Optimal Consensus Set for Annulus Fitting

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Abstract. An annulus is defined as a set of points contained between two circles. This paper presents a method for fitting an annulus to a given set of points in a 2D images in the presence of noise by maximizing the number of inliers, namely the consensus set, while fixing the thickness. We present a deterministic algorithm that searches the optimal solution(s) within a time complexity of $O(N^4)$, N being the number of points.
keywords: Digital geometry; shape fitting; consensus set; outliers; digital arc; annulus.

1 Introduction

Detection of basic geometric properties such as lines, planes and circles are essential tasks in the field of image analysis and computer vision. The main objective of this paper is annulus fitting to a given set of points. For instance, annulus fitting is useful for shape approximation [4] and image segmentation [6]. In this paper, we present a novel method that, given an arbitrary 2D point cloud, finds annuli of a given width that minimizes the number of outliers, or alternatively maximizes the number of inliers. One of the possible application of our method in the digital space is the fitting of discrete analytical circle of Andres [2]. The set of points which do fit the model is also called consensus set. The idea of using such consensus sets was proposed for the RANdom Sample Consensus (RANSAC) method [9], which is one of the most widely used in the field of computer vision. However RANSAC is inherently probabilistic in its approach and does not guarantee any optimality while our method is both deterministic and optimal in the size of the consensus set.

Different algorithms detecting annuli have been proposed. Most of these algorithms minimize the thickness of the annuli (digital circle). Among them, some consider that no noise is present in the image, and concentrate only on the problem of recognition instead of the fitting problem [16, 5, 1]. Some algorithms deal with outliers [7, 10, 11] but the number of outliers is usually predefined [10, 11] and the problem consists in minimizing the width.

One frequently used approach in annulus recognition stems from the O'Rourke et al. [13] disk recognition method that transformed it into a problem of circular separability; they use a mapping that raises every point (x, y) to the paraboloid $z = x^2 + y^2$. Using this mapping, every circle in the primal space corresponds

to a plane cutting the paraboloid in the dual space and thus circular separability is transformed to a plane separability in the dual space. This separability problem is then solved in linear time using Megiddo's or Dyer linear algorithms [8, 12]. Recently, in the discrete geometry community, a simple online linear-time algorithm for recognition of digital circular arcs has been proposed in [15] based on the idea of O'Rourke et al. [13]; this algorithm can be used in an incremental way, with a complexity $O(n^{4/3})$ whenever a new point is added by using an optimization proposed in [3]. Using O'Rourke's mapping, an annuli of fixed area in the primal space, corresponds to two parallel planes of fixed vertical thickness in the dual space. The vertical thickness of the two parallel planes corresponding to an annulus is however not fixed when the thickness w of the annuli and not the area of the annuli is fixed, as is the case of our fitting problem. In this case, the dual approach does not seem to simplify the problem.

To our knowledge there is no work neither in the continuous domain nor in the discrete domain that treated the problem of optimal annulus fitting with fixed width. Our main contribution is that we find exact solution(s) by minimizing the number of outliers.

The rest of the paper is organized as follows: in section 2 we expose the problem of annulus fitting, present some properties of annuli and explain how we can find all the consensus sets. In sections 3 we show how to build an annulus from three points. Section 4 provides the algorithm for finding the optimal annulus and shows some results. Finally Section 5 states some conclusions and perspectives.

2 Annulus fitting

An annulus A of width ω and radius R centered at $C(C_x, C_y)$, is defined by the set of points in R^2 satisfying two inequalities:

$$S = \left\{(P_x, P_y) \in \mathbb{R}^2 : R^2 \leq (P_x - C_x)^2 + (P_y - C_y)^2 \leq (R + \omega)^2\right\} \quad (1)$$

where $C(C_x, C_y) \in \mathbb{R}^2$ and $R, \omega \in \mathbb{R}_+$.

Using the above annulus model, our fitting problem is then described as follows: given a finite set $S = \left\{(P_x, P_y) \in \mathbb{R}^2\right\}$ of n points such that $n \geq 3$, and given a width ω we would like to find an annulus A of width ω such that it contains the maximum number of points in S. Points $(P_x, P_y) \in S \cap A$ are called inliers; otherwise they are called outliers. It should be noted that $n \geq 3$ since if S contains less than 3 points, the fitting problem has an infinity number of solutions.

We denote B_i (resp. B_e) that we call the internal (resp. external) border of the annulus, the set of points located at distance R (resp. $R + \omega$) from C.

2.1 Annuli and their consensus sets

Our approach is focused on inlier sets, also called consensus sets. Since S is finite, it is obvious that the number of different consensus sets for the annulus is finite as well. Thus, if we can find all different consensus sets $\mathcal{C}$ from a given set S, we just need to verify the size of each $\mathcal{C}$ and find the maximum one as the optimal solution. Then the following question comes up naturally: is it possible, given a width ω to find all the consensus sets of S? if the answer is positive, how can we do it? In the section 2.2, we will answer both questions by giving some properties related to annuli.

2.2 Annular characterizations

The following theorem states that given a width ω, and given an annulus A covering a set of points S, there exists at least another annulus A' of same width, that covers S and passes through at least 3 points of S.

Theorem 1. *Let S be a set of $n (n \geq 3)$ points in $\mathbb{R}^2$. Let $A = (C(C_x, C_y), R, \omega)$ be the annulus of center $C(C_x, C_y)$, of internal radius R and of width ω such that $\forall (P_x, P_y) \in S,\ R^2 \leq (P_x - C_x)^2 + (P_y - C_y)^2 \leq (R + \omega)^2$. Then it exists $A' = (C'(C'_x, C'_y), R', \omega)$ such that:*

$$\exists P_0, P_1, P_2 \in S,\ \forall i \in [0, 2], P_i \in B_i \text{ or } P_i \in B_e$$

Proof. Let S be a set of n ($n \geq 3$) points in $\mathbb{R}^2$. Let $A = (C(C_x, C_y), R, \omega)$ be the annulus of center $C(C_x, C_y)$ with internal radius R and width ω such that A covers S: i.e. $\forall (P_x, P_y) \in S,\ R^2 < (P_x - C_x)^2 + (P_y - C_y) < (R + \omega)^2$. Our first assumption is that no point of S is on the annulus borders.

The theorem proof is given in three steps: First we show that we can always **decrease the radius** so as to put at least one point of S on the annulus **external** borders. We note this point P_0. In the second step, we show that a rotation centered on P_0 allows to put another point P_1 of S on one border. The last step is more delicate since it consists in changing both the radius and the center position so that a third point P_2 is now on one of the borders of the annulus. Of course, if in any of these steps we put more than one point on the borders than we just skip one or more steps.

A First step: the radius decreasing. Let R_0 be the distance between the farthest point P_0 of S from the center C of A. The annulus $A_0 = (C(C_x, C_y), (R_0 - \omega), \omega)$ is such that P_0 is on its external border B_e.

B Second step: the rotation centered on P_0. Since the rotation center is on the external border (see [A]), we can continuously rotate A_0 to reach a second point P_1 of S.
We note P_1 the first point reached. Let $Rot_{P_0,\theta}$ the rotation centered on P_0 with angle θ, such that θ is the smallest angle verifying:

$$\exists \theta \in [0, 2\pi],\ \exists P_1 \in S,\ R^2 \leq (P_{1_x} - C_x)^2 + (P_{1_y} - C_y)^2 \leq (R + \omega)^2$$

We denote $A_1 = (Rot_{P_0,\theta}(C), R, \omega)$.

C Third step: the radius variation. After the first two steps, we obtain an annulus A_1 that has two points P_0 and P_1 of S on its borders. **Two** configurations can appear: either both points are on the border B_e, or the points are each on a different border (case iii).

i. Both points P_0 and P_1 are on B_e. In this case, any modification on the radius R involves a move of the center along the bisector of segment P_0 and P_1. When the center C passes exactly between both points, some problems may occur and we have to separate two cases:

* $d(P_0P_1) \geq 2\omega$: in this case, the center can pass between the points and there always exists an annulus of width ω passing through both points. All the points of S can be reached this way and we necessarily encounter a third point of S. In Fig. 1a, we can see the annulus with the two extreme cases, when the radius is $^+_-\infty$. Fig. 1b, shows an inital set of points with an initial annulus of center C passing by two external border points P_0 and P_1 (colored black), and the rotated annulus (colored red) by moving the center toward $-\infty$ from C to C_2 in order to reach a third points P_2. Fig. 1c shows the case when the center moves toward $+\infty$ in order to reach a third points P_2. The movement of the center between $^+_-\infty$ allows to reach a third point.
* $d(P_0P_1) < 2\omega$: when the points P_0, P_1 are on B_e, the circle having the two points on B_e must have at least a radius of ω since if the radius is less than ω, no internal circle and so no annulus can be built. However when $d(P_0P_1) < 2\omega$, the movement of the center from C to C_2 to reach a third points P_3 cannot be done in all the cases. For example in Fig. 1d, $d(P_0P_1) = \omega$ and in this case, the center cannot move along the green line since $\frac{\omega^2}{4} + b^2 >= \omega^2$, i.e. b must be greater than $\frac{\sqrt{3}\omega}{2}$. Therefore there exist some points of S that cannot be reached. Fig. 1e shows a case where no annulus having three border points can be constructed from the annulus having P_0 and P_1 on B_e. When this is the case, we have to change the configuration: we choose as point P_2 the point closest to the limit and we build the annulus of width ω with internal circle passing through P_0, P_1 and P_2 (Fig. 1f).

ii. Both points are on different borders.
In Fig. 2a, b, c, there are three possible configurations of two points on different border of the annulus A. In each of these configurations, modifying the radius allows to reach almost all the initial set. If a point P_2 of S is inside the green circle Fig. 2d, e , inside the annulus, it has to be reached by one of the annulus border. However, there exists an area that is not reached by such a variation (in dark in Fig. 2d, e).
If the points of S are all in this dark part, we have to change our strategy: we choose as point P_2 of S the closest to one of the extreme

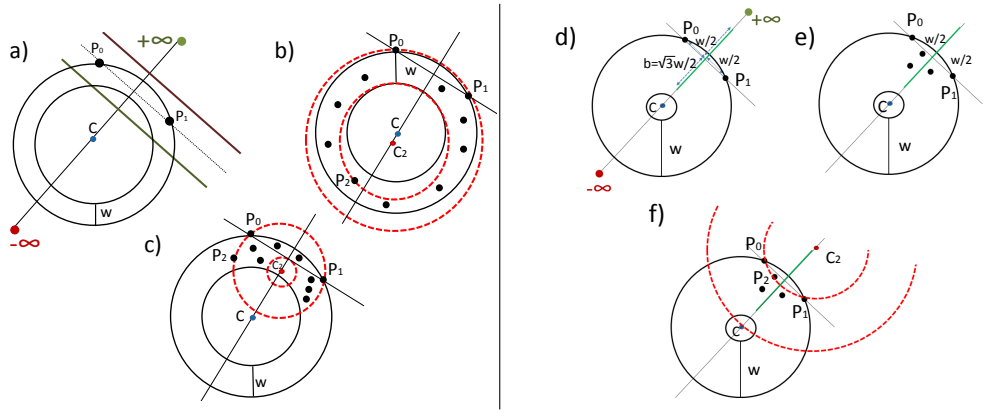

Fig. 1. Third step: a), b) and c), shows case ii where both points P_0 and P_1 are on the external border of A and the distance is greater than 2ω. The center C colored blue must be moved along the line $-\infty$ in b) and $+\infty$ in c) in order to reach a third point P_2. The new center is C_2 and is colored in red, d), e), f) shows case ii where both points P_0 and P_1 are on the external border of A and the distance between them is less than 2ω. The configuration must be changed by choosing a point P_2 closest to the limit and the new annulus is the one colored in red.

lines and we build the annulus of width ω such that the three points P_0, P_1 and P_2 are on B_e (see Fig. 2g, h). This annulus thus covers the whole dark area where all the other points of S are and it passes through 3 points of S.
In Fig. 2c, f, where both points are at distance ω; we have to perform a rotation centered on one of both points until reaching another point P' on the border. If we rotate using P_0, the point P_1 in Fig. 2f no longer belongs to the border of the annulus. In this case the configuration can be viewed as the same configurations we already adressed: either P' and P_0 are on the internal border or P' is on the external border but not at the distance ω from P_0 (otherwise it would have been reached by a circle centered on P_0 with radius ω passing through P_1 and thus we would already have found the annulus we were searching for). Both cases leads to a third point.

In all cases, if an annulus of width ω covers S, then it is possible to build an annulus of same width that passes through 3 points of S. □

3 Building an annulus of width ω from three points

The following theorem states that we can build a finite number of annuli of width ω from 3 points (see Fig. 3).

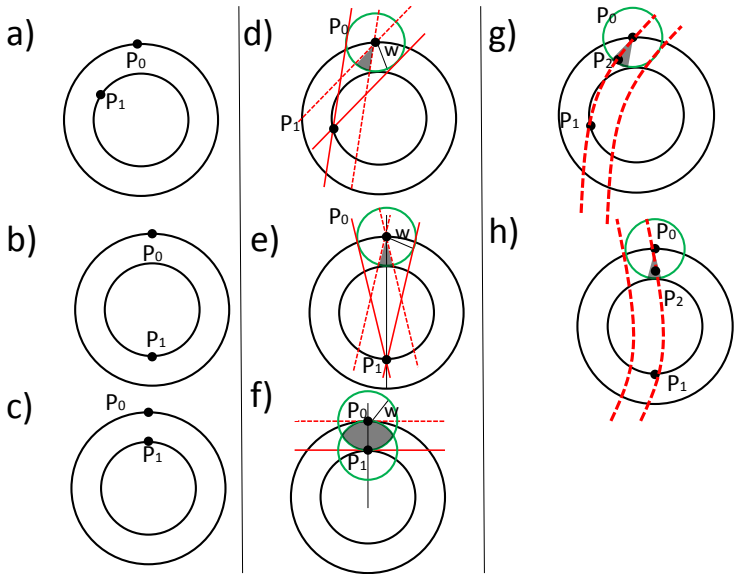

Fig. 2. Third step: case iii, both P_0 and P_1 are on different border of A.

Theorem 2. *There are at most* 8 *annuli of a given width* ω *passing through* 3 *given points* P_1, P_2 *and* P_3 *of* S.

Proof.

- If the 3 points are on the same circle: depending on the radius of the circle one or two annuli can be build. If the radius of the circle is greater than ω then 2 annular are built, the first having the points on the internal border and the other having the points on its external border. When the radius is less than ω then only one annulus having the three points on its internal border can be built.
- If 2 of the 3 points are on one border and the third one on the other: three possible configurations exists either (P_1, P_2 or P_3 is alone on one border) and for each of them we can buid at most two annuli (the lonely point is either on B_i or on B_e). The principle of the building process is to build the two circles C' and C'' that passes through the 2 points on the same border and tangent (from the outside or inside) to the circle c of radius ω centered on the third point. (see [14] for more details about the constuction).
  Fig. 3 shows the different cases of building the annulus, we assume that P_0 and P_2 are on one border and P_1 on the other. Fig. 3a shows the construction method; the circle C of radius ω centered on P_2 is drawn (blue circle), then the two circles C' and C'' passing through P_0 and P_2 and tangent to C are drawn (red circles). Fig. 3b presents the particular case where P_0 is exactly at distance ω from P_1 (i.e. on the circle of center P_1 and radius ω). In this case there is a unique circle that passes through P_0 and P_2 and

tangent to C. Finally, there is a particular case when both P_0 and P_2 are inside the circle C (Fig. 3c), in this case it should be stated than no annulus having P_0 and P_2 on one border and P_1 on the other can be found.

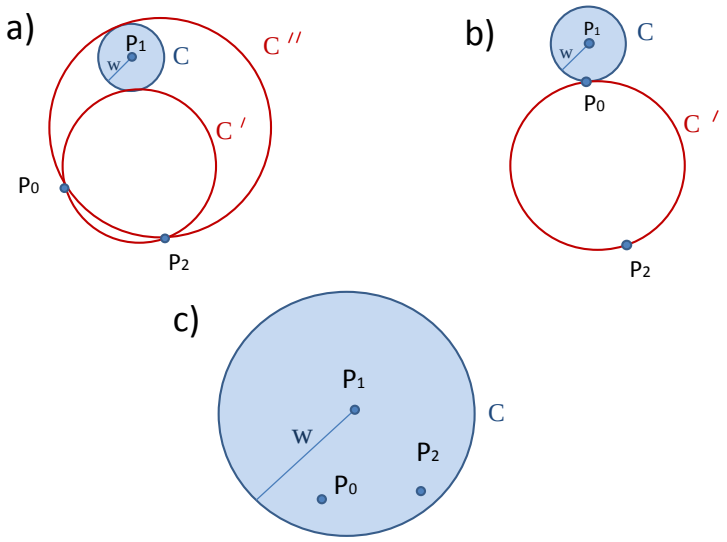

Fig. 3. Building of an annulus of width ω from 3 points: a) P_0 and P_2 are outside the circle C of radius ω centered on P_1, b) P_0 is on the circle C, c) P_0 and P_2 are inside the circle C.

4 Finding the optimal fitting annulus of width ω.

The principle of this naïve algorithm is to test all the possible triplets of points in S. For each triplet, we compute all the possible annuli (see theorem 2) and count the number of inliers. The optimal annulus (or the annuli) is the one that encloses the maximum number of inliers.

4.1 Algorithm

We now present Algorithm 1. Input is a set S of N discrete points and a thickness w of our digital circle model. Output is a set $\mathbf{V}$ of parameter values $(C_x^{op}, C_y^{op}, R^{op})$ corresponding to the fitted annuli that give the optimal consensus sets. The time complexity of the algorithm is $O(N^4)$ because we have N points in S and every combination of three points defines each of the 8 circles,

Algorithm 1: Annuli fitting while fixing the thickness

input : A set $S = \{P_i\}_{i\in[1,n]}$ of N points and a thickness ω
output: A list $\mathbf{V}$ of parameter values $[(C_x^{op}, C_y^{op}, R^{op}), ...]$ of the best fitted annuli A

```
1  begin
2      initialize Max = 0;
3      foreach i ∈ [1, n − 2] do
4          foreach j ∈ [i + 1, n − 1] do
5              foreach k ∈ [j + 1, n]] do
6                  L=list of the existing annuli passing through P_i, P_j, P_k;
7                  foreach element of L do
8                      initialize nb_inliers = 0;
9                      for m = 1, ..., N do
10                         if dist(C, P_m) >= R and dist(C, P_m) <= R + ω then
11                             nb_inliers = nb_inliers + 1;
12                     if nb_inliers > Max then
13                         clear(V);
14                         Max = nb_inliers;
15                         set C_x^op = C_x, C_y^op = C_y, R^op = R;
16                         put (C_x^op, C_y^op, R^op) in V;
17                     else if nb_inliers = Max then
18                         set C_x^op = C_x, C_y^op = C_y, R^op = R;
19                         append (C_x^op, C_y^op, R^op) in V;
20     return V;
```

so taking the points three by three has a complexity of $O(N^3)$, then checking how many inliers and outliers we have for a given annulus is done in $O(N)$ time.

4.2 Experiments

We applied our method for 2D noisy digital Andres circles as shown in Fig. 4, 6, 7. For each of these set of points, an annulus of width $\omega = 3$, $\omega = 1$ and $\omega = 1$ is used respectively. Table 1 shows the number of points, the optimal consensus set size as well as the center position and the radius R of the inner circle obtained after the fitting. It should be noted that in Fig. 7, two optimal consensus sets are possible, since two annulus having the same number of inliers can be fitted (two circles in Fig. 7). This proves that our method is capable of detecting all optimal consensus sets. Our method is also applied to an Andres arc of width 1 as shown in Fig. 5, we can see that the arc of width $\omega = 1$ is detected.

Table 1. The number of points and the optimal consensus set size for each of Fig. 4, 5, 6, 7.

Figures	Number of points	Center position	R	Thickness w	Opt. consensus set size
Fig. 4	289	(31,31)	13	3	286
Fig. 5	121	(101.581,102.226)	86	1	118
Fig. 6	119	(31,31)	14	1	65
Fig. 7	309	(31,31) (49,49)	19	1	114

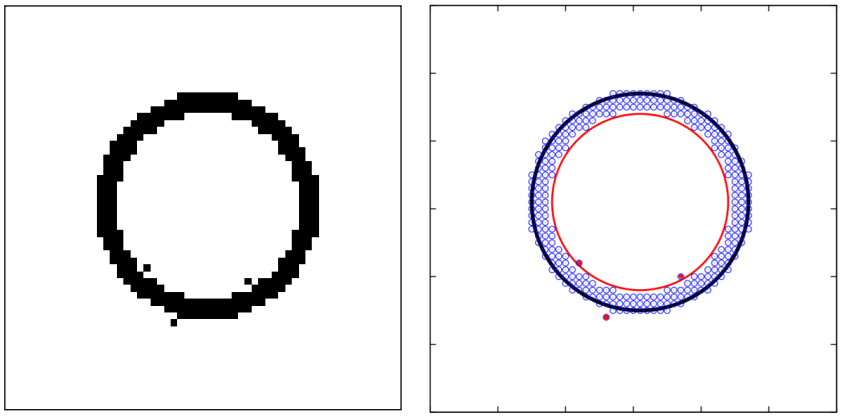

Fig. 4. Annulus fitting for a noisy digital Andres circle of width 3.

5 Conclusion and Perspectives

In this paper we have presented a new method for annulus fitting for annuli of fixed width. Our approach is costly in terms of computation time however, its main advantage is that it guarantees optimal result from the point of view of maximal consensus set: we are guaranteed to fit an annulus with the least amount of outliers. This is the first time, to the author's best knowledge that and axact optimal methods dealing with the problem of annulus fitting with a fixed width with outliers has been proposed. One of the future work concerns the complexity of our approach. Our approach is rather brute-force and we obtain a $O(N^4)$ complexity. The question of improving this complexity is open. We have wondered if this complexity may not be optimal. Indeed, it should be noted that fitting a digital plane (corresponding to two parallel continuous planes) to a given set of points in the presence of noise by maximizing the

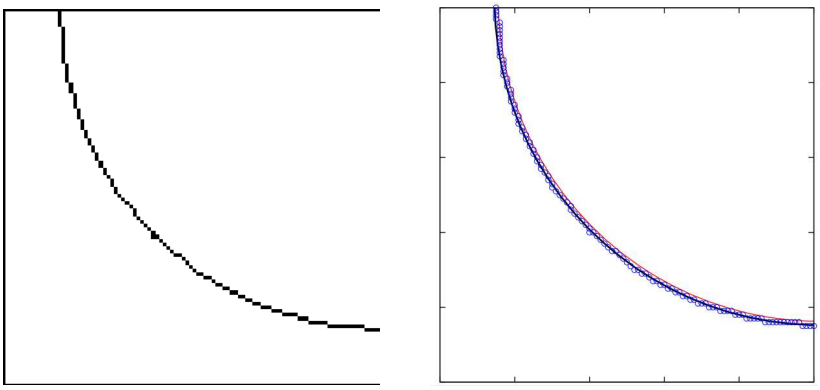

Fig. 5. Annulus fitting for a noisy digital Andres arc of width 1 .

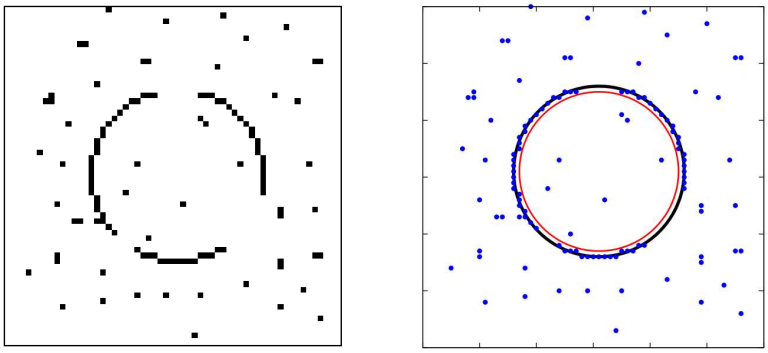

Fig. 6. Annulus fitting for a noisy image of a digital Andres circle of width 1.

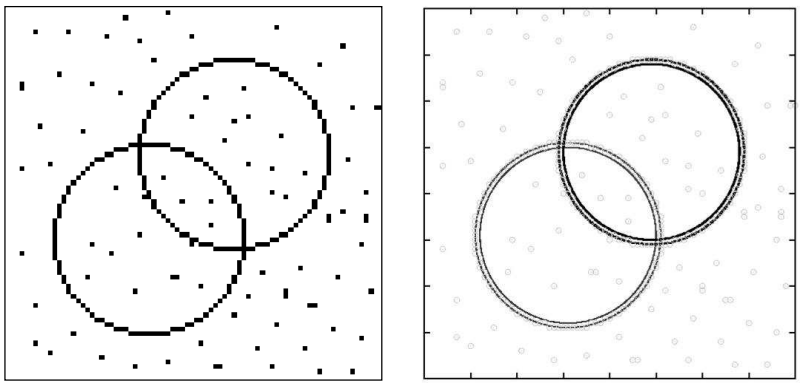

Fig. 7. Annulus fitting for two noisy digital Andres circle of width 1.

number of inliers is solved in $O(N^3 \log N)$ [17]. As we have explained in the introduction, there does not seem to be an immediate way of transforming our problem into a plane fitting problem with a O'Rourke type mapping [13]. However, if such a way exists, we would still face a $O(N^3 \log N)$ lower complexity limit. One of the most immediate planned extensions is the fitting of 3D annuli.

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