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POLYHEDRAL DIVISORS AND TORUS ACTIONS OF COMPLEXITY ONE OVER ARBITRARY FIELDS

KEVIN LANGLOIS

Key words: multigraded ring, polyhedral divisor, algebraic torus action.
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ABSTRACT. We show that the presentation of affine \mathbb{T} -varieties of complexity one in terms of polyhedral divisors holds over an arbitrary field. We also describe a class of multigraded algebras over Dedekind domains. We study how the algebra associated to a polyhedral divisor changes when we extend the scalars. As another application, we provide a combinatorial description of affine \mathbf{G} -varieties of complexity one over a field, where \mathbf{G} is a (not necessarily split) torus, by using elementary facts on Galois descent. This class of affine \mathbf{G} -varieties is described via a new combinatorial object, which we call (Galois) invariant polyhedral divisor.

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INTRODUCTION

In this paper, we are interested in a combinatorial description of multigraded normal affine algebras of complexity one. From a geometrical viewpoint, these algebras are related to the classification of algebraic torus actions of complexity one on affine varieties. Let k be a field. Consider a split algebraic torus \mathbb{T} over k . Recall that a \mathbb{T} -variety is a normal variety over k endowed with an effective \mathbb{T} -action. There exist several combinatorial descriptions of \mathbb{T} -varieties in term of the convex geometry. See [Dol75, Pin77, Dem88, FZ03] for the Dolgachev-Pinkham-Demazure (D.P.D.) presentation, [KKMS73, Tim97, Tim08] for toric case and complexity one case, and [AH06, AHS08, AOPSV12] for higher complexity. Most classical works on \mathbb{T} -varieties require the ground field k to be algebraically closed of characteristic zero. It is worthwhile mentioning that the description of affine \mathbb{G}_m -varieties [Dem88] due to Demazure holds over any field.

Let us list the most important results of the paper.

- The Altmann-Hausen presentation of affine \mathbb{T} -varieties of complexity one in terms of polyhedral divisor holds over an arbitrary field, see Theorem 4.3.

- This description holds as well for an important class of multigraded algebras over Dedekind domains, see Theorem 2.5.

- We study how the algebra associated to a polyhedral divisor changes when we extend the scalars, see 2.12 and 3.9.

- As another application, we provide a combinatorial description of affine \mathbf{G} -varieties of complexity one, where \mathbf{G} is a (not necessarily split) torus over k , by using elementary facts on Galois descent. This class of affine \mathbf{G} -varieties is classified via a new combinatorial object, which we call a (Galois) invariant polyhedral divisor, see Theorem 5.10.

Let us discuss these results in more details. We start with a simple case of varieties with an action of a split torus. Recall that a split algebraic torus \mathbb{T} of dimension n defined over k is an algebraic group over k isomorphic to \mathbb{G}_m^n , where $\mathbb{G}_m = \mathbb{G}_{m,k}$ is the multiplicative algebraic group $\text{Spec } k[t, t^{-1}]$. Let $M = \text{Hom}(\mathbb{T}, \mathbb{G}_m)$ be the character lattice of the torus \mathbb{T} . Then defining a \mathbb{T} -action on an affine variety X is equivalent to fixing an M -grading on the algebra $A = k[X]$, where $k[X]$ is the coordinate ring of X . Following the classification of affine \mathbb{G}_m -surfaces [FK91], we say as in [Lie13, 1.1] that the M -graded algebra A is *elliptic* if the graded piece A_0 is reduced to k .

Multigraded affine algebras are classified via a numerical invariant called complexity. This invariant was introduced in [LV83] for the classification of homogeneous spaces under the action of a connected reductive group. Consider the field $k(X)$ of rational functions on X and its subfield K_0 of \mathbb{T} -invariant functions. The complexity of the \mathbb{T} -action on X is the transcendence degree of K_0 over the field k . Note that for the situation where k is algebraically closed, the complexity is also the codimension of the general \mathbb{T} -orbit in X (see [Ros63]).

In order to describe affine \mathbb{T} -varieties of complexity one, we have to consider combinatorial objects coming from convex geometry and from the geometry of algebraic curves. Let C be a regular curve over k . A point of C is assumed to be a closed point, and in particular, not necessarily rational. Thus, the residue field extension of k at any point of C has finite degree.

To reformulate our first result, we recall some notation introduced in [AH06, Section 1]. Denote by $N = \text{Hom}(\mathbb{G}_m, \mathbb{T})$ the lattice of one-parameter subgroups of the torus \mathbb{T} which is the dual of the lattice M . Let $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$, $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$ be the associated dual \mathbb{Q} -vector spaces of M, N , respectively, and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. We can define as in [AH06] a Weil divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ with σ -polyhedral coefficients in $N_{\mathbb{Q}}$, called polyhedral divisor of Altmann-Hausen. More precisely, each $\Delta_z \subset N_{\mathbb{Q}}$ is a polyhedron with a tail cone σ (see 2.1) and $\Delta_z = \sigma$ for all but finitely many points $z \in C$. Denoting by κ_z the residue field of the point $z \in C$ and by $[\kappa_z : k] \cdot \Delta_z$ the image of Δ_z under the homothety of ratio $[\kappa_z : k]$, the sum

$$\text{deg } \mathfrak{D} = \sum_{z \in C} [\kappa_z : k] \cdot \Delta_z$$

is a polyhedron in $N_{\mathbb{Q}}$. This sum may be seen as the finite Minkowski sum of all polyhedra $[\kappa_z : k] \cdot \Delta_z$ different from σ . Considering the dual cone $\sigma^{\vee} \subset M_{\mathbb{Q}}$ of σ , we define an evaluation function

$$\sigma^{\vee} \rightarrow \text{Div}_{\mathbb{Q}}(C), \quad m \mapsto \mathfrak{D}(m) = \sum_{z \in C} \min_{l \in \Delta_z} \langle m, l \rangle \cdot z$$

with values in the vector space $\text{Div}_{\mathbb{Q}}(C)$ of Weil \mathbb{Q} -divisors over C . As in the classical case [AH06, 2.12] we introduce the technical condition of properness for the polyhedral divisor \mathfrak{D} (see 2.2, 3.4, 4.2) that we recall thereafter.

Definition 0.1. A σ -polyhedral divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is called *proper* if it satisfies one of the following conditions.

- (i) C is affine.
- (ii) C is projective and $\deg \mathfrak{D}$ is strictly contained in the cone σ . Furthermore, if $\deg \mathfrak{D}(m) = 0$, then m belongs to the boundary of σ^\vee and some non-zero integral multiple of $\mathfrak{D}(m)$ is principal.

For instance, if $C = \mathbb{P}_k^1$ is the projective line, then the polyhedral divisor \mathfrak{D} is proper if and only if $\deg \mathfrak{D}$ is strictly included in σ .

For every affine variety X with an effective \mathbb{T} -action, we will call *multiplicative system* of $k(X)$ a sequence $(\chi^m)_{m \in M}$, where each χ^m is a homogeneous element of $k(X)$ of degree m satisfying the conditions $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ for all $m, m' \in M$, and $\chi^0 = 1$. One of the main results of this paper can be stated as follows.

Theorem 0.2. *Let k be a field.*

- (i) *If \mathfrak{D} is a proper σ -polyhedral divisor on a regular curve C over k , then the M -graded algebra $A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma^\vee \cap M} A_m$, where*

$$A_m = H^0(C, \mathcal{O}_C(\lfloor \mathfrak{D}(m) \rfloor)),$$

is the coordinate ring of an affine \mathbb{T} -variety of complexity one over k .

- (ii) *Conversely, to any affine \mathbb{T} -variety $X = \text{Spec } A$ of complexity one over k , one can associate a pair $(C_X, \mathfrak{D}_{X, \gamma})$ as follows.*

(a) *C_X is the abstract regular curve over k defined by the conditions $k[C_X] = k[X]^\mathbb{T}$ and $k(C_X) = k(X)^\mathbb{T}$.*

(b) *$\mathfrak{D}_{X, \gamma}$ is a proper σ_X -polyhedral divisor over C_X , which is uniquely determined by X and by a multiplicative system $\gamma = (\chi^m)_{m \in M}$ of $k(X)$.*

We have a natural identification $A = A[C_X, \mathfrak{D}_{X, \gamma}]$ of M -graded algebras with the property that every homogeneous element $f \in A$ of degree m is equal to $f_m \chi^m$, for a unique global section f_m of the sheaf $\mathcal{O}_{C_X}(\lfloor \mathfrak{D}_{X, \gamma}(m) \rfloor)$.

In the proof of assertion (ii), we use an effective calculation from [Lan13]. We divide the proof into two cases. In the *non-elliptic case* we show that the assertion holds more generally in the context of Dedekind domains. More precisely, we give a perfect dictionary similar to 0.2(i), (ii) for M -graded algebras defined by a polyhedral divisor over a Dedekind ring (see 2.2, 2.3 and Theorem 2.5). We deal in 2.6 with an example of a polyhedral divisor over $\mathbb{Z}[\sqrt{-5}]$. In the *elliptic case*, we consider an elliptic M -graded algebra A over k satisfying the assumptions of 0.2(ii). By a well-known result (see [EGA II, 7.4]), we can construct a regular projective curve arising from the algebraic function field $K_0 = (\text{Frac } A)^\mathbb{T}$. In this construction, the points of C are identified with the places of K_0 . Then we show that the M -graded algebra is described by a polyhedral divisor over C (see Theorem 3.5).

Let us pass further to the general case of varieties with an action of a non necessarily split torus. The reader may consult [Bry79, CTHS05, Vos82, ELST12] for the theory of non-split toric varieties and [Hur11] for the spherical embeddings. Let \mathbf{G} be a torus over k ; then \mathbf{G} splits in a finite Galois extension E/k . Let $\text{Var}_{\mathbf{G}, E}(k)$ be the

category of affine \mathbf{G} -varieties of complexity one splitting in E/k (see 5.4). For an object $X \in \text{Var}_{\mathbf{G},E}(k)$ let $[X]$ be the isomorphism class and $X_E = X \times_{\text{Spec } k} \text{Spec } E$ be its extension of X over the field extension. Fixing $X \in \text{Var}_{\mathbf{G},E}(k)$, as an application of our previous results, we study the pointed set

$$(\{[Y] \mid Y \in \text{Var}_{\mathbf{G},E}(k) \text{ and } X_E \simeq_{\text{Var}_{\mathbf{G},E}(E)} Y_E\}, [X])$$

of isomorphism classes of E/k -forms of X that is in bijection with the first pointed set $H^1(E/k, \text{Aut}_{\mathbf{G}_E}(X_E))$ of non abelian Galois cohomology. By elementary arguments (see 5.7) the latter pointed sets are described by all possible homogeneous semi-linear $\mathfrak{G}_{E/k}$ -actions on the multigraded algebra $E[X_E]$, where $\mathfrak{G}_{E/k}$ is the Galois group of E/k . Translating this to the language of polyhedral divisors, we obtain a combinatorial description of E/k -forms of X , see Theorem 5.10. This theorem can be viewed as a first step towards the study of the forms of \mathbf{G} -varieties of complexity one.

Let us give a brief summary of the contents of each section. In the first section, we recall how to extend the D.P.D. presentation of parabolic graded algebra to the context of Dedekind domain. This fact has been mentioned in [FZ03] and firstly treated by Nagat Karroum in a master dissertation [Kar04]. In the second and the third sections, we study respectively a class of multigraded algebras over Dedekind domains and a class of elliptic multigraded algebras over a field. In the fourth section, we classify affine \mathbb{T} -varieties of complexity one. The last section is devoted to the non-split case.

0.3. All considered rings are assumed to be commutative and unitary. Let k be a field. Given a lattice M we let $k[M]$ be the semigroup algebra

$$\bigoplus_{m \in M} k\chi^m, \text{ where } \chi^{m+m'} = \chi^m \cdot \chi^{m'} \text{ and } \chi^0 = 1.$$

Recall that a \mathbb{Q} -divisor on a scheme Y is a Weil divisor on Y with rational coefficients. By a *variety* X over k we mean an integral separated scheme of finite type over k ; one assumes in addition that k is algebraically closed in the field of rational functions $k(X)$. In particular, X is geometrically irreducible.

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1. GRADED ALGEBRAS OVER DEDEKIND DOMAINS

In this section, we recall how to generalize the Dolgachev-Pinkham-Demazure (D.P.D.) presentation in [FZ03, Section 3] to the context of Dedekind domains (see Lemma 1.6). This generalization concerns in particular an algebraic description of affine normal parabolic complex \mathbb{C}^* -surfaces. Let us recall the definition of a Dedekind domain.

1.1. An integral domain A_0 is called a *Dedekind domain* (or a Dedekind ring) if it is not a field and if it satisfies the following conditions.

- (i) The ring A_0 is noetherian.
- (ii) The ring A_0 is integrally closed in its field of fractions.
- (iii) Every nonzero prime ideal is a maximal ideal.

Let us mention several classical examples of Dedekind domains.

Example 1.2. Let K be a number field. Then the ring \mathbb{Z}_K of integers of K is a Dedekind ring.

Let A be a finitely generated normal algebra of dimension one over a field k . This means that the scheme $C = \text{Spec } A$ is a regular affine curve. Then the coordinate ring $A = k[C]$ is Dedekind.

The algebra of power series $k[[t]]$ in one variable over the field k is a Dedekind domain. More generally, every principal ideal domain (and so every discrete valuation ring) that is not a field is a Dedekind domain.

1.3. Let A_0 be an integral domain, and let K_0 be its field of fractions. Recall that a *fractional ideal* \mathfrak{b} is a finitely generated nonzero A_0 -submodule of K_0 . Actually, every fractional ideal is of the form $\frac{1}{f} \cdot \mathfrak{a}$, where $f \in A_0$ is nonzero and \mathfrak{a} is a nonzero ideal of A_0 . If \mathfrak{b} is equal to $u \cdot A_0$ for some nonzero element $u \in K_0$, then we say that \mathfrak{b} is a *principal fractional ideal*.

The following result gives a description of fractional ideals of A_0 in terms of Weil divisors on $Y = \text{Spec } A_0$, where A_0 is a Dedekind domain. This assertion is well known, and so the proof is omitted.

Lemma 1.4. *Let A_0 be a Dedekind ring with field of fractions K_0 . Let $Y = \text{Spec } A_0$. Then the map*

$$\text{Div}_Z(Y) \rightarrow \text{Id}(A_0), \quad D \mapsto H^0(Y, \mathcal{O}_Y(D))$$

is a bijection between the set of integral Weil divisors on Y and the set of fractional ideals of A_0 . Every fractional ideal is locally free of rank 1 as A_0 -module and the natural map

$$H^0(Y, \mathcal{O}_Y(D)) \otimes H^0(Y, \mathcal{O}_Y(D')) \rightarrow H^0(Y, \mathcal{O}_Y(D + D'))$$

is surjective. A Weil divisor D on Y is principal (resp. effective) if and only if the corresponding fractional ideal is principal (resp. contains A_0).

Notation 1.5. Let A_0 be a Dedekind domain. For a \mathbb{Q} -divisor D on the affine scheme $Y = \text{Spec } A_0$ we denote by $A_0[D]$ the graded algebra

$$\bigoplus_{i \in \mathbb{N}} H^0(Y, \mathcal{O}_Y(\lfloor iD \rfloor)) t^i,$$

where t is a variable over the field K_0 . Note that $A_0[D]$ is normal as intersection of discrete valuation rings with field of fractions $K_0(t)$ (see the argument for [Dem88, 2.7]).

The next lemma provides a D.P.D. presentation for a class of graded subrings of $K_0[t]$. It will be used in the next section. Here we give an elementary proof using the description in 1.4 of fractional ideals.

Lemma 1.6. *Let A_0 be a Dedekind ring with the field of fractions K_0 . Let*

$$A = \bigoplus_{i \in \mathbb{N}} A_i t^i \subset K_0[t]$$

be a normal graded subalgebra of finite type over A_0 , where every A_i is contained in K_0 . Assume that the field of fractions of A is $K_0(t)$. Then there exists a unique \mathbb{Q} -divisor D on $Y = \text{Spec } A_0$ such that $A = A_0[D]$. Furthermore we have $Y = \text{Proj } A$.

Proof. Theorem 1.4 and Lemma 2.2 in [GY83] imply that every nonzero module A_i can be written as

$$A_i = H^0(Y, \mathcal{O}_Y(D_i))$$

for some $D_i \in \text{Div}_{\mathbb{Z}}(Y)$. By Proposition 3 in [Bou72, III.3] there exists a positive integer d such that the subalgebra

$$A^{(d)} := \bigoplus_{i \geq 0} A_{di} t^{di}$$

is generated by $A_d t^d$. Proceeding by induction, for any $i \in \mathbb{N}$ we have $D_{di} = iD_d$. Let $D = D_d/d$. Then using the normality of A and $A_0[D]$, we obtain for any homogenous element $f \in K_0[t]$ the equivalences

$$f \in A_0[D] \Leftrightarrow f^d \in A_0[D] \Leftrightarrow f^d \in A \Leftrightarrow f \in A.$$

This yields $A = A_0[D]$.

Let D' be another \mathbb{Q} -divisor on Y such that $A = A_0[D']$. Comparing the graded pieces of $A_0[D]$ and of $A_0[D']$, it follows that $[iD] = [iD']$ for any $i \in \mathbb{N}$. Hence $D = D'$ and so the decomposition is unique.

It remains to show the equality $Y = \text{Proj } A$. Let $V = \text{Proj } A$. By Exercice 5.13 in [Har77, II] and Proposition 3 in [Bou72, III.1], we may assume that $A = A_0[D]$ is generated by $A_1 t$. Since the sheaf $\mathcal{O}_Y(D)$ is locally free of rank one over \mathcal{O}_Y , there exist $g_1, \dots, g_s \in A_0$ such that

$$Y = \bigcup_{j=1}^s Y_{g_j}, \quad \text{where } Y_{g_j} = \text{Spec } (A_0)_{g_j}$$

and such that for $e = 1, \dots, s$,

$$A_1 \otimes_{A_0} (A_0)_{g_e} = \mathcal{O}_Y(D)(Y_{g_e}) = h_e \cdot A_0$$

for some $h_e \in K_0^*$. Let $\pi : V \rightarrow Y$ be the natural morphism induced by the inclusion $A_0 \subset A$. The preimage of the open subset Y_{g_e} under π is

$$\text{Proj } A \otimes_{A_0} (A_0)_{g_e} = \text{Proj } (A_0)_{g_e} [A_1 \otimes_{A_0} (A_0)_{g_e} t] = \text{Proj } (A_0)_{g_e} [h_e t] = Y_{g_e}.$$

Hence π is the identity map and so $Y = V$, as required. \square

As an immediate consequence we obtain the following. The reader can see that the proof of [FZ03, 3.9] is applicable word by word to positively graded 2-dimensional normal algebras of finite type over a Dedekind domain.

Lemma 1.7. *Let A_0 be a Dedekind ring with the field of fractions K_0 , and let t be a variable over K_0 . Consider the subalgebra*

$$A = A_0[f_1 t^{m_1}, \dots, f_r t^{m_r}] \subset K_0[t],$$

where m_1, \dots, m_r are positive integers and $f_1, \dots, f_r \in K_0^*$ are such that the field of fractions of A is $K_0(t)$. Then the normalization of A is equal to $A_0[D]$, where D is the

\mathbb{Q} -divisor¹

$$D = - \min_{1 \leq i \leq r} \frac{\operatorname{div} f_i}{m_i}.$$

2. MULTIGRADED ALGEBRAS OVER DEDEKIND DOMAINS

Let A_0 be a Dedekind ring and let K_0 be its field of fractions. Given a lattice M , the purpose of this section is to study normal noetherian M -graded A_0 -subalgebras of $K_0[M]$. We show below that these subalgebras admit a description in terms of polyhedral divisors. We start by recalling some necessary notation from convex geometry, see [AH06, Section 1].

2.1. Let N be a lattice, and let $M = \operatorname{Hom}(N, \mathbb{Z})$ be its dual lattice. Denote by $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$ and $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$ the associated dual \mathbb{Q} -linear spaces, respectively. For any linear form $m \in M_{\mathbb{Q}}$ and for any vector $v \in N_{\mathbb{Q}}$ set $\langle m, v \rangle = m(v)$. A polyhedral cone $\sigma \subset N_{\mathbb{Q}}$ is called *strongly convex* if it admits a vertex. This is equivalent to saying that the dual cone

$$\sigma^{\vee} = \{ m \in M_{\mathbb{Q}} \mid \forall v \in \sigma, \langle m, v \rangle \geq 0 \}$$

is of full dimension.

Recall that for a nonzero strongly convex polyhedral cone $\sigma \subset N_{\mathbb{Q}}$ the *Hilbert basis* $\mathcal{H}_{\sigma} = \mathcal{H}_{\sigma, N}$ of σ in the lattice N is the subset of all irreducible elements

$$\{ v \in \sigma_N - \{0\} \mid \forall v_1, v_2 \in \sigma_N - \{0\}, v = v_1 + v_2 \Rightarrow v = v_1 \text{ or } v = v_2 \}.$$

It is known that the set \mathcal{H}_{σ} is finite and generates the semigroup $(\sigma \cap N, +)$. Furthermore, it is minimal for these latter properties. The cone σ is said *regular* if \mathcal{H}_{σ} is contained in a basis of N .

Let us fix a strongly convex polyhedral cone $\sigma \subset N_{\mathbb{Q}}$. A subset $Q \subset N_{\mathbb{Q}}$ is a *polytope* if Q is the convex hull of a non-empty finite subset of vectors. We define $\operatorname{Pol}_{\sigma}(N_{\mathbb{Q}})$ to be the set of polyhedra which can be written as the Minkowski sum $P = Q + \sigma$ with Q a polytope of $N_{\mathbb{Q}}$. An element of $\operatorname{Pol}_{\sigma}(N_{\mathbb{Q}})$ is called a polyhedron with *tailed cone* σ .

Next we introduce the notion of polyhedral divisors over Dedekind domains.

Definition 2.2. Consider the subset Z of closed points of the affine scheme $Y = \operatorname{Spec} A_0$. A σ -polyhedral divisor \mathfrak{D} over A_0 is a formal sum

$$\mathfrak{D} = \sum_{z \in Z} \Delta_z \cdot z,$$

where Δ_z belongs to $\operatorname{Pol}_{\sigma}(N_{\mathbb{Q}})$ and $\Delta_z = \sigma$ for all but finitely many z in Z . Let z_1, \dots, z_r be elements of Z such that $\{z \in Z \mid \Delta_z \neq \sigma\} \subset \{z_1, \dots, z_r\}$. If the meaning of A_0 is clear from the context, then we write

$$\mathfrak{D} = \sum_{i=1}^r \Delta_{z_i} \cdot z_i.$$

¹Let D_1, \dots, D_r be \mathbb{Q} -divisors on a scheme Y . We define the minimum of D_1, \dots, D_r by letting

$$\min_{1 \leq i \leq r} D_i = \sum_{H \subset Y} \min_{1 \leq i \leq r} \{a_{i,H}\} \cdot H,$$

where for $i = 1, \dots, r$, the number $a_{i,H}$ is the coefficient of D_i corresponding to the prime divisor $H \subset Y$.

In the sequel, we let $\omega_M = \omega \cap M$ for a polyhedral cone $\omega \subset M_{\mathbb{Q}}$. Starting with a σ -polyhedral divisor \mathfrak{D} we can build an M -graded algebra over A_0 with weight cone σ^{\vee} in the same way as in [AH06, Section 3].

2.3. Let $m \in \sigma^{\vee}$. Then for any $z \in Z$ the expression

$$h_z(m) = \min_{v \in \Delta_z} \langle m, v \rangle$$

is well defined. The function h_z on the cone σ^{\vee} is upper convex and positively homogeneous. It is identically zero if and only if $\Delta_z = \sigma$. The *evaluation* of \mathfrak{D} in a vector $m \in \sigma^{\vee}$ is the \mathbb{Q} -divisor

$$\mathfrak{D}(m) = \sum_{z \in Z} h_z(m) \cdot z.$$

In analogy with the notation of [FZ03] we denote by $A_0[\mathfrak{D}]$ the M -graded subring

$$\bigoplus_{m \in \sigma_M^{\vee}} A_m \chi^m \subset K_0[M], \text{ where } A_m = H^0(Y, \mathcal{O}_Y([\mathfrak{D}(m)])).$$

Notation 2.4. Let

$$f = (f_1 \chi^{m_1}, \dots, f_r \chi^{m_r})$$

be an r -tuple of homogeneous elements of $K_0[M]$. Assume that the vectors m_1, \dots, m_r generate the cone σ^{\vee} . We denote by $\mathfrak{D}[f]$ the σ -polyhedral divisor

$$\sum_{z \in Z} \Delta_z[f] \cdot z, \text{ where } \Delta_z[f] = \{v \in N_{\mathbb{Q}} \mid \langle m_i, v \rangle \geq -\text{ord}_z f_i, i = 1, 2, \dots, r\}.$$

In section 3, we use a similar notation for polyhedral divisors over a regular projective curve.

The main result of this section is the following theorem. For a proof of part (iii) we refer the reader to the argument of Theorem 2.4 in [Lan13].

Theorem 2.5. *Let A_0 be a Dedekind domain with field of fractions K_0 and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. Then the following hold.*

- (i) *If \mathfrak{D} is a σ -polyhedral divisor over A_0 , then the algebra $A_0[\mathfrak{D}]$ is normal, noetherian, and has the same field of fractions as $K_0[M]$.*
- (ii) *Conversely, let*

$$A = \bigoplus_{m \in \sigma_M^{\vee}} A_m \chi^m$$

be a normal noetherian M -graded A_0 -subalgebra of $K_0[M]$ with weight cone σ^{\vee} and $A_m \subset K_0$, for all $m \in \sigma_M^{\vee}$. Assume that the rings A and $K_0[M]$ have the same field of fractions². Then there exists a unique σ -polyhedral divisor \mathfrak{D} over A_0 such that $A = A_0[\mathfrak{D}]$.

- (iii) *More explicitly, if*

$$f = (f_1 \chi^{m_1}, \dots, f_r \chi^{m_r})$$

²This condition is equivalent to ask that the weight semigroup of A generates M .

is an r -tuple of homogeneous elements of $K_0[M]$ with nonzero vectors m_1, \dots, m_r generating the lattice M , then the normalization of the ring

$$A = A_0[f_1\chi^{m_1}, \dots, f_r\chi^{m_r}]$$

is equal to $A_0[\mathfrak{D}[f]]$ (see 2.4).

Let us give an example related to the ring of integers of a number field.

Example 2.6. For a number field K , the group of classes $\text{Cl } K$ is the quotient of the group of fractional ideals of K by the subgroup of principal fractional ideals. In other words, $\text{Cl } K = \text{Pic } Y$, where $Y = \text{Spec } \mathbb{Z}_K$ is the affine scheme associated to the ring of integers of K . It is known that the group $\text{Cl } K$ is finite. Furthermore \mathbb{Z}_K is a principal ideal domain if and only if $\text{Cl } K$ is trivial.

Let $K = \mathbb{Q}(\sqrt{-5})$. Then $\mathbb{Z}_K = \mathbb{Z}[\sqrt{-5}]$ and the group $\text{Cl } K$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. A set of representatives in $\text{Cl } K$ is given by the fractional ideals $\mathfrak{a} = (2, 1 + \sqrt{-5})$ and \mathbb{Z}_K . Given x, y two independent variables over K , consider the \mathbb{Z}^2 -graded ring

$$A = \mathbb{Z}_K [3x^2y, 2y, 6x].$$

Let us describe the normalization \bar{A} of A . Denoting respectively by $\mathfrak{b}, \mathfrak{c}$ the prime ideals $(3, 1 + \sqrt{-5})$ and $(3, 1 - \sqrt{-5})$, we have the decompositions

$$(2) = \mathfrak{a}^2, \quad (3) = \mathfrak{b} \cdot \mathfrak{c}.$$

Observe that the ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are distinct. Thus we have

$$\text{div } 2 = 2 \cdot [\mathfrak{a}] \quad \text{and} \quad \text{div } 3 = [\mathfrak{b}] + [\mathfrak{c}],$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are seen as closed points of $Y = \text{Spec } \mathbb{Z}_K$. Let \mathfrak{D} be the polyhedral divisor over \mathbb{Z}_K given by $\Delta_{\mathfrak{a}} \cdot [\mathfrak{a}] + \Delta_{\mathfrak{b}} \cdot [\mathfrak{b}] + \Delta_{\mathfrak{c}} \cdot [\mathfrak{c}]$ with the polyhedra

$$\Delta_{\mathfrak{a}} = \{(v_1, v_2) \in \mathbb{Q}^2 \mid 2v_1 + v_2 \geq 0, v_2 \geq -2, v_1 \geq -2\} \quad \text{and}$$

$$\Delta_{\mathfrak{b}} = \Delta_{\mathfrak{c}} = \{(v_1, v_2) \in \mathbb{Q}^2 \mid 2v_1 + v_2 \geq -1, v_2 \geq 0, v_1 \geq -1\}.$$

By Theorem 2.5 we obtain $\bar{A} = A_0[\mathfrak{D}]$, where $A_0 = \mathbb{Z}_K$. The weight cone of A is the first quadrant $\omega = (\mathbb{Q}_{\geq 0})^2$. An easy computation shows that

$$A_0[\mathfrak{D}] = \mathbb{Z}_K [2y, 6xy, 3(1 + \sqrt{-5})xy, 3x^2y, 6x].$$

The proof of Theorem 2.5 needs some preparations. We start by a well-known result [GY83, Theorem 1.1] yielding an equivalence between noetherian and finitely generated properties of multigraded algebras. Note that this result does not hold for algebras graded by an arbitrary abelian group; a counterexample is given in [GY83, 3.1].

Lemma 2.7. *Let G denote a finitely generated abelian group and let A be a G -graded ring. Then the following statements are equivalent.*

- (i) *The ring A is noetherian.*
- (ii) *The graded piece A_0 corresponding to the neutral element of G is a noetherian ring and the A_0 -algebra A is finitely generated.*

The next lemma will enable us to show that the ring $A_0[\mathfrak{D}]$, coming from a polyhedral divisor \mathfrak{D} over a Dedekind domain A_0 , is noetherian.

Lemma 2.8. *For any \mathbb{Q} -divisors D_1, \dots, D_r on $Y = \text{Spec } A_0$, the A_0 -algebra*

$$B = \bigoplus_{(m_1, \dots, m_r) \in \mathbb{N}^r} H^0 \left(Y, \mathcal{O}_Y \left(\left[\sum_{i=1}^r m_i D_i \right] \right) \right)$$

is finitely generated.

Proof. Let d be a positive integer such that for $i = 1, \dots, r$, the divisor dD_i is integral. Consider the lattice polytope

$$Q = \{ (m_1, \dots, m_r) \in \mathbb{Q}^r \mid 0 \leq m_i \leq d, i = 1, \dots, r \}.$$

The subset $Q \cap \mathbb{N}^r$ being finite, the A_0 -module

$$E := \bigoplus_{(m_1, \dots, m_r) \in \mathbb{N}^r \cap Q} H^0 \left(Y, \mathcal{O}_Y \left(\left[\sum_{i=1}^r m_i D_i \right] \right) \right)$$

is finitely generated (see 1.4). Let (m_1, \dots, m_r) be an element of \mathbb{N}^r . Write $m_i = dq_i + r_i$, where $q_i, r_i \in \mathbb{N}$ are such that $0 \leq r_i < d$. The equality

$$\left[\sum_{i=1}^r m_i D_i \right] = \sum_{i=1}^r q_i [dD_i] + \left[\sum_{i=1}^r r_i D_i \right]$$

implies that every homogeneous element of B can be expressed as a polynomial in E . If f_1, \dots, f_s generate the A_0 -module E , then we have $A = A_0[f_1, \dots, f_s]$, proving our statement. \square

Next we give a proof of the first part of Theorem 2.5.

Proof. Let $A = A_0[\mathfrak{D}]$. Since the cone σ^\vee is full dimensional, by Lemma 1.4 the algebras A and $K_0[M]$ have the same field of fractions. Let us show that A is a normal ring. Given a closed point $z \in Z$ and an element of $v \in \Delta_z$, we define the map

$$\nu_{z,v} : K_0[M] - \{0\} \rightarrow \mathbb{Z}$$

as follows. Let $\alpha \in K_0[M]$ be nonzero. We may decompose it as a sum of homogeneous elements

$$\alpha = \sum_{i=1}^r f_i \chi^{m_i}, \quad \text{where } f_i \in K_0^\star.$$

We let

$$\nu_{z,v}(\alpha) = \min_{1 \leq i \leq r} \{ \text{ord}_z f_i + \langle m_i, v \rangle \}.$$

The map $\nu_{z,v}$ defines a discrete valuation on $\text{Frac } A$. Denote by $\mathcal{O}_{v,z}$ the associated local ring. By the definition of the algebra $A_0[\mathfrak{D}]$ we have

$$A = K_0[M] \cap \bigcap_{z \in Z} \bigcap_{v \in \Delta_z} \mathcal{O}_{v,z}.$$

This shows that A is normal as an intersection of normal rings with the same field of fractions $\text{Frac } A$.

It remains to show that A is noetherian. By Hilbert's Basis Theorem, it suffices to show that A is finitely generated. Let $\lambda_1, \dots, \lambda_e$ be full dimensional regular subcones of σ^\vee , which define a subdivision of σ^\vee . Assume that for any i the evaluation map

$$\sigma^\vee \rightarrow \text{Div}_{\mathbb{Q}}(Y), \quad m \mapsto \mathfrak{D}(m)$$

is linear on λ_i . Fix $i \in \mathbb{N}$ such that $1 \leq i \leq e$. Consider the distinct elements v_1, \dots, v_n of the Hilbert basis of λ_i . Denote by A_{λ_i} the algebra

$$\bigoplus_{m \in \lambda_i \cap M} H^0(Y, \mathcal{O}_Y(\lfloor \mathfrak{D}(m) \rfloor)) \chi^m.$$

Then the vectors v_1, \dots, v_n form a basis of the lattice M and so

$$A_{\lambda_i} \simeq \bigoplus_{(m_1, \dots, m_n) \in \mathbb{N}^n} H^0 \left(Y, \mathcal{O}_Y \left(\left\lfloor \sum_{i=1}^n m_i \mathfrak{D}(v_i) \right\rfloor \right) \right).$$

By Lemma 2.8, the algebra A_{λ_i} is finitely generated over A_0 . The surjective map

$$A_{\lambda_1} \otimes \dots \otimes A_{\lambda_e} \rightarrow A$$

shows that A is also finitely generated. \square

For the second part of Theorem 2.5 we need the following lemma.

Lemma 2.9. *Assume that A verifies the assumptions of 2.5 (ii). Then the following statements hold.*

- (i) *For any $m \in \sigma_M^\vee$ we have $A_m \neq \{0\}$. In other words, the weight semigroup of the M -graded algebra A is σ_M^\vee .*
- (ii) *If $L = \mathbb{Q}_{\geq 0} \cdot m'$ is a half-line contained in σ^\vee , then the ring*

$$A_L := \bigoplus_{m \in L \cap M} A_m \chi^m$$

is normal and noetherian.

Proof. Let

$$S = \{m \in \sigma_M^\vee, A_m \neq \{0\}\}$$

be the weight semigroup of A . Assume that $S \neq \sigma_M^\vee$. Then there exist $e \in \mathbb{Z}_{>0}$ and $m \in M$ such that $m \notin S$ and $e \cdot m \in S$. Since A is a noetherian ring, by [GY83, Lemma 2.2] the A_0 -module A_{em} is a fractional ideal of A_0 . By Lemma 1.4 we obtain

$$A_{em} = H^0(Y, \mathcal{O}_Y(D_{em}))$$

for some integral divisor $D_{em} \in \text{Div}_{\mathbb{Z}}(Y)$. Let f be a nonzero section of

$$H^0 \left(Y, \mathcal{O}_Y \left(\left\lfloor \frac{D_{em}}{e} \right\rfloor \right) \right).$$

This element exists by virtue of Lemma 1.4. We have the inequalities

$$\text{div } f^e \geq -e \left\lfloor \frac{D_{em}}{e} \right\rfloor \geq -D_{em}.$$

The normality of A implies that $f \in A_m$. This contradicts our assumption and gives (i). For the second assertion we notice that A_L is noetherian by 2.7 and by the argument of [AH06, Lemma 4.1].

It remains to show that A_L is normal. Let $\alpha \in \text{Frac } A_L$ be an integral element over A_L . By normality of A and $K_0[\chi^m]$ we obtain that $\alpha \in A \cap K_0[\chi^m] = A_L$ and so A_L is normal. \square

Let us introduce the following notation.

Notation 2.10. Let

$$(m_i, e_i), \quad i = 1, \dots, r$$

be elements of $M \times \mathbb{Z}$ such that the vectors m_1, \dots, m_r are nonzero and generate the lattice M . Then the cone $\omega = \text{Cone}(m_1, \dots, m_r)$ is full dimensional in $M_{\mathbb{Q}}$. Consider the ω^{\vee} -polyhedron

$$\Delta = \{v \in N_{\mathbb{Q}}, \langle m_i, v \rangle \geq -e_i, \quad i = 1, 2, \dots, r\}.$$

Let $L = \mathbb{Q}_{\geq 0} \cdot m$ be a half-line contained in ω with a primitive vector m . In other words, the element m generates the semigroup $L \cap M$. Denote by \mathcal{H}_L the Hilbert basis in the lattice \mathbb{Z}^r of the nonzero cone

$$p^{-1}(L) \cap (\mathbb{Q}_{\geq 0})^r, \quad \text{where } p: \mathbb{Q}^r \rightarrow M_{\mathbb{Q}}$$

is the \mathbb{Q} -linear map sending the canonical basis onto (m_1, \dots, m_r) . We let

$$\mathcal{H}_L^* = \left\{ (s_1, \dots, s_r) \in \mathcal{H}_L, \quad \sum_{i=1}^r s_i \cdot m_i \neq 0 \right\}.$$

For any vector $(s_1, \dots, s_r) \in \mathcal{H}_L^*$ there exists a unique $\lambda(s_1, \dots, s_r) \in \mathbb{Z}_{>0}$ such that

$$\sum_{i=1}^r s_i \cdot m_i = \lambda(s_1, \dots, s_r) \cdot m.$$

The proof of the following lemma uses only some elementary facts of commutative algebra and of convex geometry. This lemma is the key point in order to obtain the Altmann-Hausen presentation of Theorem 2.5 (ii).

Lemma 2.11. *Let $\min \langle m, \Delta \rangle = \min_{v \in \Delta} \langle m, v \rangle$. Under the assumptions of 2.10 we have*

$$\min \langle m, \Delta \rangle = - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^*} \frac{\sum_{i=1}^r s_i \cdot e_i}{\lambda(s_1, \dots, s_r)}.$$

Proof. Consider the M -graded subalgebra

$$A = \mathbb{C}[t][t^{e_1} \chi^{m_1}, \dots, t^{e_r} \chi^{m_r}] \subset \mathbb{C}(t)[M],$$

where t is a variable. The field of fractions of A is the same as that of $\mathbb{C}(t)[M]$. By the results of [Hoc72], the normalization of A is

$$\bar{A} = \mathbb{C}[\omega_0 \cap (M \times \mathbb{Z})], \quad \text{where } \omega_0 \subset M_{\mathbb{Q}} \times \mathbb{Q}$$

is the rational cone generated by $(0, 1), (m_1, e_1), \dots, (m_r, e_r)$. A routine calculation shows that

$$\omega_0 = \{(w, -\min \langle w, \Delta \rangle + e) \mid w \in \omega, e \in \mathbb{Q}_{\geq 0}\},$$

and so

$$\bar{A} = \bigoplus_{m \in \omega \cap M} H^0(\mathbb{A}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1}([\min \langle m, \Delta \rangle] \cdot (0))) \chi^m,$$

where $\mathbb{A}_{\mathbb{C}}^1 = \text{Spec } \mathbb{C}[t]$.

The sublattice $G \subset M$ generated by $p(\mathcal{H}_L^\star)$ is a subgroup of $\mathbb{Z} \cdot m$. Therefore there exists a unique integer $d \in \mathbb{Z}_{>0}$ such that $G = d\mathbb{Z} \cdot m$. For an element $m' \in \omega \cap M$, we denote by $A_{m'}$ (resp. $\bar{A}_{m'}$) the graded piece of A (resp. \bar{A}) corresponding to m' . Then the normalization $\bar{A}_L^{(d)}$ of the algebra

$$A_L^{(d)} := \bigoplus_{s \geq 0} A_{sdm} \chi^{sdm} \quad \text{is} \quad B_L := \bigoplus_{s \geq 0} \bar{A}_{sdm} \chi^{sdm}.$$

Furthermore

$$A_L = \bigoplus_{s \geq 0} A_{sm} \chi^{sm}$$

is generated over $\mathbb{C}[t]$ by the elements

$$f_{(s_1, \dots, s_r)} := \prod_{i=1}^r (t^{e_i} \chi^{m_i})^{s_i} = t^{\sum_{i=1}^r s_i e_i} \chi^{\lambda(s_1, \dots, s_r)m},$$

where (s_1, \dots, s_r) runs over \mathcal{H}_L^\star . By the choice of the integer d we have $A_L^{(d)} = A_L$. Considering the G -graduation of $A_L^{(d)}$, for any $(s_1, \dots, s_r) \in \mathcal{H}_L^\star$ the element $f_{(s_1, \dots, s_r)}$ of the graded ring $A_L^{(d)}$ has degree

$$\deg f_{(s_1, \dots, s_r)} := \frac{\lambda(s_1, \dots, s_r)}{d}.$$

Letting

$$D = - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^\star} \frac{\text{div } f_{(s_1, \dots, s_r)}}{\deg f_{(s_1, \dots, s_r)}} = - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^\star} d \cdot \frac{\sum_{i=1}^r s_i e_i}{\lambda(s_1, \dots, s_r)} \cdot (0),$$

by Lemma 1.7 we obtain

$$\bar{A}_L^{(d)} = \bigoplus_{s \geq 0} H^0(\mathbb{A}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1}(\lfloor sD \rfloor)) \chi^{sdm}.$$

The equality $\bar{A}_L^{(d)} = B_L$ implies that for any integer $s \geq 0$

$$H^0(\mathbb{A}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1}(\lfloor \min \langle sd \cdot m, \Delta \rangle \rfloor \cdot (0))) = H^0(\mathbb{A}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1}(\lfloor sD \rfloor)).$$

Hence by Lemma 1.6 we have

$$D = \min \langle d \cdot m, \Delta \rangle \cdot (0).$$

Dividing by d , we obtain the desired formula. \square

Let A be an M -graded algebra satisfying the assumptions of 2.5 (ii). Using the D.P.D. presentation on each half line of the weight cone σ^\vee (see Lemma 1.6), we can define a map

$$\sigma^\vee \rightarrow \text{Div}_{\mathbb{Q}}(Y), \quad m \mapsto D_m.$$

It is upper convex, positively homogeneous, and verifies, for any $m \in \sigma_M^\vee$, the equality

$$A_m = H^0(C, \mathcal{O}_C(\lfloor D_m \rfloor)).$$

By Lemma 2.11, this map is piecewise linear (see [AH06, 2.11]). Equivalently, $m \mapsto D_m$ is the evaluation map of a polyhedral divisor. The following proof will specify this idea.

Proof of 2.5 (ii). By 2.7 we may consider a system of homogeneous generators

$$f = (f_1\chi^{m_1}, \dots, f_r\chi^{m_r})$$

of A , with nonzero vectors $m_1, \dots, m_r \in M$. We use the same notation as in 2.4. Denote by \mathfrak{D} the σ -polyhedral divisor $\mathfrak{D}[f]$. Let us show that $A = A_0[\mathfrak{D}]$. Let $L = \mathbb{Q}_{\geq 0} \cdot m$ be a half-line contained in $\omega = \sigma^\vee$, with m being the primitive vector of L . By Lemma 2.9, the graded subalgebra

$$A_L := \bigoplus_{m' \in L \cap M} A_{m'}\chi^{m'} \subset K_0[\chi^m]$$

is normal, noetherian, and has the same field of fractions as that of $K_0[\chi^m]$. Furthermore, with the same notation as in 2.10, the algebra A_L is generated by the set

$$\left\{ \prod_{i=1}^r (f_i\chi^{m_i})^{s_i}, (s_1, \dots, s_r) \in \mathcal{H}_L^* \right\}.$$

By Lemma 1.7, if

$$D_m := - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^*} \frac{\sum_{i=1}^r s_i \operatorname{div} f_i}{\lambda(s_1, \dots, s_r)},$$

then $A_L = A_0[D_m]$ with respect to the variable χ^m . By Lemma 2.11, for any closed point $z \in Z$ we have

$$h_z[f](m) = \min \langle m, \Delta_z[f] \rangle = - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^*} \frac{\sum_{i=1}^r s_i \operatorname{ord}_z f_i}{\lambda(s_1, \dots, s_r)}.$$

Hence $\mathfrak{D}(m) = D_m$. Since this equality holds for all primitive vectors belonging to σ^\vee , we may conclude that $A = A_0[\mathfrak{D}]$. The uniqueness of \mathfrak{D} is straightforward (see Lemma 1.4 and [Lan13, 2.2]). \square

Using well-known facts about Dedekind domains we obtain the following result.

Proposition 2.12. *Let A_0 be a Dedekind domain and let B_0 be the integral closure of A_0 in a finite separable extension L_0/K_0 , where $K_0 = \operatorname{Frac} A_0$. Let $\mathfrak{D} = \sum_{z \in Z} \Delta_z \cdot z$ be a polyhedral divisor over A_0 , where $Z \subset Y = \operatorname{Spec} A_0$ is the subset of closed points. Let $Y' = \operatorname{Spec} B_0$ and consider the natural projection $p : Y' \rightarrow Y$. Then B_0 is a Dedekind domain and we have the formula*

$$A_0[\mathfrak{D}] \otimes_{A_0} B_0 = B_0[p^*\mathfrak{D}] \quad \text{with} \quad p^*\mathfrak{D} = \sum_{z \in Z} \Delta_z \cdot p^*(z).$$

Example 2.13. Consider the polyhedral divisor

$$\mathfrak{D} = \Delta_{(t)} \cdot (t) + \Delta_{(t^2+1)} \cdot (t^2 + 1)$$

over the Dedekind ring $A_0 = \mathbb{R}[t]$, where the coefficients are

$$\Delta_{(t)} = (-1, 0) + \sigma, \quad \Delta_{(t^2+1)} = [(0, 0), (0, 1)] + \sigma,$$

and where $\sigma \subset \mathbb{Q}^2$ is the rational cone generated by $(1, 0)$ and $(1, 1)$. An easy computation shows that

$$A_0[\mathfrak{D}] = \mathbb{R} [t, -t\chi^{(1,0)}, \chi^{(0,1)}, t(t^2 + 1)\chi^{(1,-1)}] \simeq \frac{\mathbb{R}[x_1, x_2, x_3, x_4]}{((1 + x_1^2)x_2 + x_3x_4)},$$

where x_1, x_2, x_3, x_4 are independent variables over \mathbb{R} . Let $\zeta = \sqrt{-1}$. Considering the natural projection $p : \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{R}}^1$ we obtain

$$p^*\mathfrak{D} = \Delta_0 \cdot 0 + \Delta_{(t^2+1)} \cdot \zeta + \Delta_{(t^2+1)} \cdot (-\zeta).$$

Letting $B_0 = \mathbb{C}[t]$ one concludes that $A_0[\mathfrak{D}] \otimes_{\mathbb{R}} \mathbb{C} = B_0[p^*\mathfrak{D}]$.

3. MULTIGRADED ALGEBRAS AND ALGEBRAIC FUNCTION FIELDS

In this section, we study another type of multigraded algebras. They are described by a proper polyhedral divisor over an algebraic function field in one variable. Fix an arbitrary field k . Recall that an *algebraic function field* (in one variable) over k is a finitely generated field extension K_0/k of transcendence degree one with the property that k is algebraically closed in K_0 .

3.1. By virtue of our convention, a regular projective curve C over k yields an algebraic function field K_0/k , where $K_0 = k(C)$. As an application of the valuative criterion of properness (see [EGA II, Section 7.4]), every algebraic function field K_0/k is the field of rational functions of a unique (up to isomorphism) regular projective curve C over k .

In the next paragraph, we recall the construction of the curve C starting from an algebraic function field K_0 .

3.2. A *valuation ring* of K_0 is a proper subring $\mathcal{O} \subset K_0$ strictly containing k and such that for any nonzero element $f \in K_0$, either $f \in \mathcal{O}$ or $\frac{1}{f} \in \mathcal{O}$. By [Sti93, 1.1.6] every valuation ring of K_0 is the ring associated to a discrete valuation of K_0/k . A subset $P \subset K_0$ is called a *place* of K_0 if there is some valuation ring \mathcal{O} of K_0 such that P is the maximal ideal of \mathcal{O} . We denote by $\mathcal{R}_k K_0$ the set of places of K_0 . The latter is called the *Riemann surface* of K_0 . By [EGA II, 7.4.18] the set $\mathcal{R}_k K_0$ can be identified with the (closed) points of a regular projective curve C over the field k such that $K_0 = k(C)$.

In the sequel we consider $C = \mathcal{R}_k K_0$ as a geometrical object with its structure of scheme. By convention an element z belonging to C is a closed point. We write P_z the place associated to a point $z \in C$. Note that we keep the notation of convex geometry introduced in 2.1.

3.3. Let M, N be a pair of dual lattices, and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. A σ -*polyhedral divisor* over K_0 (or over C) is a formal sum $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ with $\Delta_z \in \text{Pol}_{\sigma}(N_{\mathbb{Q}})$ and $\Delta_z = \sigma$ for all but finitely many $z \in C$. Again we let

$$\mathfrak{D}(m) = \sum_{z \in C} \min_{v \in \Delta_z} \langle m, v \rangle \cdot z$$

be the *evaluation* in $m \in \sigma^{\vee}$; that is, a \mathbb{Q} -divisor over the curve C . We let $\kappa(P) = \mathcal{O}/P$, where \mathcal{O} is the valuation ring of a place P . The field $\kappa(P)$ is a finite extension of k [Sti93, 1.1.15]; we call it the *residue field* of P . We denote by $[\kappa(P) : k]$ the dimension of the k -vector space $\kappa(P)$; the latter is also called the degree of the place P . Recall that we define the *degree* of a \mathbb{Q} -divisor $D = \sum_{z \in C} a_z \cdot z$ to be the rational number

$$\deg D = \sum_{z \in C} [\kappa(P_z) : k] \cdot a_z.$$

Similarly, the *degree* of the polyhedral divisor \mathfrak{D} is the Minkowski sum

$$\deg \mathfrak{D} = \sum_{z \in C} [\kappa(P_z) : k] \cdot \Delta_z.$$

Given $m \in \sigma^\vee$ we have naturally the relation $(\deg \mathfrak{D})(m) = \deg \mathfrak{D}(m)$.

We can now introduce the notion of properness for polyhedral divisors (see [AH06, 2.7, 2.12]).

Definition 3.4. A σ -polyhedral divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is called *proper* if it satisfies the following conditions.

- (i) The polyhedron $\deg \mathfrak{D}$ is strictly contained in the cone σ .
- (ii) If $\deg \mathfrak{D}(m) = 0$, then m belongs to the boundary of σ^\vee and a multiple of $\mathfrak{D}(m)$ is principal.

Our next main result gives a description similar to that in 2.5 for algebraic function fields. For a proof of 3.5 (iii) we refer to the argument of [Lan13, 2.4].

Theorem 3.5. *Let k be a field, and let $C = \mathcal{R}_k K_0$ be the Riemann surface of an algebraic function field K_0/k . Then the following statements hold.*

- (i) *Let*

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m$$

be an M -graded normal noetherian k -subalgebra of $K_0[M]$ with weight cone σ^\vee and $A_0 = k$. We assume that for all $m \in \sigma_M^\vee$, $A_m \subset K_0$. If A and $K_0[M]$ have the same field of fractions, then there exists a unique proper σ -polyhedral divisor \mathfrak{D} over C such that $A = A[C, \mathfrak{D}]$, where

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} H^0(C, \mathcal{O}_C(\lfloor \mathfrak{D}(m) \rfloor)) \chi^m.$$

- (ii) *Let \mathfrak{D} be a proper σ -polyhedral divisor over C . Then the algebra $A[C, \mathfrak{D}]$ is M -graded, normal, and finitely generated with weight cone σ^\vee . Furthermore it has the same field of fractions as $K_0[M]$.*
- (iii) *Let*

$$A = k[f_1 \chi^{m_1}, \dots, f_r \chi^{m_r}]$$

be an M -graded subalgebra of $K_0[M]$, where the $f_i \chi^{m_i}$ is a homogeneous element of nonzero degree m_i . Let $f = (f_1 \chi^{m_1}, \dots, f_r \chi^{m_r})$. Assume that A and $K_0[M]$ have the same field of fractions. Then $\mathfrak{D}[f]$ is the proper σ -polyhedral divisor such that the normalization of A is $A[C, \mathfrak{D}[f]]$ (see 2.4).

For the proof of 3.5 we need some preliminary results. We begin by collecting some properties of an M -graded algebra A as in 3.5 (i) to some graded subring A_L .

Lemma 3.6. *Let A be an M -graded algebra satisfying the assumptions of 3.5 (i). Given a half-line $L = \mathbb{Q}_{\geq 0} \cdot m \subset \sigma^\vee$ with a primitive vector m , consider the subalgebra*

$$A_L = \bigoplus_{m' \in L \cap M} A_{m'} \chi^{m'}.$$

Let

$$Q(A_L)_0 = \left\{ \frac{a}{b} \mid a \in A_{sm}, b \in A_{sm}, b \neq 0, s \geq 0 \right\}.$$

Then the following assertions hold.

- (i) The algebra A_L is finitely generated and normal.
- (ii) Either $Q(A_L)_0 = k$ or $Q(A_L)_0 = K_0$.
- (iii) If $Q(A_L)_0 = k$, then $A_L = k[\beta\chi^{dm}]$ for some $\beta \in K_0^*$ and some $d \in \mathbb{Z}_{>0}$.

Proof. The proof of (i) is similar to that of 2.9 (ii) and so we omitted it.

The field $Q(A_L)_0$ is an extension of k contained in K_0 . If the transcendence degree of $Q(A_L)_0$ over k is zero, then by normality of A_L we have $Q(A_L)_0 = k$. Otherwise the extension $K_0/Q(A_L)_0$ is algebraic. Let α be an element of K_0 . Then there exist $a_1, \dots, a_d \in Q(A_L)_0$ with $a_d \neq 0$ such that

$$\alpha^d = \sum_{j=1}^d a_j \alpha^{d-j}.$$

Let

$$I = \{i \in \{1, \dots, d\}, a_i \neq 0\}.$$

For any $i \in I$ we write $a_i = \frac{p_i}{q_i}$ with $p_i, q_i \in A_L$ being homogeneous of the same degree. Considering $q = \prod_{i \in I} q_i$ we obtain the equality

$$(\alpha q)^d = \sum_{j=1}^d a_j q^j (\alpha q)^{d-j}.$$

The normality of A_L gives $\alpha q \in A_L$. Thus $\alpha = \alpha q / q \in Q(A_L)_0$.

To show (iii) we let $S \subset \mathbb{Z} \cdot m$ be the weight semigroup of the graded algebra A_L . Since L is contained in the weight cone σ^\vee , S is nonzero. Therefore if G is the subgroup generated by S , then there exists $d \in \mathbb{Z}_{>0}$ such that $G = \mathbb{Z}d \cdot m$. Letting $u = \chi^{dm}$ we can write

$$A_L = \bigoplus_{s \geq 0} A_{sdm} u^s.$$

Thus for any homogeneous elements $a_1 u^l, a_2 u^l \in A_L$ of the same degree we have $\frac{a_1}{a_2} \in Q(A_L)_0^* = k^*$ so that

$$A_L = \bigoplus_{s \in S'} k f_s u^s,$$

where $S' := \frac{1}{d} S$ and $f_s \in k(C)^*$. Let us fix homogeneous generators $f_{s_1} u^{s_1}, \dots, f_{s_r} u^{s_r}$ of the G -graded algebra A_L . Consider $d' := \text{g.c.d.}(s_1, \dots, s_r)$. If $d' > 1$, then the inclusion $S \subset dd' \mathbb{Z} \cdot m$ yields a contradiction. So $d' = 1$ and there are integers l_1, \dots, l_r such that $1 = \sum_{i=1}^r l_i s_i$. The element

$$\beta u = \prod_{i=1}^r (f_{s_i} u^{s_i})^{l_i}$$

verifies

$$\frac{(\beta u)^{s_1}}{f_{s_1} u^{s_1}} \in Q(A_L)_0^* = k^*.$$

By normality of A_L , $\beta u \in A_L$ and so $A_L = k[\beta u] = k[\beta \chi^{dm}]$. Now (iii) follows. \square

The following lemma is well known. For the main argument we refer the reader to [Dem88, Section 3], [Liu02, §7.4.1, Proposition 4.4], or [AH06, 9.1].

Lemma 3.7. *Let D_1, D_2, D be \mathbb{Q} -divisors on C . Then the following hold.*

- (i) *If D has positive degree, then there exists $d \in \mathbb{Z}_{>0}$ such that the invertible sheaf $\mathcal{O}_C(\lfloor dD \rfloor)$ of \mathcal{O}_C -modules is very ample. Furthermore, the graded algebra*

$$B = \bigoplus_{l \geq 0} H^0(C, \mathcal{O}_C(\lfloor lD \rfloor)) t^l,$$

where t is a variable over $k(C)$, is finitely generated. The field of fractions of B is $k(C)(t)$.

- (ii) *Assume that for $i = 1, 2$ we have either $\deg D_i > 0$ or rD_i is principal for some $r \in \mathbb{Z}_{>0}$. If for any $s \in \mathbb{N}$ the inclusion*

$$H^0(C, \mathcal{O}_C(\lfloor sD_1 \rfloor)) \subset H^0(C, \mathcal{O}_C(\lfloor sD_2 \rfloor))$$

holds, then we have $D_1 \leq D_2$.

In the next corollary, we keep the notation of Lemma 3.6. Using Demazure's Theorem for normal graded algebras, we show that each A_L admits a D.P.D. presentation with the same regular projective curve C .

Corollary 3.8. *There exists a unique \mathbb{Q} -divisor D on C such that*

$$A_L = \bigoplus_{s \geq 0} H^0(C, \mathcal{O}_C(\lfloor sD \rfloor)) \chi^{sm}$$

and the following hold.

- (i) *If $Q(A_L)_0 = k$, then $D = \frac{\operatorname{div} f}{d}$ for some $f \in K_0^*$ and some $d \in \mathbb{Z}_{>0}$.*
(ii) *If $Q(A_L)_0 = K_0$, then $\deg D > 0$.*
(iii) *If $f_1 \chi^{s_1 m}, \dots, f_r \chi^{s_r m}$ are homogeneous generators of the algebra A_L , then*

$$D = - \min_{1 \leq i \leq r} \frac{\operatorname{div} f_i}{s_i}.$$

Proof. (i) Assume that $Q(A_L)_0 = k$. By Lemma 3.6, $A_L = k[\beta \chi^{dm}]$ for some $\beta \in K_0^*$ and some $d \in \mathbb{Z}_{>0}$. Thus, we can take $D = \frac{\operatorname{div} \beta^{-1}}{d}$. The uniqueness in this case is easy. This gives assertion (i).

(ii) The field of rational functions of the normal variety $\operatorname{Proj} A_L$ is $K_0 = Q(A_L)_0$. Since $\operatorname{Proj} A_L$ is a regular projective curve over $A_0 = k$, we may identify its points with the places of K_0 . Therefore the existence and the uniqueness of D follow from Demazure's Theorem (see [Dem88, Theorem 3.5]). Furthermore $Q(A_L)_0 \neq k$ implies that $\dim_k A_{sm} \geq 2$, for some $s \in \mathbb{Z}_{>0}$. Hence by [Sti93, 1.4.12] we obtain $\deg D > 0$.

The proof of (iii) follows from 3.7 and from the argument in [FZ03, 3.9]. \square

As a consequence of Corollary 3.8, again we can apply the formula of 2.11 to obtain the existence of the polyhedral divisor \mathfrak{D} as in the statement of 3.5 (i).

Proof of 3.5 (i). Let us adopt the notation introduced in 2.4 and 2.10. Let

$$f = (f_1\chi^{m_1}, \dots, f_r\chi^{m_r})$$

be a system of homogeneous generators of A . Consider a half-line

$$L = \mathbb{Q}_{\geq 0} \cdot m \subset \sigma^\vee$$

with primitive vector $m \in M$. By Corollary 3.8

$$A_L = \bigoplus_{s \geq 0} H^0(C, \mathcal{O}_C(\lfloor sD_m \rfloor))\chi^{sm}$$

for a unique \mathbb{Q} -divisor D_m on C . By the proof of [AH06, Lemma 4.1] the algebra A_L is generated by

$$\prod_{i=1}^r (f_i\chi^{m_i})^{s_i}, \text{ where } (s_1, \dots, s_r) \in \mathcal{H}_L^*.$$

By Corollary 3.8 (iii) and Lemma 2.11 we have $\mathfrak{D}[f](m) = D_m$ and so $A = A[C, \mathfrak{D}[f]]$.

It remains to show that $\mathfrak{D} = \mathfrak{D}[f]$ is proper; the uniqueness of \mathfrak{D} will then follow by Lemma 3.7 (ii). Denote by $S \subset C$ the union of the supports of the divisors $\text{div } f_i$, for $i = 1, \dots, r$. Let $v \in \text{deg } \mathfrak{D}$. We can write

$$v = \sum_{z \in S} [\kappa(P_z) : k] \cdot v_z$$

for some $v_z \in \Delta_z[f]$. Therefore for any i we have

$$\langle m_i, \sum_{z \in S} [\kappa(P_z) : k] \cdot v_z \rangle \geq - \sum_{s \in S} [\kappa(P_z) : k] \cdot \text{ord}_z f_i = -\text{deg } \text{div } f_i = 0$$

and so $\text{deg } \mathfrak{D} \subset \sigma$. If $\text{deg } \mathfrak{D} = \sigma$, then one concludes that $\text{Frac } A$ is different from $\text{Frac } K_0[M]$, contradicting our assumption. Hence $\text{deg } \mathfrak{D} \neq \sigma$. Let $m \in \sigma_M^\vee$ be such that $\text{deg } \mathfrak{D}(m) = 0$. Then m belongs to the boundary of σ^\vee . Consider the half-line L generated by m . Applying Corollary 3.8 (i) to the algebra A_L , we deduce that a multiple of $\mathfrak{D}(m)$ is principal, proving that \mathfrak{D} is proper. \square

Proof of 3.5 (ii). Let us show that the algebras $A = A[C, \mathfrak{D}]$ and $K_0[M]$ have the same field of fractions. Let $L = \mathbb{Q}_{\geq 0} \cdot m$ be a half-line intersecting σ^\vee with its relative interior and having m for primitive vector. Since $\text{deg } \mathfrak{D}(m) > 0$ by Lemma 3.7 (i) we have $\text{Frac } A_L = K_0(\chi^m)$, yielding our first claim. As a consequence, σ^\vee is the weight cone of the M -graded algebra A . The proof of the normality is similar to that of 2.5 (i).

Let us show further that A is finitely generated. First we may consider a subdivision of σ^\vee by regular strongly convex polyhedral cones $\omega_1, \dots, \omega_s$ such that for any i we have $\omega_i \cap \text{relint } \sigma^\vee \neq \emptyset$, ω_i is full dimensional, and \mathfrak{D} is linear on ω_i . Fix $1 \leq i \leq s$ and a positive integer μ . Let (e_1, \dots, e_n) be a basis of M generating the cone ω_i and such that $e_1 \in \text{relint } \sigma^\vee$. By properness of \mathfrak{D} , there exists $d \in \mathbb{Z}_{>0}$ such that every $\mathfrak{D}(de_j)$ is a globally generated integral divisor. Letting

$$A_{\omega_i, \mu} = \bigoplus_{(a_1, \dots, a_n) \in \mathbb{Z}^n} H^0 \left(C, \mathcal{O} \left(\left\lfloor \sum_{i=1}^n a_i \mu e_i \right\rfloor \right) \right) \chi^{\sum_i a_i \mu e_i}$$

we consider homogeneous elements $f_1\chi^{m_1}, \dots, f_r\chi^{m_r} \in A_{\omega_i, d}$ obtained by taking generators of the space of global sections of every $\mathcal{O}(\mathfrak{D}(de_j))$ and homogeneous generators of the graded algebra

$$B = \bigoplus_{l \geq 0} H^0(C, \mathcal{O}_C(\mathfrak{D}(dle_1)))\chi^{lde_1},$$

see Lemma 3.7 (i). Using Theorem 3.5 (iii) the normalization of $k[f_1\chi^{m_1}, \dots, f_r\chi^{m_r}]$ is $A_{\omega_i, d}$. So by Theorem 2 in [Bou72, V3.2] the algebra $A_{\omega_i} = A_{\omega_i, 1}$ is finitely generated. One concludes by considering the surjection $A_{\omega_1} \otimes \dots \otimes A_{\omega_s} \rightarrow A$. \square

In the next assertion, we study how the algebra associated to a polyhedral divisor over a regular projective curve changes, when we extend the scalars passing to the algebraic closure of the ground field k . The first claims are classical for the theory of algebraic function fields, and so the proofs are omitted.

Theorem 3.9. *Assume that k is a perfect field, and let \bar{k} be an algebraic closure of k . Let $C = \mathcal{R}_k K_0$ be the Riemann surface of an algebraic function field K_0/k . The absolute Galois group of $\mathfrak{S}_{\bar{k}/k}$ acts on the closed points of curve*

$$C_{\bar{k}} = C \times_{\text{Spec } k} \text{Spec } \bar{k}$$

which can be identified with the set of the \bar{k} -rational points of $C(\bar{k})$. The orbit space $C(\bar{k})/\mathfrak{S}_{\bar{k}/k}$ can be identified with C . We denote by $\pi : C(\bar{k}) \rightarrow C$ the quotient map. In terms of the places, π is defined as the map

$$\mathcal{R}_{\bar{k}} K_0 \otimes_k \bar{k} \rightarrow \mathcal{R}_k K_0, \quad P \mapsto P \cap K_0.$$

If $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is a proper σ -polyhedral divisor over C , then

$$A[C, \mathfrak{D}] \otimes_k \bar{k} = A[C(\bar{k}), \mathfrak{D}_{\bar{k}}],$$

where $\mathfrak{D}_{\bar{k}}$ is the proper σ -polyhedral divisor over $C(\bar{k})$ defined by

$$\mathfrak{D}_{\bar{k}} = \sum_{z \in C} \Delta_z \cdot \pi^*(z) \quad \text{with} \quad \pi^*(z) = \sum_{z' \in \pi^{-1}(z)} z'.$$

Proof. Given a Weil \mathbb{Q} -divisor D over C , by [Sti93, Theorem 3.6.3] we obtain

$$H^0(C, \mathcal{O}_C([D])) \otimes_k \bar{k} = H^0(C(\bar{k}), \mathcal{O}_{C(\bar{k})}([\pi^* D])).$$

The proof reduces to a computation of $A[C, \mathfrak{D}] \otimes_k \bar{k}$. The properness of $\mathfrak{D}_{\bar{k}}$ is given for instance by 3.5 (i). \square

Remark 3.10. It is well known that every finitely generated extension of a perfect field is separable. In the non-perfect case, we may consider the inseparable algebraic function field of one variable

$$K_0 = \text{Frac} \frac{k[X, Y]}{(tX^2 + s + Y^2)},$$

where $k = \mathbb{F}_2(s, t)$ is the rational function field in two variables. Consequently, for any proper polyhedral divisor \mathfrak{D} over $C = \mathcal{R}_k K_0$, the ring $A[C, \mathfrak{D}] \otimes_k \bar{k}$ contains a nonzero nilpotent element.

4. SPLIT AFFINE \mathbb{T} -VARIETIES OF COMPLEXITY ONE

As an application of the results in the previous sections, we can give now a combinatorial description of affine \mathbb{T} -varieties of complexity one over any field k .

4.1. Let \mathbb{T} be a split algebraic torus over k . Denote by M and N its dual lattices of characters and of one-parameter subgroups, respectively. Let $X = \text{Spec } A$ be an affine variety over k . Assume that \mathbb{T} acts on X , respectively. Then the associated morphism $A \rightarrow A \otimes_k k[\mathbb{T}]$ endows A with an M -grading. Conversely, an M -grading on the algebra A yields naturally a \mathbb{T} -action on X . We say that X is a \mathbb{T} -variety if X is normal and if the \mathbb{T} -action on X is effective³. This latter is equivalent to the condition that A is normal and the set of its weights generates the lattice M .

Definition 4.2. (1) Given an effective \mathbb{T} -action on the variety X , a *multiplicative system* of $k(X)$ is a sequence $(\chi^m)_{m \in M}$, where each χ^m is a homogeneous element of $k(X)$ of degree m such that $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$, for all $m, m' \in M$, and $\chi^0 = 1$.

(2) Let C be a regular curve over k , and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. A σ -polyhedral divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is called *proper* if it satisfies one of the following conditions.

(i) C is affine. In particular, \mathfrak{D} is a polyhedral divisor over the Dedekind ring $A_0 = k[C]$.

(ii) C is projective, and \mathfrak{D} is a proper polyhedral divisor in the sense of 3.4. We denote by $A[C, \mathfrak{D}]$ the associated M -graded algebra.

Combining 2.5 and 3.5 one can describe an affine \mathbb{T} -variety of complexity one over an arbitrary field via a proper polyhedral divisor.

Theorem 4.3.

- (i) To any affine \mathbb{T} -variety $X = \text{Spec } A$ of complexity one over k , one can associate a pair $(C_X, \mathfrak{D}_{X, \gamma})$, where
- (a) C_X is the abstract regular curve over k defined by the conditions $k[C_X] = k[X]^{\mathbb{T}}$ and $k(C_X) = k(X)^{\mathbb{T}}$.
- (b) $\mathfrak{D}_{X, \gamma}$ is a proper σ_X -polyhedral divisor over C_X , which depends only on X and on a multiplicative system $\gamma = (\chi^m)_{m \in M}$ of $k(X)$. For each primitive vector $m \in M$ in the relative interior of σ_X^{\vee} , we denote by $\pi_m : V_m \rightarrow C_X$ the quotient map for the \mathbb{G}_m -action on V_m , where

$$V_m = \text{Spec } A_{(m)} \setminus \mathbb{V}\left(\bigoplus_{r \in \mathbb{Z}_{>0}} A_{rm}\right)$$

and $A_{(m)} = \bigoplus_{r \geq 0} A_{rm}$. The polyhedral divisor $\mathfrak{D}_{X, \gamma}$ is given by the relation

$$\mathfrak{D}_{X, \gamma}(m) = \sum_{Z \subset V_m} \frac{p_Z}{q_Z} (\pi_m)_*(\bar{Z}),$$

³Seeing \mathbb{T} as a representable group functor, this means that the kernel of the natural transformation of group functors $\mathbb{T} \rightarrow \text{Aut } X$ is trivial.

where $\operatorname{div}_{V_m}(\chi^m) = \sum_{Z \subset V_m} p_Z \cdot Z$, the prime divisor \bar{Z} is the closure of Z in $\operatorname{Spec} A_{(m)}$, and⁴

$$\begin{cases} q_Z = \text{g.c.d.}\{r \in \mathbb{Z}_{>0} \mid (A_{(m)}/I(\bar{Z}))_{rm} \neq \{0\}\} & \text{if } p_Z \neq 0, \text{ and} \\ q_Z = 1 & \text{otherwise.} \end{cases}$$

Moreover, we have a natural identification $A = A[C_X, \mathfrak{D}_{X,\gamma}]$ of M -graded algebras with the property that every homogeneous element $f \in A$ of degree m is equal to $f_m \chi^m$ for a unique global section f_m of the sheaf $\mathcal{O}_{C_X}([\mathfrak{D}_{X,\gamma}(m)])$.

- (ii) Conversely, if \mathfrak{D} is a proper σ -polyhedral divisor on a regular curve C over k , then $X = \operatorname{Spec} A[C, \mathfrak{D}]$ defines an affine \mathbb{T} -variety of complexity one over k .

Proof. (i) Let $\sigma_X \subset N_{\mathbb{Q}}$ be the dual of the weight cone of A . Choosing a multiplicative system $\gamma = (\chi^m)_{m \in M}$, we have an embedding

$$A \subset \bigoplus_{m \in M} k(X)^{\mathbb{T}} \chi^m = k(X)^{\mathbb{T}}[M].$$

Furthermore, A and $k(X)^{\mathbb{T}}[M]$ have the same field of fractions. The graded piece A_0 is the algebra of \mathbb{T} -invariants. Denote by K_0 the field of fractions of A_0 . Assume that $A_0 \neq k$. Then we have $K_0 = k(X)^{\mathbb{T}}$. Indeed, by assumption every algebraic element of K_0 over k belongs to k . Therefore the transcendence degree of K_0/k is equal to 1 so that $k(X)^{\mathbb{T}}/K_0$ is algebraic. Using the normality of A_0 one concludes that $K_0 = k(X)^{\mathbb{T}}$. Remark further that the ring A_0 is a Dedekind domain. By Theorem 2.5 (ii), we obtain that $A = A[C_X, \mathfrak{D}_{X,\gamma}]$ for a unique σ_X -polyhedral divisor $\mathfrak{D}_{X,\gamma}$ over A_0 . If $A_0 = k$, then one concludes by Theorem 3.5 (i). The characterization of $\mathfrak{D}_{X,\gamma}(m)$ in term of the principal divisor $\operatorname{div}_{V_m}(\chi^m)$, for every primitive vector $m \in \sigma_X^{\vee} \cap M$, is a consequence of [Dem88, 3.5]. Assertion (ii) follows immediately from 2.5 (i) and 3.5 (ii). \square

4.4. By a *principal σ -polyhedral divisor* \mathfrak{F} over C we mean a pair (φ, \mathfrak{D}) with a semi-group morphism $\varphi : \sigma_M^{\vee} \rightarrow k(C)^*$ and a σ -polyhedral divisor \mathfrak{D} over C such that for any $m \in \sigma_M^{\vee}$ we have

$$\mathfrak{D}(m) = \operatorname{div}_C \mathfrak{F}(m).$$

Starting with \mathfrak{F} and choosing a finite generating set of σ_M^{\vee} one can easily construct \mathfrak{D} satisfying the equalities as before. Usually we denote \mathfrak{F} and \mathfrak{D} by the same symbol.

The following result provides a description of equivariant isomorphisms between two affine \mathbb{T} -varieties of complexity one over an arbitrary field. See [AH06, Section 8, 9] for higher complexity when the ground field is algebraically closed of characteristic zero.

Proposition 4.5. *Let X_1, X_2 be two affine \mathbb{T} -varieties of complexity one over k described by the respective pairs $(C_{X_1}, \mathfrak{D}_{X_1, \gamma_1})$ and $(C_{X_2}, \mathfrak{D}_{X_2, \gamma_2})$ (see 4.3). Then the \mathbb{T} -varieties X_1 and X_2 are \mathbb{T} -isomorphic if and only if there exist an isomorphism $\phi : C_{X_1} \rightarrow C_{X_2}$ of algebraic curves over k , an automorphism⁵ of lattices $F : N \rightarrow N$*

⁴Note that every closed subscheme $\bar{Z} \subset \operatorname{Spec} A_{(m)}$ such that Z is a prime divisor contained in the support of $\operatorname{div}(\chi^m)$ is \mathbb{G}_m -stable. Hence $I(\bar{Z})$ is a graded ideal of $A_{(m)}$ and the number q_Z is well defined.

⁵For any morphism $F : N \rightarrow N$, we denote by $F_{\mathbb{Q}} : N_{\mathbb{Q}} \rightarrow N_{\mathbb{Q}}$ the induced \mathbb{Q} -linear map of F .

satisfying $F_{\mathbb{Q}}(\sigma_{X_1}) = \sigma_{X_2}$, and a principal σ_{X_2} -polyhedral divisor \mathfrak{F} over C_{X_1} such that for all $m \in \sigma_{X_2}^{\vee} \cap M$,

$$\phi^*(\mathfrak{D}_{X_2, \gamma_2})(m) = F_*(\mathfrak{D}_{X_1, \gamma_1})(m) + \phi^*(\mathfrak{F})(m),$$

where

$$\mathfrak{D}_{X_1, \gamma_1} = \sum_{z \in C_{X_1}} \Delta_z^1 \cdot z, \quad \mathfrak{D}_{X_2, \gamma_2} = \sum_{z \in C_{X_2}} \Delta_z^2 \cdot z, \quad \text{and}$$

$$F_*(\mathfrak{D}_{X_1, \gamma_1}) = \sum_{z \in C_{X_1}} (F_{\mathbb{Q}}(\Delta_z^1) + \sigma_{X_2}) \cdot z, \quad \phi^*(\mathfrak{D}_{X_2, \gamma_2}) = \sum_{z \in C_{X_2}} \Delta_z^2 \cdot \phi^{-1}(z).$$

Proof. Let $\psi : k[X_2] \rightarrow k[X_1]$ be an isomorphism of M -graded algebras over k . Then ψ induces two natural maps

$$\begin{aligned} \phi^* : k(C_{X_2}) = k(X_2)^{\mathbb{T}} &\rightarrow k(C_{X_1}) = k(X_1)^{\mathbb{T}}, \\ \phi^* : k[C_{X_2}] = k[X_2]^{\mathbb{T}} &\rightarrow k[C_{X_1}] = k[X_1]^{\mathbb{T}}, \end{aligned}$$

which yield a morphism of algebraic curves $\phi : C_{X_1} \rightarrow C_{X_2}$. Moreover, we define an automorphism $F : N \rightarrow N$ such that $F_{\mathbb{Q}}(\sigma_{X_1}) = \sigma_{X_2}$ and a principal σ_{X_2} -polyhedral divisor \mathfrak{F} as follows. If $\gamma_1 = (\chi^m)_{m \in M}$ and $\gamma_2 = (\xi^m)_{m \in M}$, then for every homogeneous element $f\xi^m \in k[X_2]$ there exists a unique vector $F^{\vee}(m) \in \sigma_{X_1}^{\vee} \cap M$ such that

$$(1) \quad \psi(f\xi^m) = \phi^*(f) \cdot \phi^*(f_m) \chi^{F^{\vee}(m)}.$$

The map F is the dual morphism of F^{\vee} . Extending ψ to an isomorphism $\bar{\psi} : k(X_2) \rightarrow k(X_1)$, we can define the semigroup morphism

$$\sigma_{X_2}^{\vee} \cap M \rightarrow k(C_{X_2})^*, \quad m \mapsto f_m^{-1},$$

which gives the principal polyhedral divisor \mathfrak{F} . Now we conclude by the equivalences

$$\begin{aligned} \operatorname{div}(\phi^*(f)) + \phi^*(\mathfrak{D}_{X_2, \gamma_2})(m) &\geq 0 \\ \Leftrightarrow \operatorname{div}(f) + \mathfrak{D}_{X_2, \gamma_2}(m) &\geq 0 \\ \Leftrightarrow \operatorname{div}(\phi^*(f) \cdot \phi^*(f_m)) + \mathfrak{D}_{X_1, \gamma_1}(F^{\vee}(m)) &\geq 0 \\ \Leftrightarrow \operatorname{div}(\phi^*(f)) + F_*(\mathfrak{D}_{X_1, \gamma_1})(m) + \phi^*(\mathfrak{F})(m) &\geq 0, \end{aligned}$$

which imply the equality $\phi^*(\mathfrak{D}_{X_2, \gamma_2})(m) = F_*(\mathfrak{D}_{X_1, \gamma_1})(m) + \phi^*(\mathfrak{F})(m)$, for every $m \in \sigma_{X_2}^{\vee} \cap M$. Conversely, starting with the triple (F, ϕ, \mathfrak{F}) , we define a morphism $\psi : k[X_2] \rightarrow k[X_1]$ by the equality (1), where $\phi^* : k(C_{X_2}) \rightarrow k(C_{X_1})$ is the comorphism of ϕ , F is the dual of F^{\vee} , and \mathfrak{F} is given by the morphism $m \mapsto f_m^{-1}$. A straightforward verification shows that ψ is well defined and gives an isomorphism of M -graded algebras. \square

5. NON-SPLIT CASE VIA GALOIS DESCENT

In view of the result in the previous section, we provide a combinatorial description of affine normal varieties endowed with a (non-necessary split) torus action of complexity one (see 5.4 for a precise definition). This can be compared with well-known descriptions for toric and spherical varieties, see [Bry79, CTHS05, Vo82, ELST12, Hur11].

5.1. For a field extension F/k and a scheme X over k we let

$$X_F = X \times_{\mathrm{Spec} k} \mathrm{Spec} F.$$

This is a scheme over F . An algebraic *torus* of dimension n is an algebraic group \mathbf{G} over k such that there exists a finite Galois extension E/k yielding an isomorphism of algebraic groups $\mathbf{G}_E \simeq \mathbb{G}_{m,E}^n(\star)$, where \mathbb{G}_m is the multiplicative group scheme over k . We say that the torus \mathbf{G} *splits in the extension E/k* if we have an isomorphism similar to (\star) . For more details concerning the theory of non-split reductive group the reader may consult [BT65, Spr98].

Let \mathbf{G} be a torus over k that splits in a finite Galois extension E/k . Denote by $\mathfrak{G}_{E/k}$ the Galois group of E/k . Consider also the dual lattices M and N , respectively, of characters and of one-parameter subgroups of the split torus \mathbf{G}_E . In the sequel most of our varieties will be defined over the field E . We start by recalling the following classical notion.

Definition 5.2.

- (i) A $\mathfrak{G}_{E/k}$ -action on a variety V over E is called *semi-linear* if $\mathfrak{G}_{E/k}$ acts by scheme automorphisms, and if for any $g \in \mathfrak{G}_{E/k}$ the diagram

$$\begin{array}{ccc} V & \xrightarrow{g} & V \\ \downarrow & & \downarrow \\ \mathrm{Spec} E & \xrightarrow{g} & \mathrm{Spec} E \end{array}$$

is commutative.

- (ii) Let B be an algebra over E . A *semi-linear* $\mathfrak{G}_{E/k}$ -action on B is an action by ring automorphisms such that for all $a \in B$, $\lambda \in E$, and $g \in \mathfrak{G}_{E/k}$

$$g \cdot (\lambda a) = g(\lambda)g \cdot a.$$

If V is affine, then defining a semi-linear $\mathfrak{G}_{E/k}$ -action on V is equivalent to defining a semi-linear $\mathfrak{G}_{E/k}$ -action on the algebra $E[V]$.

Next, we recall a well-known description of algebraic tori related to finite groups actions on lattices.

5.3. The Galois group $\mathfrak{G}_{E/k}$ acts naturally on the torus

$$\mathbf{G}_E = \mathbf{G} \times_{\mathrm{Spec} k} \mathrm{Spec} E$$

by action on the second factor. The corresponding action on $E[M]$ is determined by a linear $\mathfrak{G}_{E/k}$ -action on M (see e.g. [ELST12, Proposition 2.5], [Vos82, Section 1]) permuting the Laurent monomials.

Conversely, given a linear $\mathfrak{G}_{E/k}$ -action on M we have a semi-linear action on $E[M]$ defined by

$$g \cdot (\lambda \chi^m) = g(\lambda) \chi^{g \cdot m},$$

where $g \in \mathfrak{G}_{E/k}$, $\lambda \in E$ and $m \in M$, respects the Hopf algebra structure. As a consequence of the Speiser's Lemma, we obtain a torus \mathbf{G} over k that splits in E/k . In addition, the semi-linear action that we have defined on $\mathbf{G}_E = \mathbf{G} \times_{\mathrm{Spec} k} \mathrm{Spec} E$ is exactly the natural semi-linear action on the second factor.

Let us further introduce our category of \mathbf{G} -varieties.

Definition 5.4. A \mathbf{G} -variety of complexity d (splitting in E/k) is a normal variety over k with a \mathbf{G} -action and such that X_E is a \mathbf{G}_E -variety of complexity d in the sense of Section 4. A \mathbf{G} -morphism between \mathbf{G} -varieties X, Y over k is a morphism $f : X \rightarrow Y$ of varieties over k such that

$$\begin{array}{ccc} \mathbf{G} \times X & \xrightarrow{\text{id} \times f} & \mathbf{G} \times Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative.

An important class of semi-linear actions is provided by the actions respecting a split torus action. The $\mathfrak{S}_{E/k}$ -action on \mathbf{G}_E is defined as in 5.3.

Definition 5.5.

- (i) Let B be an M -graded algebra over E . A semi-linear $\mathfrak{S}_{E/k}$ -action on B is called *homogeneous* if it sends homogeneous elements into homogeneous elements.
- (ii) A semi-linear $\mathfrak{S}_{E/k}$ -action on a \mathbf{G}_E -variety V *respects the \mathbf{G}_E -action* if the following diagram

$$\begin{array}{ccc} \mathbf{G}_E \times V & \xrightarrow{g \times g} & \mathbf{G}_E \times V \\ \downarrow & & \downarrow \\ V & \xrightarrow{g} & V \end{array}$$

commutes, where g runs $\mathfrak{S}_{E/k}$.

Assuming that V is affine, a semi-linear $\mathfrak{S}_{E/k}$ -action on the variety V respecting the \mathbf{G}_E -action corresponds to a homogeneous semi-linear $\mathfrak{S}_{E/k}$ -action on the algebra $E[V]$.

The following result is classically stated for the category of quasi-projective varieties (see the proof of [Hur12, 1.10]). In the setting of affine \mathbf{G} -varieties we include a short argument.

Lemma 5.6. *Let V be an affine \mathbf{G}_E -variety of complexity d over E with a semi-linear $\mathfrak{S}_{E/k}$ -action. Then the quotient $X = V/\mathfrak{S}_{E/k}$ is an affine \mathbf{G} -variety of complexity d . We have a natural isomorphism of \mathbf{G}_E -varieties $X_E \simeq V$ respecting the $\mathfrak{S}_{E/k}$ -actions.*

Proof. It is known that $R = B^{\mathfrak{S}_{E/k}}$ is finitely generated. Let us show that R is normal. Letting L be the field of fractions of R and considering an element $f \in L$ integral over R , by the normality of B , we have $f \in B \cap L = R$. This proves that R is normal. Using the above definition, the variety X is endowed with a \mathbf{G} -action. The rest of the proof follows from Speiser's Lemma. \square

Fixing an affine \mathbf{G} -variety X of complexity d over E , an E/k -form of X is an affine \mathbf{G} -variety Y over k such that we have a \mathbf{G}_E -isomorphism $X_E \simeq Y_E$. Our aim is to give a combinatorial description of E/k -forms of X . Let us recall first in this context some notion of non-abelian Galois cohomology (see e.g. [Ser94, III Section 1] for the category of varieties).

5.7. Let Y, Y' be E/k -forms of the fixed affine \mathbf{G} -variety X . The Galois group $\mathfrak{G}_{E/k}$ acts on the set of \mathbf{G}_E -isomorphisms between Y_E and Y'_E . Consequently, it acts also by group automorphisms on the group of \mathbf{G}_E -automorphisms $\text{Aut}_{\mathbf{G}_E}(X_E)$ of X_E . More precisely, recall that for any $g \in \mathfrak{G}_{E/k}$ and any \mathbf{G}_E -isomorphism $\varphi : Y_E \rightarrow Y'_E$ one defines $g(\varphi)$ by the following commutative diagram

$$\begin{array}{ccc} Y_E & \xrightarrow{g(\varphi)} & Y'_E \\ g \downarrow & & \downarrow g \\ Y_E & \xrightarrow{\varphi} & Y'_E \end{array} .$$

Note that this $\mathfrak{G}_{E/k}$ -action depends on the choice of the E/k -forms Y, Y' . Now given a \mathbf{G}_E -isomorphism $\psi : X_E \rightarrow Y_E$ the map

$$a : \mathfrak{G}_{E/k} \rightarrow \text{Aut}_{\mathbf{G}_E}(X_E), \quad g \mapsto a_g = \psi^{-1} \circ g(\psi)$$

is a 1-cocycle. This means that for all $g, g' \in \mathfrak{G}_{E/k}$ we have

$$a_g \circ g(a_{g'}) = \psi^{-1} \circ g(\psi) \circ g(\psi^{-1} \circ g'(\psi)) = a_{gg'} .$$

Let $\phi : Y \rightarrow Y'$ be a \mathbf{G} -isomorphism and consider a \mathbf{G}_E -isomorphism $\varphi : X_E \rightarrow Y'_E$ yielding a 1-cocycle b as above. The diagram

$$\begin{array}{ccc} X_E & \xrightarrow{\psi} & Y_E \\ \alpha \downarrow & & \downarrow \phi' = \phi \times \text{id} \\ X_E & \xrightarrow{\varphi} & Y'_E \end{array}$$

is commutative, where $\alpha \in \text{Aut}_{\mathbf{G}_E}(X_E)$ and ϕ' is the extension ϕ . Since for any $g \in \mathfrak{G}_{E/k}$ we have $g(\phi') = \phi'$, it follows that

$$b_g = \alpha \circ a_g \circ g(\alpha^{-1}) .$$

In this case we say that the cocycles a and b are *cohomologous*. We obtain as well a map Φ between the pointed set of isomorphism classes of E/k -forms of X and the pointed set

$$H^1(E/k, \text{Aut}_{\mathbf{G}_E}(X_E))$$

of cohomology classes of 1-cocycles $a : \mathfrak{G}_{E/k} \rightarrow \text{Aut}_{\mathbf{G}_E}(X_E)$.

Conversely, starting with a cocycle a the map

$$\mathfrak{G}_{E/k} \rightarrow \text{Aut}_{\mathbf{G}_E}(X_E), \quad g \mapsto a_g \circ g$$

is a semi-linear action on X_E respecting the \mathbf{G}_E -action. According to Lemma 5.6 one can obtain an E/k -form W of X by taking the quotient $X_E/\mathfrak{G}_{E/k}$. Changing a by a cohomologous 1-cocycle gives an E/k -form of X isomorphic to W . Thus we deduce that the map Φ is bijective.

Moreover, let γ be a semi-linear $\mathfrak{G}_{E/k}$ -action on X_E . Remark that

$$\begin{array}{ccc} X_E & \xrightarrow{\gamma(g')} & X_E \\ g^{-1} \downarrow & & \downarrow g^{-1} \\ X_E & \xrightarrow{g(\gamma(g'))} & X_E \end{array}$$

commutes for all $g, g' \in \mathfrak{G}_{E/k}$. Hence the equality $a_g = \gamma(g) \circ g^{-1}$ defines a 1-cocycle a . A straightforward verification shows that $H^1(E/k, \text{Aut}_{\mathbf{G}_E}(X_E))$ is also in bijection with the pointed set of conjugacy classes of semi-linear $\mathfrak{G}_{E/k}$ -actions on X_E respecting the \mathbf{G}_E -action.

As explained in the above paragraph, classifying the pointed set of E/k -forms of X is equivalent to classifying all possible semi-linear $\mathfrak{G}_{E/k}$ -actions on X_E . Thus generalizing the notion of proper polyhedral divisors, we consider the combinatorial counterpart of this classification.

Definition 5.8. Let C be a regular curve over E and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex cone. A $\mathfrak{G}_{E/k}$ -invariant σ -polyhedral divisor over C is a 4-tuple $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ verifying the following conditions.

- (i) \mathfrak{D} (resp. \mathfrak{F}) is a proper (resp. principal) σ -polyhedral divisor over C .
- (ii) The curve C is endowed with a semi-linear $\mathfrak{G}_{E/k}$ -action

$$\mathfrak{G}_{E/k} \times C \rightarrow C, \quad (g, z) \mapsto g \star z.$$

This yields naturally an action on the space of Weil \mathbb{Q} -divisors over C . More precisely, given $g \in \mathfrak{G}_{E/k}$ and a \mathbb{Q} -divisor D over C we let

$$g \star D = \sum_{z \in C} a_{g^{-1} \star z} \cdot z, \quad \text{where } D = \sum_{z \in C} a_z \cdot z.$$

- (iii) The lattice M is endowed with a linear $\mathfrak{G}_{E/k}$ -action

$$\mathfrak{G}_{E/k} \times M \rightarrow M, \quad (g, m) \mapsto g \cdot m$$

preserving the subset σ_M^{\vee} .

The 4-tuple $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ satisfies in addition the equality

$$g \star (\mathfrak{D}(m) + \mathfrak{F}(m)) = \mathfrak{D}(g \cdot m) + \mathfrak{F}(g \cdot m),$$

where $m \in \sigma_M^{\vee}$ and $g \in \mathfrak{G}_{E/k}$.

The following result is a direct consequence of Hilbert's Theorem 90. For the convenience of the reader we give a short argument.

Lemma 5.9. Let E_0/K_0 be a finite Galois extension with Galois group \mathfrak{G}_{E_0/K_0} . Assume that \mathfrak{G}_{E_0/K_0} acts linearly on M . For any $g \in \mathfrak{G}_{E_0/K_0}$ consider a morphism of groups $f_g : M \rightarrow E_0^{\star}$ satisfying the equalities

$$f_{gh}(m) = g(f_h(m)) f_g(h \cdot m),$$

where $g, h \in \mathfrak{G}_{E_0/K_0}$ and $m \in M$. Then there exists a morphism of groups $b : M \rightarrow E_0^{\star}$ such that for all $g \in \mathfrak{G}_{E_0/K_0}$, $m \in M$ we have

$$f_g(m) = b(g \cdot m) g(b(m))^{-1}.$$

Proof. The opposite of \mathfrak{G}_{E_0/K_0} is the group H with underlying set \mathfrak{G}_{E_0/K_0} and the multiplication law defined by $g \star h = hg$, where $g, h \in H$. For $g \in H$ we denote by $a_g : M \rightarrow E_0^*$ the morphism of groups defined by

$$a_g(m) = g^{-1}(f_g(m)),$$

where $m \in M$. We can also define an H -action by group automorphisms on the abelian group

$$T = \text{Hom}(M, E_0^*)$$

over E_0 by letting $(g \cdot \alpha)(m) = g^{-1}(\alpha(g \cdot m))$, where $\alpha \in T$, $g \in H$, and $m \in M$. Considering $g, h \in H$ we obtain

$$a_{h \star g}(m) = (gh)^{-1}(f_{gh}(m)) = (gh)^{-1}(g(f_h(m))f_g(h \cdot m)) = a_h(m)(h \cdot a_g)(m)$$

so that $g \mapsto a_g$ is a 1-cocycle. By the Hilbert Theorem 90 one has

$$H^1(H, T) \simeq H^1(E_0/K_0, T) = 1.$$

Hence there exists $b \in T$ such that for any $g \in H$ we have $a_g = b \cdot (g \cdot b^{-1})$. The latter equalities imply the result. \square

The next theorem yields a classification of affine \mathbf{G} -varieties of complexity one in terms of invariant polyhedral divisors.

Theorem 5.10. *Let \mathbf{G} be a torus over k splitting in a finite Galois extension E/k . Denote by $\mathfrak{G}_{E/k}$ the Galois group of E/k .*

- (i) *Every affine \mathbf{G} -variety of complexity one splitting in E/k is described by a $\mathfrak{G}_{E/k}$ -invariant proper polyhedral divisor over a regular curve.*
- (ii) *Conversely, let C be a regular curve over E . For a $\mathfrak{G}_{E/k}$ -invariant proper σ -polyhedral divisor $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ over C one can endow the algebra $A[C, \mathfrak{D}]$ with a homogeneous semi-linear $\mathfrak{G}_{E/k}$ -action and associate an affine \mathbf{G} -variety of complexity one over k splitting in E/k by letting $X = \text{Spec } A$, where*

$$A = A[C, \mathfrak{D}]^{\mathfrak{G}_{E/k}}.$$

Proof. (i) Let X be an affine \mathbf{G} -variety of complexity one over k . According to Theorem 4.3 we may suppose that $B = A[C, \mathfrak{D}]$ is the coordinate ring of X_E for some proper σ -polyhedral divisor \mathfrak{D} over a regular curve C . The algebra B is endowed with a homogeneous semi-linear $\mathfrak{G}_{E/k}$ -action. Let $E_0 = E(C)$. Extending this action on $E_0[M]$ we notice that E_0 and $E[C]$ are preserved. Thus we obtain a semi-linear $\mathfrak{G}_{E/k}$ -action on C . If C is projective, then one defines the $\mathfrak{G}_{E/k}$ -action on C in the following way ; given a place $P \subset E_0$ we let

$$g \star P = \{g \star f \mid f \in P\}.$$

In the case where C is arbitrary, the Speiser Lemma gives the equality

$$E_0 = E \cdot K_0, \quad \text{where } K_0 = E_0^{\mathfrak{G}_{E/k}}.$$

The finite extension E_0/K_0 is Galois. We have a natural identification $\mathfrak{G}_{E/k} \simeq \mathfrak{G}_{E_0/K_0}$ with the Galois group of E_0/K_0 . For all $m \in M$, $g \in \mathfrak{G}_{E/k}$ we have

$$(2) \quad g \cdot (f\chi^m) = g(f)f_g(m)\chi^{\Gamma(g,m)}$$

for some element f_g of the abelian group $T = \text{Hom}(M, E_0^*)$ and some $\Gamma(g, m) \in M$. We observe that Γ is a linear action on M . Denote by $g \cdot m$ the lattice vector $\Gamma(g, m)$. For all $g, h \in \mathfrak{G}_{E/k}$ we have

$$f_{gh}(m)\chi^m = gh \cdot \chi^m = g \cdot (h \cdot \chi^m) = g(f_h(m))f_g(h \cdot m)\chi^{gh \cdot m}.$$

Using Lemma 5.9 there exists $b \in T$ such that for all $m \in M$, $g \in \mathfrak{G}_{E/k}$ we have $f_g(m) = b(g \cdot m)/g(b(m))$. We let \mathfrak{F} be the principal σ -polyhedral divisor associated to b .

It remains to show the equalities

$$(3) \quad g \star (\mathfrak{D}(m) + \mathfrak{F}(m)) = \mathfrak{D}(g \cdot m) + \mathfrak{F}(g \cdot m), \quad \forall m \in \sigma_M^\vee, \forall g \in \mathfrak{G}_{E/k}.$$

First of all, we remark that if $f \in E_0^*$ and $g \in \mathfrak{G}_{E/k}$, then $g \star \text{div } f = \text{div } g(f)$. Let $f\chi^m \in B$ be homogeneous of degree m . The transformation of $f\chi^m$ by g is an element of B of degree $g \cdot m$ and so

$$\text{div } g(f)f_g(m) + \mathfrak{D}(g \cdot m) \geq 0.$$

This implies that

$$g \star (-\text{div } f + \mathfrak{F}(m)) \leq \mathfrak{F}(g \cdot m) + \mathfrak{D}(g \cdot m).$$

According to Lemma 1.7 and Corollary 3.8 (iii) we obtain

$$g \star (\mathfrak{D}(m) + \mathfrak{F}(m)) \leq \mathfrak{D}(g \cdot m) + \mathfrak{F}(g \cdot m).$$

The converse inequality uses a similar argument. One concludes that $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ is an invariant polyhedral divisor.

(ii) If $b \in T$ corresponds to \mathfrak{F} , then by virtue of (3) one defines a homogeneous semi-linear $\mathfrak{G}_{E/k}$ -action on $A[C, \mathfrak{D}]$ by letting $f_g(m) = b(g \cdot m)/g(b(m))$ and by the equality (2). The rest of the proof is a consequence of Lemma 5.6. \square

Let us provide the following elementary example.

Example 5.11. Consider the σ -polyhedral divisor \mathfrak{D} over $\mathbb{A}_{\mathbb{C}}^1 = \text{Spec } \mathbb{C}[t]$ defined by

$$((1, 0) + \sigma) \cdot \zeta + ((0, 1) + \sigma) \cdot (-\zeta) + ((1, -1) + \sigma) \cdot 0,$$

where σ is the first quadrant $\mathbb{Q}_{\geq 0}^2$ and $\zeta = \sqrt{-1}$. We endow \mathfrak{D} with a structure of $\mathfrak{G}_{\mathbb{C}/\mathbb{R}}$ -invariant polyhedral divisors by considering \mathfrak{F} given by the morphism $(m_1, m_2) \mapsto t^{m_2 - m_1}$. We have a $\mathfrak{G}_{\mathbb{C}/\mathbb{R}}$ -action

$$\mathfrak{G}_{\mathbb{C}/\mathbb{R}} \rightarrow \text{GL}_2(\mathbb{Z}), \quad g \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on the lattice \mathbb{Z}^2 , where g is the generator of $\mathfrak{G}_{\mathbb{C}/\mathbb{R}}$. The algebra $\mathbb{C}[t]$ has the natural complex conjugacy action \star of $\mathfrak{G}_{\mathbb{C}/\mathbb{R}}$. A direct computation shows that

$$A = \mathbb{C} \left[t, \frac{1}{t(t - \zeta)} \chi^{(1,0)}, \frac{t}{t + \zeta} \chi^{(0,1)} \right]$$

and so $X = \text{Spec } A$ is the affine space $\mathbb{A}_{\mathbb{C}}^3$. More concretely, the $\mathfrak{G}_{\mathbb{C}/\mathbb{R}}$ -action on the algebra A is obtained by

$$g \cdot (f(t)\chi^{(m_1, m_2)}) = \bar{f}(t)t^{2(m_1 - m_2)}\chi^{(m_2, m_1)}.$$

Letting $x = t^{-1}(1 - \zeta)^{-1}\chi^{(1,0)}$ and $y = t(1 + \zeta)^{-1}\chi^{(0,1)}$ we obtain that $A^{\mathfrak{G}_{\mathbb{C}/\mathbb{R}}} = \mathbb{R}[t, x + y, \zeta(x - y)]$. Hence $X/\mathfrak{G}_{\mathbb{C}/\mathbb{R}} \simeq \mathbb{A}_{\mathbb{R}}^3$.

Next we describe the pointed set of E/k -forms of an affine \mathbf{G} -varieties of complexity one in terms of polyhedral divisors.

Definition 5.12. The invariant σ -polyhedral divisors $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ and $(\mathfrak{D}, \mathfrak{F}', \star', \cdot')$ over C are *conjugated* if they verify the following: there exist $\varphi \in \text{Aut}(C)$, a principal σ -polyhedral divisor \mathfrak{E} over C , and a linear automorphism $F \in \text{Aut}(M)$ giving an automorphism of the E -algebra $A[C, \mathfrak{D}]$ (see 4.5) such that for any $g \in \mathfrak{G}_{E/k}$ the diagrams

$$\begin{array}{ccc} C & \xrightarrow{g\star} & C \\ \varphi \downarrow & & \downarrow \varphi \\ C & \xrightarrow{g\star'} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{g'} & M \\ F \downarrow & & \downarrow F \\ M & \xrightarrow{g'f} & M \end{array}$$

commute and for any $m \in M$ we have

$$\frac{\mathfrak{F}(g \cdot m)}{g \star \mathfrak{F}(m)} = \frac{g \star (\varphi^{-1})^* \mathfrak{E}(m) \cdot (\varphi^{-1})^* \mathfrak{F}'(F(g \cdot m))}{\mathfrak{E}(g \cdot m) \cdot g \star (\varphi^{-1})^* \mathfrak{F}'(F(m))}.$$

Consider an affine \mathbf{G} -variety X of complexity one described by the invariant polyhedral divisor $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$. We denote by $\mathcal{E}_X(E/k)$ the pointed set of conjugacy classes of $\mathfrak{G}_{E/k}$ -invariant σ -polyhedral divisors over C of the form $(\mathfrak{D}, \mathfrak{F}', \star', \cdot')$.

As a direct consequence of the discussion of 5.7 we obtain the following.

Corollary 5.13. *Let C be a regular curve over E . Given an affine \mathbf{G} -variety X of complexity one associated to a $\mathfrak{G}_{E/k}$ -invariant polyhedral divisor $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ over C , we have a bijection of pointed sets*

$$\mathcal{E}_X(E/k) \simeq H^1(E/k, \text{Aut}_{\mathbf{G}_E}(X_E)).$$

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