tO^{(r)}O^{(l)})^2_{j_r,j_l})}$$
$$\times e^{-c\sum_{r<l}\sigma^2_{|x_r-x_l|}(({}^tO^{(r)}O^{(l)}{}^tO^{(l)}O^{(r)})_{j_r,j_r}+({}^tO^{(l)}O^{(r)}{}^tO^{(r)}O^{(l)})_{j_l,j_l})} dO^{(1)}\cdots dO^{(k)}$$
$$= e^{(\frac{c^2N}{(1+c)}-2c)\sum_{r<l}\sigma^2_{|x_r-x_l|}} \int_{O_N(\mathbb{R})^k} (\Pi_{r=1}^k O^{(r)}_{l_{r-1},j_r} O^{(r)}_{l_r,j_r})$$
$$\times e^{(1+c)\sum_{r<l}\sigma^2_{|x_r-x_l|}({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} dO^{(1)}\cdots dO^{(k)}$$

In conclusion, we get the following equivalent:

$$\bar{Z} \underset{\epsilon\to 0}{\sim} \frac{\sigma_\epsilon^{N(N+1)k/2} e^{k\sigma_\epsilon^2/2}(1+c)^{(N-1)k}\sigma_\epsilon^{(N-1)k}}{R_N^\epsilon} \int_{\mathbb{R}^{kN}} \Pi_{r=1}^k |\Delta_1(v^{(r)})| e^{-\frac{1}{2(1+c)}\sum_{r=1}^k \sum_{i=1}^N (v_i^{(r)})^2}$$
$$\times\, e^{-\alpha\sum_{r=1}^k (\sum_{i=1}^N v_i^{(r)})^2} dv^{(1)}\cdots dv^{(k)} e^{-c\sum_{r<l}\sigma^2_{|x_r-x_l|}} F_{i,j}(x_1,\cdots,x_k)$$

where we have the following expression for $F_{i,j}(x_1,\cdots,x_k)$:

$$F_{i,j}(x_1,\cdots,x_k) =$$
$$\sum_{j_1,\cdots,j_k=1}^N \sum_{\substack{l_1,\cdots,l_{k-1}=1 \\ l_0=i; l_k=j}}^N \int_{O_N(\mathbb{R})^k} (\Pi_{r=1}^k O^{(r)}_{l_{r-1},j_r} O^{(r)}_{l_r,j_r}) e^{(1+c)\sum_{r<l}\sigma^2_{|x_r-x_l|}({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} dO^{(1)}\cdots dO^{(k)}$$
$$= \sum_{j_1,\cdots,j_k=1}^N \int_{O_N(\mathbb{R})^k} O^{(1)}_{i,j_1} ({}^tO^{(1)}O^{(2)})_{j_1,j_2}\cdots({}^tO^{(k-1)}O^{(k)})_{j_{k-1},j_k} O^{(k)}_{j,j_k}$$
$$\times\, e^{(1+c)\sum_{r<l}\sigma^2_{|x_r-x_l|}({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} dO^{(1)}\cdots dO^{(k)}$$

We get thus the following expression:

$$\bar{Z}/(Zc_\epsilon^k) \underset{\epsilon\to 0}{\sim} C_{k,N} e^{-c\sum_{r<l}\sigma^2_{|x_r-x_l|}} F_{i,j}(x_1,\cdots,x_k), \tag{13}$$

where $C_{k,N}$ is a constant which depends only on k, N (we can compute this constant but it is tedious and will not be necessary for the purpose of this paper).

3.4.4 Computation of the moment of order k and of the structure functions

From relation (13), we get the following expression:

$$E(trM^\epsilon(A)^k) \underset{\epsilon\to 0}{\to} C_{k,N} \int_{A^k} e^{-c\sum_{r<l}\sigma^2_{|x_r-x_l|}} \sum_{i=1}^N F_{i,i}(x_1,\cdots,x_k) dx_1\cdots dx_k \tag{14}$$

The main difficulty is to study the functions $F_{i,j}$. If one takes the trace, we get:

$$\sum_{i=1}^N F_{i,i}(x_1,\cdots,x_k)$$
$$= \sum_{j_1,\cdots,j_k=1}^N \int_{O_N(\mathbb{R})^k} ({}^tO^{(1)}O^{(2)})_{j_1,j_2}\cdots({}^tO^{(k-1)}O^{(k)})_{j_{k-1},j_k} ({}^tO^{(k)}O^{(1)})_{j_k,j_1}$$
$$\times\, e^{(1+c)\sum_{r<l}\sigma^2_{|x_r-x_l|}({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} dO^{(1)}\cdots dO^{(k)}. \tag{15}$$

In particular, for $k=2$, we recover that:

$$\sum_{i=1}^N F_{i,i}(x_1,x_2) = N^2 \int_{O_N(\mathbb{R})^2} ({}^tO^{(1)}O^{(2)})^2_{1,1} e^{(1+c)\sigma^2_{|x_2-x_1|}({}^tO^{(1)}O^{(2)})^2_{1,1}} dO^{(1)} dO_2.$$

In order to prove (3), we want to study the asymptotic of $\sum_{i=1}^N F_{i,i}(\ell x_1, \cdots, \ell x_k)$ as $\ell \to 0$ with $(x_1, \cdots, x_k)$ fixed. Since $\sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \leqslant \frac{k(k-1)}{2}$, it is obvious that:

$$\overline{\lim_{\ell \to 0}} \frac{\ln(\sum_{i=1}^N F_{i,i}(\ell x_1, \cdots, \ell x_k))}{\ln \frac{1}{\ell}} \leqslant (1+c)\gamma^2 \frac{k(k-1)}{2} \tag{16}$$

To get an inverse inequality in (16), we will study the asymptotic of each term in the sum (15). Here we suppose that $L = 1$ and $m = 0$ to simplify the presentation. We fix $(j_1, \cdots, j_k)$ and ϵ, δ small such that $\epsilon < \delta$. We have:

$$\begin{aligned}
&\int_{O_N(\mathbb{R})^k} ({}^tO^{(1)}O^{(2)})_{j_1,j_2} \cdots ({}^tO^{(k-1)}O^{(k)})_{j_{k-1},j_k} ({}^tO^{(k)}O^{(1)})_{j_k,j_1} \\
&\times e^{(1+c)\gamma^2 \ln \frac{1}{\ell} \sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} e^{(1+c)\sum_{r<l} \sigma^2_{|x_r - x_l|} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} dO^{(1)} \cdots dO^{(k)} \\
&= A_\epsilon + A_{\epsilon,\delta} + A_\delta
\end{aligned}$$

where

$$A_\epsilon = \int_{\sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \geqslant \frac{k(k-1)}{2} - \epsilon} \cdots, \quad A_{\epsilon,\delta} = \int_{\frac{k(k-1)}{2} - \delta \leqslant \sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \leqslant \frac{k(k-1)}{2} - \epsilon} \cdots$$

and A_δ is the $\sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \leqslant \frac{k(k-1)}{2} - \delta$ part of the integral. On the event $\sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \geqslant \frac{k(k-1)}{2} - \epsilon$, each $|{}^tO^{(r)}O^{(l)})_{j_r,j_l}|$ is greater or equal to $\sqrt{1-\epsilon}$. In particular, we have that $|({}^tO^{(r)}O^{(r+1)})_{j_r,j_{r+1}}| \geqslant \sqrt{1-\epsilon}$ for all $r \leqslant k-1$. Notice that $({}^tO^{(k)}O^{(1)})_{j_k,j_1} = ({}^tO^{(1)}O^{(2)})_{j_1,j_2} \cdots ({}^tO^{(k-1)}O^{(k)})_{j_{k-1},j_k} + O(\epsilon)$. Therefore, we can conclude that $({}^tO^{(1)}O^{(2)})_{j_1,j_2} \cdots ({}^tO^{(k-1)}O^{(k)})_{j_{k-1},j_k}$ and $({}^tO^{(1)}O^{(2)})_{j_1,j_2}$ have the same sign. Thus, we get:

$$\begin{aligned}
&A_\epsilon \\
&\geqslant e^{(1+c)\gamma^2 \ln \frac{1}{\ell} (\frac{k(k-1)}{2} - \epsilon)} \\
&\times \int_{\sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \geqslant \frac{k(k-1)}{2} - \epsilon} ({}^tO^{(1)}O^{(2)})_{j_1,j_2} \cdots ({}^tO^{(k-1)}O^{(k)})_{j_{k-1},j_k} ({}^tO^{(k)}O^{(1)})_{j_k,j_1} \\
&\times e^{(1+c)\sum_{r<l} \sigma^2_{|x_r - x_l|} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} dO^{(1)} \cdots dO^{(k)} \\
&\geqslant e^{(1+c)\gamma^2 \ln \frac{1}{\ell} (\frac{k(k-1)}{2} - \epsilon)} ((1-\epsilon)^k + O(\epsilon)) \\
&\times \int_{\sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \geqslant \frac{k(k-1)}{2} - \epsilon} e^{(1+c)\sum_{r<l} \sigma^2_{|x_r - x_l|} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} dO^{(1)} \cdots dO^{(k)}.
\end{aligned}$$

The only thing to check is that $\sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \geqslant \frac{k(k-1)}{2} - \epsilon$ has a positive probability but this can be seen easily by setting one chosen element of each $O^{(r)}$, say $O^{(r)}_{1,j_r}$, very close to one. Now, one can choose δ larger than ϵ such that $|A_{\epsilon,\delta}| \leqslant \frac{A_\epsilon}{2}$. Finally, for these choices of ϵ, δ, we have:

$$\begin{aligned}
|A_\delta| &\leqslant e^{(1+c)\gamma^2 \ln \frac{1}{\ell} (\frac{k(k-1)}{2} - \delta)} \\
&\times \int_{\sum_{r<l} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l} \leqslant \frac{k(k-1)}{2} - \delta} e^{(1+c)\sum_{r<l} \sigma^2_{|x_r - x_l|} ({}^tO^{(r)}O^{(l)})^2_{j_r,j_l}} dO^{(1)} \cdots dO^{(k)}.
\end{aligned}$$

We thus get the following:

$$\varliminf_{\ell\to 0}\frac{\ln(A_\epsilon + A_{\epsilon,\delta} + A_\delta)}{\ln\frac{1}{\ell}} \geqslant (1+c)\gamma^2(\frac{k(k-1)}{2}-\epsilon)$$

Since this is valid for all ϵ, by combining with (16), we get that:

$$\lim_{\ell\to 0}\frac{\ln(\sum_{i=1}^N F_{i,i}(\ell x_1,\cdots,\ell x_k))}{\ln\frac{1}{\ell}} = (1+c)\gamma^2\frac{k(k-1)}{2} \tag{17}$$

The desired result (3) is then a consequence of (17) and of the limit (14).

A. Appendix

A.1 Discussion about the construction of kernels

In this subsection, we discuss in further details the construction of the kernel K as summarized in remark 2. In dimension 1 and 2, it is plain to see that

$$\ln_+\frac{L}{|x|} = \int_0^{+\infty}(t-|x|)_+\nu_L(dt) \tag{18}$$

where the measure ν_L is given by (δ_u stands for the Dirac mass at u):

$$\nu_L(dt) = \mathbf{1}_{[0,L]}(t)\frac{dt}{t^2} + \frac{1}{L}\delta_L(dt).$$

Hence, for every $\mu > 0$, we have:

$$\ln_+\frac{L}{|x|} = \frac{1}{\mu}\ln_+\frac{L^\mu}{|x|^\mu} = \int_0^{+\infty}(t-|x|^\mu)_+\nu_{L^\mu}(dt).$$

We are therefore led to consider $\mu > 0$ such that the function $x \mapsto (1-|x|^\mu)_+$ is positive definite, the so-called Kuttner-Golubov problem (see [14]).

For $d = 1$, it is straightforward to check that $(1-|x|)_+$ is positive definite. We can thus consider a Gaussian process X^ϵ with covariance kernel given by

$$K_\epsilon(x) = \gamma^2\int_\epsilon^L (t-|x|)_+\nu_L(dt).$$

Notice that

$$\forall x\neq 0,\quad \gamma^2\ln_+\frac{L}{|x|} = \lim_{\epsilon\to 0}K_\epsilon(x) \tag{19}$$

and

$$\forall \epsilon < |x| \leqslant L,\quad K_\epsilon(x) = \gamma^2\int_{|x|}^L (t-|x|)_+\nu_L(dt) = \gamma^2\ln_+\frac{L}{|x|}. \tag{20}$$

In dimension 2, we can use the same strategy since Pasenchenko [21] proved that the mapping $x \mapsto (1-|x|^{1/2})_+$ is positive definite over $\mathbb{R}^2$. We can thus consider a Gaussian process X^ϵ with covariance kernel given by

$$K_\epsilon(x) = 2\gamma^2\int_{\epsilon^{1/2}}^{L^{1/2}}(t-|x|^{1/2})_+\nu_{L^{1/2}}(dt),$$

sharing the same properties (22) and (23).

In dimension 3, it is not known whether the mapping $x \mapsto \ln_+ \frac{L}{|x|}$ admits an integral representation of the type explained above. Nevertheless it is positive definite so that we can use the convolution techniques developed in [24]. In dimension 4, it is not positive definite [24] so that another construction is required. We explain the methods in [23]. We set the dimension d to be larger than $d \geqslant 3$. Let us denote by S the sphere of $\mathbb{R}^d$ and σ the surface measure on the sphere such that $\sigma(S) = 1$. Remind that this measure is invariant under rotations. We define the function

$$\forall x \in \mathbb{R}^d, \quad F(x) = \gamma^2 \int_S \ln_+ \frac{L}{|\langle x, s\rangle|} \sigma(ds). \tag{21}$$

It is plain to see that F is an isotropic function. Let us compute it over a neighborhood of 0: for $|x| \leqslant L$, we can write $x = |x|e_x$ with $e_x \in S$. Then we have

$$F(x) = \gamma^2 \int_S \ln \frac{L}{|x||\langle e_x, s\rangle|} \sigma(ds) = \lambda^2 \ln \frac{L}{|x|} + \int_S \ln \frac{1}{|\langle e_x, s\rangle|} \sigma(ds).$$

Notice that the integral $\int_S \ln \frac{1}{|\langle e_x, s\rangle|} \sigma(ds)$ is finite (use Lemma 3 below for instance) and does not depend on x by invariance under rotations of the measure σ. By using the decomposition (18), we can thus consider a Gaussian process X^ϵ with covariance kernel given by

$$K_\epsilon(x) = \gamma^2 \int_S \int_\epsilon^L (t - |\langle x, s\rangle|)_+ \nu_L(dt) \sigma(ds),$$

sharing the properties:

$$\forall x \neq 0, \quad \lim_{\epsilon \to 0} K_\epsilon(x) = F(x) \tag{22}$$

and

$$\forall \epsilon < |x| \leqslant L, \quad K_\epsilon(x) = F(x) = \lambda^2 \ln \frac{L}{|x|} + C \tag{23}$$

for some constant C.

A.2 Auxiliary results

We give a proof of the following standard result:

Lemma 3. *If $(Z_i)_{1 \leqslant i \leqslant N}$ are i.i.d. standard Gaussian random variables then the vector*

$$V = \frac{1}{\sqrt{\sum_{i=1}^N Z_i^2}} (Z_1, \ldots, Z_N)$$

is distributed as the Haar measure on the sphere of $\mathbb{R}^N$. In particular, the density of the first entry of a random vector uniformly distributed on the sphere is given by:

$$\frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{N-1}{2})} y^{-\frac{1}{2}} (1-y)^{\frac{N-3}{2}} \mathbf{1}_{[0,1]}(y)\, dy.$$

Proof. By using the invariance under rotations of the law of the Gaussian vector $(Z_i)_{1 \leqslant i \leqslant N}$, the law of V is invariant under rotations and is supported by the sphere. By uniqueness of the Haar measure, V is distributed as the Haar measure. We have to compute the density of $\zeta_1 = \frac{Z_1^2}{\sum_{i=1}^N Z_i^2}$. Notice that

$$\zeta_1 = \frac{Y}{Y+Z}$$

where Y, Z are independent random variables with respective laws chi-squared distributions of parameters 1 et $N-1$. Therefore

$$\begin{aligned} E[f(\zeta_1)] &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f\Big(\frac{x}{x+y}\Big)\frac{1}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})}x^{-\frac{1}{2}}e^{-\frac{x}{2}}\frac{1}{2^{\frac{N-1}{2}}\Gamma(\frac{N-1}{2})}y^{\frac{N-1}{2}}e^{-\frac{y}{2}}\,dx\,dy \\ &= \frac{1}{2^{\frac{N}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{N-1}{2})}\int_0^1 f(u)\frac{1}{\sqrt{u}(1-u)^{\frac{3}{2}}}\int_{\mathbb{R}_+} e^{-\frac{y}{2(1-u)}}y^{\frac{N-2}{2}}\,dy\,du \\ &= \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{N-1}{2})}\int_0^1 f(u)u^{-\frac{1}{2}}(1-u)^{\frac{N-3}{2}}\,du. \end{aligned}$$

□

Next we characterize all the symmetric Gaussian random matrices:

Lemma 4. *Let X be a centered Gaussian symmetric random matrix of size $N \times N$. Then the diagonal terms $(X_{11}, \dots, X_{NN})$ have a covariance matrix of the form $\sigma^2(1+c)\mathrm{I}_N - c\sigma^2 P$ for some $\sigma^2 \geqslant 0$ and $c \in]-1, \frac{1}{N-1}]$, where P is the $N \times N$ matrix whose all entries are 1. The off-diagonal terms are i.i.d with variance $\sigma^2\frac{1+c}{2}$ and are independent of the diagonal terms.*

Proof. If X admits a density with respect to the Lebesgue measure dM over the set of symmetric matrices (see [1, chapter 4]), then the density of M is given by

$$e^{-f(M)}\,dM,$$

where f is a homogeneous polynomial of degree 2. By isotropy, f must be a symmetric function of the eigenvalues of M. Therefore it takes on the form

$$f(M) = \alpha \mathrm{tr}(M^2) + \beta \mathrm{tr}(M)^2$$

for some $\alpha, \beta \in \mathbb{R}$. In this case, the result follows easily.

If M does not admit a density with respect to the Lebesgue measure over the set of symmetric matrices, we can add an independent "small GOE", i.e. we consider $M + \epsilon M'$ where M' is a matrix of the GOE ensemble with a normalized variance independent of M. The matrix $M + \epsilon M'$ admits a density so that we can apply the above result. Then we pass to the limit as $\epsilon \to 0$. □

A.3 Some integral formulae

Let $\alpha, c > 0$. We want to compute the integral:

$$\int_{\mathbb{R}^N} e^{-\alpha(\sum_{i=1}^N \lambda_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N \lambda_i^2}\Pi_{i<j}|\lambda_j - \lambda_i| d\lambda.$$

We write the integrand under the form (6):

$$e^{-\alpha(\sum_{i=1}^N \lambda_i)^2 - \frac{1}{2\sigma_d^2(1+\bar{c})}\sum_{i=1}^N \lambda_i^2} \Pi_{i<j}|\lambda_j - \lambda_i|$$

where $\sigma_d^2(1+\bar{c}) = (1+c)$ and $\alpha = \frac{\bar{c}}{2\sigma_d^2(1+\bar{c})}\frac{1}{(1+\bar{c}(1-N))}$. In that case, we have $\bar{c} = \frac{2\alpha(1+c)}{1+2\alpha(1+c)(N-1)}$ and $1+\bar{c}(1-N) = \frac{1}{1+2\alpha(1+c)(N-1)}$. We deduce:

$$\int_{\mathbb{R}^N} e^{-\alpha(\sum_{i=1}^N \lambda_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N \lambda_i^2} \Pi_{i<j}|\lambda_j - \lambda_i| d\lambda = N!(2\pi)^{N/2}\left(\prod_{k=1}^N \frac{\Gamma(k/2)}{\Gamma(1/2)}\right)\frac{(1+c)^{N(N+1)/4}}{\sqrt{1+2\alpha(1+c)N}} \tag{24}$$

We also want to compute the integral:

$$\int_{\mathbb{R}^N} e^{-\alpha(\sum_{i=1}^N \lambda_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N \lambda_i^2} \Pi_{2 \leqslant i<j}|\lambda_j - \lambda_i| d\lambda.$$

We have:

$$\begin{aligned}
&\int_{\mathbb{R}^N} e^{-\alpha(\sum_{i=1}^N \lambda_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N \lambda_i^2} \prod_{2 \leqslant i<j}|\lambda_j - \lambda_i| d\lambda_1 \cdots d\lambda_N \\
&= \int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}} e^{-2\alpha\lambda_1(\sum_{i=2}^N \lambda_i) - (\alpha + \frac{1}{2(1+c)})\lambda_1^2} d\lambda_1\right) e^{-\alpha(\sum_{i=2}^N \lambda_i)^2 - \frac{1}{2(1+c)}\sum_{i=2}^N \lambda_i^2} \prod_{2 \leqslant i<j}|\lambda_j - \lambda_i| d\lambda_2 \cdots d\lambda_N \\
&= \sqrt{2\pi}\sqrt{\frac{1+c}{2\alpha(1+c)+1}} \int_{\mathbb{R}^{N-1}} e^{2\alpha^2 \frac{1+c}{2\alpha(1+c)+1}(\sum_{i=2}^N \lambda_i)^2 - \alpha(\sum_{i=2}^N \lambda_i)^2 - \frac{1}{2(1+c)}\sum_{i=2}^N \lambda_i^2} \prod_{2 \leqslant i<j}|\lambda_j - \lambda_i| d\lambda_2 \cdots d\lambda_N \\
&= \sqrt{2\pi}\sqrt{\frac{1+c}{2\alpha(1+c)+1}} \int_{\mathbb{R}^{N-1}} e^{-\frac{\alpha}{2\alpha(1+c)+1}(\sum_{i=2}^N \lambda_i)^2 - \frac{1}{2(1+c)}\sum_{i=2}^N \lambda_i^2} \prod_{2 \leqslant i<j}|\lambda_j - \lambda_i| d\lambda_2 \cdots d\lambda_N \\
&= \sqrt{2\pi}\sqrt{\frac{1+c}{2\alpha(1+c)+1}} \int_{\mathbb{R}^{N-1}} e^{-\frac{\bar{c}}{2\sigma_d^2(1+\bar{c})}\frac{1}{(1+\bar{c}(2-N))}(\sum_{i=2}^N \lambda_i)^2 - \frac{1}{2\sigma_d^2(1+\bar{c})}\sum_{i=2}^N \lambda_i^2} \prod_{2 \leqslant i<j}|\lambda_j - \lambda_i| d\lambda_2 \cdots d\lambda_N
\end{aligned}$$

for $\sigma_d^2(1+\bar{c}) = 1+c$ and $\bar{c} = \frac{2\alpha(1+c)}{2\alpha(1+c)(N-1)+1}$ (or equivalently, $1+\bar{c}(2-N) = \frac{1+2\alpha(1+c)}{1+2\alpha(1+c)(N-1)}$ and $1+\bar{c} = \frac{1+2\alpha(1+c)N}{1+2\alpha(1+c)(N-1)}$). This leads to the following:

$$\int_{\mathbb{R}^{N-1}} e^{-\frac{\bar{c}}{2\sigma_d^2(1+\bar{c})}\frac{1}{(1+\bar{c}(2-N))}(\sum_{i=2}^N \lambda_i)^2 - \frac{1}{2\sigma_d^2(1+\bar{c})}\sum_{i=2}^N \lambda_i^2} \prod_{2 \leqslant i<j}|\lambda_j - \lambda_i| d\lambda_2 \cdots d\lambda_N = \bar{Z}_{N-1}$$

where $\bar{Z}_{N-1} = (N-1)!(2\pi)^{(N-1)/2}(\prod_{k=1}^{N-1}\frac{\Gamma(k/2)}{\Gamma(1/2)})\sigma_d^{N(N-1)/2}(1+\bar{c})^{(N-2)(N+1)/4}\sqrt{1+\bar{c}(2-N)}$.
In conclusion, we get:

$$\begin{aligned}
&\int_{\mathbb{R}^N} e^{-\alpha(\sum_{i=1}^N \lambda_i)^2 - \frac{1}{2(1+c)}\sum_{i=1}^N \lambda_i^2} \prod_{2 \leqslant i<j}|\lambda_j - \lambda_i| d\lambda_1 \cdots d\lambda_N \\
&= \sqrt{1+c}(N-1)!(2\pi)^{N/2}\left(\prod_{k=1}^{N-1} \frac{\Gamma(k/2)}{\Gamma(1/2)}\right)\frac{(1+c)^{N(N-1)/4}}{\sqrt{1+2\alpha(1+c)N}}
\end{aligned} \tag{25}$$

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