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► **To cite this version:**

Francois Chapon, Romain Couillet, Walid Hachem, Xavier Mestre. On the isolated eigenvalues of large Gram random matrices with a fixed rank deformation. 2012. hal-00713811v1

**HAL Id: hal-00713811**

**<https://hal.science/hal-00713811v1>**

Preprint submitted on 2 Jul 2012 (v1), last revised 22 Jan 2013 (v2)

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# ON THE ISOLATED EIGENVALUES OF LARGE GRAM RANDOM MATRICES WITH A FIXED RANK DEFORMATION

FRANÇOIS CHAPON, ROMAIN COUILLET,  
WALID HACHEM AND XAVIER MESTRE

ABSTRACT. Consider the matrix  $\Sigma_n = n^{-1/2}X_n D_n^{1/2} + P_n$  where the matrix  $X_n \in \mathbb{C}^{N \times n}$  has Gaussian standard independent elements,  $D_n$  is a deterministic diagonal nonnegative matrix, and  $P_n$  is a deterministic matrix with fixed rank. Under some known conditions, the spectral measures of  $\Sigma_n \Sigma_n^*$  and  $n^{-1}X_n D_n X_n^*$  both converge towards a compactly supported probability measure  $\mu$  as  $N$  and  $n$  converge to infinity at the same rate. In this paper, we prove that finitely many eigenvalues of  $\Sigma_n \Sigma_n^*$  may stay outside the support of  $\mu$  in the large dimensional regime. The existence and locations of these eigenvalues in any compact interval outside the support of  $\mu$  are studied. The fluctuations of the extreme isolated eigenvalues of  $\Sigma_n \Sigma_n^*$  are also analyzed. The results find applications in the fields of signal processing and radio communications.

## 1. INTRODUCTION

**1.1. The model and the literature.** Consider a sequence of  $N \times n$  matrices  $Y_n$ ,  $n = 1, 2, \dots$ , of the form  $Y_n = X_n D_n^{1/2}$  where  $X_n$  is a  $N \times n$  random matrix whose coefficients  $X_{ij}$  are independent and identically distributed (iid) complex Gaussian random variables such that  $\Re(X_{11})$  and  $\Im(X_{11})$  are independent, each with mean zero and variance  $1/2$ , and where  $D_n$  is a deterministic nonnegative diagonal  $n \times n$  matrix. Writing  $D_n = \text{diag}(d_j^n)_{j=1, \dots, n}$  and denoting by  $\delta$  the Dirac measure, it is assumed that the spectral measure  $\nu_n = n^{-1} \sum_{j=1}^n \delta_{d_j^n}$  of  $D_n$  converges weakly to a compactly supported probability measure  $\nu$  when  $n \rightarrow \infty$ . It is also assumed that the maximum of the distances from the diagonal elements of  $D_n$  to the support  $\text{supp}(\nu)$  of  $\nu$  goes to zero as  $n \rightarrow \infty$ . Assume that  $N/n \rightarrow c$  when  $n \rightarrow \infty$ , where  $c$  is a positive constant. Then it is known that with probability one, the spectral measure of the Gram matrix  $n^{-1}Y_n Y_n^*$  converges weakly to a compactly supported probability measure  $\mu$  (see [26], [16], [35], [36]) and, with probability one,  $n^{-1}Y_n Y_n^*$  has no eigenvalues in any compact interval outside  $\text{supp}(\mu)$  for large  $n$  [3].

Let  $r$  be a given positive integer and consider a sequence of deterministic  $N \times n$  matrices  $P_n$ ,  $n = 1, 2, \dots$ , such that  $\text{rank}(P_n) = r$  and  $\sup_n \|P_n\| < \infty$  where  $\|\cdot\|$  is the spectral norm. Consider the matrix  $\Sigma_n = n^{-1/2}Y_n + P_n$ . Since the additive deformation  $P_n$  has a fixed rank, the spectral measure of  $\Sigma_n \Sigma_n^*$  still converges to  $\mu$  (see, e.g., [2, Lemma 2.2]). However, a finite number of “isolated” eigenvalues of  $\Sigma_n \Sigma_n^*$

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*Date:* 20 June 2012.

*2000 Mathematics Subject Classification.* Primary 15A52, Secondary 15A18, 60F15.

*Key words and phrases.* Random Matrix Theory, Stieltjes Transform, Fixed rank deformation, Extreme eigenvalues, Gaussian fluctuations.

The work of the first three authors was supported by the French Ile-de-France region, DIM LSC fund, Digiteo project DESIR.

might stay outside the support of  $\mu$ . The purpose of this paper is to characterize the conditions under which this phenomenon occurs, to determine the locations of the isolated eigenvalues, and to study the fluctuations of the largest among them.

The study of the isolated eigenvalues of large random matrices has a wide range of applications. These include detection and estimation of wireless communication signals [20], fault diagnosis in complex systems [14], financial portfolio management [34], or chemometrics [29]. In the statistics literature, one of the first contributions to deal with this subject was [23]. It raised the question of the behavior of the extreme eigenvalues of a sample covariance matrix when the population covariance matrix has all but finitely many of its eigenvalues equal to one (leading to a multiplicative fixed rank deformation). This problem has been studied thoroughly in [5, 6, 32]. Other contributions (see [11]) study the isolated eigenvalues of a Wigner matrix subject to a fixed rank additive deformation. The asymptotic fluctuations of the isolated eigenvalues have been addressed in [5, 33, 32, 1, 11, 12, 7].

Recently, Benaych-Georges and Nadakuditi proposed in [8, 9] a generic method for characterizing the behavior of the isolated eigenvalues for a large palette of random matrix models. For our model, this method shows that the limiting locations as well as the fluctuations of the isolated eigenvalues are intimately related to the asymptotic behavior of certain bilinear forms involving the resolvents  $(n^{-1}Y_n Y_n^* - xI_N)^{-1}$  and  $(n^{-1}Y_n^* Y_n - xI_n)^{-1}$  of the undeformed matrix at real values of  $x$ . When  $D_n = I_n$ , the asymptotic behavior of these bilinear forms can be simply identified (see [9]) thanks to the fact that the probability law of  $Y_n$  is invariant by left or right multiplication by unitary deterministic matrices. For general  $D_n$ , other tools need to be used. In this paper, these bilinear forms are studied with the help of an integration by parts formula for functionals of Gaussian vectors and the Poincaré-Nash inequality. These tools belong to the arsenal of random matrix theory, as shown in the recent monograph [31] and in the references therein. In order to be able to use them in our context, we make use of a regularizing function ensuring that the moments of the bilinear forms exist for certain  $x \in \mathbb{R}_+ = [0, \infty)$ .

Before entering the subject, we remark that the results of the paper can be straightforwardly generalized to the case where  $D_n$  is replaced by a nonnegative Hermitian matrix  $R_n$ , which is a more practical model in many applications. Indeed, write  $R_n = V_n D_n V_n^*$  where  $V_n$  is a unitary matrix, and let  $R_n^{1/2} = V_n D_n^{1/2} V_n^*$ . Then the singular values of  $\Sigma_n = n^{-1/2} X_n D_n^{1/2} + P_n$  coincide with those of  $n^{-1/2} (X_n V_n^*) R_n^{1/2} + P_n V_n^*$ . Since  $X_n$  and  $X_n V_n^*$  are equal in law, the vector of singular values of  $\Sigma_n$  and the vector of singular values of  $n^{-1/2} X_n R_n^{1/2} + P_n'$  where  $P_n' = P_n V_n^*$  are also equal in law.

The paper is organized as follows. The assumptions and the main results are provided in Section 2. Section 3 is devoted to the basic mathematical tools needed for the proofs. These proofs are provided in Sections 4 and 5, which concern respectively the first order (convergence) and the second order (fluctuations) behavior of the isolated eigenvalues.

## 2. PROBLEM DESCRIPTION AND MAIN RESULTS

Given a sequence of integers  $N = N(n)$ ,  $n = 1, 2, \dots$ , we consider the sequence of  $N \times n$  matrices  $\Sigma_n = n^{-1/2}Y_n + P_n = n^{-1/2}X_n D_n^{1/2} + P_n$  with the following assumptions:

**Assumption 1.** *The ratio  $c_n = N(n)/n$  converges to a positive constant  $c$  as  $n \rightarrow \infty$ .*

**Assumption 2.** *The matrix  $X_n = [X_{ij}]_{i,j=1}^{N,n}$  is a  $N \times n$  random matrix whose coefficients  $X_{ij}$  are iid complex random variables such that  $\Re(X_{11})$  and  $\Im(X_{11})$  are independent, each with probability distribution  $\mathcal{N}(0, 1/2)$ .*

**Assumption 3.** *The sequence of  $n \times n$  deterministic diagonal nonnegative matrices  $D_n = \text{diag}(d_j^n)_{j=1}^n$  satisfies the following:*

- (1) *The probability measure  $\nu_n = n^{-1} \sum_{j=1}^n \delta_{d_j^n}$  converges weakly to a probability measure  $\nu$  with compact support.*
- (2) *The distances  $\mathbf{d}(d_j^n, \text{supp}(\nu))$  from  $d_j^n$  to  $\text{supp}(\nu)$  satisfy*

$$\max_{j \in \{1, \dots, n\}} \mathbf{d}(d_j^n, \text{supp}(\nu)) \xrightarrow{n \rightarrow \infty} 0.$$

This assumption implies that  $\mathbf{d}_{\max} = \sup_n \|D_n\| < \infty$ .

As is usual in random matrix theory, a central role will be played here by the so called Stieltjes Transform. The Stieltjes Transform of a positive finite measure  $\mu$  over the Borel sets of  $\mathbb{R}$  is the function

$$m(z) = \int_{\mathbb{R}} \frac{1}{t-z} \mu(dt) \quad (1)$$

analytic on  $\mathbb{C} - \text{supp}(\mu)$ . It is straightforward to check that  $\Im m(z) \geq 0$  when  $z \in \mathbb{C}_+ = \{z : \Im(z) > 0\}$ , and  $\sup_{y>0} |ym(iy)| < \infty$ . Conversely, any analytic function  $m(z)$  on  $\mathbb{C}_+$  that has these two properties admits the integral representation (1) where  $\mu$  is a positive finite measure. Furthermore, for any continuous real function  $\varphi$  with compact support in  $\mathbb{R}$ ,

$$\int \varphi(t) \mu(dt) = \frac{1}{\pi} \lim_{y \downarrow 0} \int \varphi(x) \Im m(x + iy) dx \quad (2)$$

which implies that the measure  $\mu$  is uniquely defined by its Stieltjes Transform. Finally, if  $\Im(zm(z)) \geq 0$  when  $z \in \mathbb{C}_+$ , then  $\mu((-\infty, 0)) = 0$  [25].

These facts can be generalized to Hermitian matrix-valued nonnegative finite measures [10, 15]. Let  $m(z)$  be a  $\mathbb{C}^{r \times r}$ -valued analytic function on  $z \in \mathbb{C}_+$ . Letting  $\Im X = (X - X^*)/(2i)$ , assume that  $\Im m(z) \geq 0$  and  $\Im(zm(z)) \geq 0$  in the order of the Hermitian matrices for any  $z \in \mathbb{C}_+$ , and that  $\sup_{y>0} \|ym(iy)\| < \infty$ . Then  $m(z)$  admits the representation (1) where  $\mu$  is now a  $r \times r$  matrix-valued nonnegative finite measure such that  $\mu((-\infty, 0)) = 0$ . One can also check that  $\mu([0, \infty)) = -\lim_{y \rightarrow \infty} iy m(-iy)$ .

The following theorem characterizes the asymptotic behavior of the eigenvalues of  $n^{-1}Y_n Y_n^*$ :

**Theorem 2.1.** *Under Assumptions 1, 2 and 3, the following hold true:*

(1) For any  $z \in \mathbb{C}_+$ , the equation

$$\mathbf{m} = \left( -z + \int \frac{t}{1 + \mathbf{c} \mathbf{m} t} \nu(dt) \right)^{-1} \quad (3)$$

admits a unique solution  $\mathbf{m} \in \mathbb{C}_+$ . The function  $\mathbf{m} = \mathbf{m}(z)$  so defined on  $\mathbb{C}_+$  is the Stieltjes Transform of a probability measure  $\mu$  whose support is a compact set of  $\mathbb{R}_+$ .

Let  $(\lambda_i^n)_{i=1, \dots, N}$  be the eigenvalues of  $n^{-1} Y_n Y_n^*$ , and let  $\theta_n = N^{-1} \sum_{i=1}^N \delta_{\lambda_i^n}$  be the spectral measure of this matrix. Then for every bounded and continuous real function  $f$ ,

$$\int f(t) \theta_n(dt) \xrightarrow[n \rightarrow \infty]{a.s.} \int f(t) \mu(dt). \quad (4)$$

(2) For any interval  $[x_1, x_2] \subset \mathbb{R} - \text{supp}(\mu)$ ,

$$\#\{i : \lambda_i^n \in [x_1, x_2]\} = 0 \text{ with probability 1 for all large } n.$$

The first part of this theorem has been shown in [26, 36], and the second part in [3].

Let  $Q_n(z) = (n^{-1} Y_n Y_n^* - z I_N)^{-1}$  and  $\tilde{Q}_n(z) = (n^{-1} Y_n^* Y_n - z I_n)^{-1}$  be the resolvents of  $n^{-1} Y_n Y_n^*$  and  $n^{-1} Y_n^* Y_n$  respectively. The first part of Theorem 2.1 can also be deduced from the following result that will be used in our subsequent derivations:

**Proposition 2.1** ([36, 22, 18]). *Assume that  $D_n$  is a  $n \times n$  diagonal nonnegative matrix. Then, for any  $n$ , the equation*

$$m_n = \left[ -z \left( 1 + \frac{1}{n} \text{Tr} D_n \tilde{T}_n \right) \right]^{-1} \quad \text{where} \quad \tilde{T}_n = [-z (I_n + c_n m_n D_n)]^{-1}$$

admits a unique solution  $m_n \in \mathbb{C}_+$  for any  $z \in \mathbb{C}_+$ . The function  $m_n = m_n(z)$  so defined on  $\mathbb{C}_+$  is the Stieltjes Transform of a probability measure  $\mu_n$  whose support is a compact set of  $\mathbb{R}_+$ . Moreover, the  $n \times n$  diagonal matrix-valued function  $\tilde{T}_n(z) = [-z (I_n + c_n m_n(z) D_n)]^{-1}$  is analytic on  $\mathbb{C}_+$  and  $n^{-1} \text{Tr} \tilde{T}_n(z)$  coincides with the Stieltjes Transform of  $c_n \mu_n + (1 - c_n) \delta_0$ .

Let Assumption 2 hold true, and assume that  $\sup_n \|D_n\| < \infty$ , and  $0 < \liminf c_n \leq \limsup c_n < \infty$ . Then

$$\frac{1}{N} \text{Tr} (Q_n(z) - m_n(z) I_N) \xrightarrow[n \rightarrow \infty]{a.s.} 0 \quad \text{and} \quad \frac{1}{n} \text{Tr} (\tilde{Q}_n(z) - \tilde{T}_n(z)) \xrightarrow[n \rightarrow \infty]{a.s.} 0 \quad (5)$$

for any  $z \in \mathbb{C}_+$ . When Assumptions 1 and 3 hold true,  $m_n(z)$  converges to  $\mathbf{m}(z)$  provided in the statement of Theorem 2.1 uniformly on the compact subsets of  $\mathbb{C}_+$ .

The function  $m_n(z) = \left( -z + \int t (1 + c_n m_n(z) t)^{-1} \nu_n(dt) \right)^{-1}$  is a ‘‘finite horizon’’ analogue of  $\mathbf{m}(z)$ . Since  $N^{-1} \text{Tr} Q_n(z)$  is the Stieltjes Transform of the spectral measure  $\theta_n$ , Convergence (4) stems from the first convergence in (5). Note that  $n^{-1} \text{Tr} \tilde{Q}_n(z)$  is the Stieltjes Transform of  $c_n \theta_n + (1 - c_n) \delta_0$ . Hence Convergence (4) can also be deduced from the second convergence in (5).

We now consider the additive deformation  $P_n$ :

**Assumption 4.** *The deterministic  $N \times n$  matrices  $P_n$  have a fixed rank equal to  $r$ . Moreover,  $\mathbf{p}_{\max} = \sup_n \|P_n\| < \infty$ .*

In order for some of the eigenvalues of  $\Sigma_n \Sigma_n^*$  to converge to values outside  $\text{supp}(\mu)$ , an extra assumption involving in some sense the interaction between  $P_n$  and  $D_n$  is needed.

Let  $P_n = U_n R_n^*$  be a factorization of  $P_n$  where  $U_n$  is an isometry matrix of size  $N \times r$ , e.g., a QR factorization. Define the  $r \times r$  Hermitian nonnegative matrix-valued measure  $\Lambda_n$  as

$$\Lambda_n = R_n^* \begin{bmatrix} \delta_{d_1^n} & & \\ & \ddots & \\ & & \delta_{d_r^n} \end{bmatrix} R_n. \quad (6)$$

It is clear that the support of  $\Lambda_n$  is included in  $[0, \mathbf{d}_{\max}]$  and that  $\Lambda_n([0, \mathbf{d}_{\max}]) \leq \mathbf{p}_{\max}^2 I_r$ . Since the sequence  $\Lambda_n([0, \mathbf{d}_{\max}])$  is bounded in norm, for every sequence of integers increasing to infinity, there is a subsequence  $n_k$  and a nonnegative finite measure  $\Lambda_*$  such that  $\int f d\Lambda_{n_k} \rightarrow \int f d\Lambda_*$  for every function  $f \in \mathcal{C}([0, \mathbf{d}_{\max}])$ , with  $\mathcal{C}([0, \mathbf{d}_{\max}])$  the set of continuous functions on  $[0, \mathbf{d}_{\max}]$ . This fact is a straightforward extension of its analogue for scalar measures.

**Assumption 5.** *Any two accumulation points  $\Lambda_1$  and  $\Lambda_2$  of the sequences  $\Lambda_n$  satisfy  $\Lambda_1(dx) = U \Lambda_2(dx) U^*$  where  $U$  is a unitary matrix.*

An equivalent statement of this assumption, perhaps more easily verifiable in some applications, is the following: there exists a sequence of factorizations  $P_n = U_n R_n^*$  such that for every function  $f \in \mathcal{C}([0, \mathbf{d}_{\max}])$ , the sequence of measures  $\Lambda_n$  associated to these factorizations by (6) satisfies  $\int f d\Lambda_n \rightarrow \int f d\Lambda_*$  where  $\Lambda_*$  is a matrix-valued nonnegative finite measure.

It is shown in [37] that the limiting spectral measure  $\mu$  has a continuous density on  $\mathbb{R}^* = \mathbb{R} - \{0\}$  (see Prop. 3.1 below). Our first order result addresses the problem of the presence of isolated eigenvalues of  $\Sigma_n \Sigma_n^*$  in any compact interval outside the support of this density. Of prime importance will be the  $r \times r$  matrix functions

$$H_*(z) = \int \frac{\mathbf{m}(z)}{1 + \mathbf{cm}(z)t} \Lambda_*(dt)$$

where  $\Lambda_*$  is an accumulation point of a sequence  $\Lambda_n$ . Since  $|1 + \mathbf{cm}(z)t| = |z(1 + \mathbf{cm}(z)t)/z| \geq |\Im(z(1 + \mathbf{cm}(z)t))|/|z| \geq \Im(z)/|z|$  on  $\mathbb{C}_+$ , the function  $H_*(z)$  is analytic on  $\mathbb{C}_+$ . It is further easy to show that  $\Im(H_*(z)) \geq 0$  and  $\Im(zH_*(z)) \geq 0$  on  $\mathbb{C}_+$ , and  $\sup_{y>0} \|yH_*(iy)\| < \infty$ . Hence  $H_*(z)$  is the Stieltjes Transform of a matrix-valued nonnegative finite measure carried by  $[0, \infty)$ . Note also that, under Assumption 5, the eigenvalues of  $H_*(z)$  remain unchanged if  $\Lambda_*$  is replaced by another accumulation point.

**Theorem 2.2.** *Let Assumptions 1, 2 and 3 hold true. Denote by  $(\hat{\lambda}_i^n)_{i=1, \dots, N}$  the eigenvalues of  $\Sigma_n \Sigma_n^*$ . Let  $(a, b)$  be an interval in  $\text{supp}(\mu)^c = \mathbb{R} - \text{supp}(\mu)$  such that  $a$  belongs to the boundary  $\partial \text{supp}(\mu)$  or  $a = 0$ , and  $b \in \partial \text{supp}(\mu)$  or  $b = \infty$ . Then the following facts hold true:*

- (1) *Let  $(P_n)$  be a sequence satisfying Assumptions 4 and 5. Given an accumulation point  $\Lambda_*$  of a sequence  $\Lambda_n$ , let  $H_*(z) = \int \mathbf{m}(z)(1 + \mathbf{cm}(z)t)^{-1} \Lambda_*(dt)$ . Then  $H_*(z)$  can be analytically extended to  $(a, b)$  where its values are Hermitian matrices, and the extension is increasing in the order of Hermitian*

matrices on  $(a, b)$ . The function  $\mathcal{D}(x) = \det(H_*(x) + I_r)$  has at most  $r$  zeros on  $(a, b)$ . Let  $\rho_1, \dots, \rho_k$ ,  $k \leq r$  be these zeros counting multiplicities. If  $k \geq 1$ , let  $[a', b'] \subset (a, b)$  be such that  $[\rho_1, \rho_k] \subset (a', b')$ . If  $k = 0$ , let  $[a', b']$  be any closed interval in  $(a, b)$ . Define the functions  $C(x)$  and  $\widehat{C}_n(x)$  on  $[a', b']$  as

$$C(x) = \#\{i : \rho_i \leq x\} \quad \text{and} \quad \widehat{C}_n(x) = \#\{i : \hat{\lambda}_i^n \in [a', b'] \text{ and } \hat{\lambda}_i^n \leq x\}.$$

Then

$$\int_{a'}^{b'} \left| \widehat{C}_n(x) - C(x) \right| dx \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

- (2) If  $b = \inf(\text{supp}(\mu) - \{0\})$ , then for any positive  $b' < b$  and for any sequence of matrices  $(P_n)$  satisfying Assumption 4,

$$\#\{i : \hat{\lambda}_i^n \in (0, b']\} = 0 \text{ with probability 1 for large } n.$$

- (3) Assume  $a > \inf(\text{supp}(\mu) - \{0\})$ . Then for any  $\rho_1 \leq \dots \leq \rho_r$  in  $(a, b)$ , there exists a sequence of matrices  $P_n$  satisfying Assumptions 4 and 5 for which  $\int_{a'}^{b'} |\widehat{C}_n(x) - C(x)| dx \xrightarrow{a.s.} 0$ , where  $[a', b'] \subset (a, b)$  and  $[\rho_1, \rho_r] \subset (a', b')$ , and where  $C(x)$  and  $\widehat{C}_n(x)$  are defined as above.

Hence, for  $n$  large,  $\Sigma_n \Sigma_n^*$  cannot have isolated eigenvalues before the first ‘‘bulk’’ of eigenvalues, *i.e.*, the first interval of the support of  $\mu$ . Alternatively, between any two bulks or after the last bulk of this support, the number of isolated eigenvalues of  $\Sigma_n \Sigma_n^*$  can reach the rank of the additive deformation.

It would be useful to complete the results of this theorem by specifying the indices of the isolated eigenvalues that appear between the bulks. This study (not done in this paper) may be done by following the ideas of [11] or [38] relative to the so called separation of the eigenvalues of  $\Sigma_n \Sigma_n^*$ . Another approach dealing with the same kind of problem is developed in [4].

In a few words, the proof of Theorem 2.2 consists in showing that the isolated eigenvalues of  $\Sigma_n \Sigma_n^*$  in  $(a, b)$  are close for large  $n$  to the zeros of  $\det(G_n(x) + I_N)$  in this interval, where  $G_n(x)$  is the analytic extension of  $G_n(z) = m_n(z)P_n(I + c_n m_n(z)D_n)^{-1}P_n^*$  to  $(a, b)$ . Performing a factorization  $P_n = U_n R_n^*$  where  $U_n$  is an isometry matrix of size  $N \times r$ , we observe that  $G_n(z)$  can be represented in an orthonormal basis of  $\mathbb{C}^N$  whose first vectors are the columns of  $U_n$  as  $\begin{bmatrix} H_n(z) & 0 \\ 0 & 0 \end{bmatrix}$  where  $H_n(z) = m_n(z)R_n^*(I + c_n m_n(z)D_n)^{-1}R_n$ . These functions can be written as

$$H_n(z) = \int \frac{m_n(z)}{1 + c_n m_n(z)t} \Lambda_n(dt), \quad (7)$$

and we shall show that they admit the  $H_*(z)$  as uniform limits on the compact subsets of  $\mathbb{C} - \text{supp}(\mu)$ . In practice, the isolated eigenvalues for a given large  $n$  can be approximated by the zeros of  $\det(m_n(x)R_n^*(I + c_n m_n(x)D_n)^{-1}R_n + I_r)$  in  $(a, b)$ .

A scenario of practical importance at least in the fields of signal processing and wireless communications is provided by the following assumption. Recall that  $\nu$  is the probability measure given by Assumption 3:

**Assumption 6.** *The accumulation points  $\Lambda_*$  are of the form  $\nu(dt) \times U\Omega U^*$  where*

$$\Omega = \begin{bmatrix} \omega_1^2 I_{j_1} & & \\ & \ddots & \\ & & \omega_t^2 I_{j_t} \end{bmatrix} > 0, \quad \omega_1^2 > \dots > \omega_t^2, \quad j_1 + \dots + j_t = r$$

and where  $U$  is a unitary matrix.

In wireless communications, the  $\omega_i^2$  typically represent the powers of radio sources transmitting signals to an antenna array.

Observe from Proposition 2.1 that  $\int (1 + \mathbf{cm}(z)t)^{-1} \nu(dt) = -z \lim(n^{-1} \text{Tr} \tilde{T}_n(z)) = -cz\mathbf{m}(z) + 1 - c$ . Consequently, in this particular case,  $H_*(x)$  is unitarily equivalent to  $-\mathbf{m}(x)(cx\mathbf{m}(x) - 1 + c)\Omega$  on  $(a, b)$ . This brings the following corollary of Theorem 2.2:

**Corollary 2.1.** *Assume the setting of Theorem 2.2-(1), and let Assumption 6 hold true. Then the function  $g(x) = \mathbf{m}(x)(cx\mathbf{m}(x) - 1 + c)$  is decreasing on  $(a, b)$ . Depending on the value of  $\omega_\ell^2$ ,  $\ell = 1, \dots, t$ , the equation  $\omega_\ell^2 g(x) = 1$  has either zero or one solution in  $(a, b)$ . Denoting by  $\rho_1, \dots, \rho_s$  these solutions counting multiplicities,  $\int_{a'}^{b'} |\widehat{C}_n(x) - C(x)| dx \xrightarrow{a.s.} 0$ , where  $a', b', \widehat{C}_n(x)$  and  $C(x)$  are built as in the statement of Theorem 2.2-(1).*

We now turn to the second order result. This result will be stated in the simple and practical framework of Assumption 6. Actually, a stronger assumption is needed:

**Assumption 7.** *The following facts hold true:*

$$\sup_n \sqrt{n} |c_n - c| < \infty,$$

$$\limsup_n \sqrt{n} \left| \int \frac{1}{t-x} \nu_n(dt) - \int \frac{1}{t-x} \nu(dt) \right| < \infty \text{ for all } x \in \mathbb{R} - \text{supp}(\nu).$$

Moreover, there exists a sequence of factorizations of  $P_n$  such that the measures  $\Lambda_n$  associated with these factorizations by (6) converge to  $\nu(dt) \times \Omega$  and such that

$$\limsup_n \sqrt{n} \left\| \int \frac{1}{t-x} \Lambda_n(dt) - \int \frac{1}{t-x} \nu(dt) \times \Omega \right\| < \infty \text{ for all } x \in \mathbb{R} - \text{supp}(\nu)$$

We recall that a GUE matrix (i.e., a matrix taken from the Gaussian Unitary Ensemble) is a random Hermitian matrix  $G$  such that  $G_{ii} \sim \mathcal{N}(0, 1)$ ,  $\Re(G_{ij}) \sim \mathcal{N}(0, 1/2)$  and  $\Im(G_{ij}) \sim \mathcal{N}(0, 1/2)$  for  $i < j$ , and such that all these random variables are independent.

**Theorem 2.3.** *Let Assumptions 1-7 hold true. Let  $g$  be the function defined in the statement of Corollary 2.1 and let  $B_\mu = \sup(\text{supp}(\mu))$ . Assume that the equations  $\omega_\ell^2 g(x) = 1$  with  $x \in (B_\mu, \infty)$ , have a solution for each  $\ell \in \{1, \dots, p\}$ ,  $p \leq 1$ , denoted  $\rho_1 > \dots > \rho_p$  (with multiplicities  $j_1, \dots, j_p$ ), respectively. Let  $\Lambda_n$  be any sequence of measures satisfying the third item of Assumption 7, and let  $H_n(z)$  be the matrix function associated with  $\Lambda_n$  by (7). Denote by  $H_{1,n}(z), \dots, H_{p,n}(z)$  the first  $p$  upper left diagonal blocks of  $H_n(z)$ , where  $H_{i,n}(z) \in \mathbb{C}^{j_i \times j_i}$ . Then the following facts hold true:*

- $\Delta(\rho_i) = 1 - c \int \left( \frac{\mathbf{m}(\rho_i)t}{1 + \mathbf{cm}(\rho_i)t} \right)^2 \nu(dt)$  is positive for every  $i = 1, \dots, p$ .



- $\limsup_n \|\sqrt{n}(H_{i,n}(\rho_i) + I_{j_i})\| < \infty$  for every  $i = 1, \dots, p$ .

For  $i = 1, \dots, p$ , let

$$M_i^n = \sqrt{n} \left( \begin{bmatrix} \hat{\lambda}_{j_1+\dots+j_{i-1}+1}^n \\ \vdots \\ \hat{\lambda}_{j_1+\dots+j_i}^n \end{bmatrix} - \rho_i \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)$$

where  $j_0 = 0$  and where the eigenvalues  $\hat{\lambda}_i^n$  of  $\Sigma_n \Sigma_n^*$  are arranged in decreasing order. Let  $G_1, \dots, G_p$  be independent GUE matrices such that  $G_i$  is a  $j_i \times j_i$  matrix. Then, for any bounded continuous  $f: \mathbb{R}^{j_1+\dots+j_p} \rightarrow \mathbb{R}$ ,

$$\mathbb{E} [f(M_1^n, \dots, M_p^n)] - \mathbb{E} [f(\Xi_1^n, \dots, \Xi_p^n)] \xrightarrow{n \rightarrow \infty} 0$$

where  $\Xi_i^n \in \mathbb{R}^{j_i}$  is the random vector of the decreasingly ordered eigenvalues of the matrix

$$\frac{1}{\omega_i^2 g(\rho_i)'} (\alpha_i G_i + \sqrt{n} (H_{i,n}(\rho_i) + I_{j_i})),$$

where

$$\alpha_i^2 = \frac{\mathbf{m}^2(\rho_i)}{\Delta(\rho_i)} \left[ \int \frac{t^2 + 2\omega_i^2 t}{(1 + \mathbf{cm}(\rho_i)t)^2} \nu(dt) + c \left( \int \frac{\rho_i \omega_i^2 \mathbf{m}(\rho_i)t}{(1 + \mathbf{cm}(\rho_i)t)^2} \nu(dt) \right)^2 \right].$$

Before turning to the proofs, some comments can be useful. We shall see that the results of Theorems 2.2 and 2.3 are intimately related to the first and second order behaviors of bilinear forms of the type  $u_n^* Q_n(x) v_n$ ,  $\tilde{u}_n^* \tilde{Q}_n(x) \tilde{v}_n$ , and  $n^{-1/2} u_n^* Y_n \tilde{Q}_n(x) \tilde{v}_n$  where  $u_n$ ,  $v_n$ ,  $\tilde{u}_n$  and  $\tilde{v}_n$  are deterministic vectors of bounded norm and of appropriate dimensions, and where  $x$  is a real number lying outside the support of  $\mu$ . In fact, it is possible to generalize Theorem 2.2 to the case where the elements of  $X_n$  are not necessarily Gaussian. This can be made possible by using the technique of [21] to analyze the first order behavior of these bilinear forms. On the other hand, the Gaussian assumption plays a central role in Theorem 2.3. Indeed, the proof of this theorem is based on the fact that these bilinear forms asymptotically fluctuate like Gaussian random variables when centered and scaled by  $\sqrt{n}$ . Take  $u_n = e_{1,N}$  and  $\tilde{v}_n = e_{1,n}$  where  $e_{k,m}$  is the  $k^{\text{th}}$  canonical vector of  $\mathbb{R}^m$ . Denoting by  $\tilde{q}_{ij}^n$  and by  $\tilde{t}_{ij}^n$  the  $(i, j)$  elements of  $\tilde{Q}_n(x)$  and  $\tilde{T}_n(x)$  respectively, we can informally write

$$e_{1,N}^* Y_n \tilde{Q}_n(x) e_{1,n} = \sum_{j=1}^n (d_j^n)^{1/2} \tilde{q}_{j1}^n X_{1j} \approx (d_1^n)^{1/2} \tilde{t}_{11}^n X_{11} + \sum_{j=2}^n (d_j^n)^{1/2} \tilde{q}_{j1}^n X_{1j}$$

following [21]. It can be shown furthermore that  $\tilde{t}_{11}^n = \mathcal{O}(1)$  for large  $n$  and that the sum  $\sum_{j=2}^n$  is tight. Hence,  $e_{1,N}^* Y_n \tilde{Q}_n(x) e_{1,n}$  is tight. However, when  $X_{11}$  is not Gaussian, we infer that  $e_{1,N}^* Y_n \tilde{Q}_n(x) e_{1,n}$  does not converge in general towards a Gaussian random variable. In this case, when  $P_n = \omega^2 e_{1,N} e_{1,n}^*$  (see Section 5), Theorem 2.3 no longer holds. Yet, we conjecture that the results of this Theorem can be recovered when  $e_{1,N}$  and  $e_{1,n}$  are replaced with vectors whose elements are “spread enough”, see [12] which deals with a similar problem.

**A word about the notations.** In the remainder of the paper, we shall often drop the subscript or the superscript  $n$  when there is no ambiguity. A constant bound that might change from an inequality to another but which is independent of  $n$  will always be denoted  $K$ . Element  $(i, j)$  of matrix  $M$  is denoted  $M_{ij}$  or  $[M]_{ij}$ . Element  $i$  of vector  $x$  is denoted  $[x]_i$ . As usual,  $A^*$  means the adjoint of a matrix or a vector  $A$ , and for a complex number  $z$ , we will use either  $\bar{z}$  or  $z^*$  to denote the complex conjugate of  $z$ . Convergence in probability will be denoted  $\xrightarrow{\mathcal{P}}$ , and convergence in distribution  $\xrightarrow{\mathcal{D}}$ .

### 3. MATHEMATICAL TOOLS AND USEFUL RESULTS

We start this section with some basic mathematical tools.

**3.1. Differentiation formulas.** Let  $\partial/\partial z = (\partial/\partial x - i\partial/\partial y)/2$  and  $\partial/\partial \bar{z} = (\partial/\partial x + i\partial/\partial y)/2$  for  $z = x + iy$ . Given a Hermitian matrix  $X$  with a spectral decomposition  $X = \sum_{\ell} \lambda_{\ell} v_{\ell} v_{\ell}^*$ , let  $\text{adj}(X) = \sum_k (\prod_{\ell \neq k} \lambda_{\ell}) v_k v_k^*$  be the classical adjoint of  $X$ , *i.e.*, the transpose of its cofactor matrix. Let  $\psi$  be a continuously differentiable real-valued function on  $\mathbb{R}$ . Then

$$\frac{\partial \det \psi(n^{-1}YY^*)}{\partial \bar{Y}_{ij}} = \frac{1}{n} [\text{adj}(\psi(n^{-1}YY^*)) \psi'(n^{-1}YY^*) y_j]_i$$

where  $y_j$  is column  $j$  of  $Y$ , see [19, Lemma 3.9] for a proof.

We shall also need the expressions of the following derivatives of the elements of the resolvents  $Q$  and  $\tilde{Q}$  (see [18]):

$$\frac{\partial Q_{pq}}{\partial \bar{Y}_{ij}} = -\frac{1}{n} [QY]_{pj} Q_{iq}, \quad \frac{\partial \tilde{Q}_{pq}}{\partial \bar{Y}_{ij}} = -\frac{1}{n} \tilde{Q}_{pj} [Y\tilde{Q}]_{iq}.$$

**3.2. Gaussian tools.** Our analysis fundamentally relies on two mathematical tools which are often used in the analysis of large random matrices with Gaussian elements. The first is the so called Integration by Parts (IP) formula for functionals of Gaussian vectors introduced in random matrix theory in [24, 30]. Let  $\Gamma : \mathbb{R}^{2Nn} \rightarrow \mathbb{C}$  be a continuously differentiable function polynomially bounded together with its partial derivatives. Then

$$\mathbb{E}(Y_{ij}\Gamma(Y)) = d_j \mathbb{E} \left[ \frac{\partial \Gamma(Y)}{\partial \bar{Y}_{ij}} \right]$$

for any  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, n\}$ . The second tool is the Poincaré-Nash inequality (see for instance [13]). In our situation, it states that the variance  $\text{Var}(\Gamma(Y))$  satisfies

$$\text{Var}(\Gamma(Y)) \leq \sum_{i=1}^N \sum_{j=1}^n d_j \mathbb{E} \left[ \left| \frac{\partial \Gamma(Y)}{\partial \bar{Y}_{ij}} \right|^2 + \left| \frac{\partial \Gamma(Y)}{\partial Y_{ij}} \right|^2 \right].$$

The results of Silverstein and Choi [37] relative to the support of  $\mu$  will be needed in the proof and recalled in the following paragraph for completeness. Close results can be found in [27] and in [28].

### 3.3. Analysis of the support of $\mu$ .

**Proposition 3.1** ([37], Th.1.1). *For all  $x \in \mathbb{R}^*$ ,  $\lim_{z \in \mathbb{C}_+ \rightarrow x} \mathbf{m}(z)$  exists. The limit that we denote  $\mathbf{m}(x)$  is continuous on  $\mathbb{R}^*$ . Moreover,  $\mu$  has a continuous density  $f$  on  $\mathbb{R}^*$  given by  $f(x) = \pi^{-1} \Im \mathbf{m}(x)$ .*

In [37], the support of  $\mu$  is also identified. Since  $\mathbf{m}(z)$  is the unique solution in  $\mathbb{C}_+$  of (3) for  $z \in \mathbb{C}_+$ , it has a unique inverse on  $\mathbb{C}_+$  given by

$$z(\mathbf{m}) = -\frac{1}{\mathbf{m}} + \int \frac{t}{1 + c\mathbf{m}t} \nu(dt)$$

The characterization of the support of  $\mu$  is based on the following idea. On any open interval of  $\text{supp}(\mu)^c$ ,  $\mathbf{m}(x) = \int (t-x)^{-1} \mu(dt)$  is a real, continuous and increasing function. Consequently, it has a real, continuous and increasing inverse. In [37], it is shown that the converse is also true. More precisely, let  $B = \{\mathbf{m} : \mathbf{m} \neq 0, -(c\mathbf{m})^{-1} \in \text{supp}(\nu)^c\}$ , and let

$$\begin{aligned} x &: B \longrightarrow \mathbb{R} \\ \mathbf{m} &\longmapsto x(\mathbf{m}) = -\frac{1}{\mathbf{m}} + \int \frac{t}{1 + c\mathbf{m}t} \nu(dt). \end{aligned} \quad (8)$$

Then the following proposition holds:

**Proposition 3.2** ([37], Th. 4.1 and 4.2). *For any  $x_0 \in \text{supp}(\mu)^c$ , let  $\mathbf{m}_0 = \mathbf{m}(x_0)$ . Then  $\mathbf{m}_0 \in B$ ,  $x_0 = x(\mathbf{m}_0)$ , and  $x'(\mathbf{m}_0) > 0$ . Conversely, let  $\mathbf{m}_0 \in B$  such that  $x'(\mathbf{m}_0) > 0$ . Then  $x_0 = x(\mathbf{m}_0) \in \text{supp}(\mu)^c$ , and  $\mathbf{m}(x_0) = \mathbf{m}_0$ .*

The following proposition will also be useful:

**Proposition 3.3** ([37], Th. 4.4). *Let  $[\mathbf{m}_1, \mathbf{m}_2]$  and  $[\mathbf{m}_3, \mathbf{m}_4]$  be two disjoint intervals of  $B$  satisfying  $\forall \mathbf{m} \in (\mathbf{m}_1, \mathbf{m}_2) \cup (\mathbf{m}_3, \mathbf{m}_4)$ ,  $x'(\mathbf{m}) > 0$ . Then  $[x_1, x_2]$  and  $[x_3, x_4]$  are disjoint where  $x_i = x(\mathbf{m}_i)$ .*

The following result is also proven in [37]:

**Proposition 3.4.** *Assume that  $\nu(\{0\}) = 0$ . Then  $\mu(\{0\}) = \max(0, 1 - c^{-1})$ .*

We shall assume hereafter that  $\nu(\{0\}) = 0$  without loss of generality (otherwise, it would be enough to change the value of  $c$ ). The two following lemmas will also be needed:

**Lemma 3.1.** *Let  $[x_1, x_2] \in \text{supp}(\mu)^c$  with  $x_1 > 0$ , and let  $D_{x_1, x_2}$  be the closed disk having  $[x_1, x_2]$  as one of its diameters. Then there exists a constant  $K$  which depends on  $x_1$  and  $x_2$  only such that*

$$\begin{aligned} \forall t \in \text{supp}(\nu), \forall z \in D_{x_1, x_2}, |1 + c\mathbf{m}(z)t| &\geq K, \quad \text{and} \\ \forall n \text{ large enough, } \forall t \in \text{supp}(\nu_n), \forall z \in D_{x_1, x_2}, |1 + c_n m_n(z)t| &\geq K. \end{aligned}$$

From the second inequality, we deduce that  $\tilde{T}_n(z)$  is analytic in a neighborhood of  $[x_1, x_2]$  for  $n$  large enough, and

$$\limsup_n \sup_{z \in D_{x_1, x_2}} \|\tilde{T}_n(z)\| < \infty. \quad (9)$$

*Proof.* When  $z \in \mathbb{C}_+$ ,  $\Im \mathbf{m}(z) > 0$  and  $\Im(-(\mathbf{cm}(z))^{-1}) > 0$ , and we have the opposite inequalities when  $\Im z < 0$ . Applying Proposition 3.2 for  $z \in [x_1, x_2]$ , we deduce that  $|\mathbf{m}(z)|$  and  $f(z) = \mathbf{d}(-(\mathbf{cm}(z))^{-1}, \text{supp}(\nu))$  are positive on  $D_{x_1, x_2}$ . Since these functions are continuous on this compact set,  $\min |\mathbf{m}(z)| = K_1 > 0$  and  $\min f(z) = K_2 > 0$  on  $D_{x_1, x_2}$ . Consequently, for any  $z \in D_{x_1, x_2}$  and any  $t \in \text{supp}(\nu)$ ,  $|1 + \mathbf{cm}(z)t| = |\mathbf{cm}(z)(-(\mathbf{cm}(z))^{-1} - t)| \geq |\mathbf{cm}(z)|f(z) \geq cK_1K_2 > 0$ . We now prove the second inequality. Denote by  $\mathbf{d}_H(A, B)$  the Hausdorff distance between two sets  $A$  and  $B$ . Let  $f_n(z) = \mathbf{d}(-(c_n m_n(z))^{-1}, \text{supp}(\nu_n))$ . We have

$$\begin{aligned} f_n(z) &\leq \mathbf{d}\left(\frac{-1}{c_n m_n(z)}, \frac{-1}{\mathbf{cm}(z)}\right) + \mathbf{d}\left(\frac{-1}{\mathbf{cm}(z)}, \text{supp}(\nu_n)\right) \\ &\leq \mathbf{d}\left(\frac{-1}{c_n m_n(z)}, \frac{-1}{\mathbf{cm}(z)}\right) + f(z) + \mathbf{d}_H(\text{supp}(\nu_n), \text{supp}(\nu)), \end{aligned}$$

and  $f(z) \leq \mathbf{d}(-(c_n m_n(z))^{-1}, -(\mathbf{cm}(z))^{-1}) + f_n(z) + \mathbf{d}_H(\text{supp}(\nu_n), \text{supp}(\nu))$  similarly. Since  $m_n(z)$  converges uniformly to  $\mathbf{m}(z)$  and  $\inf |\mathbf{m}(z)| > 0$  on  $D_{x_1, x_2}$ ,  $\mathbf{d}(-(c_n m_n(z))^{-1}, -(\mathbf{cm}(z))^{-1}) \rightarrow 0$  uniformly on this disk. By Assumption 3,  $\mathbf{d}_H(\text{supp}(\nu_n), \text{supp}(\nu)) \rightarrow 0$ . Hence  $f_n(z)$  converges uniformly to  $f(z)$  on  $D_{x_1, x_2}$  which proves the second inequality.  $\square$

**Lemma 3.2.** *In the setting of Lemma 3.1, for any sequence of vectors  $\tilde{u}_n \in \mathbb{C}^n$  such that  $\sup_n \|\tilde{u}_n\| < \infty$ , the quadratic forms  $\tilde{u}_n^* \tilde{T}_n(z) \tilde{u}_n$  are the Stieltjes Transforms of positive measures  $\gamma_n$  such that  $\sup_n \gamma_n(\mathbb{R}) < \infty$  and  $\gamma_n([x_1, x_2]) = 0$  for  $n$  large enough.*

Indeed, one can easily check the conditions that enable  $\tilde{u}_n^* \tilde{T}_n(z) \tilde{u}_n$  to be a Stieltjes Transform of a positive finite measure. The last result is obtained by analyticity in a neighborhood of  $[x_1, x_2]$ . In fact, it can be checked that  $\text{supp}(\gamma_n) \subset \text{supp}(\mu_n) \cup \{0\}$ .

**3.4. A Control over the support of  $\theta_n$ .** In this paragraph, we adapt to our case an idea developed in [11] to deal with Wigner matrices whose elements distribution satisfies a Poincaré-Nash inequality.

**Proposition 3.5.** *For any sequence of  $n \times n$  deterministic diagonal nonnegative matrices  $\tilde{U}_n$  such that  $\sup_n \|\tilde{U}_n\| < \infty$ ,*

$$\begin{aligned} \left| \frac{1}{n} \text{Tr} \mathbb{E} Q_n(z) - m_n(z) \right| &\leq \frac{P(|z|)R(|\Im(z)|^{-1})}{n^2}, \text{ and} \\ \left| \frac{1}{n} \text{Tr} \tilde{U}_n \mathbb{E} \tilde{Q}_n(z) - \frac{1}{n} \text{Tr} \tilde{U}_n \tilde{T}_n(z) \right| &\leq \frac{P(|z|)R(|\Im(z)|^{-1})}{n^2} \end{aligned}$$

for  $z \in \mathbb{C}_+$ , where  $P$  and  $R$  are polynomials with nonnegative coefficients independent of  $n$ .

This proposition is obtained from a simple extension of the results of [18, Th. 3 and Prop.5] from  $z \in (-\infty, 0)$  to  $z \in \mathbb{C}_+$ .

The following important result, due to Haagerup and Thorbjørnsen, is established in the proof of [17, Th.6.2]:

**Lemma 3.3.** *Assume that  $h(z)$  is an analytic function on  $\mathbb{C}_+$  that satisfies  $|h(z)| \leq P(|z|)R(|\Im(z)|^{-1})$  where  $P$  and  $R$  are polynomials with nonnegative coefficients.*

Then for any function  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ , the set of smooth real-valued functions with compact support in  $\mathbb{R}$ ,

$$\limsup_{y \downarrow 0} \left| \int_{\mathbb{R}} \varphi(x) h(x + iy) dx \right| < \infty.$$

Since  $N^{-1} \text{Tr} Q_n(z)$  is the Stieltjes Transform of the spectral measure  $\theta_n$ , the inversion formula (2) shows that

$$\int \varphi(t) \theta_n(dt) = \frac{1}{\pi} \lim_{y \downarrow 0} \Im \int \varphi(x) \frac{1}{N} \text{Tr} Q_n(x + iy) dx$$

for any function  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ . Using then Proposition 3.5 and Lemma 3.3, we obtain the following result:

**Proposition 3.6.** *For any function  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ ,*

$$\left| \mathbb{E} \int \varphi(t) \theta_n(dt) - \int \varphi(t) \mu_n(dt) \right| \leq \frac{K}{n^2}.$$

#### 4. FIRST ORDER ANALYSIS: PROOF OF THEOREM 2.2

In all this section,  $[x_1, x_2]$  is an interval in  $(a, b)$  where  $a$  and  $b$  are given in the statement of Theorem 2.2, and  $z$  is a complex number such that  $\Re(z) \in [x_1, x_2]$  and  $\Im(z)$  is arbitrary. Moreover,  $u_n, v_n \in \mathbb{C}^N$  and  $\tilde{u}_n, \tilde{v}_n \in \mathbb{C}^n$  are sequences of deterministic vectors such that  $\sup_n \max(\|u_n\|, \|v_n\|, \|\tilde{u}_n\|, \|\tilde{v}_n\|) < \infty$ , and  $\tilde{U}_n$  is a sequence of  $n \times n$  diagonal deterministic matrix such that  $\sup_n \|\tilde{U}_n\| < \infty$ .

We now introduce the regularization function alluded to in the introduction. Choose  $\varepsilon > 0$  small enough so that  $[x_1, x_2] \cap \mathcal{S}_\varepsilon = \emptyset$  where  $\mathcal{S}_\varepsilon = \{x \in \mathbb{R}, \mathbf{d}(x, \text{supp}(\mu) \cup \{0\}) \leq \varepsilon\}$ . Fix  $0 < \varepsilon' < \varepsilon$ , let  $\psi : \mathbb{R} \rightarrow [0, 1]$  be a continuously differentiable function such that

$$\psi(x) = \begin{cases} 1 & \text{if } x \in \mathcal{S}_{\varepsilon'} \\ 0 & \text{if } x \in \mathbb{R} - \mathcal{S}_\varepsilon \end{cases}$$

and let  $\phi_n = \det \psi(n^{-1} Y_n Y_n^*)$ . In all the subsequent derivations, quantities such as  $u_n^* Q_n(z) u_n$  or  $\tilde{u}_n^* \tilde{Q}_n(z) \tilde{u}_n$  for  $\Re(z) \in [x_1, x_2]$  will be multiplied by  $\phi_n$  in order to control their magnitudes when  $z$  is close to the real axis. By performing this regularization as is done in [19], we shall be able to define and control the moments of random variables such as  $\phi_n u_n^* Q_n(z) u_n$  or  $\phi_n \tilde{u}_n^* \tilde{Q}_n(z) \tilde{u}_n$  with the help of the Gaussian tools introduced in Section 3.2.

We start with a series of lemmas. The first of these lemmas relies on Proposition 3.6 and on the Poincaré-Nash inequality. Its detailed proof is a minor modification of the proof of [19, Lemma 3] and is therefore omitted:

**Lemma 4.1.** *Given  $0 < \varepsilon' < \varepsilon$ , let  $\varphi$  be a smooth nonnegative function equal to zero on  $\mathcal{S}_{\varepsilon'}$  and to one on  $\mathbb{R} - \mathcal{S}_\varepsilon$ . Then for any  $\ell \in \mathbb{N}$ , there exists a constant  $K_\ell$  for which*

$$\mathbb{E} \left[ \left( \text{Tr} \varphi(n^{-1} Y_n Y_n^*) \right)^\ell \right] \leq \frac{K_\ell}{n^\ell}.$$

**Remark 1.** *Notice that this lemma proves Theorem 2.1-(2). The proof provided in [3] is in fact more general, being not restricted to the Gaussian case.*

**Lemma 4.2.** *For any  $\ell \in \mathbb{N}$ , the following holds true:*

$$\mathbb{E} \left[ \left( \sum_{i,j=1}^{N,n} d_j \left| \frac{\partial \phi_n}{\partial \bar{Y}_{ij}} \right|^2 \right)^\ell \right] \leq \frac{K_\ell}{n^{2\ell}}.$$

*Proof.* Letting  $n^{-1/2}Y = W \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_N})V^*$  be a singular value decomposition of  $n^{-1/2}Y$ , we have

$$\operatorname{adj} \left( \psi \left( \frac{YY^*}{n} \right) \right) \psi' \left( \frac{YY^*}{n} \right) \frac{Y}{\sqrt{n}} = W \Xi V^* \text{ where } \Xi = \operatorname{diag} \left( \sqrt{\lambda_k} \psi'(\lambda_k) \prod_{\ell \neq k} \psi(\lambda_\ell) \right)_{k=1}^N$$

and we observe that  $\operatorname{Tr} \Xi^2 \leq K Z_n$  where  $Z_n = \#\{k : \lambda_k \in \mathcal{S}_\varepsilon - \mathcal{S}_{\varepsilon'}\}$ . Using the first identity in Section 3.1 and recalling that  $|\operatorname{Tr}(AB)| \leq \|A\| \operatorname{Tr} B$  when  $A$  is a square matrix and  $B$  is a Hermitian nonnegative matrix, we have

$$\mathbb{E} \left[ \left( \sum_{i,j=1}^{N,n} d_j \left| \frac{\partial \phi_n}{\partial \bar{Y}_{ij}} \right|^2 \right)^\ell \right] = \frac{1}{n^\ell} \mathbb{E} \left[ \left( \operatorname{Tr} \left( \operatorname{adj}(\psi) \psi' \frac{YDY^*}{n} \operatorname{adj}(\psi) \psi' \right) \right)^\ell \right] \leq \frac{K}{n^\ell} \mathbb{E} Z_n^\ell$$

and the result follows from Lemma 4.1 with a proper choice of  $\varphi$ .  $\square$

**Lemma 4.3.** *The following inequalities hold true:*

$$\begin{aligned} \mathbb{E} |\phi_n u_n^* Q_n(z) v_n - \mathbb{E}[\phi_n u_n^* Q_n(z) v_n]|^4 &\leq \frac{K}{n^2}, \\ \mathbb{E} |\phi_n \tilde{u}_n^* \tilde{Q}_n(z) \tilde{v}_n - \mathbb{E}[\phi_n \tilde{u}_n^* \tilde{Q}_n(z) \tilde{v}_n]|^4 &\leq \frac{K}{n^2}, \\ \operatorname{Var}(\phi_n \operatorname{Tr} Q_n(z)) &\leq K. \end{aligned}$$

*Proof.* We shall only prove the first inequality. By the polarization identity, this inequality is shown whenever we show that  $\mathbb{E} |\phi u^* Qu - \mathbb{E}[\phi u^* Qu]|^4 \leq K/n^2$ . Let us start by showing that  $\operatorname{Var}(\phi u^* Qu) \leq K/n$ . By the Poincaré-Nash inequality, we have

$$\begin{aligned} \operatorname{Var}(\phi u^* Qu) &\leq 2 \sum_{i,j=1}^{N,n} d_j \mathbb{E} \left| \frac{\partial \phi u^* Qu}{\partial \bar{Y}_{ij}} \right|^2 \\ &\leq 4 \sum_{i,j=1}^{N,n} d_j \mathbb{E} \left| \phi \frac{\partial u^* Qu}{\partial \bar{Y}_{ij}} \right|^2 + 4 \sum_{i,j=1}^{N,n} d_j \mathbb{E} \left| u^* Qu \frac{\partial \phi}{\partial \bar{Y}_{ij}} \right|^2. \end{aligned}$$

Using the expression of  $\partial Q_{pq}/\partial \bar{Y}_{ij}$  in Section 3.1, we have

$$\partial u^* Qu / \partial \bar{Y}_{ij} = -n^{-1} u^* Q y_j [Qu]_i,$$

hence

$$\sum_{i,j=1}^{N,n} d_j \mathbb{E} \left| \phi \frac{\partial u^* Qu}{\partial \bar{Y}_{ij}} \right|^2 = \frac{1}{n} \mathbb{E} \left[ \phi^2 u^* Q \frac{YDY^*}{n} Qu u^* Q^2 u \right] \leq \frac{K}{n}$$

since the argument of the expectation is bounded for  $\Re(z) \in [x_1, x_2]$ . From the first identity in Section 3.1,  $\sum_{i,j} d_j \mathbb{E} |u^* Qu \partial \phi / \partial \bar{Y}_{ij}|^2 \leq K \sum_{i,j} d_j \mathbb{E} |\partial \phi / \partial \bar{Y}_{ij}|^2$  which

is bounded by  $K/n^2$  by Lemma 4.2. It results that  $\text{Var}(\phi u^* Qu) \leq K/n$ . Now, writing  $\overset{\circ}{X} = X - \mathbb{E}X$ ,

$$\mathbb{E} \left| \overset{\circ}{\phi u^* Qu} \right|^4 = (\text{Var}(\phi u^* Qu))^2 + \text{Var} \left( \left( \overset{\circ}{\phi u^* Qu} \right)^2 \right) \leq K/n^2 + \text{Var} \left( \left( \overset{\circ}{\phi u^* Qu} \right)^2 \right).$$

By the Poincaré-Nash inequality,

$$\begin{aligned} \text{Var} \left( \left( \overset{\circ}{\phi u^* Qu} \right)^2 \right) &\leq 2 \sum_{i,j=1}^{N,n} d_j \mathbb{E} \left| \partial \left( \overset{\circ}{\phi u^* Qu} \right)^2 / \partial \bar{Y}_{ij} \right|^2 \\ &\leq 16 \sum_{i,j=1}^{N,n} d_j \mathbb{E} \left| \overset{\circ}{\phi u^* Qu} \phi \frac{\partial u^* Qu}{\partial \bar{Y}_{ij}} \right|^2 + 16 \sum_{i,j=1}^{N,n} d_j \mathbb{E} \left| \overset{\circ}{\phi u^* Qu} u^* Qu \frac{\partial \phi}{\partial \bar{Y}_{ij}} \right|^2 := V_1 + V_2. \end{aligned}$$

By developing the derivative in  $V_1$  similarly to above,  $V_1 \leq Kn^{-1} \mathbb{E} \left| \overset{\circ}{\phi u^* Qu} \right|^2 \leq Kn^{-2}$ . By the Cauchy-Schwarz inequality and Lemma 4.2,

$$V_2 \leq K \sum_{i,j=1}^{N,n} d_j \mathbb{E} \left| \overset{\circ}{\phi u^* Qu} \frac{\partial \phi}{\partial \bar{Y}_{ij}} \right|^2 \leq \frac{K}{n^2} \left( \mathbb{E} \left| \overset{\circ}{\phi u^* Qu} \right|^4 \right)^{1/2}.$$

Writing  $a_n = n^2 \mathbb{E} \left| \overset{\circ}{\phi u^* Qu} \right|^4$ , we have shown that  $\sqrt{a_n} \leq K/\sqrt{a_n} + K/n$ . Assume that  $a_n$  is not bounded. Then there exists a sequence  $n_k$  of integers such that  $a_{n_k} \rightarrow \infty$ , which raises a contradiction. The first inequality in the statement of this lemma is shown. The other two inequalities can be shown similarly.  $\square$

**Lemma 4.4.** *The following holds true:*

$$1 - \mathbb{E}\phi_n \leq \frac{K_\ell}{n^\ell} \quad \text{for any } \ell \in \mathbb{N}.$$

*Proof.* For  $0 < \varepsilon_1 < \varepsilon'$  where  $\varepsilon'$  is defined in the construction of  $\psi$ , let  $\varphi$  be a smooth nonnegative function equal to zero on  $\mathcal{S}_{\varepsilon_1}$  and to one on  $\mathbb{R} - \mathcal{S}_{\varepsilon'}$ . Then  $1 - \phi_n \leq (\text{Tr}(\varphi(n^{-1}YY^*)))^\ell$  for any  $\ell \in \mathbb{N}$ , and the result stems from Lemma 4.1.  $\square$

**Lemma 4.5.** *The following inequalities hold true (recall that  $\Re(z) \notin \text{supp}(\mu)$ ):*

$$\left| \mathbb{E}[\phi_n \text{Tr} Q_n(z)] - Nm_n(z) \right| \leq \frac{K}{n}, \quad \text{and} \quad \left| \text{Tr} \tilde{U}_n \left( \mathbb{E}[\phi_n \tilde{Q}_n(z)] - \tilde{T}_n(z) \right) \right| \leq \frac{K}{n}.$$

*Proof.* Let  $\varepsilon$  be defined in the construction of  $\psi$ . Choose a small  $\varepsilon_1 > \varepsilon$  in such a way that  $\mathcal{S}_{\varepsilon_1} \cap [x_1, x_2] = \emptyset$ . Let  $\zeta$  be a  $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$  nonnegative function equal to one on  $\mathcal{S}_\varepsilon$  and to zero on  $\mathbb{R} - \mathcal{S}_{\varepsilon_1}$ , so that

$$\phi \frac{1}{n} \text{Tr} Q = \phi \int \frac{\zeta(t)}{t-z} \theta_n(dt).$$

Using this equality, and recalling that  $\phi \in [0, 1]$ , we have

$$\left| \mathbb{E} \phi \frac{1}{n} \text{Tr} Q - \mathbb{E} \int \frac{\zeta(t)}{t-z} \theta_n(dt) \right| \leq \mathbb{E} \left[ (1 - \phi) \left| \int \frac{\zeta(t)}{t-z} \theta_n(dt) \right| \right] \leq \frac{1 - \mathbb{E}\phi}{d(z, \mathcal{S}_{\varepsilon_1})} \leq \frac{K_\ell}{n^\ell}$$

for any  $\ell \in \mathbb{N}$ . Moreover, we have

$$\left| \mathbb{E} \int \frac{\zeta(t)}{t-z} \theta_n(dt) - m_n(z) \right| = \left| \mathbb{E} \int \frac{\zeta(t)}{t-z} \theta_n(dt) - \int \frac{\zeta(t)}{t-z} \mu_n(dt) \right| \leq \frac{K}{n^2}$$

by Proposition 3.3, and the first inequality is proved.

By performing a spectral factorization of  $n^{-1}Y^*Y$ , one can check that  $n^{-1} \text{Tr} \tilde{U} \tilde{Q}(z)$  is the Stieltjes Transform of a positive measure  $\tau_n$  such that  $\sup_n \tau_n(\mathbb{R}) < \infty$  and  $\text{supp}(\tau_n) \subset \text{supp}(\theta_n) \cup \{0\}$ . By Lemma 3.2,  $n^{-1} \text{Tr} \tilde{U} \tilde{T}(z)$  is the Stieltjes Transform of a positive measure  $\gamma_n$  such that  $\sup_n \gamma_n(\mathbb{R}) < \infty$  and  $\gamma_n([x_1, x_2]) = 0$  for all large  $n$ . With the help of the second inequality of Proposition 3.5, we have a result similar to that of Proposition 3.6, namely that  $|\mathbb{E} \int \varphi d\tau_n - \int \varphi d\gamma_n| \leq K/n^2$  for any function  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ . We can then prove the second inequality similarly to the first one.  $\square$

**Lemma 4.6.** *The following inequalities hold true:*

$$\begin{aligned} |\mathbb{E}[\phi_n u_n^* Q_n(z) v_n] - u_n^* v_n m_n(z)| &\leq K/n, \\ |\mathbb{E}[\phi_n \tilde{u}_n^* \tilde{Q}_n(z) \tilde{v}_n] - \tilde{u}_n^* \tilde{T}_n(z) \tilde{v}_n| &\leq K/n. \end{aligned}$$

In [21], it is proven in a more general setting that  $|\mathbb{E} u_n^* Q_n(z) u_n - \|u_n\|^2 m_n(z)| \leq P(|z|) R(|\Im(z)|^{-1}) / \sqrt{n}$  for any  $z \in \mathbb{C}_+$ . Observing that  $u_n^* Q_n(z) u_n$  and  $\|u_n\|^2 m_n(z)$  are Stieltjes Transforms of positive measures, and mimicking the proof of the previous lemma, we can establish this lemma with the rate  $\mathcal{O}(n^{-1/2})$ , which is in fact enough for our purposes. However, in order to give a flavor of the derivations that will be carried out in the next section, we consider here another proof that uses the IP formula and the Poincaré-Nash inequality. To that end, we introduce new notations:

$$\begin{aligned} \beta(z) &= \phi_n \frac{1}{n} \text{Tr} Q_n(z), \quad \alpha(z) = \mathbb{E} \beta(z), \quad \hat{\beta}(z) = \beta(z) - \phi_n \alpha(z), \quad \text{and} \\ \tilde{\alpha}(z) &= \frac{1}{n} \text{Tr} D_n [-z(I_n + \alpha(z) D_n)]^{-1}. \end{aligned}$$

*Proof.* We start with the first inequality. By the IP formula, we have

$$\mathbb{E}[Q_{pi} Y_{ij} \bar{Y}_{\ell j} \phi] = -\frac{d_j}{n} \mathbb{E}[[Qy_j]_p Q_{ii} \bar{Y}_{\ell j} \phi] + \delta(\ell-i) d_j \mathbb{E}[Q_{pi} \phi] + \frac{d_j}{n} \mathbb{E}[Q_{pi} \bar{Y}_{\ell j} [\text{adj}(\psi) \psi' y_j]_i].$$

Taking the sum over  $i$ , we obtain

$$\mathbb{E}[[Qy_j]_p \bar{Y}_{\ell j} \phi] = -d_j \mathbb{E}[[Qy_j]_p \bar{Y}_{\ell j} \beta] + d_j \mathbb{E}[Q_{p\ell} \phi] + \frac{d_j}{n} \mathbb{E}[\bar{Y}_{\ell j} [Q \text{adj}(\psi) \psi' y_j]_p].$$

Writing  $\beta = \hat{\beta} + \phi \alpha$ , we get

$$\begin{aligned} \mathbb{E}[[Qy_j]_p \bar{Y}_{\ell j} \phi] &= \frac{d_j}{1 + \alpha d_j} \mathbb{E}[Q_{p\ell} \phi] - \frac{d_j}{1 + \alpha d_j} \mathbb{E}[[Qy_j]_p \bar{Y}_{\ell j} \hat{\beta}] \\ &\quad + \frac{d_j}{n(1 + \alpha d_j)} \mathbb{E}[\bar{Y}_{\ell j} [Q \text{adj}(\psi) \psi' y_j]_p]. \end{aligned}$$



Taking the sum over  $j$ , we obtain

$$\begin{aligned} \mathbb{E} \left[ \left[ Q \frac{YY^*}{n} \right]_{p\ell} \phi \right] &= -z\tilde{\alpha}\mathbb{E}[Q_{p\ell}\phi] - \mathbb{E} \left[ \hat{\beta} \left[ Q \frac{YD(I + \alpha D)^{-1}Y^*}{n} \right]_{p\ell} \right] \\ &\quad + \frac{1}{n}\mathbb{E} \left[ \left[ Q \operatorname{adj}(\psi)\psi' \frac{YD(I + \alpha D)^{-1}Y^*}{n} \right]_{p\ell} \right]. \end{aligned}$$

We now use the identity  $zQ = n^{-1}QYY^* - I$ , which results in

$$\begin{aligned} z\mathbb{E}[Q_{p\ell}\phi] &= \mathbb{E} \left[ \left[ Q \frac{YY^*}{n} \right]_{p\ell} \phi \right] - \delta(p - \ell)\mathbb{E}[\phi], \\ \mathbb{E}[Q_{p\ell}\phi] &= \frac{\delta(p - \ell)}{-z(1 + \tilde{\alpha})}\mathbb{E}[\phi] + \frac{\text{2nd and 3rd terms of next to last equation}}{z(1 + \tilde{\alpha})}. \end{aligned}$$

Multiplying each side by  $[u^*]_p[v]_\ell$  and taking the sum over  $p$  and  $\ell$ , we finally obtain

$$\begin{aligned} \mathbb{E}[u^*Qv\phi] &= \mathbb{E}[\phi] \frac{u^*v}{-z(1 + \tilde{\alpha})} - [-z(1 + \tilde{\alpha})]^{-1}\mathbb{E} \left[ \hat{\beta}u^*Q \frac{YD(I + \alpha D)^{-1}Y^*}{n} v \right] \\ &\quad + \frac{1}{n}[-z(1 + \tilde{\alpha})]^{-1}\mathbb{E} \left[ u^*Q \operatorname{adj}(\psi)\psi' \frac{YD(I + \alpha D)^{-1}Y^*}{n} v \right]. \end{aligned} \quad (10)$$

Let us evaluate the three terms at the right hand side of this equality. From Lemma 4.5, we have  $\alpha = c_n m_n + \mathcal{O}(n^{-2})$ . Using in addition the bound (9), we obtain  $\tilde{\alpha} = n^{-1} \operatorname{Tr}(D(-z(I + c_n m_n D + (\alpha - c_n m_n)D)^{-1})) = n^{-1} \operatorname{Tr} D\tilde{T} + \mathcal{O}(n^{-2})$ . Since  $m_n(z) = (-z(1 + n^{-1} \operatorname{Tr} D\tilde{T}(z)))^{-1}$ , we obtain that  $(-z(1 + \tilde{\alpha}))^{-1} = m_n(z) + \mathcal{O}(n^{-2})$ . Using in addition Lemma 4.4, we obtain that the first right hand side term of (10) is  $u^*v m_n(z) + \mathcal{O}(n^{-2})$ . Due to the presence of  $\phi$  in the expression of  $\hat{\beta}$ , the second term is bounded by  $K\mathbb{E}|\hat{\beta}|$ . Moreover,  $\hat{\beta} = n^{-1}\phi \operatorname{Tr} Q - n^{-1}\mathbb{E}[\phi \operatorname{Tr} Q] + (1 - \phi)n^{-1}\mathbb{E}[\phi \operatorname{Tr} Q]$ . By Lemmas 4.4 and 4.3,  $\mathbb{E}|\hat{\beta}| = \mathcal{O}(n^{-1})$ . The third term can be shown to be bounded by  $Kn^{-1}\mathbb{E} \operatorname{Tr} \varphi(n^{-1}YY^*) = \mathcal{O}(n^{-2})$  where  $\varphi$  is as in the statement of Lemma 4.1. This proves the first inequality in the statement of the lemma.

The second result in the statement of the lemma is proven similarly. The proof requires the second inequality of Lemma 4.5.  $\square$

The proof of the following lemma can be done along the same lines and will be omitted:

**Lemma 4.7.** *The following inequalities hold true:*

$$\begin{aligned} \left| \mathbb{E} \phi_n u_n^* Y_n \tilde{Q}_n(z) \tilde{v}_n \right| &\leq K/\sqrt{n} \\ \mathbb{E} \left| \phi_n u_n^* Y_n \tilde{Q}_n(z) \tilde{v}_n \right|^4 &\leq K. \end{aligned}$$

We now prove Theorem 2.2.

**Proof of Theorem 2.2-(1).** To prove this result, we follow the idea of [8, 9]. We start with a purely algebraic result. By Assumption 4, there exists a  $N \times r$  matrix  $A_n$  and a  $n \times r$  matrix  $B_n$ , both with rank  $r$  and bounded spectral norms, such that  $P_n = A_n B_n^*$  (at this step,  $A_n$  need not be an isometry matrix as was done before Equation (6)). Assume that  $x > 0$  is not an eigenvalue of  $n^{-1}Y_n Y_n^*$ . Then

$x$  is an eigenvalue of  $\Sigma_n \Sigma_n^*$  if and only if  $\det \widehat{S}_n(x) = 0$  where  $\widehat{S}_n(x)$  is the  $2r \times 2r$  matrix

$$\widehat{S}_n(x) = \begin{bmatrix} \sqrt{x} A_n^* Q_n(x) A_n & I_r + n^{-1/2} A_n^* Y_n \widetilde{Q}_n(x) B_n \\ I_r + n^{-1/2} B_n^* \widetilde{Q}_n(x) Y_n^* A_n & \sqrt{x} B_n^* \widetilde{Q}_n(x) B_n \end{bmatrix}$$

(for details, see the derivations in [9] or in [20, Section 3]). The intuitive idea is the following. Fix the value of  $x$ . By the results shown above,  $\widehat{S}_n(x)$  is close for large  $n$  to

$$S_n(x) = \begin{bmatrix} \sqrt{x} m_n(x) A_n^* A_n & I_r \\ I_r & \sqrt{x} B_n^* \widetilde{T}_n(x) B_n \end{bmatrix}.$$

Hence, we expect the eigenvalues of  $\Sigma_n \Sigma_n^*$  in  $(a, b)$ , when they exist, to be close for large  $n$  to the zeros in  $(a, b)$  of

$$\begin{aligned} \det S_n(x) &= \det \left( x m_n(x) A_n^* A_n B_n^* \widetilde{T}_n(x) B_n - I_r \right) \\ &= (-1)^r \det \left( I_N - x m_n(x) P_n \widetilde{T}_n(x) P_n^* \right). \end{aligned}$$

For any  $x \in (a, b)$ ,  $\det S_n(x) = (-1)^r \det(H_n(x) + I_r) \simeq (-1)^r \det(H_*(x) + I_r)$  where  $H_n$  is given by Equation (7) and  $H_*$  is provided in the statement of the theorem. Hence the zeros of  $\mathcal{D}(x)$  in  $(a, b)$  are the limits of the isolated eigenvalues of  $\Sigma_n \Sigma_n^*$ .

We now formalize this argument. Our first task is to establish the properties of  $H_*(x)$  given in the statement of Theorem 2.2-(1). We start by defining the function  $\det S_n(z)$  on  $\mathbb{C}_+$  by writing

$$\det S_n(z) = (-1)^r \det \left( I_N - z m_n(z) P_n \widetilde{T}_n(z) P_n^* \right) = (-1)^r \det \left( H_n(z) + I_r \right).$$

Let  $V$  be a small enough neighborhood of  $[x_1, x_2]$  in  $\mathbb{C}$ , where we recall that  $[x_1, x_2]$  is an arbitrary interval in  $(a, b)$ . By the construction of  $\Lambda_n$ , Proposition 2.1 and Lemma 3.1, the functions  $H_n(z)$  are analytic on  $V$  for  $n$  large enough, the functions  $H_*(z)$  are analytic on  $V$ , and from every sequence of integers increasing to infinity, one can extract a subsequence  $\varphi(n)$  along which  $H_{\varphi(n)}(z)$  converges to one of the  $H_*(z)$  uniformly on the compact subsets of  $V$ . The comments preceding Theorem 2.2 show that any  $H_*(z)$  is the Stieltjes Transform of a matrix-valued nonnegative measure  $\Gamma$ . Since  $H_*(z)$  is analytic on  $V$ , the interval  $[x_1, x_2]$  lies outside the support of  $\Gamma$ . Hence, the extension of  $H_*(z)$  to  $[x_1, x_2]$  is increasing on this interval in the order of Hermitian matrices, and the properties of this function given in the statement of Theorem 2.2-(1) are established.

In order to prove the convergence stated in Theorem 2.2-(1), it will be enough to show that for large  $n$  and with probability one,  $\Sigma \Sigma^*$  has no eigenvalues in any compact interval lying in  $(a, \rho_1), (\rho_1, \rho_2), \dots$ , or in  $(\rho_k, b)$  if  $k \geq 1$ , or in  $(a, b)$  if  $k = 0$ , and the number of its eigenvalues in any small neighborhood of any of the  $\rho_i$  is equal to the multiplicity of this zero. Let  $D_{x_1, x_2}^\circ$  be the open disk with diameter  $[x_1, x_2]$  where  $x_1$  and  $x_2$  are chosen such that  $[x_1, x_2] \subset (a, b)$  and  $\{\rho_1, \dots, \rho_k\} \cap [x_1, x_2] = \emptyset$ . Let us restrict ourselves to the probability one set where  $n^{-1} Y Y^*$  has no eigenvalues for large  $n$  in a large enough closed interval in  $(a, b)$ . We need to prove that on this set, the number of zeros of  $\det \widehat{S}_n(z)$  in  $D_{x_1, x_2}^\circ$  converges almost surely to the number of zeros of  $\det(H_*(z) + I_r)$  in that same disk. Let  $\widehat{L}_n = \#\{i : \lambda_i^n \in D_{x_1, x_2}^\circ\}$ ,  $L_n = \#\{\text{zeros of } \det S_n(z) \text{ in } D_{x_1, x_2}^\circ\}$ , and

$L = \#\{i : \rho_i \in D_{x_1, x_2}^\circ \text{ with multiplicities}\}$ . By the well known argument principle for holomorphic functions,

$$\begin{aligned}\widehat{L}_n &= \frac{1}{2i\pi} \oint_{\partial D_{x_1, x_2}^\circ} \frac{(\det \widehat{S}_n(z))'}{\det \widehat{S}_n(z)} dz, \\ L_n &= \frac{1}{2i\pi} \oint_{\partial D_{x_1, x_2}^\circ} \frac{(\det S_n(z))'}{\det S_n(z)} dz = \frac{1}{2i\pi} \oint_{\partial D_{x_1, x_2}^\circ} \frac{(\det(H_n(z) + I_r))'}{\det(H_n(z) + I_r)} dz \quad \text{and} \\ L &= \frac{1}{2i\pi} \oint_{\partial D_{x_1, x_2}^\circ} \frac{(\det(H_*(z) + I_r))'}{\det(H_*(z) + I_r)} dz\end{aligned}$$

where  $\partial D_{x_1, x_2}^\circ$  is seen as a positively oriented contour.

For any  $1 \leq k, \ell \leq r$ , let  $h_{n,k,\ell}(z) = [A_n^*(Q_n(z) - m_n(z)I_N)A_n]_{k,\ell}$ . Let  $V$  be a small neighborhood of  $D_{x_1, x_2}$ , the closure of  $D_{x_1, x_2}^\circ$ . Let  $z_m$  be a sequence of complex numbers in  $V$  having an accumulation point in  $V$ . By Lemmas 4.1, 4.3 and 4.6 and the Borel Cantelli lemma,  $h_{n,k,\ell}(z_m) \xrightarrow{\text{as}} 0$  as  $n \rightarrow \infty$  for every  $m$ . Moreover, for  $n$  large, the  $h_{n,k,\ell}$  are uniformly bounded on any compact subset of  $V$ . By the normal family theorem, every  $n$ -sequence of  $h_{n,k,\ell}$  contains a further subsequence which converges uniformly on the compact subsets of  $V$  to a holomorphic function  $h_*$ . Since  $h_*(z_m) = 0$  for every  $m$ , we obtain that almost surely,  $h_{n,k,\ell}$  converges uniformly to zero on the compact subsets of  $V$ , and the same can be said about  $\|A_n^*(Q_n(z) - m_n(z)I_N)A_n\|$ . Using in addition Lemmas 3.1 and 4.7 we obtain the same result for  $\|B_n^*(\widetilde{Q}_n(z) - \widetilde{T}_n(z))B_n\|$  and  $n^{-1/2}\|A_n^*Y_n\widetilde{Q}_n(z)B_n\|$ .

Since  $\det X$  is a polynomial in the elements of matrix  $X$ ,  $\det \widehat{S}_n(z) - \det S_n(z)$  converges almost surely to zero on  $\partial D_{x_1, x_2}^\circ$ , and this convergence is uniform. By analyticity, the same can be said about the derivatives of these quantities. Moreover,  $\det S_n(z)$  converges to  $(-1)^r \det(H_*(z) + I_r)$  (which is the same for all accumulation points  $\Lambda_*$ ) uniformly on  $\partial D_{x_1, x_2}^\circ$ . Similarly,  $(\det S_n(z))'$  converges to  $(-1)^r (\det(H_*(z) + I_r))'$  uniformly on  $\partial D_{x_1, x_2}^\circ$ . Furthermore, by construction of the interval  $[x_1, x_2]$ , we have  $\inf_{z \in \partial D_{x_1, x_2}^\circ} |\det(H_*(z) + I_r)| > 0$  which implies that  $\liminf_n \inf_{z \in \partial D_{x_1, x_2}^\circ} |\det S_n(z)| > 0$ . It follows that  $\widehat{L}_n - L_n \xrightarrow{\text{as}} 0$  and  $L_n \rightarrow L$  as  $n \rightarrow \infty$ . This concludes the proof of Theorem 2.2-(1).

**Proof of Theorem 2.2-(2) and (3).** We start with the following lemma:

**Lemma 4.8.** *Let  $A = \inf(\text{supp}(\mu) - \{0\})$ . Let  $I = [x_1, x_2] \subset \text{supp}(\mu)^c$ . Then the following hold true:*

- (i) *If  $x_2 < A$ , then  $\mathbf{m}(x)(1 + c\mathbf{m}(x)t)^{-1} > 0$  for all  $x \in I$  and all  $t \in \text{supp}(\nu)$ .*
- (ii) *Alternatively, if  $x_1 > A$ , then there exists a measurable set  $E \subset \mathbb{R}_+$  such that  $\nu(\partial E) = 0$ ,  $\nu(E) > 0$  and  $\mathbf{m}(x)(1 + c\mathbf{m}(x)t)^{-1} < 0$  for all  $x \in I$  and all  $t \in E$ .*

*Proof.* To have an illustration of some of the proof arguments, the reader might refer to Figures 1 and 2 which provide typical plots of  $x(\mathbf{m})$  for  $c < 1$  and  $c > 1$  respectively. We start with a preliminary result:

Since  $\mathbf{m}(z)$  is the Stieltjes Transform of a probability measure supported by  $\mathbb{R}_+$ , the function  $\mathbf{m}(x)$  decreases to zero as  $x \rightarrow -\infty$ . From Proposition 3.2,  $x(\mathbf{m}) \rightarrow -\infty$  as  $\mathbf{m} \rightarrow 0^+$ , and

$$\mathbf{m} > 0 \Rightarrow x(\mathbf{m}) < A.$$

From Proposition 3.2,  $J = \mathbf{m}(I)$  is an interval  $[\mathbf{m}_1, \mathbf{m}_2] \subset B$ . Let  $E = [0, -(\mathbf{c}\mathbf{m}_1)^{-1}]$  with the convention  $E = \emptyset$  when  $\mathbf{m}_1 > 0$ . We already assumed that  $\nu(\{0\}) = 0$ . Since  $-(\mathbf{c}\mathbf{m}_1)^{-1} \in \text{supp}(\nu)^c$  by Proposition 3.2,  $\nu(\{-(\mathbf{c}\mathbf{m}_1)^{-1}\}) = 0$ . The main part of the proof consists in showing that

$$\nu(E) > 0 \Leftrightarrow x_1 > A. \quad (11)$$

To see why (11) proves the lemma, consider first  $x_1 > A$ . Then  $\mathbf{m}_1 < 0$  and  $\mathbf{m}_2 < 0$ . For any  $t \in E$  and any  $\mathbf{m} \in J$ ,  $1 + \mathbf{c}\mathbf{m}t \geq 1 - \mathbf{c}\mathbf{m}(\mathbf{c}\mathbf{m}_1)^{-1} > 0$ . As  $\mathbf{m} < 0$ , (ii) is true. Assume now that  $x_2 < A$ . In the case  $J \subset (0, \infty)$ , (i) is immediate. Assume  $J \subset B \cap (-\infty, 0)$ . Then, since  $\nu(E) = 0$ , we get that  $\forall t \in \text{supp}(\nu), t \geq -(\mathbf{c}\mathbf{m}_1)^{-1}$ , therefore  $t \geq -(\mathbf{c}\mathbf{m}_2)^{-1}$ . Consequently, for any  $t \in \text{supp}(\nu)$  and any  $\mathbf{m} \in J$ ,  $1 + \mathbf{c}\mathbf{m}t \leq 1 - \mathbf{c}\mathbf{m}(\mathbf{c}\mathbf{m}_2)^{-1} = 1 - \mathbf{m}/\mathbf{m}_2 < 0$ , and (i) is true.

Let us first prove (11) in the  $\Leftarrow$  direction. When  $x_1 > A$ , we have seen that  $\mathbf{m}_1 < 0$ . Assume that  $\nu(E) = 0$ . Then  $(-\infty, \mathbf{m}_1] \subset B$ . Since  $t \geq -(\mathbf{c}\mathbf{m}_1)^{-1}$  in the integral in (8),  $x(\mathbf{m}) \rightarrow 0$  as  $\mathbf{m} \rightarrow -\infty$  by the dominated convergence theorem. By Propositions 3.1, 3.2 and 3.3,  $x(\mathbf{m})$  should be increasing from 0 to  $x_1$  on  $(-\infty, \mathbf{m}_1]$ . This contradicts  $x_1 > A$ .

We now prove (11) in the  $\Rightarrow$  direction. To that end, we consider in turn the cases  $c < 1$ ,  $c > 1$  and  $c = 1$ .

Assume  $c < 1$ . We have seen that  $x(\mathbf{m}) \rightarrow -\infty$  as  $\mathbf{m} \rightarrow 0^+$ . From (8) we notice that  $\mathbf{m}x(\mathbf{m}) \rightarrow (1-c)/c > 0$  as  $\mathbf{m} \rightarrow \infty$ , hence  $x(\mathbf{m})$  reaches a positive maximum on  $(0, \infty)$ . By Propositions 3.2 and 3.3, this maximum is  $A$ , and we have  $x < A \Rightarrow \mathbf{m}(x) > 0$ . Therefore,  $x_1 < A \Rightarrow \nu(E) = 0$ .

Consider now the case  $c > 1$ . We shall also show that  $x_1 < A \Rightarrow \nu(E) = 0$ . By Proposition 3.4, the measure  $\mu$  has a Dirac at zero with weight  $1-c^{-1}$ . Hence, either  $x_1 \leq x_2 < 0$ , or  $A > 0$  and  $0 < x_1 \leq x_2 < A$ . Since  $\mathbf{m}(z)$  is the Stieltjes Transform of a probability measure supported by  $\mathbb{R}_+$ , it holds that  $x < 0 \Rightarrow \mathbf{m}(x) > 0$ . Hence,  $\nu(E) = 0$  when  $x_1 < 0$ . We now consider the second case. Since  $(0, x_1] \subset \text{supp}(\mu)^c$ , the image of this interval by  $\mathbf{m}$  belongs to  $B$ . By Proposition 3.4,  $\lim_{x \rightarrow 0^+} \mathbf{m}(x) = -\infty$ . Hence this image is  $(-\infty, x_1]$ . This implies that  $\nu(E) = 0$ .

We finally consider the case  $c = 1$ . We show here that  $A = 0$ , which will result in  $x < A \Rightarrow \mathbf{m}(x) > 0 \Rightarrow \nu(E) = 0$  as above. Assume  $A > 0$  and let  $x_0 \in (0, A)$ . By Proposition 3.4,  $\mu(\{0\}) = 0$  hence  $\mathbf{m}(x_0) = \int (t - x_0)^{-1} \mu(dt) > 0$ . But from (8), we observe that  $x(\mathbf{m})$  increases from  $-\infty$  to 0 as  $\mathbf{m}$  increases from 0 to  $\infty$ , which raises a contradiction. This concludes the proof of Lemma 4.8.  $\square$

This lemma shows that for any  $x < \inf(\text{supp}(\mu) - \{0\})$ ,  $H_*(x) \geq 0$ , hence  $\mathcal{D}(x) > 0$  for those  $x$ . This proves Theorem 2.2-(2).

Turning to Theorem 2.2-(3), choose an interval  $[x_1, x_2] \subset (a, b)$  such that  $[\rho_1, \rho_r] \subset (x_1, x_2)$ . Let  $E$  be the set associated with  $[x_1, x_2]$  by Lemma 4.8-(ii). By Lemma 4.8-(ii) and the properties of a Stieltjes Transform, the function

$$q(x) = \int \frac{\mathbf{m}(x)}{1 + \mathbf{c}\mathbf{m}(x)t} 1_E(t) \nu(dt)$$

is negative and increasing on  $[x_1, x_2]$ . Let  $\mathbf{\Omega} = \text{diag}(\omega_1^2, \dots, \omega_r^2)$  where  $\omega_k^2 = -1/q(\rho_k)$ . Then it is clear that function  $\mathcal{D}(x) = \det(q(x)\mathbf{\Omega} + I_r)$  has  $r$  roots in  $[x_1, x_2]$  which coincide with the  $\rho_k$ . Theorem 2.2-(3) will be proved if we find a sequence of matrices  $P_n$  such that one of the  $H_n(z)$  associated to  $P_n$  as in (7) converges to  $q(z)\mathbf{\Omega}$  uniformly in the compact subsets of a neighborhood of  $[x_1, x_2]$ .

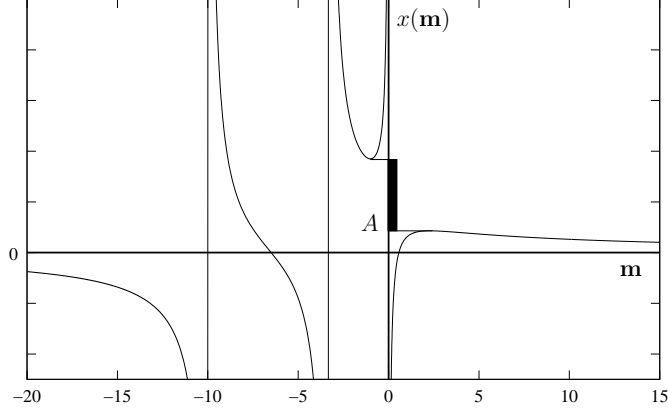


FIGURE 1. Plot of  $x(\mathbf{m})$  for  $c = 0.1$  and  $\nu = 0.5(\delta_1 + \delta_3)$ . The thick segment represents  $\text{supp}(\mu)$ .

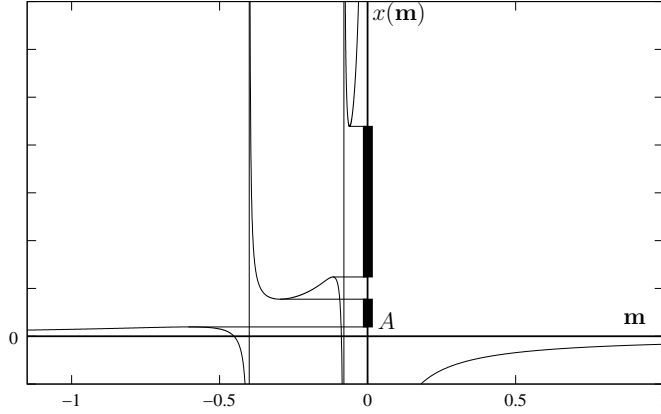


FIGURE 2. Plot of  $x(\mathbf{m})$  for  $c = 5$  and  $\nu = 0.5(\delta_{1/2} + \delta_{5/2})$ . The thick segments represent  $\text{supp}(\mu)$ .

Rearrange the elements of  $D_n$  in such a way that all the  $d_j^n$  which belong to  $E$  are in the top left corner of this matrix. Let  $M_n = [M_{ij}^n]$  be a random  $[n\nu(E)] \times r$  matrix with iid elements such that  $\sqrt{n}M_{11}^n$  has mean zero and variance one. Let  $Z_n$  be the  $n \times r$  matrix obtained by adding  $n - [n\nu(E)]$  rows of zeros below  $M_n$ . Then the law of large numbers shows in conjunction with a normal family theorem argument that there is a set of probability one over which  $zm_n(z)Z_n^*\tilde{T}_n(z)Z_n$  converges to  $q(z)I_r$  uniformly on the compact subsets of a neighborhood of  $[x_1, x_2]$ . Consequently, there exists a sequence of deterministic matrices  $B_n$  such that  $zm_n(z)B_n^*\tilde{T}_n(z)B_n \rightarrow q(z)I_r$  uniformly on these compact subsets. Matrix  $P_n = A_nB_n^*$  with  $A_n = \begin{bmatrix} \Omega^{1/2} \\ 0_{(N-r) \times r} \end{bmatrix}$  satisfies the required property. Theorem 2.2-(3) is proved.

## 5. SECOND ORDER ANALYSIS

In all this section we shall work on a sequence of factorizations  $P_n = U_n R_n^*$  such that  $\Lambda_n$  satisfies the third item of Assumption 7. We also write  $U_n = [U_{1,n} \cdots U_{t,n}]$  and  $R_n = [R_{1,n} \cdots R_{t,n}]$  where  $U_{i,n} \in \mathbb{C}^{N \times j_i}$  and  $R_{i,n} \in \mathbb{C}^{n \times j_i}$ .

We start by briefly showing the first two facts stated by Theorem 2.3. For any  $i = 1, \dots, p$ , it is clear that  $\mathbf{m}(\rho_i)^2 > 0$  and  $\mathbf{m}'(\rho_i) > 0$ . An immediate calculus then gives  $\mathbf{m}'(\rho_i)\Delta(\rho_i) = \mathbf{m}^2(\rho_i)$  which shows that  $\Delta(\rho_i) > 0$ .

To prove the second fact, we shall establish more generally that  $\limsup \sqrt{n} \|H_n(\rho_i) + g(\rho_i)\Omega\| < \infty$ . Invoking Equation (3) and its analogue  $m_n(z) = (-z + \int t(1 + c_n m_n(z)t)^{-1} \nu_n(dt))^{-1}$ , taking the difference and doing some straightforward derivations, we get that  $(m_n(\rho_i) - \mathbf{m}(\rho_i))(\Delta(\rho_i) + \varepsilon_1) = \varepsilon_2$  where  $\varepsilon_1 \rightarrow 0$  and where  $|\varepsilon_2| \leq K/\sqrt{n}$  thanks to the first two items of Assumption 7. Hence  $|m_n(\rho_i) - \mathbf{m}(\rho_i)| \leq K/\sqrt{n}$ . Now we have

$$\begin{aligned} H_n(\rho_i) + g(\rho_i)\Omega &= \int \left( \frac{m_n(\rho_i)}{1 + c_n m_n(\rho_i)t} - \frac{\mathbf{m}(\rho_i)}{1 + c\mathbf{m}(\rho_i)t} \right) \Lambda_n(dt) \\ &\quad + \int \frac{\mathbf{m}(\rho_i)}{1 + c\mathbf{m}(\rho_i)t} \Lambda_n(dt) - \int \frac{\mathbf{m}(\rho_i)}{1 + c\mathbf{m}(\rho_i)t} \nu(dt) \times \Omega. \end{aligned}$$

which shows thanks to Assumption 7 that  $\limsup \sqrt{n} \|H_n(\rho_i) + g(\rho_i)\Omega\| < \infty$ .

We now enter the core of the proof Theorem 2.3. The following preliminary lemmas are proven in the appendix:

**Lemma 5.1.** *Let  $s$  be a fixed integer, and let  $Z_N = [Z_{ij}]$  be a  $N \times s$  complex matrix with iid elements with independent  $\mathcal{N}(0, 1/2)$  real and imaginary parts. Let  $\Upsilon_N = [\Upsilon_{ij}]$  be a deterministic Hermitian  $N \times N$  matrix such that  $\text{Tr } \Upsilon_N = 0$ , and let  $F_N = [F_{ij}]$  be a complex deterministic  $N \times s$  matrix. Assume that  $F_N^* F_N \rightarrow \varsigma^2 I_s$ , that  $\limsup_N \|\Upsilon_N\| < \infty$ , and that  $N^{-1} \text{Tr } \Upsilon_N^2 \rightarrow \sigma^2$  as  $N \rightarrow \infty$ . Let  $M$  be a  $s \times s$  complex matrix with iid elements with independent  $\mathcal{N}(0, 1/2)$  real and imaginary parts, and let  $G$  be a  $s \times s$  GUE matrix independent of  $M$ . Then*

$$\left( N^{-1/2} Z_N^* \Upsilon_N Z_N, Z_N^* F_N \right) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} (\sigma G, \varsigma M).$$

**Lemma 5.2.** *For  $x \in \text{supp}(\mu)^c$ ,*

$$\mathbb{E} \left[ \phi_n \tilde{u}_n^* \tilde{Q}_n(x) (n^{-1} Y_n^* Y_n) \tilde{Q}_n(x) \tilde{u}_n \right] = c_n \frac{x^2 m_n(x)^2 \tilde{u}_n^* D_n \tilde{T}_n^2(x) \tilde{u}_n}{1 - c_n x^2 m_n(x)^2 \frac{1}{n} \text{Tr } D_n^2 \tilde{T}_n^2(x)} + \mathcal{O}(n^{-1})$$

and

$$\text{Var} \left( \phi_n \tilde{u}_n^* \tilde{Q}_n(x) (n^{-1} Y_n^* Y_n) \tilde{Q}_n(x) \tilde{u}_n \right) \leq \frac{K}{n}.$$

**Lemma 5.3.** *For  $i = 1, \dots, p$ , let  $A_i$  be a deterministic Hermitian  $j_i \times j_i$  matrix independent of  $n$ , where  $p$  and the  $j_i$  are as in the statement of Theorem 2.3. For  $i = 1, \dots, p$ , let  $M_{i,n}$  a  $n \times j_i$  matrix such that  $\sup_n \|M_{i,n}\| < \infty$ . Then for any  $t \in \mathbb{R}$ ,*

$$\mathbb{E} \left[ \exp \left( i\sqrt{nt} \sum_{i=1}^p \text{Tr } A_i M_{i,n}^* (\tilde{Q}_n(\rho_i) - \tilde{T}_n(\rho_i)) M_{i,n} \right) \right] = \exp(-t^2 \tilde{\sigma}_n^2 / 2) + \mathcal{O}(n^{-1/2})$$

where

$$\begin{aligned} \tilde{\sigma}_n^2 &= \sum_{i,k=1}^p c_n \rho_i \rho_k m_n(\rho_i) m_n(\rho_k) \\ &\quad \times \frac{\text{Tr } A_i M_{i,n}^* \tilde{T}_n(\rho_i) D_n \tilde{T}_n(\rho_k) M_{k,n} A_k M_{k,n}^* \tilde{T}_n(\rho_k) D_n \tilde{T}_n(\rho_i) M_{i,n}}{1 - c_n \rho_i \rho_k m_n(\rho_i) m_n(\rho_k) \frac{1}{n} \text{Tr } D_n \tilde{T}_n(\rho_i) D_n \tilde{T}_n(\rho_k)}. \end{aligned}$$

Replacing the  $M_{i,n}$  with the blocks  $R_{i,n}$  of  $R_n$  in the statement of Lemma 5.3 and observing that

$$R_n^* \tilde{T}_n(\rho_i) D_n \tilde{T}_n(\rho_k) R_n = \int \frac{t}{\rho_i \rho_k (1 + c_n m_n(\rho_i) t) (1 + c_n m_n(\rho_k) t)} \Lambda_n(dt),$$

we obtain from the third item of Assumption 7 that  $\tilde{\sigma}_n^2 \rightarrow \sum_{i=1}^p \tilde{\sigma}_i^2 \text{Tr } A_i^2$  where

$$\tilde{\sigma}_i^2 = \frac{c\omega_i^4}{\Delta(\rho_i)} \left( \int \frac{\mathbf{m}(\rho_i)t}{(1 + c\mathbf{m}(\rho_i)t)^2} \nu(dt) \right)^2.$$

Invoking the Cramer-Wold device, this means that the  $p$ -uple of random matrices

$$\sqrt{n} \left( R_{i,n}^* (\tilde{Q}_n(\rho_i) - \tilde{T}_n(\rho_i)) R_{i,n} \right)_{i=1}^p$$

converges in distribution towards  $(\tilde{\sigma}_i \tilde{G}_i)_{i=1}^p$  where  $\tilde{G}_1, \dots, \tilde{G}_p$  are independent GUE matrices with  $\tilde{G}_i \in \mathbb{C}^{j_i \times j_i}$ .

Lemmas 5.1–5.3 lead to the following result which plays a central role in the proof of Theorem 2.3:

**Lemma 5.4.** *Consider the  $3p$ -uple of random matrices*

$$L_n = \sqrt{n} \times \left( \frac{U_{i,n}^* Y_n \tilde{Q}_n(\rho_i) R_{i,n}}{\sqrt{n}}, U_{i,n}^* (\tilde{Q}_n(\rho_i) - m_n(\rho_i) I_N) U_{i,n}, R_{i,n}^* (\tilde{Q}_n(\rho_i) - \tilde{T}_n(\rho_i)) R_{i,n} \right)_{i=1}^p.$$

Define the following quantities

$$\begin{aligned} \varsigma_i^2 &= \frac{\omega_i^2}{\Delta(\rho_i)} \int \frac{\mathbf{m}^2(\rho_i)t}{(1 + c\mathbf{m}(\rho_i)t)^2} \nu(dt) \\ \sigma_i^2 &= \frac{1}{\Delta(\rho_i)} \int \frac{\mathbf{m}^4(\rho_i)t^2}{(1 + c\mathbf{m}(\rho_i)t)^2} \nu(dt) \\ \tilde{\sigma}_i^2 &= \frac{c\omega_i^4}{\Delta(\rho_i)} \left( \int \frac{\mathbf{m}(\rho_i)t}{(1 + c\mathbf{m}(\rho_i)t)^2} \nu(dt) \right)^2. \end{aligned}$$

Let  $M_1, \dots, M_p$  be random matrices such that  $M_i \in \mathbb{C}^{j_i \times j_i}$  and has independent elements with independent  $\mathcal{N}(0, 1/2)$  real and imaginary parts. Let  $G_1, \tilde{G}_1, \dots, G_p, \tilde{G}_p$  be GUE matrices such that  $G_i, \tilde{G}_i \in \mathbb{C}^{j_i \times j_i}$ . Assume in addition that  $M_1, G_1, \tilde{G}_1, \dots, M_p, G_p, \tilde{G}_p$  are independent. Then

$$L_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \left( \varsigma_i M_i, \sigma_i G_i, \tilde{\sigma}_i \tilde{G}_i \right)_{i=1}^p.$$

*Proof.* Let  $\alpha_n(\rho) = N^{-1} \text{Tr} Q_n(\rho)$ . By Lemmas 4.3 and 4.5,  $\sqrt{n}(\alpha_n(\rho_i) - m_n(\rho_i)) \xrightarrow{\mathcal{P}} 0$ . Therefore, we can replace the  $m_n(\rho_i)$  in the expression of  $L_n$  by  $\alpha_n(\rho_i)$ , as we shall do in the rest of the proof.

Write  $s = j_1 + \dots + j_p$  and let  $Z_n$  be a  $N \times s$  complex matrix with iid elements with independent  $\mathcal{N}(0, 1/2)$  real and imaginary parts. Assume that  $Z_n$  and  $X_n$  are independent. Write  $Z_n = [Z_{1,n} \dots Z_{p,n}]$  where the block  $Z_{i,n}$  is  $N \times j_i$ . Let  $n^{-1/2} X_n = W_n \Delta_n \widetilde{W}_n^*$  be a singular value decomposition of  $n^{-1/2} X_n$ . By assumption 2, the square matrices  $W_n$  and  $\widetilde{W}_n$  are Haar distributed over their respective unitary groups, and moreover,  $W_n$ ,  $\Delta_n$  and  $\widetilde{W}_n$  are independent. Let

$$\overline{L}_n = \sqrt{n} \left( \frac{U_n^* Y_n \widetilde{Q}_n(\rho_i) R_{i,n}}{\sqrt{n}}, U_n^* (Q_n(\rho_i) - \alpha_n(\rho_i) I_N) U_n, R_{i,n}^* (\widetilde{Q}_n(\rho_i) - \widetilde{T}_n(\rho_i)) R_{i,n} \right)_{i=1}^p.$$

We have

$$\overline{L}_n \stackrel{\mathcal{L}}{=} \left( \sqrt{N} (Z_n^* Z_n)^{-1/2} Z_n^* F_{i,n}, N^{1/2} (Z_n^* Z_n)^{-1/2} Z_n^* \Upsilon_{i,n} Z_n (Z_n^* Z_n)^{-1/2}, \sqrt{n} R_{i,n}^* (\widetilde{Q}_n(\rho_i) - \widetilde{T}_n(\rho_i)) R_{i,n} \right)_{i=1}^p$$

where  $F_{i,n} = c_n^{-1/2} \Delta_n \widetilde{W}_n^* D_n^{1/2} \widetilde{Q}_n(\rho_i) R_{i,n}$  and  $\Upsilon_{i,n} = c_n^{-1/2} ((\Delta_n \widetilde{W}_n^* D_n \widetilde{W}_n \Delta_n - \rho_i)^{-1} - \alpha_n(\rho_i) I_N)$ . We shall now show that the term  $\sqrt{N} (Z_n^* Z_n)^{-1/2} Z_n^* F_{i,n}$  can be replaced with  $Z_n^* F_{i,n}$ . By the law of large numbers, we have  $N^{-1} Z_n^* Z_n \xrightarrow{\text{as}} I_s$ . By the independence of  $Z_n$  and  $(\Delta_n, \widetilde{W}_n)$ , we have  $\mathbb{E}[\text{Tr} Z_n^* F_{i,n} F_{i,n}^* Z_n \mid (\Delta_n, \widetilde{W}_n)] = s c_n^{-1} \text{Tr} R_{i,n}^* \widetilde{Q}_n(\rho_i) (n^{-1} Y_n^* Y_n) \widetilde{Q}_n(\rho_i) R_{i,n}$  whose limit superior is bounded with probability one. Hence  $Z_n^* F_{i,n}$  is tight, proving that the replacement can be done.

By deriving the variances of the elements of  $N^{-1/2} Z_n^* \Upsilon_{i,n} Z_n$  with respect to the law of  $Z_n$ , and by recalling that  $\limsup_n \|\Upsilon_{i,n}\|$  is bounded with probability one, we obtain that these elements are also tight. It results that we can replace  $L_n$  with

$$\underline{L}_n = \left( Z_{i,n}^* F_{i,n}, \frac{Z_{i,n}^* \Upsilon_{i,n} Z_{i,n}}{\sqrt{N}}, \sqrt{n} R_{i,n}^* (\widetilde{Q}_n(\rho_i) - \widetilde{T}_n(\rho_i)) R_{i,n} \right)_{i=1}^p.$$

For  $i = 1, \dots, p$ , let  $A_i$  and  $B_i$  be deterministic Hermitian  $j_i \times j_i$  matrices and let  $C_i$  be deterministic complex  $j_i \times j_i$  matrices, all independent of  $n$ . The lemma will be established if we prove that

$$\begin{aligned} & \mathbb{E} \left\{ \exp \left( i \sqrt{nt} \sum_{i=1}^p \text{Tr} A_i R_{i,n}^* (\widetilde{Q}_n(\rho_i) - \widetilde{T}_n(\rho_i)) R_{i,n} \right) \right. \\ & \quad \times \mathbb{E} \left[ \exp \left( it \sum_{i=1}^p N^{-1/2} \text{Tr} B_i Z_{i,n}^* \Upsilon_{i,n} Z_{i,n} + \Re(\text{Tr} C_i Z_{i,n}^* F_{i,n}) \right) \mid (\Delta_n, \widetilde{W}_n) \right] \Big\} \\ & \xrightarrow[n \rightarrow \infty]{} \prod_{i=1}^p \exp(-t^2 (\widetilde{\sigma}_i^2 \text{Tr} A_i^2 + \sigma_i^2 \text{Tr} B_i^2 + \frac{1}{2} \varsigma_i^2 \text{Tr} C_i C_i^*) / 2). \end{aligned} \quad (12)$$



In addition to the boundedness of  $\|\Upsilon_{i,n}\|$  w.p. one, we have  $\text{Tr } \Upsilon_{i,n} = 0$ , and

$$\begin{aligned} \frac{1}{N} \text{Tr } \Upsilon_{i,n}^2 &= \frac{1}{c_n N} \sum_{\ell=1}^N ((\lambda_\ell^n - \rho_i)^{-1} - \alpha_n(\rho_i))^2 \\ &\xrightarrow[n \rightarrow \infty]{\text{a.s.}} c^{-1} (\mathbf{m}'(\rho_i) - \mathbf{m}(\rho_i)^2) = c^{-1} \mathbf{m}(\rho_i)^2 (\mathbf{\Delta}(\rho_i)^{-1} - 1) = \boldsymbol{\sigma}_i^2. \end{aligned}$$

Moreover, using Lemma 5.2 in conjunction with Assumption 6, we obtain

$$F_{i,n}^* F_{i,n} = c_n^{-1} R_{i,n}^* \left( \tilde{Q}_n(\rho_i) \frac{1}{n} Y_n^* Y_n \tilde{Q}_n(\rho_i) \right) R_{i,n} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} \boldsymbol{\varsigma}_i^2 I_{j_i}.$$

From any sequence of integers increasing to infinity, there exists a subsequence along which this convergence in probability holds in the almost sure sense. Applying Lemma 5.1, we get that the inner expectation at the left hand side of (12) converges almost surely along this subsequence towards  $\prod_{i=1}^p \exp(-t^2 (\boldsymbol{\sigma}_i^2 \text{Tr } B_i^2 + \frac{1}{2} \boldsymbol{\varsigma}_i^2 \text{Tr } C_i C_i^*) / 2)$ . Using in addition Lemma 5.3 along with the dominated convergence theorem, we obtain that Convergence (12) holds true along this subsequence. Since the original sequence is arbitrary, we obtain the required result.  $\square$

The remainder of the proof of Theorem 2.3 is an adaptation of the approach of [7].

**Lemma 5.5.** *For a given  $x \in \mathbb{R}$  and a given  $i \in \{1, \dots, p\}$ , let  $y = \rho_i + n^{-1/2}x$ , and let*

$$\hat{S}_n(y) = \begin{bmatrix} \sqrt{y} U_n^* \tilde{Q}_n(y) U_n & I_r + n^{-1/2} U_n^* Y_n \tilde{Q}_n(y) R_n \\ I_r + n^{-1/2} R_n^* \tilde{Q}_n(y) Y_n^* U_n & \sqrt{y} R_n^* \tilde{Q}_n(y) R_n \end{bmatrix}$$

Let

$$\begin{aligned} \chi_n^{(i)}(x) &= n^{\frac{j_i}{2}} \left[ \det \hat{S}_n(y) - \prod_{k \neq i} [\omega_k^2 g(\rho_i) - 1]^{j_k} \right. \\ &\times \det \left( \frac{\sqrt{n} U_{i,n}^* (\tilde{Q}_n(\rho_i) - m_n(\rho_i) I_N) U_{i,n}}{\mathbf{m}(\rho_i)} + \rho_i \mathbf{m}(\rho_i) \sqrt{n} R_{i,n}^* (\tilde{Q}_n(\rho_i) - \tilde{T}_n(\rho_i)) R_{i,n} \right. \\ &\left. \left. - 2 \Re \left[ U_{i,n}^* Y_n \tilde{Q}_n(\rho_i) R_{i,n} \right] - \sqrt{n} (H_{i,n}(\rho_i) + I_{j_i}) - x H'_{i,n}(\rho_i) \right] \right] \end{aligned} \quad (13)$$

where  $\Re(M) = (M + M^*)/2$  for a square matrix  $M$ . Then

$$(\chi_n^{(i)}(x_1), \dots, \chi_n^{(i)}(x_p)) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0$$

for every finite sequence  $\{x_1, \dots, x_p\}$ .

*Proof.* We show the result for  $i = 1$ , the same procedure being valid for the other values of  $i$ . The notation  $X_n = o_P(1)$  means that the random variable  $X_n$  converges to zero in probability, while  $X_n = \mathcal{O}_P(n^{-\ell})$  means that  $n^\ell X_n$  is tight. Write  $U = [U_1, \bar{U}_1]$  and  $R = [R_1, \bar{R}_1]$  where  $\bar{U}_1 = [U_2, \dots, U_t]$  and  $\bar{R}_1 = [R_2, \dots, R_t]$ .

Writing

$$\begin{aligned}
A &= \begin{bmatrix} \sqrt{y}U_1^*Q(y)U_1 & \sqrt{y}U_1^*Q(y)\bar{U}_1 \\ \sqrt{y}\bar{U}_1^*Q(y)U_1 & \sqrt{y}\bar{U}_1^*Q(y)\bar{U}_1 \end{bmatrix} := \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}, \\
B &= \begin{bmatrix} I_{j_1} + n^{-1/2}U_1^*Y\tilde{Q}(y)R_1 & n^{-1/2}U_1^*Y\tilde{Q}(y)\bar{R}_1 \\ n^{-1/2}\bar{U}_1^*Y\tilde{Q}(y)R_1 & I_{r-j_1} + n^{-1/2}\bar{U}_1^*Y\tilde{Q}(y)\bar{R}_1 \end{bmatrix} := \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \text{ and} \\
C &= \begin{bmatrix} \sqrt{y}R_1^*\tilde{Q}(y)R_1 & \sqrt{y}R_1^*\tilde{Q}(y)\bar{R}_1 \\ \sqrt{y}\bar{R}_1^*\tilde{Q}(y)R_1 & \sqrt{y}\bar{R}_1^*\tilde{Q}(y)\bar{R}_1 \end{bmatrix} := \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix},
\end{aligned}$$

we have

$$\det \hat{S} = \det \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \det \begin{bmatrix} A_{11} & B_{11} & A_{12} & B_{12} \\ B_{11}^* & C_{11} & B_{21}^* & C_{12} \\ A_{12}^* & B_{21} & A_{22} & B_{22} \\ B_{12}^* & C_{12}^* & B_{22}^* & C_{22} \end{bmatrix} := \det \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}$$

after a row and column permutation. Hence  $n^{j_1/2} \det \hat{S} = \det M_{22} \times n^{j_1/2} \det(M_{11} - M_{12}M_{22}^{-1}M_{12}^*)$ . Write  $\Omega = \text{diag}(\omega_1^2 I_{j_1}, \Omega_2)$ . From the first order analysis we get that

$$M_{22} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \begin{bmatrix} \sqrt{\rho_1} \mathbf{m}(\rho_1) I_{r-j_1} & I_{r-j_1} \\ I_{r-j_1} & \sqrt{\rho_1} (\mathbf{cm}(\rho_1) - \rho_1^{-1}(1-c)) \Omega_2 \end{bmatrix}$$

which is invertible since  $\det M_{22} \xrightarrow{\text{as}} \prod_{k>1} (\omega_k^2 g(\rho_1) - 1)^{j_k} \neq 0$ . Moreover,  $\|M_{12}\| = \mathcal{O}_P(n^{-1/2})$ . To see this, consider for instance the term  $\sqrt{n}C_{12} = \sqrt{ny}R_1^*(\tilde{Q} - \tilde{T})\bar{R}_1 + \sqrt{ny}R_1^*\tilde{T}\bar{R}_1$ . The first term is tight by Lemma 5.4 while the second is bounded by Assumption 7. The other terms are treated similarly. It results that  $\|M_{12}M_{22}^{-1}M_{12}^*\| = \mathcal{O}_P(1/n)$ .

In addition,  $\det(y^{-1/2}C_{11}) \xrightarrow{\text{as}} [\omega_1^2(\mathbf{cm}(\rho_1) - \rho_1^{-1}(1-c))]^{j_1} = (\rho_1 \mathbf{m}(\rho_1))^{-j_1}$  by the definition of  $\rho_1$ . From these observations we get that

$$\begin{aligned}
n^{j_1/2} \det \hat{S} &= \left( \prod_{k>1} (\omega_k^2 g(\rho_1) - 1)^{j_k} + o_P(1) \right) \left( (\rho_1 \mathbf{m}(\rho_1))^{-j_1} + o_P(1) \right) \\
&\quad \times \det \left( \sqrt{ny}A_{11} - \sqrt{ny}B_{11}C_{11}^{-1}B_{11}^* + \mathcal{O}_P(n^{-1/2}) \right).
\end{aligned}$$

Now we make the expansion

$$\sqrt{ny}A_{11} - \sqrt{ny}B_{11}C_{11}^{-1}B_{11}^* \tag{14}$$

$$\begin{aligned}
&= y\sqrt{n}U_1^*(Q(y) - m_n(y)I_N)U_1 + y\sqrt{nm}m_n(y)I_{j_1} \\
&+ \sqrt{n} \left( I_{j_1} + U_1^* \frac{Y\tilde{Q}(y)}{\sqrt{n}} R_1 \right) (R_1^* \tilde{Q}(y) R_1)^{-1} (R_1^* (\tilde{Q}(y) - \tilde{T}(y)) R_1) (R_1^* \tilde{T}(y) R_1)^{-1} \\
&\times \left( I_{j_1} + R_1^* \frac{\tilde{Q}(y)Y^*}{\sqrt{n}} U_1 \right) \\
&- \sqrt{n} \left( I_{j_1} + U_1^* \frac{Y\tilde{Q}(y)}{\sqrt{n}} R_1 \right) (R_1^* \tilde{T}(y) R_1)^{-1} \left( I_{j_1} + R_1^* \frac{\tilde{Q}(y)Y^*}{\sqrt{n}} U_1 \right). \tag{15}
\end{aligned}$$

To go further, remark that

$$\begin{aligned}
& y\sqrt{n}m_n(y)I_{j_1} - \sqrt{n}(R_1^*\tilde{T}(y)R_1)^{-1} \\
&= (R_1^*\tilde{T}(y)R_1)^{-1} \left[ \sqrt{n} \left( ym_n(y)R_1^*\tilde{T}(y)R_1 - \rho_1m_n(\rho_1)R_1^*\tilde{T}(\rho_1)R_1 \right) \right. \\
&\quad \left. + \sqrt{n} \left( \rho_1m_n(\rho_1)R_1^*\tilde{T}(\rho_1)R_1 - I_{j_1} \right) \right] \\
&= \rho_1\mathbf{m}(\rho_1) \left[ -xH_1'(\rho_1) - \sqrt{n}(H_1(\rho_1) + I_{j_1}) \right] + o(1)
\end{aligned}$$

where we recall that  $y = \rho_1 + xn^{-\frac{1}{2}} \rightarrow \rho_1$  and that  $R_1^*\tilde{T}(y)R_1 \rightarrow (\rho_1\mathbf{m}(\rho_1))^{-1}I_{j_1}$ . Recall from Lemma 5.4 that  $\sqrt{n}U_1^*(Q - m_nI)U_1$ ,  $\sqrt{n}R_1^*(\tilde{Q} - \tilde{T})R_1$ , and  $U_1^*Y\tilde{Q}R_1$  are tight. Keeping the non negligible terms, we can write (14) under the form

$$\begin{aligned}
& \sqrt{ny}A_{11} - \sqrt{ny}B_{11}C_{11}^{-1}B_{11}^* \\
&= \rho_1\sqrt{n}U_1^*(Q(\rho_1) - m_n(\rho_1)I_N)U_1 + (\rho_1\mathbf{m}(\rho_1))^2\sqrt{n}R_1^*(\tilde{Q}(\rho_1) - \tilde{T}(\rho_1)I_n)R_1 \\
&\quad - 2\rho_1\mathbf{m}(\rho_1)\Re \left[ U_1^*Y\tilde{Q}(\rho_1)R_1 \right] - \rho_1\mathbf{m}(\rho_1) \left( xH_1'(\rho_1) + \sqrt{n}(H_1(\rho_1) + I_{j_1}) \right) + o_P(1).
\end{aligned}$$

Plugging this expression at the right hand side of the expression of  $n^{j_1/2} \det \hat{S}$  and observing that  $H_{1,n}'(\rho_1) \rightarrow -\omega_{1,n}^2 g(\rho_1)' I_{j_1}$  concludes the proof.  $\square$

For  $i = 1, \dots, p$ , take  $x_1(i) > y_1(i) > x_2(i) > y_2(i) > \dots > y_{j_i}(i)$  fixed sequences of real numbers. Call  $J_n = (\sqrt{n}(\hat{\lambda}_{k(i)+\ell}^n - \rho_i), i = 1, \dots, p, \ell = 1, \dots, j_i)$ , with  $k(i) = \sum_{m=1}^{i-1} j_m$ . Let also  $C$  be the rectangle  $C = [x_1(1), y_1(1)] \times \dots \times [x_p(j_p), y_p(j_p)]$ . Then, for all large  $n$ , with probability one

$$\mathbb{P}(J_n \in C) = \mathbb{P} \left( \left\{ \det \hat{S}_n \left( \rho_i + \frac{x_\ell(i)}{\sqrt{n}} \right) \det \hat{S}_n \left( \rho_i + \frac{y_\ell(i)}{\sqrt{n}} \right) < 0 \right\} \right)$$

since  $\det \hat{S}_n(t)$  changes sign around  $t = \hat{\lambda}_{k(i)+\ell}^n$ , and only there (with probability one, for all large  $n$ ).

From Lemma 5.5, we see that, for growing  $n$ , the probability for the product of the determinants above to be negative for all  $i$  and  $\ell$  approaches the probability

$$\mathbb{P}(\{\det A_{x_\ell(i)} \det A_{y_\ell(i)} < 0, i = 1, \dots, p, \ell = 1, \dots, j_i\})$$

where  $A_x$  is the matrix

$$\begin{aligned}
A_x &= \frac{\sqrt{n}U_{i,n}^*(Q_n(\rho_i) - m_n(\rho_i)I_N)U_{i,n}}{\mathbf{m}(\rho_i)} + \frac{\rho_i\sqrt{n}R_{i,n}^*(\tilde{Q}_n(\rho_i) - \tilde{T}_n(\rho_i)I_{j_i})R_{i,n}}{\omega_i^2(c + c\rho_i\mathbf{m}(\rho_i) - 1)} \\
&\quad - 2\Re \left[ U_{i,n}^*Y\tilde{Q}_n(\rho_i)R_{i,n} \right] - \sqrt{n}(H_{i,n}(\rho_i) + I_{j_i}) - xH_{i,n}'(\rho_i).
\end{aligned}$$

This last probability is equal to  $\mathbb{P}(\bar{J}_n \in C)$ , where  $\bar{J}_n$  are the decreasingly ordered eigenvalues of the matrix

$$\begin{aligned}
B_i &= [H_{i,n}'(\rho_i)]^{-1} \left( \frac{\sqrt{n}U_{i,n}^*(Q_n(\rho_i) - m_n(\rho_i)I_N)U_{i,n}}{\mathbf{m}(\rho_i)} \right. \\
&\quad \left. + \frac{\rho_i\sqrt{n}R_{i,n}^*(\tilde{Q}_n(\rho_i) - \tilde{T}_n(\rho_i)I_{j_i})R_{i,n}}{\omega_i^2(c + c\rho_i\mathbf{m}(\rho_i) - 1)} - 2\Re \left[ U_{i,n}^*Y\tilde{Q}_n(\rho_i)R_{i,n} \right] - \sqrt{n}(H_{i,n}(\rho_i) + I_{j_i}) \right).
\end{aligned}$$

From Lemma 5.4,  $\{B_1, \dots, B_t\}$  asymptotically behave as scaled non-zero mean GUE matrices. Precisely, denoting  $\bar{B}_i = H'_{i,n}(\rho_i)B_i + \sqrt{n}(H_{i,n}(\rho_i) + I_{j_i})$ , from Lemma 5.4 and for all  $a, b$ ,

$$\begin{aligned} & \mathbb{E} [ |(\bar{B}_i)_{ab}|^2 ] \\ & \rightarrow \frac{\sigma_i^2}{\mathbf{m}(\rho_i)^2} + \frac{\rho_i^2 \tilde{\sigma}_i^2}{\omega_i^4 (c + c\rho_i \mathbf{m}(\rho_i) - 1)^2} + 2\zeta^2 \\ & = \frac{\sigma_i^2}{\mathbf{m}(\rho_i)^2} + \rho_i^2 \mathbf{m}(\rho_i)^2 \tilde{\sigma}_i^2 + 2\zeta^2 \\ & = \frac{\mathbf{m}^2(\rho_i)}{\mathbf{\Delta}(\rho_i)} \left[ \int \frac{t^2 \nu(dt)}{(1 + \mathbf{c}\mathbf{m}(\rho_i)t)^2} + c\rho_i^2 \omega_i^4 \left( \int \frac{\mathbf{m}(\rho_i)t\nu(dt)}{(1 + \mathbf{c}\mathbf{m}(\rho_i)t)^2} \right)^2 + \int \frac{2\omega_i^2 t\nu(dt)}{(1 + \mathbf{c}\mathbf{m}(\rho_i)t)^2} \right]. \end{aligned}$$

This concludes the proof of Theorem 2.3.

#### APPENDIX A. PROOFS OF LEMMAS 5.1 TO 5.3

**A.1. Proof of Lemma 5.1.** Given a  $s \times s$  deterministic Hermitian matrix  $A$  and a  $s \times s$  deterministic complex matrix  $B$ , let  $\Gamma_N = N^{-1/2} \text{Tr} AZ_N^* \Upsilon_N Z_N + \Re(\text{Tr} BZ_N^* F_N)$  where  $\Re(M) = (M + M^*)/2$  for any square matrix  $M$ . We shall show that for any  $t \in \mathbb{R}$ ,

$$\varphi_N(t) := \mathbb{E}[\exp(it\Gamma_N)] \xrightarrow{N \rightarrow \infty} \exp\left(-t^2 \frac{\sigma^2 \text{Tr} A^2 + \zeta^2 \text{Tr} BB^*/2}{2}\right) := \exp\left(-\frac{t^2 \mathbf{v}^2}{2}\right).$$

The result will follow by invoking the Cramér-Wold device. To establish this convergence, we show that the derivative  $\varphi'_N(t)$  satisfies  $\varphi'_N(t) = -t\mathbf{v}^2\varphi_N(t) + \varepsilon_N(t)$  where  $\varepsilon_N(t) \rightarrow 0$  as  $N \rightarrow \infty$  uniformly on any compact interval of  $\mathbb{R}$ . That being true, the function  $\psi_N(t) = \varphi_N(t) \exp(t^2\mathbf{v}^2/2)$  satisfies  $\psi_N(t) = 1 + \int_0^t \varepsilon_N(u) \exp(u^2\mathbf{v}^2/2) du \rightarrow 1$  which proves the lemma.

By the IP formula, we get

$$\begin{aligned} \varphi'(t) &= i\mathbb{E}[\Gamma \exp(it\Gamma)] \\ &= i\mathbb{E}\left[\left(\sum_{i,j=1}^s \sum_{k,\ell=1}^N \frac{A_{ij} Z_{kj}^* \Upsilon_{k\ell} Z_{\ell i}}{\sqrt{N}} + \sum_{i,j=1}^s \sum_{k=1}^N \frac{B_{ij} Z_{kj}^* F_{ki} + F_{ki}^* Z_{kj} B_{ij}^*}{2}\right) \right. \\ &\quad \left. \times \exp(it\Gamma)\right] \\ &= i\mathbb{E}\left[\sum_{i,j,k,\ell} \frac{A_{ij} \Upsilon_{k\ell}}{\sqrt{N}} \frac{\partial(Z_{\ell i} \exp(it\Gamma))}{\partial Z_{kj}} \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j,k} B_{ij} F_{ki} \frac{\partial \exp(it\Gamma)}{\partial Z_{kj}} + F_{ki}^* B_{ij}^* \frac{\partial \exp(it\Gamma)}{\partial Z_{kj}^*}\right]. \end{aligned}$$

We obtain after a small calculation

$$\begin{aligned} \frac{\partial \exp(it\Gamma)}{\partial Z_{kj}} &= it \left( \frac{[AZ^* \Upsilon]_{jk}}{\sqrt{N}} + \frac{1}{2} [B^* F^*]_{jk} \right) \exp(it\Gamma), \\ \frac{\partial \exp(it\Gamma)}{\partial Z_{kj}^*} &= it \left( \frac{[\Upsilon Z A]_{kj}}{\sqrt{N}} + \frac{1}{2} [FB]_{kj} \right) \exp(it\Gamma) \end{aligned}$$

which leads to

$$\begin{aligned} \varphi'(t) &= -t\mathbb{E}[N^{-1} \operatorname{Tr} A^2 Z^* \Upsilon^2 Z \exp(it\Gamma)] - (t/2) \operatorname{Tr}(BB^* F^* F) \varphi(t) \\ &\quad + iN^{-1/2} \operatorname{Tr} A \operatorname{Tr} \Upsilon \varphi(t) \\ &\quad - t\mathbb{E}[N^{-1/2} \operatorname{Tr} AB^* F^* \Upsilon Z \exp(it\Gamma)] - (t/2)\mathbb{E}[N^{-1/2} \operatorname{Tr} Z^* \Upsilon FBA \exp(it\Gamma)]. \end{aligned}$$

Let us consider the first term at the right hand side of this equation. We have  $\mathbb{E}[N^{-1} \operatorname{Tr} A^2 Z^* \Upsilon^2 Z] = N^{-1} \operatorname{Tr} A^2 \operatorname{Tr} \Upsilon^2$ . Applying the Poincaré-Nash inequality, we obtain after some calculations that  $\operatorname{Var}(N^{-1} \operatorname{Tr} A^2 Z^* \Upsilon^2 Z) \leq 2N^{-2} \operatorname{Tr} A^4 \operatorname{Tr} \Upsilon^4 = \mathcal{O}(N^{-1})$  since  $\|\Upsilon\|$  is bounded. It results that  $\mathbb{E}[N^{-1} \operatorname{Tr} A^2 Z^* \Upsilon^2 Z \exp(it\Gamma)] = N^{-1} \operatorname{Tr} A^2 \operatorname{Tr} \Upsilon^2 \varphi(t) + \mathcal{O}(N^{-1/2})$  by Cauchy-Schwarz inequality. The third term is zero by hypothesis. Finally,  $N^{-1}\mathbb{E}|\operatorname{Tr} Z^* \Upsilon FBA|^2 = N^{-1} \operatorname{Tr} \Upsilon^2 FBA^2 B^* F^* \leq N^{-1} \|\Upsilon\|^2 \operatorname{Tr} FBA^2 B^* F^* = \mathcal{O}(N^{-1})$ . Hence, the last two terms are  $\mathcal{O}(N^{-1/2})$  by Cauchy-Schwarz inequality, which proves Lemma 5.1.

**A.2. An intermediate result.** The following lemma will be needed in the proof of Lemma 5.2:

**Lemma A.1.** *For  $x, y \in \operatorname{supp}(\mu)^c$ ,*

$$\begin{aligned} \mathbb{E} \left[ \phi_n \frac{1}{n} \operatorname{Tr} \tilde{Q}_n(x) D \tilde{Q}_n(y) D \right] &= \frac{\frac{1}{n} \operatorname{Tr} D \tilde{T}_n(x) D \tilde{T}_n(y)}{1 - c_n x m_n(x) y m_n(y) \frac{1}{n} \operatorname{Tr} D \tilde{T}_n(x) D \tilde{T}_n(y)} + \mathcal{O}(n^{-1}) \\ \mathbb{E} \left[ \phi_n \tilde{u}_n^* \tilde{Q}_n(x) D \tilde{Q}_n(y) \tilde{v}_n \right] &= \frac{\tilde{u}_n^* \tilde{T}_n(x) D \tilde{T}_n(y) \tilde{v}_n}{1 - c_n x m_n(x) y m_n(y) \frac{1}{n} \operatorname{Tr} D \tilde{T}_n(x) D \tilde{T}_n(y)} + \mathcal{O}(n^{-1}). \end{aligned}$$

*Proof.* We denote here  $\tilde{Q}_x = \tilde{Q}(x)$  and drop all unnecessary indices. Using the integration by parts formula, we obtain

$$\begin{aligned} \frac{1}{n} \mathbb{E} \left[ \phi Y_{ia}^* Y_{ij} \tilde{Q}_{x,jp} d_p \tilde{Q}_{y,pq} \right] &= \frac{d_p d_j}{n} \left( \delta(a-j) \mathbb{E} \left[ \phi \tilde{Q}_{x,jp} \tilde{Q}_{y,pq} \right] \right. \\ &\quad \left. - \frac{1}{n} \mathbb{E} \left[ \phi \tilde{Q}_{x,jj} [Y \tilde{Q}_x]_{ip} Y_{ia}^* \tilde{Q}_{y,pq} \right] - \frac{1}{n} \mathbb{E} \left[ \phi Y_{ia}^* \tilde{Q}_{x,jp} \tilde{Q}_{y,pj} [Y \tilde{Q}_y]_{iq} \right] \right. \\ &\quad \left. + \mathbb{E} \left[ \frac{1}{n} [\operatorname{adj}(\psi) \psi' Y]_{ij} Y_{ia}^* \tilde{Q}_{x,jp} \tilde{Q}_{y,pq} \right] \right). \end{aligned}$$

Summing over  $i, p$ , and  $j$ , this is

$$\begin{aligned} \frac{1}{n} \mathbb{E} \left[ \phi [Y^* Y \tilde{Q}_x D \tilde{Q}_y]_{aq} \right] &= \frac{1}{n^2} \mathbb{E} \left[ [Y^* \operatorname{adj}(\psi) \psi' Y D \tilde{Q}_x D \tilde{Q}_y]_{aq} \right] + c_n d_a \mathbb{E} \left[ \phi [\tilde{Q}_x D \tilde{Q}_y]_{aq} \right] \\ &\quad - \frac{1}{n} \mathbb{E} \left[ \phi \frac{1}{n} \operatorname{Tr} D \tilde{Q}_x [Y^* Y \tilde{Q}_x D \tilde{Q}_y]_{aq} \right] - \frac{1}{n} \mathbb{E} \left[ \phi \frac{1}{n} \operatorname{Tr} \tilde{Q}_x D \tilde{Q}_y D [Y^* Y \tilde{Q}_y]_{aq} \right]. \end{aligned}$$

Using the relation  $\frac{1}{n} Y^* Y \tilde{Q}_x = x \tilde{Q}_x + I_n$  and appropriately gathering the terms on each side gives

$$\begin{aligned} &\mathbb{E} \left[ \phi [\tilde{Q}_x D \tilde{Q}_y]_{aq} \left( x - c_n d_a + x \frac{1}{n} \operatorname{Tr} D \tilde{Q}_x \right) \right] \\ &= -\mathbb{E} \left[ \phi [D \tilde{Q}_y]_{aq} \left( 1 + \frac{1}{n} \operatorname{Tr} D \tilde{Q}_x \right) \right] - \mathbb{E} \left[ \phi \frac{1}{n} \operatorname{Tr} \tilde{Q}_x D \tilde{Q}_y D (\delta(a-q) + y [\tilde{Q}_y]_{aq}) \right] \\ &\quad + \mathbb{E} \left[ \frac{1}{n^2} [Y^* \operatorname{adj}(\psi) \psi' Y D \tilde{Q}_x D \tilde{Q}_y]_{aq} \right]. \end{aligned} \tag{16}$$

Introducing the term  $\tilde{\beta}_x = \phi \frac{1}{n} \text{Tr} D\tilde{Q}_x$  and  $\hat{\beta}_x = \tilde{\beta}_x - \phi \mathbb{E}[\tilde{\beta}_x]$ , we have

$$\begin{aligned}
& \mathbb{E} \left[ \phi [\tilde{Q}_x D \tilde{Q}_y]_{aq} \right] (x - c_n d_a + x \mathbb{E}[\tilde{\beta}_x]) \\
&= -\mathbb{E} \left[ \phi [D \tilde{Q}_y]_{aq} \right] (1 + \mathbb{E}[\tilde{\beta}_x]) - \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_y D \right] (\delta(a - q) + y \mathbb{E}[[\tilde{Q}_y]_{aq}]) \\
&- \mathbb{E} \left[ [D \tilde{Q}_y]_{aq} \hat{\beta}_x \right] - \mathbb{E} \left[ [\tilde{Q}_x D \tilde{Q}_y]_{aq} x \hat{\beta}_x \right] - \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_y D y ([\tilde{Q}_y]_{aq} - \mathbb{E}[\tilde{Q}_y]_{aq}) \right] \\
&+ \mathbb{E} \left[ \frac{1}{n^2} [Y^* \text{adj}(\psi) \psi' Y D \tilde{Q}_x D \tilde{Q}_y]_{aq} \right]. \tag{17}
\end{aligned}$$

At this point, we can prove both results for the trace and for the quadratic form. We start by dividing each side by  $x - c_n d_a + x \mathbb{E}[\tilde{\beta}_x]$ . We begin with the trace result. Multiplying the resulting left- and right-hand sides by  $d_a$ , summing over  $a = q$  and normalizing by  $1/n$ , we obtain

$$\begin{aligned}
\mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_x D \right] &= -(1 + \mathbb{E}[\tilde{\beta}_x]) \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} D \tilde{Q}_y D A_x \right] \\
&- \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_y D \right] \left( y \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} D A_x \tilde{Q}_y \right] + \frac{1}{n} \text{Tr} D A_x \right) + \varepsilon_n
\end{aligned}$$

where we denoted  $A_x = (x(1 + \mathbb{E}[\tilde{\beta}_x])I_n - c_n D)^{-1}$  and where

$$\begin{aligned}
\varepsilon_n &= \mathbb{E} \left[ \text{Tr} \frac{Y^* \text{adj}(\psi) \psi' Y}{n^3} D \tilde{Q}_x D \tilde{Q}_y D A_x \right] - \mathbb{E} \left[ \frac{1}{n} \text{Tr} D \tilde{Q}_y D A_x \hat{\beta}_x \right] \\
&- \mathbb{E} \left[ \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_y D A_x \hat{\beta}_x \right] \\
&- \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_y D y \left( \frac{1}{n} \text{Tr} \tilde{Q}_y D A_x - \mathbb{E} \left( \frac{1}{n} \text{Tr} \tilde{Q}_y D A_x \right) \right) \right]. \tag{18}
\end{aligned}$$

From Lemma 4.5,  $\mathbb{E}[\tilde{\beta}_x] = \tilde{\delta}_x + \mathcal{O}(n^{-2})$ , where we denoted  $\tilde{\delta}_x = \frac{1}{n} \text{Tr} D \tilde{T}_x$ . Also, it is easily observed that

$$\left( I_n (1 + \tilde{\delta}_x) x - c_n D \right)^{-1} = -\frac{1}{1 + \tilde{\delta}_x} \tilde{T}_x \tag{19}$$

with  $\tilde{T}_x = \tilde{T}(x)$ . Therefore, along with Lemma 4.5, we now have

$$\begin{aligned}
& \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_x D \right] \\
&= \frac{1}{n} \text{Tr} D \tilde{T}_x D \tilde{T}_y + \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_x D \right] \frac{y \frac{1}{n} \text{Tr} D \tilde{T}_x \tilde{T}_y + \frac{1}{n} \text{Tr} D \tilde{T}_x}{1 + \tilde{\delta}_x} + \varepsilon_n + \mathcal{O}(n^{-2}).
\end{aligned}$$

Using now the fact that  $y \tilde{T}_y + I_n = c_n \frac{1}{1 + \tilde{\delta}_y} D \tilde{T}_y$ , we conclude

$$\mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_x D \right] = \frac{\frac{1}{n} \text{Tr} D \tilde{T}_x D \tilde{T}_y}{1 - c_n (1 + \tilde{\delta}_x)^{-1} (1 + \tilde{\delta}_y)^{-1} \frac{1}{n} \text{Tr} D \tilde{T}_x D \tilde{T}_y} + \varepsilon_n + \mathcal{O}(n^{-2}).$$

It therefore remains to prove that  $\varepsilon_n = \mathcal{O}(n^{-1})$ . Due to the presence of  $\phi$  in the expression of  $\hat{\beta}_x$ , and using Lemma 4.3 and Cauchy-Schwarz inequality, one can see that the last three terms in the expression of  $\varepsilon_n$  are  $\mathcal{O}(n^{-1})$ . As for the first term, it is treated in a similar manner as in the proof of Lemma 4.6, and is  $\mathcal{O}(n^{-2})$ .

In order to prove the result on the quadratic form, we start again from (17). Dividing each side again by  $x - c_n d_a + x\mathbb{E}[\tilde{\beta}_x]$ , introducing  $[\tilde{u}]_a$ ,  $[\tilde{v}]_q$ , and summing over the indices, we obtain

$$\begin{aligned} & \mathbb{E} \left[ \phi \tilde{u}^* \tilde{Q}_x D \tilde{Q}_y \tilde{v} \right] \\ &= -\mathbb{E} \left[ \phi \tilde{u}^* A_x D \tilde{Q}_y \tilde{v} \right] - \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_y D \right] \left( \tilde{u}^* A_x (y \mathbb{E}[\phi \tilde{Q}_y] + I_n) \tilde{v} \right) + \varepsilon'_n \end{aligned} \quad (20)$$

where  $\varepsilon'_n$  is very similar to  $\varepsilon_n$  and is shown to be  $\mathcal{O}(n^{-1})$  with the same line of arguments. Using Lemma 4.3, (19), and the previous result on  $\mathbb{E}[\phi \frac{1}{n} \text{Tr} \tilde{Q}_x D \tilde{Q}_y D]$ , we finally obtain

$$\begin{aligned} & \mathbb{E} \left[ \phi \tilde{u}^* \tilde{Q}_x D \tilde{Q}_y \tilde{v} \right] \\ &= \tilde{u}^* \tilde{T}_x D \tilde{T}_y \tilde{v} \left( 1 + \frac{c_n(1 + \tilde{\delta}_x)^{-1}(1 + \tilde{\delta}_y)^{-1} \frac{1}{n} \text{Tr} D \tilde{T}_x D \tilde{T}_y}{1 - c_n(1 + \tilde{\delta}_x)^{-1}(1 + \tilde{\delta}_y)^{-1} \frac{1}{n} \text{Tr} D \tilde{T}_x D \tilde{T}_y} \right) + \mathcal{O}(n^{-1}). \end{aligned}$$

from which

$$\mathbb{E} \left[ \phi \tilde{u}^* \tilde{Q}_x D \tilde{Q}_y \tilde{v} \right] = \frac{\tilde{u}^* \tilde{T}_x D \tilde{T}_y \tilde{v}}{1 - c_n(1 + \tilde{\delta}_x)^{-1}(1 + \tilde{\delta}_y)^{-1} \frac{1}{n} \text{Tr} D \tilde{T}_x D \tilde{T}_y} + \mathcal{O}(n^{-1}).$$

We conclude with the remark  $xm_n(x) = -(1 + \tilde{\delta}_x)^{-1}$ .  $\square$

**A.3. Proof of Lemma 5.2.** The line of proof closely follows the proof of Lemma A.1. We provide here its main steps. By the integration by parts formula, we have

$$\begin{aligned} \mathbb{E}[\phi \tilde{Q}_{pk} Y_{\ell k}^* Y_{\ell m} \tilde{Q}_{mr}] &= -\frac{d_m}{n} \mathbb{E}[\phi \tilde{Q}_{pk} Y_{\ell k}^* \tilde{Q}_{mm} [Y \tilde{Q}]_{\ell r}] + \delta(k - m) d_m \mathbb{E}[\phi \tilde{Q}_{pk} \tilde{Q}_{mr}] \\ &\quad - \frac{d_m}{n} \mathbb{E}[\phi Y_{\ell k}^* \tilde{Q}_{mr} \tilde{Q}_{pm} [Y \tilde{Q}]_{\ell k}] \\ &\quad + \frac{d_m}{n} \mathbb{E}[\tilde{Q}_{pk} Y_{\ell k}^* \tilde{Q}_{mr} [\text{adj}(\psi) \psi' Y]_{\ell m}] \end{aligned}$$

Taking the sum over  $m$ , we obtain

$$\begin{aligned} \mathbb{E}[\phi \tilde{Q}_{pk} Y_{\ell k}^* [Y \tilde{Q}]_{\ell r}] &= \frac{d_k}{1 + \mathbb{E}[\tilde{\beta}]} \mathbb{E}[\phi \tilde{Q}_{pk} \tilde{Q}_{kr}] - \frac{1}{1 + \mathbb{E}[\tilde{\beta}]} \frac{1}{n} \mathbb{E}[\phi Y_{\ell k}^* [\tilde{Q} D \tilde{Q}]_{pr} [Y \tilde{Q}]_{\ell k}] \\ &\quad + \frac{1}{1 + \mathbb{E}[\tilde{\beta}]} \frac{1}{n} \mathbb{E}[\tilde{Q}_{pk} Y_{\ell k}^* [\text{adj}(\psi) \psi' Y D \tilde{Q}]_{\ell r}] - \mathbb{E}[\hat{\beta} \tilde{Q}_{pk} Y_{\ell k}^* [Y \tilde{Q}]_{\ell r}] \end{aligned}$$

where  $\tilde{\beta}(x) = \phi \frac{1}{n} \text{Tr} D \tilde{Q}(x)$  and  $\hat{\beta}(x) = \tilde{\beta}(x) - \phi \mathbb{E}[\tilde{\beta}(x)]$  as in the proof of Lemma A.1. Taking the sum over  $\ell$  then over  $k$ , we obtain

$$\begin{aligned} \mathbb{E}[\phi [\tilde{Q} \frac{Y^* Y}{n} \tilde{Q}]_{pr}] &= c_n \frac{1}{1 + \mathbb{E}[\tilde{\beta}]} \mathbb{E}[\phi [\tilde{Q} D \tilde{Q}]_{pr}] - \frac{1}{1 + \mathbb{E}[\tilde{\beta}]} \mathbb{E}[\phi [\tilde{Q} D \tilde{Q}]_{pr} \frac{1}{n} \text{Tr}(\frac{Y^* Y}{n} \tilde{Q})] \\ &\quad + \frac{1}{1 + \mathbb{E}[\tilde{\beta}]} \frac{1}{n} \mathbb{E}[[\tilde{Q} \frac{Y^* \text{adj}(\psi) \psi' Y}{n} D \tilde{Q}]_{pr}] - \mathbb{E}[\hat{\beta} [\tilde{Q} \frac{Y^* Y}{n} \tilde{Q}]_{pr}] \end{aligned}$$

Observing that  $(1 + \mathbb{E}[\tilde{\beta}(x)])^{-1} = -xm_n(x) + \mathcal{O}(n^{-2})$  and making the usual approximations, we get

$$\mathbb{E}[\phi \tilde{u}^* \tilde{Q} \frac{Y^* Y}{n} \tilde{Q} \tilde{u}] = \left( xm_n(x) \frac{1}{n} \text{Tr}(\mathbb{E}[\phi \frac{Y^* Y}{n} \tilde{Q}]) - c_n xm_n(x) \right) \mathbb{E}[\phi \tilde{u}^* \tilde{Q} D \tilde{Q} \tilde{u}] + \mathcal{O}(n^{-1})$$

Observing that  $n^{-1} \text{Tr}[\mathbb{E}[\phi(n^{-1}Y^*Y)\tilde{Q}(x)]] = Nn^{-1}xm_n(x) + Nn^{-1} + \mathcal{O}(n^{-2})$  and invoking Lemma A.1, we obtain the desired result.

**A.4. Proof of Lemma 5.3.** As in the previous proofs, we discard unnecessary indices. We also denote  $\tilde{Q}_i = \tilde{Q}(\rho_i)$ . For readability, we also write  $\tilde{M}_i = M_{i,n}A_i$  and use the shortcut notation  $\Gamma = \sqrt{n} \sum_{i=1}^p \text{Tr} M_i^* \tilde{Q}_i \tilde{M}_i$ . We focus first on the term in  $\rho_1$ . The line of proof closely follows that of Lemma 5.1 with the exception that we need to introduce the regularization function  $\phi$  to ensure the existence of all the quantities under study. That is, with  $\varphi_N(t) = \mathbb{E}[\exp(it\phi\Gamma)]$ , we only need to show that  $\varphi'_N(t) = -t\tilde{\sigma}_n^2\varphi_N(t) + \mathcal{O}(1/\sqrt{n})$ . Using  $|\varphi_N(t)| \leq 1$  and Lemma 4.4,  $|\mathbb{E}[\exp(it\Gamma)] - \varphi_N(t)| \leq 1 - \mathbb{E}[\phi] \rightarrow 0$  as  $N \rightarrow \infty$ , from which the result unfolds.

Using the integration by parts formula, we first obtain

$$\begin{aligned} & \mathbb{E} \left[ \phi \left[ \frac{Y^*Y}{n} \tilde{Q}_1 \right]_{pq} e^{it\phi\Gamma} \right] \\ &= c_n \mathbb{E} \left[ \phi [D\tilde{Q}_1]_{pq} e^{it\phi\Gamma} \right] - \mathbb{E} \left[ \phi \frac{1}{n} \text{Tr} D\tilde{Q}_1 \left[ \frac{Y^*Y}{n} \tilde{Q}_1 \right]_{pq} e^{it\phi\Gamma} \right] \\ & - \mathbb{E} \left[ ite^{it\phi\Gamma} \phi^2 \frac{1}{\sqrt{n}} \sum_{j=1}^p \sum_{a=1}^r \left[ (\tilde{M}_j)_a^* \tilde{Q}_j D\tilde{Q}_1 \right]_q \left[ \frac{Y^*Y}{n} \tilde{Q}_j (M_j)_a \right]_p \right] + \varepsilon_{n,pq} \end{aligned}$$

where

$$\varepsilon_{n,pq} = \mathbb{E} \left[ \frac{1}{n} \left[ \frac{Y^* \text{adj}(\psi)\psi'Y}{n} D\tilde{Q}_1 \right]_{pq} e^{it\phi\Gamma} \right] + \mathbb{E} \left[ \phi \frac{1}{n} \left[ \frac{Y^* \text{adj}(\psi)\psi'Y}{n} D\tilde{Q}_1 \right]_{pq} it\Gamma e^{it\phi\Gamma} \right]$$

and where we denoted  $X_a$  the column  $a$  of matrix  $X$ ,  $X_a^*$  being the row vector  $(X_a)^*$ .

With  $\tilde{\beta}_j = \phi \frac{1}{n} \text{Tr} D\tilde{Q}_j$ ,  $\hat{\beta}_j = \tilde{\beta}_j - \phi \mathbb{E}[\tilde{\beta}_j]$ , and with the relation  $n^{-1}Y^*Y\tilde{Q}_1 = I_n + \rho_1\tilde{Q}_1$ , we obtain

$$\begin{aligned} & \left( \rho_1(1 + \mathbb{E}[\tilde{\beta}_1]) - c_n d_p \right) \mathbb{E} \left[ \phi [\tilde{Q}_1]_{pq} e^{it\phi\Gamma} \right] = -\delta(p-q)(1 + \mathbb{E}[\tilde{\beta}_1]) \mathbb{E} \left[ \phi e^{it\phi\Gamma} \right] \\ & - \mathbb{E} \left[ ite^{it\phi\Gamma} \phi^2 \frac{1}{\sqrt{n}} \sum_{j=1}^p \sum_{a=1}^r \left[ (\tilde{M}_j)_a^* \tilde{Q}_j D\tilde{Q}_1 \right]_q \left[ \frac{Y^*Y}{n} \tilde{Q}_j (M_j)_a \right]_p \right] + \varepsilon'_{n,pq} \end{aligned}$$

where

$$\varepsilon'_{n,pq} = \varepsilon_{n,pq} - \mathbb{E} \left[ \hat{\beta}_1 \left[ \frac{Y^*Y}{n} \tilde{Q}_1 \right]_{pq} e^{it\phi\Gamma} \right].$$

Dividing each side by  $\rho_1(1 + \mathbb{E}[\tilde{\beta}_1]) - c_n d_p$ , then multiplying by  $(\tilde{M}_1)_p$  and  $(M_1)_q$ , and summing over  $p, q$  gives

$$\begin{aligned} \mathbb{E}[\phi \text{Tr}(\tilde{M}_1^* \tilde{Q}_1 M_1) e^{it\phi\Gamma}] &= -(1 + \mathbb{E}[\tilde{\beta}_1]) \mathbb{E}[\phi e^{it\Gamma}] \text{Tr} \left( \tilde{M}_1^* A_{\rho_1} M_1 \right) \\ & - it \mathbb{E} \left[ \phi^2 e^{it\phi\Gamma} \frac{1}{\sqrt{n}} \sum_{j=1}^p \text{Tr} \tilde{M}_1^* A_{\rho_1} \frac{Y^*Y}{n} \tilde{Q}_j M_j \tilde{M}_j^* \tilde{Q}_j D\tilde{Q}_1 \right] + \varepsilon'_n \end{aligned}$$



with  $A_{\rho_i} = (\rho_i(1 + \mathbb{E}[\tilde{\beta}_i])I_n - c_n D)^{-1}$ , and

$$\varepsilon'_n = \text{Tr } \tilde{M}_1^* A_{\rho_1} E' M_1$$

with  $(E')_{pq} = \varepsilon'_{pq}$ . From (19), the identity  $n^{-1} Y^* Y \tilde{Q}_j = I_n + \rho_j \tilde{Q}_j$ , and Lemma 4.5, we finally obtain

$$\begin{aligned} & \mathbb{E} \left[ \phi \text{Tr}(\tilde{M}_1^* \tilde{Q}_1 M_1) e^{t\phi\Gamma} \right] - \mathbb{E}[\phi e^{t\phi\Gamma}] \text{Tr } \tilde{M}_1 \tilde{T}_1 M_1 \\ &= it \mathbb{E}[\phi e^{t\phi\Gamma}] \frac{1}{\sqrt{n}} \sum_{j=1}^p \frac{\text{Tr } \tilde{M}_1^* \frac{\tilde{T}_1}{1+\tilde{\delta}_1} \frac{c_n D \tilde{T}_j}{1+\tilde{\delta}_j} M_j \tilde{M}_j^* \tilde{T}_j D \tilde{T}_1 M_1}{1 - c_n(1 + \tilde{\delta}_1)^{-1}(1 + \tilde{\delta}_j)^{-1} \frac{1}{n} \text{Tr } D \tilde{T}_1 D \tilde{T}_j} + \varepsilon'_n + \mathcal{O}(n^{-2}) \end{aligned}$$

with  $\tilde{T}_i = \tilde{T}(\rho_i)$ , from which

$$\begin{aligned} & \mathbb{E} \left[ \phi \text{Tr}(\tilde{M}_1^* \tilde{Q}_1 M_1) e^{t\phi\Gamma} \right] - \mathbb{E}[\phi e^{t\phi\Gamma}] \text{Tr } \tilde{M}_1 \tilde{T}_1 M_1 \\ &= \frac{it \mathbb{E}[\phi e^{t\phi\Gamma}]}{\sqrt{n}} \sum_{j=1}^p \frac{c_n \rho_1 m_n(\rho_1) \rho_j m_n(\rho_j) \text{Tr } \tilde{M}_1^* \tilde{T}_1 D \tilde{T}_j M_j \tilde{M}_j^* \tilde{T}_j D \tilde{T}_1 M_1}{1 - c_n \rho_1 m_n(\rho_1) \rho_j m_n(\rho_j) \frac{1}{n} \text{Tr } D \tilde{T}_1 D \tilde{T}_j} + \varepsilon'_n + \mathcal{O}(n^{-2}). \end{aligned}$$

It remains to show that  $\varepsilon'_n = \mathcal{O}(n^{-1})$ . We have explicitly

$$\begin{aligned} \varepsilon'_n &= \mathbb{E} \left[ \frac{1}{n} \text{Tr} \left( \tilde{M}_1^* A_{\rho_1} \frac{Y^* \text{adj}(\psi) \psi' Y}{n} D \tilde{Q}_1 M_1 \right) (1 + \phi t \Gamma) e^{t\phi\Gamma} \right] \\ &\quad - \mathbb{E} \left[ \phi \hat{\beta}_1 \text{Tr} \left( \tilde{M}_1^* A_{\rho_1} \frac{Y^* Y}{n} \tilde{Q}_1 M_1 \right) e^{t\phi\Gamma} \right]. \end{aligned}$$

Using the fact that  $|e^{t\phi\Gamma}| = 1$  and the relation  $n^{-1} Y^* Y \tilde{Q}_1 = \rho_1 \tilde{Q}_1 + I_n$ , the second term is easily shown to be  $\mathcal{O}(n^{-1})$  from the Cauchy-Schwarz inequality and Lemma 4.3. If it were not for the factor  $\Gamma$ , the convergence of the first term would unfold from similar arguments as in the proof of Lemma 4.6. We only need to show here that  $\mathbb{E}[|\phi\Gamma|^2] = \mathcal{O}(1)$ . But this follows immediately from Lemma 4.3 and Lemma 4.5.

The generalization to  $\sum_i \mathbb{E}[\phi \text{Tr}(\tilde{M}_i^* \tilde{Q}_i M_i) e^{t\phi\Gamma}]$  is then immediate and we have the expected result.

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FRANÇOIS CHAPON,  
Université Paul Sabatier,  
Institut de Mathématiques de Toulouse,  
118 route de Narbonne,  
31062 Toulouse Cedex 9,  
France.  
e-mail: [francois.chapon@math.univ-toulouse.fr](mailto:francois.chapon@math.univ-toulouse.fr)

ROMAIN COUILLET,  
Supélec,  
Plateau de Moulon,  
91192 Gif-sur-Yvette Cedex,  
France.  
e-mail: [romain.couillet@supelec.fr](mailto:romain.couillet@supelec.fr)

WALID HACHEM,  
CNRS, Télécom Paristech  
46, rue Barrault,  
75634 Paris Cedex 13,  
France.  
e-mail: [walid.hachem@telecom-paristech.fr](mailto:walid.hachem@telecom-paristech.fr)

XAVIER MESTRE,  
Centre Tecnològic de Telecomunicacions de Catalunya (CTTC),  
Parc Mediterrani de la Tecnologia (PMT) - Building B4,  
Av. Carl Friedrich Gauss 7,  
08860 - Castelldefels  
Barcelona, Spain.  
e-mail: [xavier.mestre@cttc.cat](mailto:xavier.mestre@cttc.cat)