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Abstract. Multi-resolution analysis and numerical precision problems are very important subjects in fields like image analysis or geometrical modeling. In the continuation of our previous works, we propose to apply the method of Ω-arithmetization to ellipses. We obtain a discrete multi-resolution representation of arcs of ellipses. The corresponding algorithms are completely constructive and thus, can be exactly translated into functional computer programs. Moreover, we give a global condition for the connectivity of the discrete curves generated by the method at every scale.

Keywords: discrete geometry, multi-resolution analysis, nonstandard analysis.

1 Introduction

Discrete analytical geometry has been somewhat accidently founded in 1988 by Jean-Pierre Reveillès [18] when he proposed the analytical description of a discrete straight line. This was an unexpected result that came out of theoretical research in nonstandard analysis (NSA). Nonstandard analysis [20, 17] provides an explicit framework for the manipulation of infinitely large and infinitely small numbers. The authors, in this paper and several previous papers, have decided to go back to the roots of Reveillès' discovery: the arithmetization method. The arithmetization process is a way to discretize a continuous curve that is a solution of a differential equation. The general idea is to transform a classical approximation scheme (such as the Euler scheme for instance) of the continuous solution of a differential equation defining a curve, into an equivalent discrete scheme. This is possible because, given an infinitely large (nonstandard) number ω (the global scale), it is possible to establish an equivalence between the set of limited real numbers and a subset $\mathcal{HR}_\omega$ of $\mathbb{Z}$. The set $\mathcal{HR}_\omega$, with an additional structure, is called the Harthong-Reeb line. The intuitive idea is that, in some way, the real line $\mathbb{R}$ is similar to the discrete line $\mathbb{Z}$ seen from far away.

In a previous work, the authors re-examined the circle of Holin [9, 10]. The corresponding arithmetization process was based on infinitely large integers that had only an axiomatic status. Hence, the method was not constructive in nature and it was impossible to give an exact numerical representation of the result. In a past paper [5], we tried to tackle the issue of the constructivity of our model by using Ω-numbers of Laugwitz and Schmieden [12]. Roughly speaking, an Ω-number (natural, integer or rational) is a serie of numbers of same nature, with an adapted equality relation. The sets of Ω-numbers are extending the corresponding sets of usual numbers with the added advantage of providing naturally infinitely large integer numbers: for instance, an Ω-integer α represented by a sequence (α_n) of integers is such that $\alpha \approx +\infty$ if $lim_{n\rightarrow+\infty}\alpha_n = +\infty$ in the usual meaning. Clearly, these infinite numerical entities are constructive . It was a pleasant surprise to discover that the corresponding Ω-arithmetization method is also a discrete multi-scale representation of the real function on which the method is applied.

The goal of the present paper is to apply the Ω-arithmetization to the case of an ellipse and to study some connectivity property of the corresponding discrete curve. The first result is a constructive algorithm which gives an exact discrete multi-resolution representation of an arc of ellipse. In the Figure 1, we give a graphical illustration of this kind of representation. In fact, this multi-resolution aspect is a normal consequence of the Ω-arithmetization: this is in relation with thenature of the scaling parameter ω of the method. Since ω is now an infinitely large Ω-integer, it encodes an infinity of increasing scales. The arithmetization process gives simultaneously a discretization of the initial real function at each of these scales. Since nowadays many developments in image analysis, geometrical modelling, etc. comprise multi-resolution approaches and must deal with numerical precision problems, the Ω-arithmetization is a new tool which has the interesting property of taking into account these two aspects. The second main result of this paper is about the discrete connectivity: we show that the connectivity of the corresponding discrete ellipse arcs is a global property and that there is a rectangle within these curves are connected at every scale. Such global properties for a step-by-step integration process is unexpected. It extends a similar result we already had for circles but here, the property extends through all the scales. The paper is organized as follows: in Section 2, we introduce the Ω-numbers and the associated Harthong-Reeb line. In Section 3, we propose an Ω-arithmetization of ellipse arcs. Properties and graphical representation are discussed in Section 4. We conclude and provide perspectives for this work in Section 5.

2 Theoretical Basis : the Ω-numbers and the Associated Harthong-Reeb Line

The aim of this section is to present the basis of the nonstandard theory of Laugwitz and Schmieden [12, 13]. Our goal is to implement such a theory using the Ocaml language [11] and use it to build conic arcs.

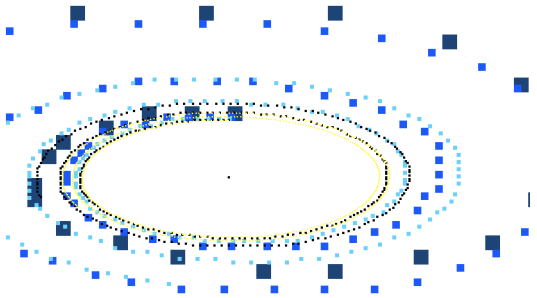

Fig. 1. Graphical representations of the multiresolution aspects of the Ω-arithmetization of an arc of ellipse. (Full explanation in Section 3).

Globally, this theory permits the definition of a nonstandard model of the integer number set that can be used to build a discrete model of real numbers, this discrete model of the continuum is called the Harthong-Reeb line [8]. The main interest of this nonstandard model (compared to other nonstandard theories such as, for instance, Internal Set Theory [17]) is that it is constructive [3]. Therefore, the implementation in a programming language is possible. In this section, we will not describe the whole theory but only introduce the basic notions that are essential to understand the Harthong-Reeb line. For more details about our approach please refer to [5].

2.1 The Ω-numbers of Laugwitz and Schmieden

To extend a theory of integer numbers, Laugwitz and Schmieden introduce a new symbol, Ω to the classical ones $(0, 3, 9, +, /, ...)$. The only thing that we know about it is that Ω verifies the following propriety named the *Basic Definition* and called (BD) :

Definition 1. *Let $S(n)$ be a statement in $\mathbb{N}$ depending of $n \in \mathbb{N}$. If $S(n)$ is true for almost $n \in \mathbb{N}$, then $S(\Omega)$ is true.*

We specify that here and in all this article, the expression "*almost* $n \in \mathbb{N}$" means "for all $n \in \mathbb{N}$ from some level N", i.e. "$(\exists N \in \mathbb{N})$ such that $(\forall n \in \mathbb{N})$ with $n > N$". Since Ω can be substituted to any natural number, it denotes an Ω-number which is the first example of Ω-integer. Immediately, we can verify that Ω is infinitely large, i.e. greater than every element of $\mathbb{N}$. Indeed, for $p \in \mathbb{N}$, we apply (BD) to the statement $p < n$ which is true for almost $n \in \mathbb{N}$; thus $p < \Omega$ for each $p \in \mathbb{N}$. And Ω is the sequence $(n)_{n\in\mathbb{N}}$.

Hence, each element a of this theory will be declined as a sequence $(a_n)_{n\in\mathbb{N}}$. To compare such Ω-numbers, we put the following equivalence relation:

Definition 2. *Let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ be two Ω-numbers, a and b are equal if it exists $N \in \mathbb{N}$ such that for all $n > N$, $a_n = b_n$.*

The definition of the operations and relations between $\mathbb{Z}_\Omega$, the set of Ω-numbers are the following:

Definition 3. *Let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ two Ω-numbers,*

- $a + b =_{def} (a_n + b_n)_{n\in\mathbb{N}}$ *and* $-a =_{def} (-a_n)$ *and* $a \times b =_{def} (a_n \times b_n)_{n\in\mathbb{N}}$;
- $a > b =_{def} [(\exists N \forall n > N)\ a_n > b_n]$ *and* $a \geqslant b =_{def} [(\exists N \forall n > N)\ a_n \geqslant b_n]$;
- $|a| =_{def} (|a_n|)$.

As in all nonstandard theories, there exist two classes of elements, the standard one and the nonstandard one. We recall that in classical IST nonstandard theory, this second class exists only in an axiomatic way: there are the infinitely small and large numbers. Here two classes of elements which can be distinguished:

- the class of element *standard* which are the elements $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ which verify $\exists p \in \mathbb{Z}$ such that $\exists N \in \mathbb{N},\ \forall n > N, \alpha_n = p$ (example: $(2)_{n\in\mathbb{N}}$).
- the class of element *nonstandard* which are all the other element of $\mathbb{Z}_\Omega$ (examples : $(-1)^n_{n\in\mathbb{N}}$,$(n)_{n\in\mathbb{N}}$)

Among the nonstandard element, we focus *infinitely large* numbers which are the sequences $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ such that $lim_{n\to+\infty}\alpha_n = +\infty$ (example: $(n)_{n\in\mathbb{N}}$).

2.2 The Harthong-Reeb Line Based on Ω-numbers

The Harthong-Reeb line ($\mathcal{HR}_\omega$) is built upon the usual nonstandard axiomatic theory IST [8]. This kind of formalism was introduced by Diener in [6]. It is defined as a scaling on the integer numbers. We consider a new unit which is an infinitely large integer named $\omega = (\omega_n)_{n\in\mathbb{N}}$, which can be Ω himself. This scaling strongly contracts $\mathbb{Z}$ so that the result looks like $\mathbb{R}$ [7].

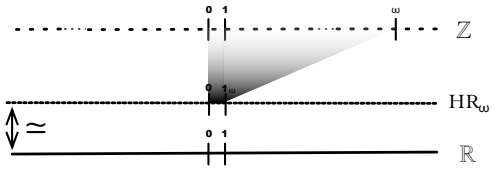

Fig. 2. Intuitive representation of the Harthong-Reeb line.

More formally, we defined the Harthong-Reeb with Ω-numbers as follows.

Definition 4. *We consider the set*

$$\mathcal{HR}_\omega = \{x \in \mathbb{Z}_\Omega,\ \exists p \in \mathbb{N},\ |x| \leq p\,\omega\}$$

and the relations, operations and constants on $\mathcal{HR}_\omega$ described by the following definitional equalities: for all $(x, y) \in \mathcal{HR}^2_\omega$, we set

- $(x =_\omega y) =_{def} (\forall p \in \mathbb{N})\,(p|x-y| \leq \omega)$;
- $(x >_\omega y) =_{def} (\exists p \in \mathbb{N})\,(p(x-y) \geq \omega)$;
- $(x \neq_\omega y) =_{def} (x >_\omega y) \vee (x <_\omega y)$;
- $(x \leq_\omega y) =_{def} (\forall z \in \mathcal{HR}_\omega)\,(z <_\omega x \Rightarrow z <_\omega y)$;
- $(x +_\omega y) =_{def} (x+y)$ *and* $0_\omega =_{def} 0$ *and* $-_\omega x =_{def} -x$;
- $(x \times_\omega y) =_{def} ((x \times y) \div \omega)$ *and* $1_\omega =_{def} \omega$ *and* $x^{(-1)_\omega} =_{def} (\omega^2 \div x)$ *for* $x \neq_\omega 0$.

Then, the Harthong-Reeb line is the numerical system $(\mathcal{HR}_\omega, =_\omega, \leq_\omega, +_\omega, \times_\omega)$.

We can say that $\mathcal{HR}_\omega$ is the set of Ω-integers which are limited at the scale ω. We can describe all these relations and operations with integers, for instance, $x =_\omega y \Longleftrightarrow \forall p \in \mathbb{N}\ \exists M_p \in \mathbb{N}\ \forall n \geq M_p \quad p|x_n - y_n| \leq \omega_n$

The goal of the construction of this line is to obtain a discrete model of the continuum. To understand that the Harthong-Reeb line is a kind of continuum, we can compare it to $(\mathcal{R}_\Omega^{lim}, \simeq, \lesssim, +, \times)$. This is the set of limited Ω-rational numbers of Laugwitz and Schmieden defined considering sequences of rational numbers. The two following maps:

$$\left\{ \begin{array}{rcl} \varphi_\omega : \mathcal{HR}_\omega & \to & \mathcal{R}_\Omega^{lim} \\ x & \mapsto & x/\omega \end{array} \right\} \text{ and } \left\{ \begin{array}{rcl} \psi_\omega : \mathcal{R}_\Omega^{lim} & \to & \mathcal{HR}_\omega \\ u & \mapsto & (\lfloor \omega u \rfloor) \end{array} \right\}$$

are the isomorphic maps necessary to pass from the classical real world to the discrete one. The following section uses the Harthong-Reeb line to define the *arithmetization* process based on the well known Euler scheme.

3 The Discrete Ellipse Arcs

In this section we revisit and extend recent work about the arithmetization method. The arithmetization is basically a way of transforming a continuous Euler scheme into a discrete one. This leads to a step-by-step generation algorithm of a discrete object. Recently the authors have arithmetized differential equations defining a circular arc [19]. This was done with an axiomatic definition of infinitely large integers. In another recent paper, the authors introduced the Ω-arithmetization method based on Ω-numbers which allows a constructive representation of infinitely large integers. In this section we apply the Ω-arithmetization to circles and more generally to ellipses.

We consider an axis-aligned ellipse of equation

$$\frac{x(t)^2}{a'^2} + \frac{y(t)^2}{b'^2} = 1 \tag{1}$$

with a' and $b' \in \mathbb{Q}$. The parametric form of this set is

$$\left\{ \begin{array}{l} x = a' \cos(t) \\ y = b' \sin(t) \end{array} \right. \tag{2}$$

and is the solution of the differential system

$$\begin{cases} x' = -a'/b'\ y(t) \\ y' = b'/a'\ x(t). \end{cases} \tag{3}$$

As a' and b' are in $\mathbb{Q}$, so there exist $p, q, \in \mathbb{Z}$ and $r, s \in \mathbb{N}^*$ such that $a' = p/r$ and $b' = q/s$. We can define two integer numbers a an b thus that $a/b = a'/b'$ and $b/a = b'/a'$. So, now only integers are manipulated in the differential system.

To obtain a solution we use the well known Euler method. In this case, we have:

$$\begin{cases} (x_0, y_0) & = (0, b') \\ (x_{n+1}, y_{n+1}) = (x_n, y_n) + (-\frac{a}{b}h\ y_n, \frac{b}{a}h\ x_n). \end{cases} \tag{4}$$

We know that the smaller the integration step h, better is the approximation. In some meaning, the better choice is h infinitesimal.

We want to embed this scheme in $\mathcal{HR}_\Omega$ line. Using the idea of the isomorphism ψ_ω

$$\begin{aligned} \psi_\omega : \mathcal{R}_{lim} &\longrightarrow \mathcal{HR}_\omega \\ x &\longmapsto \lfloor \omega x \rfloor \end{aligned}$$

and replace the step h by $1/\beta$ we can write the following scheme

$$\begin{cases} (x_0, y_0) & = (0, \lfloor \omega b' \rfloor) \\ (x_{n+1}, y_{n+1}) = (x_n, y_n) + ((-ay_n) \div (b\beta), bx_n \div (a\beta)) \end{cases} \tag{5}$$

where

- $\exists \alpha$ such that $\alpha,\ \beta, \omega \in \mathcal{HR}_\omega$; $\alpha,\ \beta, \omega \simeq +\infty$ and $\omega = \alpha\beta$ (we say that ω is the global scale)
- $\lfloor \omega b' \rfloor = (\lfloor \omega_0 b' \rfloor, \lfloor \omega_1 b' \rfloor, \ldots)$
- $\forall u,\ v \in \mathcal{HR}_\omega, u \div v = (\lfloor u_0 \div v_0 \rfloor, \lfloor u_1 \div v_1 \rfloor, \ldots)$
- $(x_0, y_0) \in \mathcal{HR}_\omega^2$

As presented in [4, 19], the problem of this scheme is that it generates values that are infinitely far from each other and thus the corresponding discrete curve is strongly (infinitely) non-connected. To avoid this problem, we divide everything by β in order to bring the discrete points close together. It is equivalent to work at a scale α which is named the intermediary scale.[3] Let us introduce the following notations to describe the decomposition $x = \widetilde{x}\beta + \widehat{x}$ for any integer $x \in \mathcal{HR}_\omega$ so $\widetilde{x} = x \div \beta \in \mathcal{HR}_\alpha$ and $\widehat{x} = x \bmod \beta \in \{0, \ldots, \beta - 1\}$. The integer $\widetilde{x} \in \mathcal{HR}_\alpha$ is interpreted as the result of the rescaling on x. This decomposition produces the following scheme:

$$\begin{cases} (\widetilde{x}_0, \widetilde{y}_0) & = (0, \lfloor \omega b' \rfloor \div \beta) \\ (\widehat{x}_0, \widehat{y}_0) & = (0, \lfloor \omega b' \rfloor \bmod \beta) \\ (f_n^1, f_n^2) & = ((-a(\widetilde{y}_n\beta + \widehat{y}_n)) \div (b\beta), b(\widetilde{x}_n\beta + \widehat{x}_n) \div (a\beta)) \\ (\widetilde{x}_{n+1}, \widetilde{y}_{n+1}) = (\widetilde{x}_n + (\widehat{x}_n + f_n^1) \div \beta, \widetilde{y}_n + (\widehat{y}_n + f_n^2) \div \beta) \\ (\widehat{x}_{n+1}, \widehat{y}_{n+1}) = ((\widehat{x}_n + f_n^1) \bmod \beta, (\widehat{y}_n + f_n^2) \bmod \beta) \end{cases} \tag{6}$$

[3] We could of course as well work at the intermediary scale β. The equations would be slightly different because of the role of β as $1/h$ but the principle would remain the same.

the relevant variables are $\widetilde{x}$ and $\widetilde{y} \in \mathcal{HR}_\alpha$ while $\widehat{x}$ and $\widehat{x}$ only conserve the accumulated error.

Let us call $E_d^\alpha(0,a,b)$ the discrete curve defined by the solution $(\widetilde{x}_n, \widetilde{y}_n)$ of (6). $E_d^\alpha(0,a,b)$ is the arithmetization at the intermediate scale α of the initial ellipse (1).

Observe that the algorithm (6) is standard. This means that for α, β and ω standard we get a usual scheme in $\mathbb{Z}^2$. Nevertheless, in this paper, we have to consider that all objects are Ω-numbers, i.e. infinite sequences of integers operating with the relations and operations defined in Section 2, for instance, $a = (a)_{m\in\mathbb{N}}$, $b = (b)_{m\in\mathbb{N}}$[4] and $\beta = (\beta_m)_{m\in\mathbb{N}}$ and $(\widetilde{x}_n, \widetilde{y}_n)_{n\in\mathbb{N}} = ((\widetilde{x}_{m,n}), (\widetilde{y}_{m,n}))_{n\in\mathbb{N}}$.

Let us note that n represents the iteration in the algorithm and that m the level in the sequence.

4 Properties and Graphical Results

In this section we present some theoretical results on the connectivity of the elliptical arcs and some graphical illustrations of the algorithm which result of an implementation in Ocaml. Let us first start with an extension of the theorem on the connectivity of circular arcs proposed in [19] by Richard and al.

4.1 Connectivity Properties

Before proving connectivity properties, we need some definitions.

Definition 5. *An* arc of $E_d^\alpha(0,a,b)$ *is a sequence of the following form* $(\widetilde{x}_n, \widetilde{y}_n)_{k<n<k+p}$ *for* k *and* p *fixed in* $\mathbb{N}$

Then, we define the notion of connectedness with Ω-intergers and call it 8_Ω-connectedness:

Definition 6. *A curve defined by the* Ω*-points* $((\widetilde{x}_{m,n}), (\widetilde{y}_{m,n}))_{(m\in\mathbb{N},\ k\leq n\leq k+p)}$ *is* 8_Ω*-connexe if*

$$\forall n,\ \forall m,\ |x_{m,n+1} - x_{m,n}| \leq 1 \text{ and } |y_{m,n+1} - y_{m,n}| \leq 1.$$

This is equivalent to the classical 8-connectedness for each level m of the underlying sequences. This is the natural generalization to the Ω-numbers of the discrete connectivity. We also need the definition of a rectangle that is defined in $\mathbb{Z}_\Omega^2$. A *rectangle in* $\mathbb{Z}_\Omega^2$, centered in zero and with length $2l = (2l, 2l, ...)$ and width $2w = (2w, 2w, ...)$ is defined by :

$$R_{l,w} = \{(x,y) \in \mathbb{Z}_\Omega^2\ ;\ -l \leq x < l \text{ and } -w \leq y < w\}.$$

If we remember that Ω-numbers are sequences, this definition becomes:

$$R_{l,w} = \{((x_m)_{(m\in\mathbb{N})}, (y_m)_{(m\in\mathbb{N})}) \in \mathbb{Z}_\Omega^2\ ; \forall n \in \mathbb{N}\ (-l \leq x_m < l \text{ and } -w \leq y_m < w)\}.$$

[4] Here a and b are standard Ω-numbers because of φ_Ω, hence the sequences are constant ones.

The following theorem is an extension of the connectivity theorem given in [19]. This is a double extention: firstly to the axis aligned elliptical arcs and secondly to the Ω-numbers. The proof is very close to the one proposed in [19]. It is however remarkable that, since we are working with Ω-integers, the result is a multi-scale result that is valid at all scales at the same time.

Theorem 1. *Every arc of $E_d^\alpha(0,a,b)$ in the square $R_{l,w}$ is 8-Ω-connected for $l = a\beta \div b$ and $b\beta \div a$.*

Proof. Let $\Gamma = ((\widetilde{x}_n, \widetilde{y}_n))_{k \leq n \leq k+p}$ an arc of $E_d^\alpha(0,a,b)$ such that $(\widetilde{x}_n, \widetilde{y}_n) \in R_\beta$ for each $n = k, \ldots, k+p$.

The proof is in two parts: in part (a) we will give a necessary and sufficient condition for the connectedness of Γ and in part (b) we will show that the condition $\Gamma \subset R_{l,w}$ with $(l,w) = (a\beta \div b, b\beta \div a)$ is sufficient.

(a) *Equivalent conditions:* using the two schemes (5) and (6) and properties of the Euclidean division, we can see that the following conditions are equivalent:

$$\begin{aligned}
-1 \leq \widetilde{x}_{m,n+1} - \widetilde{x}_{m,n} &\leq 1 \\
-1 \leq (\widehat{x}_{m,n} + f_n^1) \div \beta_m &\leq 1 \\
-\beta_m \leq \widehat{x}_n + f_n^1 &< 2\beta_m \\
-\beta_m - \widehat{x}_{m,n} \leq (-a(\widetilde{y}_{m,n}\beta_m + \widehat{y}_{m,n})) \div (b\beta_m) &< 2\beta_m - \widehat{x}_{m,n} \\
-b\beta_m^2 - b\widehat{x}_{m,n}\beta_m \leq -a(\widetilde{y}_n\beta_m + \widehat{y}_{m,n}) &< 2b\beta_m^2 - b\widehat{x}_{m,n}\beta_m \\
-b\beta_m^2 - b\widehat{x}_{m,n}\beta_m \leq -ay_{m,n} &< 2b\beta_m^2 - b\widehat{x}_{m,n}\beta_m \\
-\frac{b}{a}(\beta_m^2 + \widehat{x}_{m,n}\beta_m) \leq -y_{m,n} &< \frac{b}{a}(2\beta_m^2 - \widehat{x}_{m,n}\beta_m)
\end{aligned}$$

Hence with a similar proof for $(\widetilde{y}_{m,n+1} - \widetilde{y}_{m,n})$, Γ is 8_Ω-connected if and only if, for each $n = k, \ldots, k+p-1$ for all m , we have:

- $\frac{b}{a}(-2\beta_m^2 + \widehat{x}_{m,n}\beta_m) < y_{m,n} \leq \frac{b}{a}(\beta_m^2 + \widehat{x}_{m,n}\beta_m)$
- $\frac{a}{b}(\beta_m^2 + \widehat{y}_{m,n}\beta_m) \leq x_{m,n} < \frac{a}{b}(2\beta_m^2 - \widehat{y}_{m,n}\beta_m)$.

(b) *Sufficient condition:* We prove here that if $\Gamma \subset R_{a\beta \div b, b\beta \div a}$, then the previous condition is verified. Since $0 \leq \widehat{x}_n \leq \beta - 1$ and $0 \leq \widehat{y}_n \leq \beta - 1$, we get the two following sequences of inequalities:

$$\begin{aligned}
&\frac{b}{a}(-2\beta_m^2 + \widehat{x}_{m,n}\beta_m) \leq \frac{b}{a}(-\beta_m^2 - \beta_m) < -\frac{b}{a}\beta_m^2 < \frac{b}{a}\beta_m^2 \leq \frac{b}{a}(\beta_m^2 + \widehat{x}_{m,n}\beta_m) \\
&-\frac{a}{b}(\beta_m^2 + \widehat{y}_{m,n}\beta_m) \leq -\frac{a}{b}\beta_m^2 < \frac{a}{b}\beta_m^2 < \frac{a}{b}(\beta_m^2 + \beta_m) \leq \frac{a}{b}(2\beta_m^2 - \widehat{y}_{m,n}\beta_m).
\end{aligned}$$

From $-\frac{b}{a}\beta_m^2 \leq y_n < \frac{b}{a}\beta_m^2$, we derive[5] $\left(-\frac{b}{a}\beta_m^2\right) \div \beta \leq y_n \div \beta < \frac{b}{a}\beta_m^2 \div \beta$ which imply $-\frac{b}{a}\beta_m \leq \widetilde{y}_n < \frac{b}{a}\beta_m$. This implies $-(b\beta_m \div a) \leq \widetilde{y}_n < b\beta_m \div a$.

For $\widetilde{x}$, applying the same property, if $-\frac{a}{b}\beta_m^2 \leq x_n < \frac{a}{b}\beta_m^2$, then $-(a\beta_m \div b) \leq \widetilde{x}_n < a\beta_m \div b$. This is the condition of the inclusion in the rectangle $R_{a\beta \div b, b\beta \div a}$ and so for the 8_Ω-connectedness. □

The section (4.2) shows some representations and interpretations of this theorem.

4.2 Graphical Illustrations

In this section, we present different representations of axis-aligned elliptical arcs which illustrate our connectivity theorem and our arithmetization method in general. The Ω-integers are implemented in Ocaml language. Firstly, some words about this language, Ocaml is a functional programming language i.e. a programming paradigm that treats computation as the evaluation of mathematical functions and avoids state and mutable data. It prefers the application of functions, contrarily to the imperative programming style, which emphasizes changes in state. Hence, objects are functions which permits the manipulation of infinite ones. For instance, Ω-integers are viewed as a function of integers: sequences which associate a value a_n to n for all n.

The multiresolution aspect is introduced by the different colors which encode the level m in the Ω-integers. We can observe that there is a correlation between the level of resolution and the size of the error. It is important to understand that the Ω-numbers contain all the scales at the same time and that the model as such is inherently multiscale. In fact, the algorithm handle entities which are infinity, but for the graphical illustration, we just extract some of them.

According to the theorem of Section 4.1, we have here two Ω-arithmetization of circle arcs. In Figure 5(a), the arc is connected and satisfies $\alpha R < \beta$, $(R, \alpha, \beta) = ((2)_n, (2n)_n, (9n)_n)$, hence we have indeed $\forall n \in \mathbb{N}, R\alpha_n = 2 * 2n = 4n < 9n$. The second example, in Figure 5(b) is not connected. Its parameters are $(R, \alpha, \beta) = ((2)_n, (4n)_n, (3n)_n)$.

The Figure 5(b). shows that all the parts (arcs) of the ellipse that are located inside the rectangle $R_{(2\beta;\lfloor 0.5\beta \rfloor)}$ are connected.

In Figure 5. we can see in (a) a multiresolution representation of a discrete ellipse $E_d^{(2n)_{n\in\mathbb{N}}}(0, 4, 2)$ with $\beta = (7n)_{n\in\mathbb{N}}$. The pictures (b), (c), (d), (e), (f) are the different scales presented separatly in Figure 6.(a). The rectangle $R_{(2\beta;\lfloor 0.5\beta \rfloor)}$ is also displayed for each picture .

[5] Usually, the division algorithm operates on integer numbers. Here we need to extend it to rational numbers. Hence we defined $u \div v$ where u and v are rational fractions by $u \div v =_{def} \lfloor u/v \rfloor$.

(a) (b)

Fig. 3. Graphical representations of the Ω-arithmetization of connected and non-connected circle arcs.

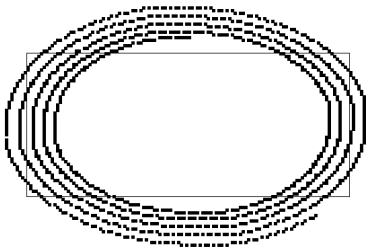

Fig. 4. Illustration of the theorem with an ellipse $E_d^{(2n)_{n\in\mathbb{N}}}(0,3,2)$ with $\beta = (80n)_{n\in\mathbb{N}}$.

5 Conclusion

In the present paper, we have applied the new concept of Ω-arithmetization to the case of an arc of ellipse. This method gives a discrete and multi-scale representation of this kind of Euclidean object. Due to the structure of the Ω-integers, we obtain a completely constructive algorithms which can be exactly translated into functional computer programs. It follows that these programs do not generate any numerical error and they provide an exact discrete multi-resolution representation of the given arc of ellipse. Furthermore, we have shown that a discrete ellipse arc is connected inside a global region of the discrete plane.

The properties of the obtained one-revolution-connected ellipse are not studied because it is due to the accumulated error of Euler scheme, and does not depend on the applied theory. Moreover, we emphasize that the goal of this work is not to define discrete objects in an intuitive way. That is why we do not com-

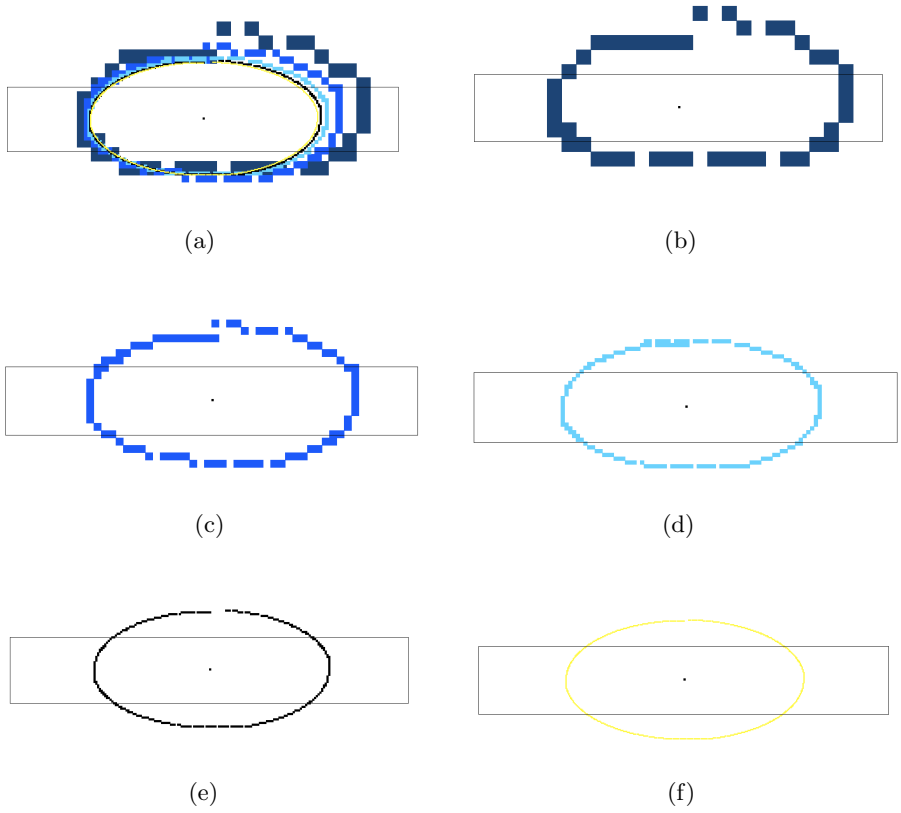

Fig. 5. Graphical representations of ellipse with the associated rectangle-connectedness.

pare our discrete ellipse definition with others like Bresenham [2] or Andres [1] definitions.

In future works on this subject, we plan to study systematically this form of multi-resolution analysis for other kinds of continuous curves such as polynomial ones and spline curves. In addition, we intend to change our general theoretical framework; we want to move from the theory of Ω-numbers of Laugwitz-Schmieden to the formalism of constructive type theory of P. Martin-Löf [14, 15]. The first reason is that this stark approach of mathematics and computer science is well suited for both developing constructive mathematics and writing programs. Furthermore, Martin-Löf has already developed a nonstandard extension of constructive type theory [16] in which we dispose of infinitely large natural numbers. Hence, in this more satisfactory context, it would be possible

to define an Harthong-Reeb line as a new model of the continuum and build a general tool for discrete multi-resolution analysis of continuous objects.

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