

# A quenched weak invariance principle

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## Abstract

In this paper we study the almost sure conditional central limit theorem in its functional form for a class of random variables satisfying a projective criterion. Applications to strongly mixing processes and non irreducible Markov chains are given. The proofs are based on the normal approximation of double indexed martingale-like sequences, a theory which has interest in itself.

*Key words:* quenched central limit theorem, weak invariance principle, Markov chains.

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## 1 Introduction

Let  $(\xi_i)_{i \geq 0}$  be a Markov chain admitting an invariant probability  $\pi$ . Let  $f$  be a real-valued function such that  $\pi(f^2) < \infty$  and  $\pi(f) = 0$ , and let  $S_n = f(\xi_1) + \dots + f(\xi_n)$ . If the central limit theorem (CLT) holds for  $n^{-1/2}S_n$  starting from the initial distribution  $\pi$ , an interesting question is to know whether it remains true for another initial distribution  $\nu$ . Maxwell and Woodroffe (2000) have given a projective criterion under which  $S_n$  satisfies the so-called conditional CLT, which implies that the CLT holds for any initial distribution having a bounded density with respect to  $\pi$ . Necessary and sufficient conditions for the conditional CLT are given in Dedecker and Merlevède (2002), and Wu and Woodroffe (2004).

The question is more delicate if  $\nu$  is a Dirac mass at point  $x$ . One says that the CLT is quenched if it holds for almost every starting point with respect to  $\pi$ . The same terminology is used for the functional central limit theorem (FCLT). For Harris recurrent Markov chain, this question is solved thanks to Proposition 17.1.6 in Meyn and Tweedie (1993): if the CLT (or FCLT) holds for the initial distribution  $\pi$ , then it holds for any initial distribution, and hence for any starting point  $x$ . In the non irreducible setting, the situation is not so clear.

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This question of the quenched CLT can be formulated in the more general context of stationary sequences: it means that, on a set of measure one, the central limit theorem holds when replacing the usual expectation by the conditional expectation with respect to the past  $\sigma$ -algebra. Some examples of stationary processes satisfying the CLT but not the quenched CLT can be found in Derriennic and Lin (2001) and Volný and Woodroffe (2010a).

The first general results on the quenched CLT and FCLT are given in Borodin and Ibragimov (1994): in the Markov chain setting, it says that the FCLT holds if there is a solution in  $\mathbb{L}^2(\pi)$  to the Poisson equation (see Gordin and Lifschitz (1978)); in a general setting it means that the FCLT is true under Gordin's condition (1969). This result has been improved by Derriennic and Lin (2001, 2003), Zhao and Woodroffe (2008), Cuny (2011), Cuny and Peligrad (2012), Volný and Woodroffe (2010b) and Merlevède *et al.* (2012). In a recent paper, Cuny and Merlevède (2012) have proved that the FCLT is quenched under the condition of Maxwell and Woodroffe (2000).

All the papers cited above use a martingale approximation in  $\mathbb{L}^2$ . Consequently, the projective condition obtained up to now are always expressed in terms of  $\mathbb{L}^2$  norms of conditional expectations. In this paper, we prove the quenched FCLT under a projective condition involving  $\mathbb{L}^1$ -norms, in the spirit of Gordin (1973). As a consequence, we obtain that the FCLT of Doukhan *et al.* (1994) for strongly mixing sequences is quenched. Note that Doukhan *et al.* (1994) have shown that their condition is optimal in some sense for the usual FCLT, so it is also sharp for the quenched FCLT. In Section 3.1, we study the example of the non irreducible Markov chain associated to an intermittent map. Once again, we shall see through this example that our condition is essentially optimal.

Our main result, Theorem 2.1 below, is a consequence of the more general Proposition 4.1, where the conditions are expressed in terms of conditional expectations of partial sums. The proof of this proposition is done via a blocking argument followed by a two step martingale decomposition. We start with a finite number of consecutive blocks of random variables. The sum in blocks are approximated by martingales. This decomposition introduces the need of studying the normal approximation for a family of double indexed martingales. This approximation has interest in itself and is presented in Section 6.

## 2 Results

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space, and  $T : \Omega \mapsto \Omega$  be a bijective bimeasurable transformation preserving the probability  $\mathbb{P}$ . An element  $A$  is said to be invariant if  $T(A) = A$ . We denote by  $\mathcal{I}$  the  $\sigma$ -algebra of all invariant sets. The probability  $\mathbb{P}$  is ergodic if each element of  $\mathcal{I}$  has measure 0 or 1.

Let  $\mathcal{F}_0$  be a  $\sigma$ -algebra of  $\mathcal{A}$  satisfying  $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$  and define the nondecreasing filtration  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$  by  $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$ . We assume that there exists a regular version  $P_{T|\mathcal{F}_0}$  of  $T$  given  $\mathcal{F}_0$ , and for any integrable random variable  $f$  from  $\Omega$  to  $\mathbb{R}$  we write  $K(f) = P_{T|\mathcal{F}_0}(f)$ . Since  $\mathbb{P}$

is invariant by  $T$ , for any integer  $k$ , a regular version  $P_{T|\mathcal{F}_k}$  of  $T$  given  $\mathcal{F}_k$  is then obtained via  $P_{T|\mathcal{F}_k}(f) = K(f \circ T^{-k}) \circ T^k$ . In the sequel, all the conditional expectations of functions of  $T$  with respect to  $\mathcal{F}_k$  are obtained through these conditional probabilities. With these notations  $\mathbb{E}(f \circ T^2|\mathcal{F}_0) = K(K(f) \circ T) = K^2(f)$ , and more generally, for any positive integer  $\ell$ ,  $\mathbb{E}(f \circ T^\ell|\mathcal{F}_0) = K^\ell(f)$ . In what follows we shall often use the notation  $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$ .

Let  $X_0$  be an  $\mathcal{F}_0$ -measurable, square integrable and centered random variable. Define the sequence  $\mathbf{X} = (X_i)_{i \in \mathbb{Z}}$  by  $X_i = X_0 \circ T^i$ . Let  $S_n = X_1 + \dots + X_n$ , and define the Donsker process  $W_n$  by  $W_n(t) = n^{-1/2}(S_{[nt]} + (nt - [nt])X_{[nt]+1})$ .

Let  $\mathcal{H}^*$  the space of continuous functions  $\varphi$  from  $(C([0, 1]), \|\cdot\|_\infty)$  to  $\mathbb{R}$  such that  $x \rightarrow |(1 + \|x\|_\infty)^{-1}\varphi(x)|$  is bounded. Our main result is the following:

**Theorem 2.1.** *Assume that*

$$\sum_{k \geq 0} \|X_0 \mathbb{E}_0(X_k)\|_1 < \infty. \quad (2.1)$$

*then the series*

$$\eta = \mathbb{E}(X_0^2|\mathcal{I}) + 2 \sum_{k > 0} \mathbb{E}(X_0 X_k|\mathcal{I}) \quad (2.2)$$

*converges in  $\mathbb{L}^1$ . Moreover, for any  $\varphi$  in  $\mathcal{H}^*$*

$$\lim_{n \rightarrow \infty} \mathbb{E}_0(\varphi(W_n)) = \int \varphi(x\sqrt{\eta})W(dx) \quad \text{almost surely and in } \mathbb{L}^1, \quad (2.3)$$

*where  $W$  is the distribution of a standard Wiener process.*

Note that the  $\mathbb{L}^1$ -convergence in (2.3) has been proved in Dedecker and Merlevède (2002). In this paper, we shall prove the almost sure convergence. Note that various classes satisfying (2.1) can be found in Dedecker and Rio (2000).

This result has an interesting interpretation in the terminology of additive functionals of Markov chains. Let  $(\xi_n)_{n \geq 0}$  be a Markov chain with values in a Polish space  $S$ , so that there exists a regular transition probability  $P_{\xi_1|\xi_0=x}$ . Let  $P$  be the transition kernel defined by  $P(f)(x) = P_{\xi_1|\xi_0=x}(f)$  for any bounded measurable function  $f$  from  $S$  to  $\mathbb{R}$ , and assume that there exists an invariant probability  $\pi$  for this transition kernel, that is a probability measure on  $S$  such that  $\pi(f) = \pi(P(f))$  for any bounded measurable function  $f$  from  $S$  to  $\mathbb{R}$ . Let then  $\mathbb{L}_0^2(\pi)$  be the set of functions from  $S$  to  $\mathbb{R}$  such that  $\pi(f^2) < \infty$  and  $\pi(f) = 0$ . For  $f \in \mathbb{L}_0^2(\pi)$  define  $X_i = f(\xi_i)$ . Notice that any stationary sequence  $(Y_k)_{k \in \mathbb{Z}}$  can be viewed as a function of a Markov process  $\xi_k = (Y_i; i \leq k)$ , for the function  $g(\xi_k) = Y_k$ .

In this setting the condition (2.1) is  $\sum_{k \geq 0} \pi(|f P^k(f)|) < \infty$ . Note that the random variable  $\eta$  defined in Theorem 2.1 is the limit almost surely and in  $\mathbb{L}^1$  of  $n^{-1}\mathbb{E}(S_n^2|\xi_0)$ , in such a way that  $\eta = \bar{\eta}(\xi_0)$ . By stationarity, it is also the limit in  $\mathbb{L}^1$  of the sequence  $n^{-1}\mathbb{E}((X_2 + \dots + X_{n+1})^2|\xi_1)$ , so that  $\bar{\eta}(\xi_0) = \bar{\eta}(\xi_1)$  almost surely. Consequently  $\bar{\eta}$  is an harmonic function for  $P$  in the sense that  $\pi$ -almost surely  $P(\bar{\eta}) = \bar{\eta}$ .

In the context of Markov chain the conclusion of Theorem 2.1 is also known under the

terminology of FCLT started at a point. To rephrase it, let  $\mathbb{P}^x$  be the probability associated to the Markov chain started from  $x$  and let  $\mathbb{E}^x$  be the corresponding expectation. Then, for any  $\varphi$  in  $\mathcal{H}^*$ ,

$$\pi \left\{ x : \mathbb{E}^x(\varphi(W_n)) \rightarrow \int \varphi(z\sqrt{\bar{\eta}(x)})W(dz) \right\} = 1 \quad (2.4)$$

and

$$\lim_{n \rightarrow \infty} \int \left| \mathbb{E}^x(\varphi(W_n)) - \int \varphi(z\sqrt{\bar{\eta}(x)})W(dz) \right| \pi(dx) = 0. \quad (2.5)$$

Note that in Theorem 2.1 no assumption of irreducibility nor of aperiodicity is imposed. Under the additional assumptions that the Markov chain is irreducible, aperiodic and positively recurrent, Chen (1999) showed that the CLT holds for the stationary Markov chain under the condition  $\sum_{k \geq 0} \pi(fP^k(f))$  is convergent, and the quenched CLT holds under the same condition by applying Proposition 17.1.6 in Meyn and Tweedie (1993).

### 3 Applications

As a consequence of Theorem 2.1, we obtain the following corollary for a class of weakly dependent sequences. We first need some definitions.

**Definition 3.1.** For a sequence  $\mathbf{Y} = (Y_i)_{i \in \mathbb{Z}}$ , where  $Y_i = Y_0 \circ T^i$  and  $Y_0$  is an  $\mathcal{F}_0$ -measurable and real-valued random variable, let for any  $k \in \mathbb{N}$ ,

$$\alpha_{\mathbf{Y}}(k) = \sup_{t \in \mathbb{R}} \left\| \mathbb{E}(\mathbf{1}_{Y_k \leq t} | \mathcal{F}_0) - \mathbb{E}(\mathbf{1}_{Y_k \leq t}) \right\|_1.$$

**Definition 3.2.** A quantile function  $Q$  is a function from  $]0, 1]$  to  $\mathbb{R}_+$ , which is left-continuous and non increasing. For any nonnegative random variable  $Z$ , we define the quantile function  $Q_Z$  of  $Z$  by  $Q_Z(u) = \inf\{t \geq 0 : \mathbb{P}(|Z| > t) \leq u\}$ .

**Definition 3.3.** Let  $\mu$  be the probability distribution of a random variable  $X$ . If  $Q$  is an integrable quantile function, let  $\text{Mon}(Q, \mu)$  be the set of functions  $g$  which are monotonic on some open interval of  $\mathbb{R}$  and null elsewhere and such that  $Q_{|g(X)|} \leq Q$ . Let  $\mathcal{F}(Q, \mu)$  be the closure in  $\mathbb{L}^1(\mu)$  of the set of functions which can be written as  $\sum_{\ell=1}^L a_\ell f_\ell$ , where  $\sum_{\ell=1}^L |a_\ell| \leq 1$  and  $f_\ell$  belongs to  $\text{Mon}(Q, \mu)$ .

**Corollary 3.4.** Let  $Y_0$  be a real-valued random variable with law  $P_{Y_0}$ , and  $Y_i = Y_0 \circ T^i$ . Let  $Q$  be a quantile function such that

$$\sum_{k \geq 0} \int_0^{\alpha_{\mathbf{Y}}(k)} Q^2(u) du < \infty. \quad (3.1)$$

Let  $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$ , where  $f$  belongs to  $\mathcal{F}(Q, P_{Y_0})$ . Then (2.1) is satisfied and consequently, for any  $\varphi$  in  $\mathcal{H}^*$ , (2.3) holds.

To prove that (3.1) implies (2.1), it suffices to apply Proposition 5.3 with  $m = q = 1$  of Merlevède and Rio (2012).

Notice that if  $(\alpha(k))_{k \geq 0}$  is the usual sequence of strong mixing coefficients of the stationary sequence  $(X_i)_{i \in \mathbb{Z}}$  as defined by Rosenblatt (1956), then it follows from Corollary 3.4 that if

$$\sum_{k \geq 0} \int_0^{\alpha(k)} Q_{|X_0|}^2(u) du < \infty, \quad (3.2)$$

then (2.3) holds for any  $\varphi$  in  $\mathcal{H}^*$ . Hence the weak invariance principle of Doukhan *et al.* (1994) is also quenched. We refer to Theorem 2 in Doukhan *et al.* (1994) and to Bradley (1997) for a discussion on the optimality of the condition (3.2).

### 3.1 Application to functions of Markov chains associated to intermittent maps

For  $\gamma$  in  $]0, 1[$ , we consider the intermittent map  $T_\gamma$  from  $[0, 1]$  to  $[0, 1]$ , which is a modification of the Pomeau-Manneville map (1980):

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[ \\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$

Recall that  $T_\gamma$  is ergodic (and even mixing in the ergodic theoretic sense) and that there exists a unique  $T_\gamma$ -invariant probability measure  $\nu_\gamma$  on  $[0, 1]$ , which is absolutely continuous with respect to the Lebesgue measure. We denote by  $L_\gamma$  the Perron-Frobenius operator of  $T_\gamma$  with respect to  $\nu_\gamma$ . Recall that for any bounded measurable functions  $f$  and  $g$ ,

$$\nu_\gamma(f \cdot g \circ T_\gamma) = \nu_\gamma(L_\gamma(f)g).$$

Let  $(Y_i)_{i \geq 0}$  be a Markov chain with transition Kernel  $L_\gamma$  and invariant measure  $\nu_\gamma$ .

**Definition 3.5.** A function  $H$  from  $\mathbb{R}_+$  to  $[0, 1]$  is a tail function if it is non-increasing, right continuous, converges to zero at infinity, and  $x \rightarrow xH(x)$  is integrable. If  $\mu$  is a probability measure on  $\mathbb{R}$  and  $H$  is a tail function, let  $\text{Mon}^*(H, \mu)$  denote the set of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  which are monotonic on some open interval and null elsewhere and such that  $\mu(|f| > t) \leq H(t)$ . Let  $\mathcal{F}^*(H, \mu)$  be the closure in  $\mathbf{L}^1(\mu)$  of the set of functions which can be written as  $\sum_{\ell=1}^L a_\ell f_\ell$ , where  $\sum_{\ell=1}^L |a_\ell| \leq 1$  and  $f_\ell \in \text{Mon}^*(H, \mu)$ .

**Corollary 3.6.** Let  $\gamma \in (0, 1/2)$  and  $(Y_i)_{i \geq 1}$  be a stationary Markov chain with transition kernel  $L_\gamma$  and invariant measure  $\nu_\gamma$ . Let  $H$  be a tail function such that

$$\int_0^\infty x(H(x))^{\frac{1-2\gamma}{1-\gamma}} dx < \infty. \quad (3.3)$$

Let  $X_i = f(Y_i) - \nu_\gamma(f)$  where  $f$  belongs to  $\mathcal{F}^*(H, \nu_\gamma)$ . Then (2.1) is satisfied and, for any  $\varphi$  in  $\mathcal{H}^*$ , (2.3) holds with

$$\eta = \nu_\gamma((f - \nu_\gamma(f))^2) + 2 \sum_{k>0} \nu_\gamma((f - \nu_\gamma(f))f \circ T_\gamma^k). \quad (3.4)$$

**Proof.** To prove this corollary, it suffices to see that (3.3) implies (3.1). For this purpose, we use Proposition 1.17 in Dedecker *et al.* (2010) stating that there exist two positive constant  $B, C$  such that, for any  $n > 0$ ,  $Bn^{(\gamma-1)/\gamma} \leq \alpha_{\mathbf{Y}}(n) \leq Cn^{(\gamma-1)/\gamma}$ , together with their computations page 817.  $\square$

In particular, if  $f$  is BV and  $\gamma < 1/2$ , we infer from Corollary 3.6 that (2.3) holds for any  $\varphi$  in  $\mathcal{H}^*$ , with  $\eta$  defined by (3.4). Note also that (3.3) is satisfied if  $H$  is such that  $H(x) \leq Cx^{-2(1-\gamma)/(1-2\gamma)}(\ln(x))^{-b}$  for  $x$  large enough and  $b > (1-\gamma)/(1-2\gamma)$ . Therefore, since the density  $h_{\nu_\gamma}$  of  $\nu_\gamma$  is such that  $h_{\nu_\gamma}(x) \leq Cx^{-\gamma}$  on  $(0, 1]$ , one can easily prove that if  $f$  is positive and non increasing on  $(0, 1)$ , with

$$f(x) \leq \frac{C}{x^{(1-2\gamma)/2} |\ln(x)|^b} \quad \text{near 0 for some } b > 1/2,$$

then (3.3) and the quenched FCLT hold. Notice that when  $f$  is exactly of the form  $f(x) = x^{-(1-2\gamma)/2}$ , Gouëzel (2004) proved that the central limit theorem holds for  $\sum_{i=1}^n (f(Y_i) - \nu_\gamma(f))$  but with the normalization  $\sqrt{n \ln(n)}$ . This shows that the condition (3.3) is essentially optimal for the quenched CLT with the normalization  $\sqrt{n}$ .

## 4 Some general results

In this section we develop sufficient conditions imposed to conditional expectations of partial sums for the validity of the quenched CLT and FCLT.

Let  $\eta$  be a  $\mathcal{F}_0$ -measurable random variable, such that  $\eta = \eta \circ T$  almost surely. For any positive integers  $i$  and  $p$ , define  $S_p^{(i)} = S_{pi} - S_{p(i-1)}$ .

## 4.1 A quenched CLT

Let us introduce the following three conditions under which the quenched central limit theorem holds:

$$\begin{aligned}
\mathbf{C}_1 \quad & \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \frac{1}{\sqrt{mp}} \sum_{i=2}^{m+1} \mathbb{E}_0 |\mathbb{E}_{(i-2)p}(S_p^{(i)})| = 0 \quad a.s. \\
\mathbf{C}_2 \quad & \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \mathbb{E}_0 \left| \sum_{i=1}^m \frac{1}{mp} \mathbb{E}_{(i-1)p} ((S_p^{(i+1)})^2) - \eta \right| = 0 \quad a.s. \\
& \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \mathbb{E}_0 \left| \sum_{i=1}^m \frac{1}{mp} \mathbb{E}_{(i-1)p} ((S_p^{(i)} + S_p^{(i+1)})^2) - 2\eta \right| = 0 \quad a.s. \\
\mathbf{C}_3 \quad & \text{for each } \varepsilon > 0 \quad \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \frac{1}{p} \mathbb{E}_0 ((S_p^{(i)})^2 \mathbf{1}_{|S_p^{(i)}|/\sqrt{p} > \varepsilon \sqrt{m}}) = 0 \quad a.s.
\end{aligned}$$

**Proposition 4.1.** *Assume that  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_3$  hold. Then, for any continuous and bounded function  $f$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_0(f(n^{-1/2} S_n)) = \int f(x \sqrt{\eta}) g(x) dx \quad a.s.$$

where  $g$  is the density of a standard normal.

This proposition is designed especially for the proof of Theorem 2.1. Notice that in the expression  $\mathbb{E}_{(i-2)p}(S_p^{(i)})$  of condition  $\mathbf{C}_1$  there is a gap of  $p$  variables between  $S_p^{(i)}$  and the variables used for conditioning. This gap is important for weakening the dependence and is essentially used in the proof of Theorem 2.1. With a similar proof, based on Theorem 6.1 instead of Proposition 4.2, the conclusion of Proposition 4.1 remains valid if we replace  $\mathbf{C}_1$  and  $\mathbf{C}_2$  by conditions involving conditional expectations without gaps:  $\mathbf{C}_1$  by

$$\mathbf{C}'_1 \quad \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \frac{1}{\sqrt{mp}} \mathbb{E}_0 \left| \sum_{i=2}^{m+1} \mathbb{E}_{(i-1)p}(S_p^{(i)}) \right| = 0 \quad a.s.$$

and  $\mathbf{C}_2$  by

$$\mathbf{C}'_2 \quad \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \mathbb{E}_0 \left| \sum_{i=1}^m \frac{1}{mp} \mathbb{E}_{(i-1)p} ((S_p^{(i)})^2) - \eta \right| = 0 \quad a.s.$$

*Proof of Proposition 4.1.* The result will follow from Proposition 4.2 below, for double indexed arrays of random variables:

**Proposition 4.2.** *Assume that  $(Y_{n,m,i})_{i \geq 1}$  is an array of random variables in  $\mathbb{L}^2$  adapted to an array  $(\mathcal{G}_{n,m,i})_{i \geq 1}$  of nested sigma fields. Let  $\mathbb{E}_{n,m,i}$  denote the conditional expectation with respect to  $\mathcal{G}_{n,m,i}$ . Suppose that*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=2}^{m+1} \mathbb{E} |\mathbb{E}_{n,m,i-2}(Y_{n,m,i})| = 0, \tag{4.1}$$

and that there exists  $\sigma^2 \geq 0$  such that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1} (Y_{n,m,i+1}^2) - \sigma^2 \right| = 0 \quad (4.2)$$

and

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1} ((Y_{n,m,i} + Y_{n,m,i+1})^2) - 2\sigma^2 \right| = 0. \quad (4.3)$$

Assume in addition that for each  $\varepsilon > 0$

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^{m+1} \mathbb{E}(Y_{n,m,i}^2 \mathbf{1}_{|Y_{n,m,i}| > \varepsilon}) = 0. \quad (4.4)$$

Then for any continuous and bounded function  $f$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E} \left( f \left( \sum_{i=1}^m Y_{n,m,i} \right) \right) - \mathbb{E}(f(\sigma N)) \right| = 0,$$

where  $N$  is a standard Gaussian random variable.

Before proving Proposition 4.2, let us show how it leads to Proposition 4.1. Let  $m$  be a fixed positive integer less than  $n$ . Set  $p = \lfloor n/m \rfloor$ . We apply Proposition 4.2 to the sequence  $Y_{n,m,i} = S_p^{(i)} / \sqrt{mp}$  and the filtration  $\mathcal{G}_{n,m,i} = \mathcal{F}_{ip}$ . We also replace the expectation  $\mathbb{E}$  by the conditional expectation  $\mathbb{E}_0$  (recall that all the conditional expectations of functions of  $T$  with respect to  $\mathcal{F}_0$  are obtained through the regular conditional probability  $P_{T|\mathcal{F}_0}$ ), and  $\sigma^2$  by the non negative  $\mathcal{F}_0$ -measurable random variable  $\eta$ . With these notations, the conditions  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_3$  implies that (4.1), (4.2), (4.3) and (4.4) hold almost surely. It follows from Proposition 4.2 that, for any continuous and bounded function  $f$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E}_0 \left( f \left( n^{-1/2} \sum_{i=1}^{m \lfloor n/m \rfloor} X_i \right) \right) - \int f(x\sqrt{\eta})g(x)dx \right| = 0 \quad a.s.$$

where  $g$  is the density of a standard normal. Proposition 4.1 will then follow if we can prove that for any  $\varepsilon > 0$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_0 \left( \left| \sum_{i=1}^n X_i - \sum_{i=1}^{m \lfloor n/m \rfloor} X_i \right| \geq \varepsilon \sqrt{n} \right) = 0 \quad a.s. \quad (4.5)$$

With this aim, we notice that

$$\mathbb{P}_0 \left( \left| \sum_{i=1}^n X_i - \sum_{i=1}^{m \lfloor n/m \rfloor} X_i \right| \geq \varepsilon \sqrt{n} \right) \leq \mathbb{P}_0 \left( m^2 \max_{1 \leq i \leq n} X_i^2 \geq \varepsilon^2 n \right).$$



and therefore (4.5) holds by Lemma 7.1 applied to  $Z_i = X_i^2$ . It remains to prove Proposition 4.2.

*Proof of Proposition 4.2.* For any positive integer  $i$ , let

$$U_{n,m,i} = Y_{n,m,i} + \mathbb{E}_{n,m,i}(Y_{n,m,i+1}) - \mathbb{E}_{n,m,i-1}(Y_{n,m,i}). \quad (4.6)$$

To ease the notation, we shall drop the first two indexes (the pair  $n, m$ ) when no confusion is possible. With this notation,

$$Y_i = U_i - \mathbb{E}_i(Y_{i+1}) + \mathbb{E}_{i-1}(Y_i),$$

and since we have telescoping sum,

$$\sum_{i=1}^m Y_i = \sum_{i=1}^m U_i + \mathbb{E}_0(Y_1) - \mathbb{E}_m(Y_{m+1}).$$

Notice that for any  $i \in \{1, m+1\}$  and any  $\varepsilon > 0$ ,

$$\mathbb{E}(|\mathbb{E}_{i-1}(Y_i)|^2) \leq \varepsilon^2 + \mathbb{E}(Y_i^2 \mathbf{1}_{|Y_i| > \varepsilon}). \quad (4.7)$$

Therefore by condition (4.4),

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}(\mathbb{E}_{n,m,m}(Y_{n,m,m+1})^2 + (\mathbb{E}_{n,m,0}(Y_{n,m,1}))^2) = 0. \quad (4.8)$$

The theorem will be proven if we can show that the sequence  $(U_{n,m,i})_{i \geq 1}$  defined by (4.6) satisfies the conditions of Theorem 6.1. We first notice that  $\mathbb{E}_{i-1}(U_i) = \mathbb{E}_{i-1}(Y_{i+1})$ . Hence condition (6.1) is clearly satisfied under (4.1). On an other hand,

$$\begin{aligned} \text{Var}(U_i | \mathcal{G}_{i-1}) &= \mathbb{E}_{i-1}(Y_i^2 + 2Y_i \mathbb{E}_i(Y_{i+1})) + \mathbb{E}_{i-1}((\mathbb{E}_i(Y_{i+1}))^2) - (\mathbb{E}_{i-1}(Y_i))^2 \\ &\quad - 2(\mathbb{E}_{i-1}(Y_i))(\mathbb{E}_{i-1}(Y_{i+1})) - (\mathbb{E}_{i-1}(Y_{i+1}))^2. \end{aligned} \quad (4.9)$$

Notice that for any  $\varepsilon > 0$

$$\begin{aligned} \sum_{i=1}^m \mathbb{E}((\mathbb{E}_{i-1}(Y_{i+1}))^2) &\leq \varepsilon \sum_{i=1}^m \mathbb{E}|\mathbb{E}_{i-1}(Y_{i+1})| + \varepsilon \sum_{i=1}^m \mathbb{E}(|Y_{i+1}| \mathbf{1}_{|Y_{i+1}| > \varepsilon}) + \sum_{i=1}^m \mathbb{E}(Y_{i+1}^2 \mathbf{1}_{|Y_{i+1}| > \varepsilon}) \\ &\leq \varepsilon \sum_{i=1}^m \mathbb{E}|\mathbb{E}_{i-1}(Y_{i+1})| + 2 \sum_{i=2}^{m+1} \mathbb{E}(Y_i^2 \mathbf{1}_{|Y_i| > \varepsilon}). \end{aligned} \quad (4.10)$$

Similarly, for any  $\varepsilon > 0$ ,

$$\sum_{i=1}^m \mathbb{E}|(\mathbb{E}_{i-1}(Y_i))(\mathbb{E}_{i-1}(Y_{i+1}))| \leq \varepsilon \sum_{i=1}^m \mathbb{E}|\mathbb{E}_{i-1}(Y_{i+1})| + 2 \sum_{i=1}^{m+1} \mathbb{E}(Y_i^2 \mathbf{1}_{|Y_i| > \varepsilon}).$$

In addition since  $\mathbb{E}_{i-1}(Y_i^2 + 2Y_i\mathbb{E}_i(Y_{i+1})) = \mathbb{E}_{i-1}((Y_i + Y_{i+1})^2) - \mathbb{E}_{i-1}(Y_{i+1}^2)$ , the conditions (4.2) and (4.3) imply that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^{m+1} \mathbb{E}_{n,m,i-1}(Y_{n,m,i}^2 + 2Y_{n,m,i}\mathbb{E}_{n,m,i}(Y_{n,m,i+1})) - \sigma^2 \right| = 0. \quad (4.11)$$

Starting from (4.9) and considering (4.10), (4.11) and (4.11), it follows that condition (6.2) will be satisfied provided that (4.1) and (4.4) hold and

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m (\mathbb{E}_{n,m,i-1}((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2) - (\mathbb{E}_{n,m,i-1}(Y_{n,m,i}))^2) \right| = 0. \quad (4.12)$$

To prove (4.12), we first write that

$$\begin{aligned} \sum_{i=1}^m (\mathbb{E}_{i-1}((\mathbb{E}_i(Y_{i+1}))^2) - (\mathbb{E}_{i-1}(Y_i))^2) &= \mathbb{E}_m(Y_{m+1})^2 - (\mathbb{E}_0(Y_1))^2 \\ &\quad - \sum_{i=1}^m ((\mathbb{E}_i(Y_{i+1}))^2 - \mathbb{E}_{i-1}((\mathbb{E}_i(Y_{i+1}))^2)). \end{aligned}$$

By (4.8), it follows that (4.12) will hold if we can show that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m ((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2 - \mathbb{E}_{n,m,i-1}((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2)) \right| = 0. \quad (4.13)$$

This follows from an application of Lemma 6.2 with

$$d_{n,m,i} = (\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2 - \mathbb{E}_{n,m,i-1}((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2).$$

Indeed

$$\sum_{i=1}^m \mathbb{E}(|d_{n,m,i}|) \leq 2 \sum_{i=1}^{m+1} \mathbb{E}(Y_{n,m,i}^2),$$

and by Lemma 6.3, for any  $\varepsilon > 0$ ,

$$\begin{aligned} \sum_{i=1}^m \mathbb{E}(|d_{n,m,i}| \mathbf{1}_{|d_{n,m,i}| > 8\varepsilon^2}) &\leq 2 \sum_{i=1}^m \mathbb{E}((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2 \mathbf{1}_{(\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2 > 4\varepsilon^2}) \\ &\leq 2 \sum_{i=1}^m \mathbb{E}(Y_{n,m,i+1}^2 \mathbf{1}_{|\mathbb{E}_{n,m,i}(Y_{n,m,i+1})| > 2\varepsilon}) \leq 4 \sum_{i=1}^{m+1} \mathbb{E}(Y_{n,m,i}^2 \mathbf{1}_{|Y_{n,m,i}| > \varepsilon}). \end{aligned}$$

So condition (6.13) holds by using (4.2) and (4.4).

It remains to prove that (6.3) holds. Clearly this can be achieved by using (4.4) combined with Lemma 6.3.  $\square$

## 4.2 Finite dimensional convergence

For  $0 < t_1 < \dots < t_d \leq 1$ , define the function  $\pi_{t_1, \dots, t_d}$  from  $C([0, 1])$  to  $\mathbb{R}^d$  by  $\pi_{t_1, \dots, t_d}(x) = (x(t_1), x(t_2) - x(t_1), \dots, x(t_d) - x(t_{d-1}))$ . For any  $a$  in  $\mathbb{R}^d$  define the function  $f_a$  from  $\mathbb{R}^d$  to  $\mathbb{R}$  by  $f_a(x) = \langle a, x \rangle = \sum_{i=1}^d a_i x_i$ .

**Proposition 4.3.** *Let  $t_1, t_2, \dots, t_d$  be  $d$  rational numbers such that  $0 < t_1 < \dots < t_d \leq 1$ . Assume that  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_3$  hold. Then, for any continuous and bounded function  $h$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_0 \left( h \circ f_a \circ \pi_{t_1, \dots, t_d}(W_n) \right) = \int h \circ f_a \circ \pi_{t_1, \dots, t_d}(x \sqrt{\eta}) W(x) dx \quad a.s.$$

where  $W$  is the distribution of a standard Wiener process.

*Proof of Proposition 4.3.* For any  $\ell \in \{1, \dots, d\}$ , let  $t_\ell = r_\ell / s_\ell$  where  $r_\ell$  and  $s_\ell$  are positive integers. let  $c_d = \prod_{\ell=1}^d s_\ell$ . Rewrite  $t_\ell = b_\ell / c_d$ . The  $b_\ell$  are then positive integers such that  $0 < b_1 < \dots < b_d \leq c_d$ . Let  $m$  be a fixed positive integer and let  $p = \lfloor n / (m c_d) \rfloor$ . Notice that for any  $\ell \in \{1, \dots, d\}$ ,

$$\lfloor n t_\ell \rfloor - m b_\ell < m p b_\ell \leq \lfloor n t_\ell \rfloor + 1.$$

Therefore for any reals  $a_1, \dots, a_d$ , with the convention that  $t_0 = 0$  and  $b_0 = 0$ ,

$$\left| \sum_{\ell=1}^d a_\ell \sum_{i=\lfloor n t_{\ell-1} \rfloor + 1}^{\lfloor n t_\ell \rfloor} X_i - \sum_{\ell=1}^d a_\ell \sum_{i=p m b_{\ell-1} + 1}^{p m b_\ell} X_i \right| \leq \sum_{\ell=1}^d |a_\ell| \sum_{i=p m b_\ell + 1}^{(p+1) m b_\ell} |X_i|.$$

Using (7.2) of Lemma 7.1, we infer that for any  $\ell \in \{1, \dots, d\}$  and every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_0 \left( \frac{|a_\ell|}{\sqrt{n}} \sum_{i=p m b_\ell + 1}^{(p+1) m b_\ell} |X_i| > \varepsilon \right) = 0 \quad a.s.$$

In addition,

$$\left| \sum_{\ell=1}^d a_\ell (W_n(t_\ell) - W_n(t_{\ell-1})) - \sum_{\ell=1}^d a_\ell (S_{\lfloor n t_\ell \rfloor} - S_{\lfloor n t_{\ell-1} \rfloor}) \right| \leq 2 \sum_{\ell=1}^d |a_\ell| \max_{1 \leq i \leq n} |X_i|,$$

implying once again by (7.2) in Lemma 7.1 that

$$\lim_{n \rightarrow \infty} n^{-1/2} \mathbb{E}_0 \left( \left| \sum_{\ell=1}^d a_\ell (W_n(t_\ell) - W_n(t_{\ell-1})) - \sum_{\ell=1}^d a_\ell (S_{\lfloor n t_\ell \rfloor} - S_{\lfloor n t_{\ell-1} \rfloor}) \right| \right) = 0 \quad a.s. \quad (4.14)$$

From the preceding considerations, it remains to prove that for any continuous and bounded function  $f$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E}_0 \left( f \left( n^{-1/2} \sum_{\ell=1}^d a_\ell \sum_{i=p m b_{\ell-1} + 1}^{p m b_\ell} X_i \right) \right) - \mathbb{E}_0(f(\sigma_d N)) \right| = 0 \quad a.s., \quad (4.15)$$

where  $\sigma_d^2 = \eta \sum_{\ell=1}^d a_\ell^2 (t_\ell - t_{\ell-1})$  and  $N$  is a standard Gaussian random variable independent of  $\mathcal{F}_0$ . With this aim, we write that

$$\sum_{\ell=1}^d a_\ell \sum_{i=pmb_{\ell-1}+1}^{pmb_\ell} X_i = \sum_{\ell=1}^d a_\ell \sum_{i=mb_{\ell-1}+1}^{mb_\ell} S_p^{(i)} = \sum_{k=1}^{mb_d} \lambda_{m,d,k} S_p^{(k)},$$

where  $\lambda_{m,d,k} = \sum_{\ell=1}^d a_\ell \mathbf{1}_{mb_{\ell-1}+1 \leq k \leq mb_\ell}$ . Hence to prove (4.15), it suffices to apply Proposition 4.2 to the random variables  $Y_{m,n,i} = (mpc_d)^{-1/2} \lambda_{m,d,i} S_p^{(i)}$  and the filtration  $\mathcal{G}_{n,m,i} = \mathcal{F}_{ip}$ , by replacing the expectation  $\mathbb{E}$  by  $\mathbb{E}_0$ . The conditions (4.1) and (4.4) are verified by using respectively  $\mathbf{C}_1$  and  $\mathbf{C}_3$ . To verify (4.2) and (4.3) with  $\sigma^2 = \sigma_d^2 = \eta \sum_{\ell=1}^d a_\ell^2 (t_\ell - t_{\ell-1})$ , we proceed as follows. For (4.2), we write that

$$\begin{aligned} \mathbb{E}_0 \left| \sum_{i=1}^{mb_d} \mathbb{E}_{n,m,i-1}(Y_{i+1}^2) - \sigma_d^2 \right| &= \mathbb{E}_0 \left| \frac{1}{mpc_d} \sum_{\ell=1}^d a_\ell^2 \sum_{i=mb_{\ell-1}+1}^{mb_\ell} \mathbb{E}_{(i-1)p}((S_p^{(i+1)})^2) - \sigma_d^2 \right| \\ &\leq \sum_{\ell=1}^d a_\ell^2 \mathbb{E}_0 \left| \frac{1}{mpc_d} \sum_{i=mb_{\ell-1}+1}^{mb_\ell} \mathbb{E}_{(i-1)p}((S_p^{(i+1)})^2) - \eta(t_\ell - t_{\ell-1}) \right|. \end{aligned}$$

Since  $t_\ell = b_\ell/c_d$ , we obtain that

$$\begin{aligned} \mathbb{E}_0 \left| \sum_{i=1}^{mb_d} \mathbb{E}_{n,m,i-1}(Y_{i+1}^2) - \sigma_d^2 \right| &\leq \sum_{\ell=1}^d \frac{a_\ell^2 b_\ell}{c_d} \mathbb{E}_0 \left| \frac{1}{mpb_\ell} \sum_{i=1}^{mb_\ell} \mathbb{E}_{(i-1)p}((S_p^{(i+1)})^2) - \eta \right| \\ &\quad + \sum_{\ell=1}^d \frac{a_\ell^2 b_{\ell-1}}{c_d} \mathbb{E}_0 \left| \frac{1}{mpb_{\ell-1}} \sum_{i=1}^{mb_{\ell-1}} \mathbb{E}_{(i-1)p}((S_p^{(i+1)})^2) - \eta \right|. \end{aligned}$$

Condition (4.2) is then proved by using the first part of  $\mathbf{C}_2$ . Using similar arguments, we prove (4.3) by using the second part of  $\mathbf{C}_2$ .  $\square$

### 4.3 A quenched invariance principle

Let us define the maximal version of  $\mathbf{C}_3$ . For  $k \leq l$ , let  $\bar{S}_{k,l} = \max_{k \leq i \leq l} |S_i - S_k|$ .

$$\mathbf{C}_4 \quad \text{for any } \varepsilon > 0 \quad \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \frac{1}{p} \mathbb{E}_0(\bar{S}_{(i-1)p, ip}^2 \mathbf{1}_{\bar{S}_{(i-1)p, ip}/\sqrt{p} > \varepsilon \sqrt{m}}) = 0 \quad a.s.$$

**Proposition 4.4.** *Assume that  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_4$  hold. Then, for any continuous and bounded function  $f$  from  $C([0, 1])$  to  $\mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_0(f(W_n)) = \int f(x\sqrt{\eta})W(x)dx \quad \text{almost surely}$$

where  $W$  is the distribution of a standard Wiener process.

*Proof of Proposition 4.4.* In this proof,  $m$  will always denote a positive integer. Since  $\mathbf{C}_4$  implies  $\mathbf{C}_3$ , it follows that Proposition 4.3 holds. In what follows, we shall prove that the process  $\{W_n(t), t \in [0, 1]\}$  is almost surely tight, that is, for any  $\varepsilon > 0$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_0 \left( \sup_{|t-s| \leq m^{-1}} |W_n(t) - W_n(s)| > \varepsilon \right) = 0 \text{ almost surely.} \quad (4.16)$$

Note that (4.16) together with Proposition 4.3 implies that the conclusion of Proposition 4.3 also holds for any  $d$ -tuple of reals  $0 < t_1 < \dots < t_d \leq 1$  in addition of being true for any  $d$ -tuple of rationals.

According to inequality (25) in Brown (1971), to prove (4.16) it suffices to show that, for any  $\varepsilon > 0$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{P}_0 \left( \sup_{(i-1)m^{-1} < t \leq im^{-1}} |W_n(t) - W_n((i-1)m^{-1})| > \varepsilon \right) = 0 \text{ a.s.} \quad (4.17)$$

Since  $\sup_{t \in [0,1]} |W_n(t) - n^{-1/2} S_{[nt]}| = n^{-1/2} \max_{1 \leq i \leq n} |X_i|$ , by using (7.2) of Lemma 7.1, it follows that (4.17) is equivalent to

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{P}_0 \left( \sup_{(i-1)m^{-1} < t \leq im^{-1}} |S_{[nt]} - S_{[n(i-1)m^{-1}]}| > \varepsilon \sqrt{n} \right) = 0 \text{ a.s.} \quad (4.18)$$

Let  $p = [n/m]$ , and note that, for any non negative integer  $i$ ,  $[nim^{-1}] - i < pi \leq [nim^{-1}]$ . It follows that, for any integer  $i$  in  $[1, m]$ ,

$$\sup_{(i-1)m^{-1} < t \leq im^{-1}} |S_{[nt]} - S_{[n(i-1)m^{-1}]}| \leq \bar{S}_{(i-1)p, ip} + \frac{1}{\sqrt{n}} \sum_{k=[n(i-1)m^{-1}]-m}^{[n(i-1)m^{-1}]} |X_k| + \frac{1}{\sqrt{n}} \sum_{k=[nim^{-1}]-m}^{[nim^{-1}]} |X_k|.$$

Using (7.2) of Lemma 7.1, we infer that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{k=[n(i-1)m^{-1}]-m}^{[n(i-1)m^{-1}]} \mathbb{E}_0(|X_k|) = 0 \text{ a.s.}$$

Hence, (4.17) holds as soon as

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{P} \left( \bar{S}_{(i-1)p, ip} > \varepsilon \sqrt{n} \middle| \mathcal{F}_0 \right) = 0 \text{ a.s.,}$$

which holds under  $\mathbf{C}_4$ .

## 5 Proof of Theorem 2.1 and additional comments

### 5.1 Proof of Theorem 2.1

**Proposition 5.1.** *If (2.1) holds, then  $\mathbf{C}_1, \mathbf{C}_2$  and  $\mathbf{C}_4$  hold, with  $\eta$  defined in (2.2). In addition the conclusion of Proposition 4.4 also holds for  $f$  in  $\mathcal{H}^*$ .*

*Proof of Proposition 5.1.* We first prove that the following reinforced version of  $\mathbf{C}_2$  holds:

$$\begin{aligned} \mathbf{C}_2^* \quad & \text{for any integer } i \geq 1 \quad \lim_{n \rightarrow \infty} \mathbb{E}_0 \left| \frac{1}{n} \mathbb{E}_{(i-1)n} ((S_n^{(i)})^2) - \eta \right| = 0 \quad a.s. \\ & \text{for any integer } i \geq 1 \quad \lim_{n \rightarrow \infty} \mathbb{E}_0 \left| \frac{1}{n} \mathbb{E}_{(i-1)n} ((S_n^{(i)} + S_n^{(i+1)})^2) - 2\eta \right| = 0 \quad a.s. \end{aligned}$$

We only prove the first part of  $\mathbf{C}_2^*$ , the proof of the second part being similar. For any positive integer  $N$

$$\left( \frac{S_{in} - S_{(i-1)n}}{\sqrt{n}} \right)^2 = \frac{1}{n} \sum_{j=(i-1)n+1}^{in} X_j^2 + \frac{2}{n} \sum_{j=(i-1)n+1}^{in} \sum_{l=1}^{(in-j) \wedge N} X_j X_{j+l} + R_{i,l,N}. \quad (5.1)$$

Firstly,

$$\mathbb{E}_0(|\mathbb{E}_{(i-1)n}(R_{i,l,N})|) \leq \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0 \left( \sum_{l>j+N} |X_j \mathbb{E}_j(X_l)| \right).$$

Let  $Z_{j,N} = \sum_{l>j+N} |X_j \mathbb{E}_j(X_l)|$  and note that, by assumption,  $Z_{j,N} = Z_{0,N} \circ T^j$  belongs to  $\mathbb{L}^1$ . Applying the ergodic theorem in relation (7.1) of Lemma 7.1 we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(Z_{j,N}) = \mathbb{E}(Z_{0,N} | \mathcal{I}) \quad a.s.$$

Hence,

$$\limsup_{n \rightarrow \infty} \mathbb{E}_0(|\mathbb{E}_{(i-1)n}(R_{i,l,N})|) \leq \mathbb{E}(Z_{0,N} | \mathcal{I}) \quad a.s.$$

and consequently

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_0(|\mathbb{E}_{(i-1)n}(R_{i,l,N})|) = 0 \quad a.s. \quad (5.2)$$

Next, let

$$\begin{aligned} \eta_N &= \mathbb{E}(X_0^2 | \mathcal{I}) + 2 \sum_{k=1}^N \mathbb{E}(X_0 X_k | \mathcal{I}) \\ \text{and } \eta_{N,K} &= \mathbb{E}(X_0^2 \mathbf{1}_{|X_0|^2 \leq K} | \mathcal{I}) + 2 \sum_{k=1}^N \mathbb{E}(X_0 X_k \mathbf{1}_{|X_0 X_k| \leq K} | \mathcal{I}). \end{aligned}$$

By the ergodic theorem for stationary sequences,

$$\lim_{n \rightarrow \infty} \left| \eta_{N,K} - \frac{1}{n} \sum_{j=(i-1)n+1}^{in} X_j^2 \mathbf{1}_{|X_j|^2 \leq K} + \frac{2}{n} \sum_{j=(i-1)n+1}^{in-l} \sum_{l=1}^{(in-j) \wedge N} X_j X_{j+l} \mathbf{1}_{|X_j X_{j+l}| \leq K} \right| = 0 \quad a.s. \quad (5.3)$$

and by the ergodic theorem in relation (7.1) of Lemma 7.1 applied with  $Z_j = X_j^2 \mathbf{1}_{|X_j|^2 > K}$  and with  $Z_j = \sum_{l=1}^N |X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > K}$ ,

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_0 \left( \frac{1}{n} \sum_{j=(i-1)n+1}^{in} X_j^2 \mathbf{1}_{|X_j|^2 > K} + \frac{2}{n} \sum_{j=(i-1)n+1}^{in-l} \sum_{l=1}^{(in-j) \wedge N} |X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > K} \right) = 0 \quad a.s. \quad (5.4)$$

Using (5.3), (5.4) and the dominated convergence theorem, it follows that

$$\lim_{n \rightarrow \infty} \mathbb{E}_0 \left( \left| \eta_N - \frac{1}{n} \sum_{j=(i-1)n+1}^{in} X_j^2 + \frac{2}{n} \sum_{j=(i-1)n+1}^{in-l} \sum_{l=1}^{(in-j) \wedge N} X_j X_{j+l} \right| \right) = 0 \quad a.s. \quad (5.5)$$

The first part of condition  $\mathbf{C}_2^*$  follows from (5.1), (5.2) and (5.5), and the fact that  $\lim_{N \rightarrow \infty} \eta_N = \eta$  almost surely.

Next, we prove that  $\mathbf{C}_1$  holds. From the proof of Proposition 3 in Dedecker and Merlevède (2002), it suffices to prove that for any integer  $i \geq 2$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_0| | \mathcal{I}) \mathbb{E}_0 \left( \left| \mathbb{E}_{(i-2)n} \left( \frac{S_{in} - S_{(i-1)n}}{\sqrt{n}} \right) \right| \right) = 0 \quad a.s.$$

Now,

$$\begin{aligned} & \mathbb{E}_0 \left( \left| \mathbb{E}_{(i-2)n} \left( \mathbb{E}(|X_0| | \mathcal{I}) \frac{(S_{in} - S_{(i-1)n})}{\sqrt{n}} \right) \right| \right) \\ & \leq \mathbb{E}_0 \left( \left| \mathbb{E}_{(i-2)n} \left( \left( \frac{1}{n} \sum_{k=(i-2)n+1}^{(i-1)n} |X_k| - \mathbb{E}(|X_0| | \mathcal{I}) \right) \frac{(S_{in} - S_{(i-1)n})}{\sqrt{n}} \right) \right| \right) \\ & \quad + \frac{1}{n} \sum_{k=(i-2)n+1}^{(i-1)n} \mathbb{E}_0 \left( \left| \mathbb{E}_{(i-2)n} \left( |X_k| \frac{(S_{in} - S_{(i-1)n})}{\sqrt{n}} \right) \right| \right). \quad (5.6) \end{aligned}$$

Using the fact that  $\mathcal{F}_0 \subseteq \mathcal{F}_{(i-2)n}$  and applying Cauchy-Schwarz's inequality conditionally to  $\mathcal{F}_0$ , the first term on right hand in (5.6) is smaller than

$$\mathbb{E}_0^{1/2} \left( \left( \frac{1}{n} \sum_{k=(i-2)n+1}^{(i-1)n} |X_k| - \mathbb{E}(|X_0| | \mathcal{I}) \right)^2 \right) \mathbb{E}_0^{1/2} \left( \left( \frac{S_{in} - S_{(i-1)n}}{\sqrt{n}} \right)^2 \right). \quad (5.7)$$

By  $\mathbf{C}_2^*$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E}_0 \left( \left( \frac{S_{in} - S_{(i-1)n}}{\sqrt{n}} \right)^2 \right) = \eta \quad a.s. \quad (5.8)$$

Since  $X_0$  belongs to  $\mathbb{L}^2$ , proceeding as in the proof of (5.5), we obtain that

$$\lim_{n \rightarrow \infty} \mathbb{E}_0 \left( \left( \frac{1}{n} \sum_{k=(i-2)n+1}^{(i-1)n} |X_k| - \mathbb{E}(|X_0| | \mathcal{I}) \right)^2 \right) = 0 \quad a.s. \quad (5.9)$$

From (5.7), (5.8) and (5.9), we infer that the first term on right hand in (5.6) converges to 0 almost surely as  $n$  tends to infinity.

Now, for  $k > (i-2)n$ ,  $\mathcal{F}_{(i-2)n} \subseteq \mathcal{F}_k$  and

$$\begin{aligned} \mathbb{E}_0 \left( \left| \mathbb{E} \left( |X_k| \frac{(S_{in} - S_{(i-1)n})}{\sqrt{n}} \middle| \mathcal{F}_{(i-2)n} \right) \right| \right) &\leq \frac{1}{\sqrt{n}} \mathbb{E}_0 \left( \left| \mathbb{E} (|X_k| (S_{in} - S_{(i-1)n}) | \mathcal{F}_k) \right| \right) \\ &\leq \frac{1}{\sqrt{n}} \mathbb{E}_0 \left( \sum_{i=k+1}^{\infty} |X_k \mathbb{E}_k(X_i)| \right). \end{aligned}$$

Let  $Z_k = \sum_{i=k+1}^{\infty} |X_k \mathbb{E}_k(X_i)|$  and note that, by assumption,  $Z_k = Z_0 \circ T^k$  belongs to  $\mathbb{L}^1$ . It follows that that second term on right hand in (5.6) is smaller than  $n^{-3/2} \sum_{k=(i-2)n+1}^{(i-1)n} \mathbb{E}_0(Z_k)$ , which converges almost surely to 0 as  $n$  tends to infinity, by the ergodic theorem in relation (7.1) of Lemma 7.1. Hence  $\mathbf{C}_1$  is proved.

We turn now to the proof of  $\mathbf{C}_4$ . With this aim, we shall prove the following reinforcement of it:

$$\mathbf{C}_4^* \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \mathbb{E}_0 \left( \frac{\bar{S}_{(i-1)n, in}^2}{n} \left( 1 \wedge \frac{\bar{S}_{(i-1)n, in}}{\sqrt{nk}} \right) \right) = 0 \quad a.s.$$

To prove  $\mathbf{C}_4^*$ , we shall use the following maximal inequality, which is a conditional version of the inequality given in Proposition 1(a) of Dedecker and Rio (2000).

**Proposition 5.2.** *For any  $k < l$  and  $\lambda \geq 0$  let  $\Gamma_{k,l}(\lambda) = \{\bar{S}_{k,l} > \lambda\}$ . The following inequality holds*

$$\mathbb{E}_0 \left( (\bar{S}_{k,l} - \lambda)_+^2 \right) \leq 8 \sum_{i=k+1}^l \mathbb{E}_0(X_i^2 \mathbf{1}_{\Gamma_{k,i}(\lambda)}) + 16 \sum_{i=k+1}^l \mathbb{E}_0(|X_i \mathbf{1}_{\Gamma_{k,i}(\lambda)}| \mathbb{E}_i(S_l - S_i)).$$

Let us continue the proof of  $\mathbf{C}_4^*$ . Note first that

$$\mathbb{E}_0 \left( \frac{\bar{S}_{(i-1)n, in}^2}{n} \left( 1 \wedge \frac{\bar{S}_{(i-1)n, in}}{\sqrt{nk}} \right) \right) \leq 2\varepsilon \mathbb{E}_0 \left( \frac{\bar{S}_{(i-1)n, in}^2}{n} \right) + \frac{4}{n} \mathbb{E}_0 \left( (\bar{S}_{(i-1)n, in} - \varepsilon \sqrt{nk})_+^2 \right).$$

From Proposition 5.2 with  $\lambda = 0$ , we obtain that

$$\mathbb{E}_0 \left( \frac{\bar{S}_{(i-1)n, in}^2}{n} \right) \leq \frac{8}{n} \sum_{k=(i-1)n+1}^{in} \mathbb{E}_0(X_k^2) + \frac{16}{n} \sum_{k=(i-1)n+1}^{in} \mathbb{E}_0(Z_k),$$



and, by the ergodic theorem in relation (7.1) of Lemma 7.1,

$$\limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \mathbb{E}_0 \left( \frac{\bar{S}_{(i-1)n, in}^2}{n} \right) \leq 8\mathbb{E}(X_0^2 | \mathcal{I}) + 16\mathbb{E}(Z_0 | \mathcal{I}) \text{ a.s.} \quad (5.10)$$

Hence  $\mathbf{C}_4^*$  will be proved if, for any  $\varepsilon > 0$ ,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \frac{1}{n} \mathbb{E}_0 \left( (\bar{S}_{(i-1)n, in} - \varepsilon \sqrt{nk})_+^2 \right) = 0 \text{ a.s.} \quad (5.11)$$

Applying Proposition 5.2, we infer that, for any positive integer  $N$ ,

$$\begin{aligned} \mathbb{E}_0 \left( (\bar{S}_{(i-1)n, in} - \varepsilon \sqrt{nk})_+^2 \right) &\leq \frac{4}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(X_j^2 \mathbf{1}_{\Gamma_{(i-1)n, in}(\varepsilon \sqrt{nk})}) \\ &+ \frac{8}{n} \sum_{j=(i-1)n+1}^{in} \sum_{l=1}^{(in-j) \wedge N} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{\Gamma_{(i-1)n, in}(\varepsilon \sqrt{nk})}) + \frac{8}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(Z_{j,N}). \end{aligned} \quad (5.12)$$

By the the ergodic theorem in relation (7.1) of Lemma 7.1, for any positive integer  $i$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(Z_{j,N}) = \mathbb{E}(Z_{0,N} | \mathcal{I}) \text{ a.s.}$$

and consequently,

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(Z_{j,N}) = 0 \text{ a.s.} \quad (5.13)$$

Now, for any positive  $M$  and any  $0 \leq l \leq N$ ,

$$\begin{aligned} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{\Gamma_{(i-1)n, in}(\varepsilon \sqrt{nk})}) &\leq \frac{M}{\varepsilon^2 k} \mathbb{E}_0 \left( \frac{\bar{S}_{(i-1)n, in}^2}{n} \right) \\ &+ \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > M}). \end{aligned} \quad (5.14)$$

According to (5.10), we have that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \frac{M}{\varepsilon^2 k} \mathbb{E}_0 \left( \frac{\bar{S}_{(i-1)n, in}^2}{n} \right) = 0 \text{ a.s.} \quad (5.15)$$

Next, by the ergodic theorem in relation (7.1) of Lemma 7.1, for any positive integer  $i$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > M}) = \mathbb{E}(|X_0 X_l| \mathbf{1}_{|X_0 X_l| > M} | \mathcal{I}) \text{ a.s.}$$

and consequently

$$\lim_{M \rightarrow \infty} \limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > M}) = 0 \text{ a.s.} \quad (5.16)$$

Gathering (5.12), (5.13), (5.14), (5.15) and (5.16), we infer that (5.11) holds. This ends the proof of  $\mathbf{C}_4^*$ . For any continuous and bounded function  $\varphi$  from  $C([0, 1])$  to  $\mathbb{R}$ , the conclusion of Theorem 2.1 then follows from an application of Proposition 4.4. To prove that the conclusion also holds for any function  $\varphi$  in  $\mathcal{H}^*$ , it suffices to notice that since (2.1) implies  $\mathbf{C}_4^*$ , it entails in particular that almost surely, the sequence  $(n^{-1} \max_{1 \leq k \leq n} S_k^2)_{n \geq 1}$  is uniformly integrable for the conditional expectation with respect to  $\mathcal{F}_0$ .  $\square$ .

*Proof of Proposition 5.2.* It is exactly the same as to get (3.12) in the paper by Dedecker and Rio (2000), with the only difference that the expectation is replaced by the conditional expectation with respect to  $\mathcal{F}_0$ .  $\square$ .

## 5.2 Remark on martingale approximations in $\mathbb{L}^2$

The aim of this short section is to point out that the conditions  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_3$  are satisfied if there is a martingale approximation in  $\mathbb{L}^2$  for the quenched CLT. Since the conditions obtained through this martingale approach are known to be sharp in some sense, this is another way to see that our conditions  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_3$  are also sharp for the quenched CLT.

From the proof of Theorem 5.1, we see that, if  $X_1$  is a martingale difference, that is  $\mathbb{E}(X_1 | \mathcal{F}_0) = 0$ , then the conditions  $\mathbf{C}_1$ ,  $\mathbf{C}_2^*$  and  $\mathbf{C}_4^*$  are satisfied. The following claim is then easily deduced.

**Claim 5.3.** *Let  $X_0$  and  $d_0$  be two  $\mathcal{F}_0$ -measurable, centered and square integrable random variables with  $\mathbb{E}(d_0 \circ T | \mathcal{F}_0) = 0$ , and let  $X_i = X_0 \circ T^i$  and  $d_i = d_0 \circ T^i$ . Let  $S_n = X_1 + \dots + X_n$  and  $M_n = d_1 + \dots + d_n$ .*

1. *If*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_0((S_n - M_n)^2) = 0 \text{ almost surely,}$$

*then the conditions  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_3$  are satisfied.*

2. *If*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_0 \left( \max_{1 \leq k \leq n} (S_k - M_k)^2 \right) = 0 \text{ almost surely,} \quad (5.17)$$

*then the conditions  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_4$  are satisfied.*

In particular, if the condition of Maxwell and Woodroffe (2000) is satisfied

$$\sum_{n > 0} \frac{\|\mathbb{E}_0(S_n)\|_2}{n^{3/2}} < \infty,$$

then it follows from Cuny and Merlevède (2012) that (5.17) holds, so that the conditions  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_4$  are satisfied. This shows that the conditions  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_4$  are sharp for the quenched FCLT.

## 6 Normal approximation for double indexed arrays and auxiliary results

There are many situations when we are dealing with double indexed sequences of random variables. For instance at each point in the two dimensional space we start a random walk. Our motivation for this section comes from the fact that in our blocking procedure we introduce a new parameter, the number of blocks,  $m$ , that is kept fixed at the beginning.

The next theorem treats the martingale approximation for double arrays of random variables.

**Theorem 6.1.** *Assume that  $(U_{n,m,i})_{i \geq 1}$  is an array of random variables in  $\mathbb{L}^2$  adapted to an array  $(\mathcal{G}_{n,m,i})_{i \geq 1}$  of nested sigma fields. Let  $\mathbb{E}_{n,m,i}$  denote the conditional expectation with respect to  $\mathcal{G}_{n,m,i}$ . Suppose that*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1}(U_{n,m,i}) \right| = 0, \quad (6.1)$$

there exists  $\sigma^2 \geq 0$  such that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \text{Var}(U_{n,m,i} | \mathcal{G}_{n,m,i-1}) - \sigma^2 \right| = 0 \quad (6.2)$$

and for each  $\varepsilon > 0$

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{E}(U_{n,m,i}^2 \mathbf{1}_{|U_{n,m,i}| > \varepsilon}) = 0 \quad (6.3)$$

Then for any continuous and bounded function  $f$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E} \left( f \left( \sum_{i=1}^m U_{n,m,i} \right) \right) - \mathbb{E}(f(\sigma N)) \right| = 0, \quad (6.4)$$

where  $N$  is a standard Gaussian variable.

**Proof of Theorem 6.1.** For any  $i \geq 1$ , let  $d_{n,m,i} = U_{n,m,i} - \mathbb{E}_{n,m,i-1}(U_{n,m,i})$ . By condition (6.1), the theorem will follow if we can prove that (6.4) holds with  $\sum_{i=1}^m d_{n,m,i}$  replacing  $\sum_{i=1}^m U_{n,m,i}$ . If  $\sigma^2 = 0$  the theorem is trivial. So we can assume without loss of generality that  $\sigma^2 = 1$ . In the rest of the proof, in order to ease the notation, we shall drop the first two indexes  $(n, m)$ , keeping them only when it is necessary to avoid confusion. Let  $\varepsilon$  and  $M$  be

positive reals fixed for the moment. For any  $i \geq 1$ , let

$$V_i = \sum_{\ell=1}^i \mathbb{E}_{\ell-1}(d_\ell^2) \quad \text{and} \quad Y_i = d_i \mathbf{1}_{|d_i| \leq \varepsilon} \mathbf{1}_{V_i \leq M}.$$

Notice first that

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^m d_i \neq \sum_{i=1}^m Y_i\right) &\leq \mathbb{P}\left(\max_{1 \leq i \leq m} d_i > \varepsilon\right) + \mathbb{P}(V_m > M) \\ &\leq \frac{1}{\varepsilon^2} \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}) + \frac{1}{M} \left(1 + \mathbb{E}\left|\sum_{i=1}^m \text{Var}(U_i | \mathcal{G}_{i-1}) - 1\right|\right). \end{aligned}$$

Hence using Lemma 6.3, we get that

$$\mathbb{P}\left(\sum_{i=1}^m d_i \neq \sum_{i=1}^m Y_i\right) \leq \frac{8}{\varepsilon^2} \sum_{i=1}^m \mathbb{E}(U_i^2 \mathbf{1}_{|U_i| > \varepsilon/3}) + \frac{1}{M} \left(1 + \mathbb{E}\left|\sum_{i=1}^m \text{Var}(U_i | \mathcal{G}_{i-1}) - 1\right|\right).$$

Therefore using (6.3) and (6.2), it follows that

$$\lim_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\sum_{i=1}^m d_{n,m,i} \neq \sum_{i=1}^m Y_{n,m,i}\right) = 0. \quad (6.5)$$

We notice now that since  $\mathbb{E}_{i-1}(d_i) = 0$  and  $V_i$  is  $\mathcal{G}_{i-1}$ -measurable

$$\sum_{i=1}^m \mathbb{E}_{i-1}(Y_i) = \sum_{i=1}^m \mathbf{1}_{V_i \leq M} \mathbb{E}_{i-1}(d_i \mathbf{1}_{|d_i| > \varepsilon}).$$

Therefore by Lemma 6.3,

$$\mathbb{E}\left|\sum_{i=1}^m \mathbb{E}_{i-1}(Y_i)\right| \leq \frac{8}{\varepsilon} \sum_{i=1}^m \mathbb{E}(U_i^2 \mathbf{1}_{|U_i| > \varepsilon/3}),$$

implying that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}\left|\sum_{i=1}^m \mathbb{E}_{i-1}(Y_i)\right| = 0. \quad (6.6)$$

Considering (6.5) and (6.6), the theorem will follow if we can show that for any continuous bounded function  $f$ ,

$$\lim_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left|\mathbb{E}\left(f\left(\sum_{i=1}^m d_{n,m,i}^*\right)\right) - \mathbb{E}(f(N))\right| = 0, \quad (6.7)$$

where

$$d_{n,m,i}^* = Y_{n,m,i} - \mathbb{E}_{n,m,i-1}(Y_{n,m,i}).$$

By the result given in Heyde and Brown (1970), this will follow if we can show that for a  $\delta > 0$ ,

$$\lim_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{E}(|d_{n,m,i}^*|^{2+2\delta}) = 0, \quad (6.8)$$

and

$$\lim_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1}((d_{n,m,i}^*)^2) - 1 \right|^{1+\delta} = 0. \quad (6.9)$$

Notice that

$$\sum_{i=1}^m \mathbb{E}(|d_i^*|^{2+2\delta}) \leq 8(2\varepsilon)^{2\delta} \sum_{i=1}^m \mathbb{E}(d_i^2) \leq 8(2\varepsilon)^{2\delta} \left( 1 + \mathbb{E} \left| \sum_{i=1}^m \text{Var}(U_i | \mathcal{G}_{i-1}) - 1 \right| \right).$$

Hence (6.8) follows by condition (6.2). It remains to prove (6.9). With this aim, we first write that

$$\mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}((d_i^*)^2) - 1 \right|^{1+\delta} \leq 2^{2+\delta} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}(Y_i^2) - 1 \right|^{1+\delta} + 2^{2+\delta} \mathbb{E} \left| \sum_{i=1}^m (\mathbb{E}_{i-1}(Y_i))^2 \right|^{1+\delta}. \quad (6.10)$$

Now since  $V_{n,m,i}$  is  $\mathcal{G}_{n,m,i-1}$ -measurable and  $\mathbb{E}_{n,m,i-1}(d_{n,m,i}) = 0$ ,

$$\begin{aligned} \mathbb{E} \left( \sum_{i=1}^m (\mathbb{E}_{i-1}(Y_i))^2 \right)^{1+\delta} &\leq \mathbb{E} \left( \left( \sum_{i=1}^m (\mathbb{E}_{i-1}(d_i \mathbf{1}_{|d_i| > \varepsilon}))^2 \right) \left( \sum_{k=1}^m \mathbf{1}_{V_k \leq M} \mathbb{E}_{k-1}(d_k^2) \right)^\delta \right) \\ &\leq M^\delta \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}). \end{aligned} \quad (6.11)$$

On the other hand, using again the fact that  $V_{n,m,i}$  is  $\mathcal{G}_{n,m,i-1}$ -measurable and also that  $V_{n,m,i} \leq V_{n,m,i+1}$ , we derive that

$$\begin{aligned} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}(Y_i^2) - 1 \right|^{1+\delta} &\leq \mathbb{E} \left( \left( 1 + \sum_{i=1}^m \mathbf{1}_{V_i \leq M} \mathbb{E}_{i-1}(d_i^2) \right)^\delta \left| 1 - \sum_{k=1}^m \mathbf{1}_{V_k \leq M} \mathbb{E}_{k-1}(d_k^2 \mathbf{1}_{|d_k| \leq \varepsilon}) \right| \right) \\ &\leq (M+1)^\delta \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}) + (M+1)^\delta \mathbb{E} \left| 1 - \sum_{k=1}^m \mathbf{1}_{V_k \leq M} \mathbb{E}_{k-1}(d_k^2) \right| \\ &\leq (M+1)^\delta \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}) + (M+1)^\delta \mathbb{E} \left| \sum_{k=1}^m \mathbb{E}_{k-1}(d_k^2) - 1 \right| \\ &\quad + (M+1)^\delta \mathbb{E} \left| \mathbf{1}_{V_m > M} \sum_{k=1}^m \mathbb{E}_{k-1}(d_k^2) \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}(Y_i^2) - 1 \right|^{1+\delta} &\leq (M+1)^\delta \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i|>\varepsilon}) + 2(M+1)^\delta \mathbb{E} \left| \sum_{k=1}^m \mathbb{E}_{k-1}(d_k^2) - 1 \right| \\ &\quad + \frac{(M+1)^\delta}{M} \left( 1 + \mathbb{E} \left| \sum_{k=1}^m \mathbb{E}_{k-1}(d_k^2) - 1 \right| \right). \end{aligned} \quad (6.12)$$

Starting from (6.10) and considering the bounds (6.11) and (6.12) together with the fact that  $\mathbb{E}_{n,m,k-1}(d_{n,m,k}^2) = \text{Var}(U_{n,m,i} | \mathcal{G}_{n,m,i-1})$ , we then infer that (6.9) holds for any  $\delta \in ]0, 1[$ . This ends the proof of (6.7) and then of the theorem.  $\square$

**Lemma 6.2.** *Assume that  $(d_{n,m,i})_{i \geq 1}$  is an array of random variables in  $\mathbb{L}^2$  adapted to an array  $(\mathcal{G}_{n,m,i})_{i \geq 1}$  of nested sigma fields, and such that for any  $i \geq 1$ ,  $\mathbb{E}_{n,m,i-1}(d_{n,m,i}) = 0$  almost surely. Suppose that*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{E}(|d_{n,m,i}| \mathbf{1}_{|d_{n,m,i}|>\varepsilon}) = 0 \quad \text{and} \quad \sum_{i=1}^m \mathbb{E}|d_{n,m,i}| < K \quad (6.13)$$

for some positive constant  $K$ . Then

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m d_{n,m,i} \right| = 0.$$

**Proof of Lemma 6.2.** Let  $\varepsilon > 0$ , and let for any  $i \geq 1$ ,

$$d'_{n,m,i} = d_{n,m,i} \mathbf{1}_{|d_{n,m,i}| \leq \varepsilon} \quad \text{and} \quad d''_{n,m,i} = d_{n,m,i} \mathbf{1}_{|d_{n,m,i}| > \varepsilon}.$$

With this notation and since  $\mathbb{E}_{n,m,i-1}(d_{n,m,i}) = 0$  almost surely,

$$\sum_{i=1}^m d_{n,m,i} = \sum_{i=1}^m (d'_{n,m,i} - \mathbb{E}_{n,m,i-1}(d'_{n,m,i})) + \sum_{i=1}^m (d''_{n,m,i} - \mathbb{E}_{n,m,i-1}(d''_{n,m,i})).$$

Since

$$\mathbb{E} | (d''_{n,m,i} - \mathbb{E}_{n,m,i-1}(d''_{n,m,i})) | \leq 2\mathbb{E}(|d_{n,m,i}| \mathbf{1}_{|d_{n,m,i}|>\varepsilon}) = 0,$$

by using the first part of (6.13), the lemma will follow if we can prove that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m (d'_{n,m,i} - \mathbb{E}_{n,m,i-1}(d'_{n,m,i})) \right| = 0. \quad (6.14)$$

With this aim, we notice that

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^m (d'_{n,m,i} - \mathbb{E}_{n,m,i-1}(d'_{n,m,i}))\right)^2 &\leq \sum_{i=1}^m \mathbb{E}(d'_{n,m,i})^2 \leq \varepsilon \sum_{i=1}^m \mathbb{E}|d'_{n,m,i}| \\ &\leq \varepsilon \sum_{i=1}^m \mathbb{E}|d_{n,m,i}| + \sum_{i=1}^m \mathbb{E}(|d_{n,m,i}| \mathbf{1}_{|d_{n,m,i}| > \varepsilon}), \end{aligned}$$

showing that (6.14) holds under (6.13).  $\square$

**Lemma 6.3.** *Let  $X$  be a real random variable and  $\mathcal{F}$  a sigma-field. For any  $p \geq 1$  and any  $\varepsilon > 0$ ,*

$$\mathbb{E}(|X|^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon}) \leq 2 \mathbb{E}(|X|^p \mathbf{1}_{|X| > \varepsilon}), \quad (6.15)$$

and setting  $Y = X - \mathbb{E}(X|\mathcal{F})$ ,

$$\mathbb{E}(|X|^p \mathbf{1}_{|Y| > 3\varepsilon}) \leq 2 \mathbb{E}(|X|^p \mathbf{1}_{|X| > \varepsilon}) \quad \text{and} \quad \mathbb{E}(|Y|^p \mathbf{1}_{|Y| > 4\varepsilon}) \leq 3 \times 2^{p+1} \mathbb{E}(|X|^p \mathbf{1}_{|X| > \varepsilon}). \quad (6.16)$$

**Proof of Lemma 6.3.** We first write that

$$|X|^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon} \leq |X|^p \mathbf{1}_{|X| > \varepsilon} + \varepsilon^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon}. \quad (6.17)$$

Notice now that  $\{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon\} \subseteq \{|\mathbb{E}(X \mathbf{1}_{|X| > \varepsilon} | \mathcal{F})| > \varepsilon\}$ , implying that

$$\varepsilon^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon} \leq |\mathbb{E}(X \mathbf{1}_{|X| > \varepsilon} | \mathcal{F})|^p \leq \mathbb{E}(|X|^p \mathbf{1}_{|X| > \varepsilon} | \mathcal{F}), \quad (6.18)$$

Starting from (6.17), using (6.18) and taking the expectation, (6.15) follows. To prove (6.16), the only difference is that we start by writing that

$$|X|^p \mathbf{1}_{|Y| > 3\varepsilon} \leq |X|^p \mathbf{1}_{|X| > \varepsilon} + \varepsilon^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon},$$

and since for any positive reals  $a, b$  and  $\varepsilon$ ,  $(a + b)^p \mathbf{1}_{a+b > 4\varepsilon} \leq 2^{p+1} a^p \mathbf{1}_{a > 2\varepsilon} + 2^{p+1} b^p \mathbf{1}_{b > 2\varepsilon}$ ,

$$\mathbb{E}(|Y|^p \mathbf{1}_{|Y| > 4\varepsilon}) \leq 2^{p+1} \mathbb{E}(|X|^p \mathbf{1}_{|X| > 2\varepsilon}) + 2^{p+1} \mathbb{E}(|X|^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon}). \quad \square$$

## 7 Ergodic theorem

We gather below the ergodic theorems used in this paper. We keep the notations of Section 2.

**Lemma 7.1.** *Let  $Z$  be a real-valued random variable in  $\mathbb{L}^1$  and define  $Z_k = Z \circ T^k$  for any  $k$  in  $\mathbb{Z}$ . Then*

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(Z_i) \rightarrow \mathbb{E}(Z|\mathcal{I}) \quad \text{almost surely and in } \mathbb{L}^1, \quad (7.1)$$

and

$$\frac{1}{n} \mathbb{E}_0 \left( \max_{1 \leq i \leq n} |Z_i| \right) \rightarrow 0 \quad \text{almost surely and in } \mathbb{L}^1. \quad (7.2)$$

**Proof.** By definition of the operator  $K$  (see the beginning of Section 2),

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(Z_i) = \frac{1}{n} \sum_{i=1}^n K^i(Z).$$

Applying the Dunford-Schwartz ergodic theorem (see for instance Krengel (1985)) we obtain that  $(K(Z) + \dots + K^n(Z))/n$  converges almost surely and in  $\mathbb{L}^1$  to some  $g \in \mathbb{L}^1$ . Now by the usual ergodic theorem,  $(Z_1 + \dots + Z_n)/n$  converges in  $\mathbb{L}^1$  to  $\mathbb{E}(Z|\mathcal{I})$ . Therefore using the fact that  $\mathbb{E}(Z|\mathcal{I}) = \mathbb{E}_0(\mathbb{E}(Z|\mathcal{I}))$  almost surely (see Bradley (2007)) and the contraction in  $\mathbb{L}^1$ , it follows that  $g = \mathbb{E}(Z|\mathcal{I})$  almost surely.

We turn now to the proof of (7.2). With this aim, we notice that for any  $N > 0$ ,

$$\frac{1}{n} \mathbb{E}_0 \left( \max_{1 \leq i \leq n} |Z_i| \right) \leq \frac{N}{n} + \frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(|Z_i| \mathbf{1}_{|Z_i| > N}).$$

By using (7.1),  $n^{-1} \sum_{i=1}^n \mathbb{E}_0(|Z_i| \mathbf{1}_{|Z_i| > N})$  converges to  $\mathbb{E}(|Z| \mathbf{1}_{|Z| > N} | \mathcal{I})$  almost surely and in  $\mathbb{L}^1$ , as  $n$  tends to infinity. Therefore

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(|Z_i| \mathbf{1}_{|Z_i| > N}) = 0 \quad \text{almost surely and in } \mathbb{L}^1,$$

which ends the proof of (7.2).

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