Polyhedral Divisors, Dedekind Domains and Algebraic Function Fields

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Abstract. We show that the presentation of affine $\mathbb{T}$-varieties of complexity one in terms of polyhedral divisor of Altmann-Hausen holds over an arbitrary field. We describe also a class of multigraded algebras over Dedekind domains. We study how the algebra associated to a polyhedral divisor changes when we extend the scalars. As another application, we provide a combinatorial description of affine $\mathbf{G}$-varieties of complexity one over a field, where $\mathbf{G}$ is a (non-nescessary split) torus, by using elementary facts on Galois descent. This class of affine $\mathbf{G}$-varieties are described via a new combinatorial object, which we call (Galois) invariant polyhedral divisor.

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Introduction

In this paper, we are interested in a combinatorial description of multigraded normal affine algebras of complexity one. From a geometrical viewpoint, these algebras are related to the classification of algebraic tori actions of complexity one on affine varieties. Let $\mathbf{k}$ be a field and consider a split algebraic torus $\mathbb{T}$ over $\mathbf{k}$. Recall that a $\mathbb{T}$-variety is a normal variety over $\mathbf{k}$ endowed with an effective $\mathbb{T}$-action. There exist several combinatorial descriptions of $\mathbb{T}$-varieties in term of the convex geometry. See [Do],[Pi],[De],[FZ] for the Dolgachev-Pinkham-Demazure (D.P.D.) presentation, [KKMS],[Ti],[Ti2] for toric case and complexity one case, and [AH],[AHS],[AOPSV] for higher complexity. Most of these works requires the ground field $\mathbf{k}$ to be algebraically closed of characteristic zero. It is worthwile mentioning that the description of affine $\mathbb{G}_m$-varieties [De] due to Demazure holds over any field.

Let us list the most important results of the paper.

- The Altmann-Hausen presentation of affine $\mathbb{T}$-varieties of complexity one in terms of polyhedral divisor holds over an arbitrary field, see Theorem 4.3.

- This description holds as well for an important class of multigraded algebras over Dedekind domains, see Theorem 2.5.

- We study how the algebra associated to a polyhedral divisor changes when we extend the scalars, see 2.12 and 3.9.

- As another application, we provide a combinatorial description of affine $\mathbf{G}$-varieties of complexity one, where $\mathbf{G}$ is a (non-nescessary split) torus over $\mathbf{k}$, by using elementary facts on Galois descent. This class of affine $\mathbf{G}$-varieties are classifyed via a new combinatorial object, which we call a (Galois) invariant polyhedral divisor, see Theorem 5.10.

Let us now stay on these results in more details. We start with a simple case of varieties with an action of a split torus. Recall that a split algebraic torus $\mathbb{T}$ of dimension n over the field $\mathbf{k}$ is an algebraic group isomorphic to $\mathbb{G}_m^n$, where $\mathbb{G}_m$ is the multiplicative group of the filed $\mathbf{k}$. Let $M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be the character lattice of the torus $\mathbb{T}$. Then defining a $\mathbb{T}$-action on an affine variety X is equivalent to fixing an M-grading on the algebra $A = \mathbf{k}[X]$, where $\mathbf{k}[X]$ is the coordinate ring of X. Following the classification of affine $\mathbb{G}_m$-surfaces [FiKa] we say as in [Li, 1.1] that the M-graded algebra A is *elliptic* if the graded piece A_0 is reduced to $\mathbf{k}$. Multigraded affine algebras are classifyed via a numerical invariant called complexity. This invariant was introduced in [LV] for the classification of homogeneous spaces under the action of a connected reductive group. Consider the field $\mathbf{k}(X)$ of rational functions on X and its subfield K_0 of $\mathbb{T}$-invariant functions. The complexity of the $\mathbb{T}$-action on X is the trancendence degree of K_0 over the field $\mathbf{k}$. Note that for the situation where $\mathbf{k}$ is algebraically closed, the complexity is also the codimension of the general $\mathbb{T}$-orbit in X (see [Ro]).

In order to describe affine $\mathbb{T}$-schemes of complexity one, we have to consider combinatorial objects coming from convex geometry and from the geometry of algebraic curves. Let C be a smooth curve over $\mathbf{k}$. A point of C is assumed to be a closed point, and in particular, non-nescessary rational. Futhermore, the residual field extension of $\mathbf{k}$ at any point of C has finite degree.

To reformulate our first result, we need some combinatorial notion of convex geometry, see [AH, Section 1]. Denote by $N = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ the lattice of one parameter subgroups of the torus $\mathbb{T}$ wich is the dual of the lattice M. Let $M_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} M$, $N_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} N$ be the associated dual $\mathbb{Q}$-vector spaces of M, N and let $\sigma \subset N_\mathbb{Q}$ be a strongly convex polyhedral cone. We can define as in [AH] a Weil divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ with σ-polyhedral coefficients in $N_\mathbb{Q}$, called polyhedral divisor of Altmann-Hausen. More precisely, each $\Delta_z \subset N_\mathbb{Q}$ is a polyhedron with tailed cone σ (see 2.1) and $\Delta_z = \sigma$ for all but finitely many points $z \in C$. Denoting by κ_z the residual field of the point $z \in C$ and by $[\kappa_z : \mathbf{k}] \cdot \Delta_z$ the image of Δ_z under the homothety of ratio $[\kappa_z : \mathbf{k}]$, the sum

$$\deg \mathfrak{D} = \sum_{z \in C} [\kappa_z : \mathbf{k}] \cdot \Delta_z$$

is a polyhedron in $N_\mathbb{Q}$. This sum may be seen as the finite Minkowski sum of all polyhedra $[\kappa_z : \mathbf{k}] \cdot \Delta_z$ different from σ. Considering the dual cone $\sigma^\vee \subset M_\mathbb{Q}$ of σ, we define an evaluation function

$$\sigma^\vee \to \mathrm{Div}_\mathbb{Q}(C), \;\; m \mapsto \mathfrak{D}(m) = \sum_{z \in C} \min_{l \in \Delta_z} \langle m, l \rangle$$

with value in the vector space $\mathrm{Div}_{\mathbb{Q}}(C)$ of Weil $\mathbb{Q}$-divisors over C. As in the classical case [AH, 2.12] we introduce the technical condition of properness for the polyhedral divisor $\mathfrak{D}$ (see 2.2, 3.4, 4.2) that we recall thereafter.

Definition 0.1. A σ-polyhedral divisor $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ is called *proper* if it satifies one of the conditions.

(i) C is affine.
(ii) C is projective and $\deg \mathfrak{D}$ is strictly contained in the cone σ. Futhermore, if $\deg \mathfrak{D}(m) = 0$ then m belongs to the boundary of $\sigma^\vee$ and an integral multiple of $\mathfrak{D}(m)$ is principal.

For instance, if $C = \mathbb{P}^1_{\mathbf{k}}$ is the projective line then the polyhedral divisor $\mathfrak{D}$ is proper if and only if $\deg \mathfrak{D}$ is strictly included in σ. One of the main results of this paper can be stated as follows.

Theorem 0.2. *Let* $\mathbf{k}$ *be field.*

(i) *To any proper* σ*-polyhedral divisor* $\mathfrak{D}$ *on a smooth curve over* $\mathbf{k}$ *one can associate a normal finitely generated effectively* M*-graded domain of complexity one over* $\mathbf{k}$*, given by*

$$A[C,\mathfrak{D}] = \bigoplus_{m\in\sigma^\vee\cap M} A_m, \ \text{where} \ A_m = H^0(C, \mathcal{O}_C(\lfloor \mathfrak{D}(m)\rfloor)).$$

(ii) *Conversely, any normal finitely generated effectively* M*-graded domain of complexity one over* $\mathbf{k}$ *is isomorphic to* $A[C,\mathfrak{D}]$ *for some smooth curve* C *over* $\mathbf{k}$ *and some proper polyhedral divisor* $\mathfrak{D}$ *over* C*.*

In the proof of assertion (ii), we use an effective calculation from [La]. We divide the proof into two cases. In the *non-elliptic case* we show that the assertion holds more generally in the context of Dedekind domains. More precisely, we give a perfect dictionary similar to 0.2(i), (ii) for M-graded algebras defined by a polyhedral divisor over a Dedekind ring (see 2.2, 2.3 and Theorem 2.5). We deal in 2.6 as an example with a polyhedral divisor over $\mathbb{Z}[\sqrt{-5}]$. In the *elliptic case,* we consider an elliptic M-graded algebra A over $\mathbf{k}$ satisfying the assumptions of 0.2 (ii). We construct a smooth projective curve arising from the algebraic function field $K_0 = (\mathrm{Frac}\, A)^{\mathbb{T}}$; the points of C are indentifyed with the places of K_0 (see [EGA II, 7.4]). Then we show that the M-graded algebra is described by a polyhedral divisor over C (see Theorem 3.5).

Let us pass further to the general case of varieties with an action of a non nescessarily split torus. The reader may consult [Bry],[Vo],[ELST], for the theory of non-split toric varieties and [Hu] for the spherical embeddings. Let $\mathbf{G}$ be a torus over $\mathbf{k}$ that splits in a finite Galois extension $E/\mathbf{k}$. Let $\mathrm{Var}_{\mathbf{G},E}(\mathbf{k})$ be the category of affine $\mathbf{G}$-varieties of complexity one splitting in $E/\mathbf{k}$ (see 5.4). For an object $X \in \mathrm{Var}_{\mathbf{G},E}(\mathbf{k})$ we let $[X]$ be its isomorphism class and $X(E) = X \times_{\mathrm{Spec}\,\mathbf{k}} \mathrm{Spec}\, E$ be its extension of X over the field extension. Fixing $X \in \mathrm{Var}_{\mathbf{G},E}(\mathbf{k})$, as an application of our previous results, we study the pointed set

$$\left(\left\{[Y] \,|\, Y \in \mathrm{Var}_{\mathbf{G},E}(\mathbf{k}) \ \text{ and } \ \mathrm{X(E)} \simeq_{\mathrm{Var}_{\mathbf{G},E}(E)} Y(E)\right\}, [X]\right)$$

of isomorphism classes of $E/\mathbf{k}$-forms of X that is in bijection with the first pointed set $H^1(E/\mathbf{k}, \mathrm{Aut}_{\mathbf{G}(E)}(X(E)))$ of non abelian Galois cohomology. By elementary argument (see 5.7) these latter pointed sets are described by all possible homogeneous semi-linear

$\mathfrak{S}_{E/\mathbf{k}}$-action on the multigraded algebra $E[X(E)]$, where here $\mathfrak{S}_{E/\mathbf{k}}$ is the Galois group of $E/\mathbf{k}$. Translating to the language of polyhedral divisors, we obtain a combinatorial description of $E/\mathbf{k}$-forms of X, see Theorem 5.10. This theorem can be viewed as a first step towards the study of the forms of $\mathbf{G}$-varieties of complexity one.

Let us give a brief summary of the contents of each section. In the first section, we recall how to extend the D.P.D. presentation of parabolic graded algebra to context of Dedekind domain. This fact has been mentioned in [FZ] and firstly traited by a master student of Hubert Flenner [Ka]. In the second and the third section, we study respectively a class of multigraded algebras over Dedekind domains and a class of elliptic multigraded algebras over a field. In the fourth section, we classify non-split affine $\mathbb{T}$-varieties of complexity one. The last section is devoted to the non-split case.

0.3. All considered rings are commutative and unitary. Let $\mathbf{k}$ denote a field. Given a lattice M we let $\mathbf{k}[M]$ be the semigroup algebra

$$\bigoplus_{m\in M} \mathbf{k}\chi^m, \text{ where } \chi^{m+m'} = \chi^m \cdot \chi^{m'}.$$

By a *variety* X over $\mathbf{k}$ we mean an integral separated scheme of finite type over $\mathbf{k}$; one assumes in addition that $\mathbf{k}$ is algebraically closed in the field of rational functions $\mathbf{k}(X)$. In particular, X is geometrically irreducible.

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1. Graded algebras over Dedekind domains

In this section we recall how to generalize the Dolgachev-Pinkham-Demazure (D.P.D.) presentation in [FZ, Section 3] to the context of Dedekind domains (see Lemma 1.6). This generalization concerns in particular an algebraic description of affine normal parabolic complex $\mathbb{C}^*$-surfaces. Let us start by a well known definition.

1.1. An integral domain A_0 is called a *Dedekind domain* (or Dedekind ring) if it is not a field and if it satisfies the following conditions.

(i) The ring A_0 is noetherian.
(ii) The ring A_0 is integrally closed in its field of fractions.
(iii) Every nonzero prime ideal is a maximal ideal.

Let us mention several classical examples of Dedekind domains.

Exemple 1.2. Let K be a number field. If $\mathbb{Z}_K$ denotes the ring of integers of K then $\mathbb{Z}_K$ is a Dedekind ring.

Let A be a finitely generated normal algebra of dimension one over a field $\mathbf{k}$. This means that the scheme $C = \operatorname{Spec} A$ is an affine smooth curve. The coordinate ring $A = \mathbf{k}[C]$ is Dedekind.

The algebra of power series $\mathbf{k}[[t]]$ in one variable over the field $\mathbf{k}$ is a Dedekind domain. More generally every principal ideal domain (and so every discrete valuation ring) that is not a field is a Dedekind domain.

1.3. Let A_0 be an integral domain and let K_0 be its field of fractions. Recall that a *fractional ideal* $\mathfrak{b}$ is a finitely generated nonzero A_0-submodule of K_0. Actually every fractional ideal is of the form $\frac{1}{f} \cdot \mathfrak{a}$, where $f \in A_0$ is nonzero and $\mathfrak{a}$ is a nonzero ideal of A_0. If $\mathfrak{b}$ is equal to $u \cdot A_0$ for some nonzero element $u \in K_0$ then we say that $\mathfrak{b}$ is a *principal* fractional ideal.

The following result gives a description of fractional ideals of A_0 in terms of Weil divisors on $Y = \operatorname{Spec} A_0$ when A_0 is a Dedekind domain. This assertion is well known. For convenience of the reader we include a short proof.

Theorem 1.4. *Let A_0 be a Dedekind ring with field of fractions K_0. Let $Y = \operatorname{Spec} A_0$. Then the map*

$$\operatorname{Div}_{\mathbb{Z}}(Y) \to \operatorname{Id}(A_0),\ D \mapsto H^0(Y, \mathcal{O}_Y(D))$$

is a bijection between the set of integral Weil divisors on Y and the set of fractional ideals of A_0. Every fractional ideal is locally free of rank 1 as A_0-module and the natural map

$$H^0(Y, \mathcal{O}_Y(D)) \otimes H^0(Y, \mathcal{O}_Y(D')) \to H^0(Y, \mathcal{O}_Y(D + D'))$$

is surjective. A Weil divisor D on Y is principal (resp. effective) if and only if the corresponding fractional ideal is principal (resp. contains A_0).

Proof. By [Ha, II.6.11] the group of Weil divisors on Y coincides with the group of Cartier divisors. In particular, every A_0-module $H^0(Y, \mathcal{O}_Y(D))$ is of finite type [Ha, II.5.5], locally free of rank one, and so has a nonzero global section. Therefore the map

$$\operatorname{Div}_{\mathbb{Z}}(Y) \to \operatorname{Id}(A_0)$$

is well defined.

Let D, D' be divisors of $\operatorname{Div}_{\mathbb{Z}}(Y)$. Then by the previous observation the $\mathcal{O}_Y$-sheaves $\mathcal{O}_Y(D) \otimes \mathcal{O}_Y(D')$ and $\mathcal{O}_Y(D + D')$ are isomorphic. This induces an isomorphism on the level of global sections.

Every nonzero prime ideal of A_0 is the module of global sections of an invertible sheaf over $\mathcal{O}_Y$. Thus by the primary decomposition, the map $\operatorname{Div}_{\mathbb{Z}}(Y) \to \operatorname{Id}(A_0)$ is surjective.

Assume that

$$H^0(Y, \mathcal{O}_Y(D)) = H^0(Y, \mathcal{O}_Y(D'))$$

for some $D, D' \in \operatorname{Div}_{\mathbb{Z}}(Y)$. Then we can write $D = D_+ - D_-$ and $D = D'_+ - D'_-$, where D_+, D'_+, D_-, D'_+ are integral effective divisors. By tensoring we obtain the equality

$$H^0(Y, \mathcal{O}_Y(-D_- - D'_+)) = H^0(Y, \mathcal{O}_Y(-D'_- - D_+))$$

between ideals of A_0. Again using the decomposition in prime ideals we have $-D_- - D'_+ = -D'_- - D_+$ so that $D = D'$. One concludes that the map is injective.

Assume that $H^0(Y, \mathcal{O}_Y(D))$ contains A_0. Write $D = D_+ - D_-$ with D_+, D_- effective divisors having disjoint supports. Then by our assumption

$$H^0(Y, \mathcal{O}_Y(0)) = A_0 = A_0 \cap H^0(Y, \mathcal{O}_Y(D)) = H^0(Y, \mathcal{O}_Y(-D_-)).$$

This yields $D_- = 0$ and so D is effective. The rest of the proof is straightforward. □

Notation 1.5. Let A_0 be a Dedekind domain. For a $\mathbb{Q}$-divisor D on the affine scheme $Y = \operatorname{Spec} A_0$ we denote by $A_0[D]$ the graded algebra

$$\bigoplus_{i\in\mathbb{N}} H^0(Y, \mathcal{O}_Y(\lfloor iD \rfloor))\, t^i,$$

where t is a variable over the field K_0. Note that $A_0[D]$ is normal as intersection of discrete valuation rings with field of fractions $K_0(t)$ (see the argument for [De, 2.7]).

The next lemma provides a D.P.D. presentation for a class of graded subrings of $K_0[t]$. It will be useful for the next section. Here we give an elementary proof using the description in 1.4 of fractional ideals.

Lemma 1.6. *Let A_0 be a Dedekind ring with the field of fractions K_0. Let*

$$A = \bigoplus_{i\in\mathbb{N}} A_i\, t^i \subset K_0[t]$$

be a normal graded subalgebra of finite type over A_0, where every A_i is contained in K_0. Assume that the field of fractions of A is $K_0(t)$. Then there exists a unique $\mathbb{Q}$-divisor D on $Y = \operatorname{Spec} A_0$ such that $A = A_0[D]$. Futhermore we have $Y = \operatorname{Proj} A$.

Proof. Theorem 1.4 and Lemma 2.2 in [GY] imply that every nonzero module A_i can be written

$$A_i = H^0(Y, \mathcal{O}_Y(D_i))$$

for some $D_i \in \operatorname{Div}_{\mathbb{Z}}(Y)$. By Proposition 3 in [Bou, III.3] there exists a positive integer d such that the subalgebra

$$A^{(d)} := \bigoplus_{i\geq 0} A_{di}\, t^{di}$$

is generated by $A_d\, t^d$. Proceeding by induction, for any $i \in \mathbb{N}$ we have $D_{di} = iD_d$. Let $D = D_d/d$. Then using the normality of A and $A_0[D]$, we obtain for any homogenous element $f \in K_0[t]$ the following equivalences

$$f \in A_0[D] \Leftrightarrow f^d \in A_0[D] \Leftrightarrow f^d \in A \Leftrightarrow f \in A.$$

This yields $A = A_0[D]$.

Let D' be another $\mathbb{Q}$-divisor on Y such that $A = A_0[D']$. Comparing the graded pieces of $A_0[D]$ and of $A_0[D']$, it follows that $\lfloor iD \rfloor = \lfloor iD' \rfloor$ for any $i \in \mathbb{N}$. Hence $D = D'$ and so the decomposition is unique.

It remains to show the equality $Y = \operatorname{Proj} A$. Let $V = \operatorname{Proj} A$. By Exercice 5.13 in [Ha, II] and Proposition 3 in [Bou, III.1] we may assume that $A = A_0[D]$ is generated by $A_1 t$. Since the sheaf $\mathcal{O}_Y(D)$ is locally free of rank one over $\mathcal{O}_Y$ there exist $g_1, \ldots, g_s \in A_0$ such that

$$Y = \bigcup_{j=1}^{s} Y_{g_j}, \quad \text{where} \quad Y_{g_j} = \operatorname{Spec} (A_0)_{g_j}$$

and such that for $e = 1, \ldots, s$,

$$A_1 \otimes_{A_0} (A_0)_{g_e} = \mathcal{O}_Y(D)(Y_{g_e}) = h_e \cdot A_0$$

for some $h_e \in K_0^\star$. Let $\pi : V \to Y$ be the natural morphism induced by the inclusion $A_0 \subset A$. The preimage of the open subset Y_{g_e} under π is

$$\operatorname{Proj} A \otimes_{A_0} (A_0)_{g_e} = \operatorname{Proj} (A_0)_{g_e}[A_1 \otimes_{A_0} (A_0)_{g_e} t] = \operatorname{Proj} (A_0)_{g_e}[h_e t] = Y_{g_e}.$$

Hence π is the identity map and so $Y = V$, as required. □

As an immediate consequence we obtain the following. The reader can see that the proof of [FZ, 3.9] holds m.m. for positively graded 2-dimensional normal algebras of finite type over a Dedekind domain.

Corollary 1.7. *Let A_0 be a Dedekind ring with field of fractions K_0 and let t be a variable over K_0. Consider the subalgebra*

$$A = A_0[f_1 t^{m_1}, \ldots, f_r t^{m_r}] \subset K_0[t],$$

where $m_1, \ldots, m_r$ are positive integers and $f_1, \ldots, f_r \in K_0^\star$ are such that the field of fractions of A is $K_0(t)$. Then the normalization of A is equal to $A_0[D]$, where D is the $\mathbb{Q}$-divisor

$$D = -\min_{1 \le i \le r} \frac{\operatorname{div} f_i}{m_i}.$$

2. Multigraded algebras over Dedekind domains

Let A_0 be a Dedekind ring and let K_0 be its field of fractions. Given a lattice M the purpose of this section is to study normal noetherian M-graded A_0-subalgebras of $K_0[M]$. We show below that these subalgebras admit a description in terms of polyhedral divisors. We start by recalling some necessary notation from convex geometry available in [AH, Section 1].

2.1. Let N be a lattice and let $M = \operatorname{Hom}(N, \mathbb{Z})$ be its dual. Denote by $N_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} N$ and $M_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} M$ the associated dual $\mathbb{Q}$-linear spaces. To any linear form $m \in M_\mathbb{Q}$ and to any vector $v \in N_\mathbb{Q}$ we write $\langle m, v \rangle$ for the number $m(v)$. A polyhedral cone $\sigma \subset N_\mathbb{Q}$ is called *strongly convex* if it admits a vertex. This is equivalent to say that the dual cone

$$\sigma^\vee = \{\, m \in M_\mathbb{Q} \,|\, \forall v \in \sigma,\ \langle m, v \rangle \ge 0 \,\}$$

is full dimensional.

Recall that for a nonzero strongly convex polyhedral cone $\sigma \subset N_\mathbb{Q}$ the *Hilbert basis* $\mathscr{H}_\sigma = \mathscr{H}_{\sigma,N}$ of σ in the lattice N is the subset of irreducible elements

$$\{v \in \sigma_N - \{0\} \,|\, \forall v_1, v_2 \in \sigma_N - \{0\}, v = v_1 + v_2 \Rightarrow v = v_1 \text{ or } v = v_2\}.$$

It is known that the set $\mathscr{H}_\sigma$ is finite and generates the semigroup $(\sigma \cap N, +)$. Futhermore, it is minimal for these latter properties. The cone σ is said *regular* if $\mathscr{H}_\sigma$ is contained in a basis of N.

Let us fix a strongly convex polyhedral cone $\sigma \subset N_\mathbb{Q}$. A subset $Q \subset N_\mathbb{Q}$ is a *polytope* if Q is non-empty and if Q is the convex hull of a finite number of vectors. We define $\operatorname{Pol}_\sigma(N_\mathbb{Q})$ to be the set of polyhedra which can be written as the Minkowski sum $P = Q + \sigma$ with Q a polytope of $N_\mathbb{Q}$. An element of $\operatorname{Pol}_\sigma(N_\mathbb{Q})$ is called a polyhedron with *tailed cone* σ.

The following definition introduces the notion of polyhedral divisors over Dedekind domains.

Definition 2.2. Consider the subset Z of closed points of the affine scheme $Y = \operatorname{Spec} A_0$. A *$\sigma$-polyhedral divisor $\mathfrak{D}$ over A_0* is a formal sum

$$\mathfrak{D} = \sum_{z \in Z} \Delta_z \cdot z,$$

where Δ_z belongs to $\operatorname{Pol}_\sigma(N_\mathbb{Q})$ and $\Delta_z = \sigma$ for all but finitely many z in Z. Let $z_1, \ldots, z_r$ be elements of Z such that for any $z \in Z$ and for $i = 1, \ldots, r$, $z \neq z_i$ implies $\Delta_z = \sigma$. If the meaning of A_0 is clear from the context then we write

$$\mathfrak{D} = \sum_{i=1}^{r} \Delta_{z_i} \cdot z_i.$$

In the sequel, we let $\omega_M = \omega \cap M$ whether $\omega \subset M_\mathbb{Q}$ is a polyhedral cone. Starting from a σ-polyhedral divisor $\mathfrak{D}$ we can build an M-graded algebra over A_0 with weight cone $\sigma^\vee$ in the same way as in [AH, Section 3].

2.3. Let $m \in \sigma^\vee$. Then for any $z \in Z$ the expression

$$h_z(m) = \min_{v \in \Delta_z} \langle m, v \rangle$$

is well defined. The function h_z on the cone $\sigma^\vee$ is upper convex and positively homogeneous. It is identically zero if and only if $\Delta_z = \sigma$. The *evaluation* of $\mathfrak{D}$ in a vector $m \in \sigma^\vee$ is the $\mathbb{Q}$-divisor

$$\mathfrak{D}(m) = \sum_{z \in Z} h_z(m) \cdot z.$$

In analogy with the notation of [FZ] we denote by $A_0[\mathfrak{D}]$ the M-graded subring

$$\bigoplus_{m \in \sigma^\vee_M} A_m \chi^m \subset K_0[M], \quad \text{where} \quad A_m = H^0\left(Y, \mathcal{O}_Y\left(\lfloor \mathfrak{D}(m) \rfloor\right)\right).$$

Notation 2.4. Let

$$f = (f_1 \chi^{m_1}, \ldots, f_r \chi^{m_r})$$

be an r-uplet of homogeneous elements of $K_0[M]$. Assume that the vectors $m_1, \ldots, m_r$ generate the cone $\sigma^\vee$. We denote by $\mathfrak{D}[f]$ the σ-polyhedral divisor

$$\sum_{z \in Z} \Delta_z[f] \cdot z, \quad \text{where} \quad \Delta_z[f] = \left\{ v \in N_\mathbb{Q} \mid \langle m_i, v \rangle \geq -\operatorname{ord}_z f_i, \ i = 1, 2, \ldots, r \right\}.$$

In section 3, we use a similar notation for polyhedral divisors over a smooth projective curve; we remplace the set Z by a smooth projective curve C.

The main result of this section is the following theorem. For a proof of part (iii) we refer the reader to the argument of Theorem 2.4 in [La].

Theorem 2.5. *Let A_0 be a Dedekind domain with field of fractions K_0 and let $\sigma \subset N_\mathbb{Q}$ be a strongly convex polyhedral cone. Then the following hold.*

(i) *If $\mathfrak{D}$ is a σ-polyhedral divisor over A_0 then the algebra $A_0[\mathfrak{D}]$ is normal, noetherian, and has the same field of fractions as that of $K_0[M]$.*

(ii) *Conversely, let*

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m$$

be a normal noetherian M-graded A_0-subalgebra of $K_0[M]$ with weight cone $\sigma^\vee$. Assume that the rings A and $K_0[M]$ have the same field of fractions[1]*. Then there exists a unique σ-polyhedral divisor $\mathfrak{D}$ over A_0 such that $A = A_0[\mathfrak{D}]$.*

(iii) *More explicitly, if*

$$f = (f_1 \chi^{m_1}, \ldots, f_r \chi^{m_r})$$

is an r-uplet of homogeneous elements of $K_0[M]$ with nonzero vectors $m_1, \ldots, m_r$ generating the lattice M then the normalization of the ring

$$A = A_0[f_1 \chi^{m_1}, \ldots, f_r \chi^{m_r}]$$

is equal to $A_0[\mathfrak{D}[f]]$ (see 2.4).

Let us give an example related to the ring of integers of a number field.

Exemple 2.6. For a number field K, the group of classes $\operatorname{Cl} K$ is the quotient of the group of fractional ideals of K by the subgroup of principal fractional ideals. In other words, $\operatorname{Cl} K = \operatorname{Pic} Y$, where $Y = \operatorname{Spec} \mathbb{Z}_K$ is the affine scheme associated to the ring of integers of K. It is known that the group $\operatorname{Cl} K$ is finite. Futhermore $\mathbb{Z}_K$ is a principal ideal domain if and only if $\operatorname{Cl} K$ is trivial.

Let $K = \mathbb{Q}(\sqrt{-5})$. Then $\mathbb{Z}_K = \mathbb{Z}[\sqrt{-5}]$ and the group $\operatorname{Cl} K$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. A set of representatives in $\operatorname{Cl} K$ is given by the fractional ideals $\mathfrak{a} = (2,\, 1 + \sqrt{-5})$ and $\mathbb{Z}_K$. Given x, y two independent variables over K, consider the $\mathbb{Z}^2$-graded ring

$$A = \mathbb{Z}_K \left[\, 3\, x^2 y,\, 2\, y,\, 6\, x \,\right].$$

Let us describe the normalization $\bar{A}$ of A. Denoting respectively by $\mathfrak{b}$, $\mathfrak{c}$ the prime ideals $(3,\, 1 + \sqrt{-5})$ and $(3,\, 1 - \sqrt{-5})$, we have the decompositions

$$(2) = \mathfrak{a}^2, \;\; (3) = \mathfrak{b} \cdot \mathfrak{c}.$$

Observe that the ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are distincts. Thus we have

$$\operatorname{div} 2 = 2 \cdot \mathfrak{a} \;\text{ and }\; \operatorname{div} 3 = \mathfrak{b} + \mathfrak{c},$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are seen as closed points of $Y = \operatorname{Spec} \mathbb{Z}_K$. Let $\mathfrak{D}$ be the polyhedral divisor over $\mathbb{Z}_K$ given by $\Delta_\mathfrak{a} \cdot \mathfrak{a} + \Delta_\mathfrak{b} \cdot \mathfrak{b} + \Delta_\mathfrak{c} \cdot \mathfrak{c}$ with the polyhedra

$$\Delta_\mathfrak{a} = \left\{ (v_1, v_2) \in \mathbb{Q}^2 \,|\, 2v_1 + v_2 \geq 0,\, v_2 \geq -2,\, v_1 \geq -2 \right\} \;\text{ and}$$

$$\Delta_\mathfrak{b} = \Delta_\mathfrak{c} = \left\{ (v_1, v_2) \in \mathbb{Q}^2 \,|\, 2v_1 + v_2 \geq -1,\, v_2 \geq 0,\, v_1 \geq -1 \right\}.$$

By Theorem 2.5 we obtain $\bar{A} = A_0[\mathfrak{D}]$, where $A_0 = \mathbb{Z}_K$. The weight cone of A is the first quadrant $\omega = (\mathbb{Q}_{\geq 0})^2$. An easy computation shows that

$$A_0[\mathfrak{D}] = \mathbb{Z}_K \left[\, 2\, y,\; 6\, xy,\; 3(1 + \sqrt{-5})\, xy,\; 3\, x^2 y,\; 6x \,\right].$$

[1]This condition is equivalent to ask that the weight semigroup of A generates M

The proof of Theorem 2.5 needs some preparations. We start by a well known result [GY, Theorem 1.1] yielding an equivalence between noetherian and finitely generated properties of multigraded algebras. Note that this result does not hold for algebras graded by an arbitrary abelian group; a counterexample is given in [GY, 3.1].

Theorem 2.7. *Let G denote a finitely generated abelian group and let A be a G-graded ring. Then the following statements are equivalent.*

(i) *The ring A is noetherian.*
(ii) *The graded piece A_0 corresponding to the neutral element of G is a noetherian ring and the A_0-algebra A is finitely generated.*

The next lemma will enable us to show that the ring $A_0[\mathfrak{D}]$, coming from a polyhedral divisor $\mathfrak{D}$ over a Dedekind domain A_0, is noetherian.

Lemma 2.8. *Let $D_1, \ldots, D_r$ be $\mathbb{Q}$-divisors on $Y = \operatorname{Spec} A_0$. Then the A_0-algebra*

$$B = \bigoplus_{(m_1,\ldots,m_r)\in\mathbb{N}^r} H^0\left(Y, \mathcal{O}_Y\left(\left\lfloor \sum_{i=1}^r m_i D_i \right\rfloor\right)\right)$$

is finitely generated.

Proof. Let d be a positive integer such that for $i = 1, \ldots, r$, the divisor dD_i is integral. Consider the lattice polytope

$$Q = \{\, (m_1, \ldots, m_r) \in \mathbb{Q}^r \mid 0 \leq m_i \leq d,\ i = 1, \ldots, r \,\}.$$

The subset $Q \cap \mathbb{N}^r$ being finite, the A_0-module

$$E := \bigoplus_{(m_1,\ldots,m_r)\in\mathbb{N}^r\cap Q} H^0\left(Y, \mathcal{O}_Y\left(\left\lfloor \sum_{i=1}^r m_i D_i \right\rfloor\right)\right)$$

is finitely generated (see 1.4). Let $(m_1, \ldots, m_r)$ be an element of $\mathbb{N}^r$. Write $m_i = dq_i + r_i$ with $q_i, r_i \in \mathbb{N}$ such that $0 \leq r_i < d$. The equality

$$\left\lfloor \sum_{i=1}^r m_i D_i \right\rfloor = \sum_{i=1}^r q_i \lfloor dD_i \rfloor + \left\lfloor \sum_{i=1}^r r_i D_i \right\rfloor$$

implies that every homogeneous element of B can be expressed as a polynomial in E. If $f_1, \ldots, f_s$ generate the A_0-module E then we have $A = A_0[f_1, \ldots, f_s]$, proving our statement. □

Next we give a proof of the first part of Theorem 2.5.

Proof. Let $A = A_0[\mathfrak{D}]$. By Theorem 1.4 and since the cone $\sigma^\vee$ is full dimensional the algebras A and $K_0[M]$ have the same field of fractions. Let us show that A is a normal ring. Given a closed point $z \in Z$ and an element of $v \in \Delta_z$ consider the map

$$\nu_{z,v} : K_0[M] - \{0\} \to \mathbb{Z}$$

defined as follows. Let $\alpha \in K_0[M]$ be nonzero with decomposition in homogeneous elements

$$\alpha = \sum_{i=1}^r f_i \chi^{m_i}, \text{ where } f_i \in K_0^\star.$$

Then let

$$\nu_{z,v}(\alpha) = \min_{1\le i\le r} \{\operatorname{ord}_z f_i + \langle m_i, v\rangle\}\,.$$

The map $\nu_{z,v}$ defines a discrete valuation on $\operatorname{Frac} A$. Denote by $\mathcal{O}_{v,z}$ the associated local ring. By the definition of the algebra $A_0[\mathfrak{D}]$ we have

$$A = K_0[M] \cap \bigcap_{z\in Z} \bigcap_{v\in\Delta_z} \mathcal{O}_{v,z}\,,$$

proving that A is normal as intersection of normal rings with field of fractions $\operatorname{Frac} A$.

It remains to show that A is noetherian. By Hilbert's Basis Theorem, it suffices to show that A is finitely generated. Let $\lambda_1, \ldots, \lambda_e$ be full dimensional regular subcones of $\sigma^\vee$ giving a subdivision such that for any i the evaluation map

$$\sigma^\vee \to \operatorname{Div}_{\mathbb{Q}}(Y), \quad m \mapsto \mathfrak{D}(m)$$

is linear on λ_i. Fix $i \in \mathbb{N}$ such that $1 \le i \le e$. Consider the distinct elements $v_1, \ldots, v_n$ of the Hilbert basis of λ_i. Denote by A_{λ_i} the algebra

$$\bigoplus_{m\in\lambda_i\cap M} H^0(Y, \mathcal{O}_Y(\lfloor\mathfrak{D}(m)\rfloor)\chi^m.$$

Then the vectors $v_1, \ldots, v_n$ form a basis of the lattice M and so

$$A_{\lambda_i} \simeq \bigoplus_{(m_1,\ldots,m_n)\in\mathbb{N}^n} H^0\left(Y, \mathcal{O}_Y\left(\left\lfloor \sum_{i=1}^n m_i\,\mathfrak{D}(v_i)\right\rfloor\right)\right).$$

By Lemma 2.8, the algebra A_{λ_i} is finitely generated over A_0. The surjective map

$$A_{\lambda_1} \otimes \ldots \otimes A_{\lambda_e} \to A$$

shows that A is also finitely generated. □

For the second part of Theorem 2.5 we need the following lemma.

Lemma 2.9. *Assume that A verifies the assumptions of* 2.5 (ii). *Then the following statements hold.*

(i) *For any $m \in \sigma_M^\vee$ we have $A_m \neq \{0\}$. In other words, the weight semigroup of the M-graded algebra A is $\sigma_M^\vee$.*

(ii) *If $L = \mathbb{Q}_{\ge 0}\cdot m'$ is a half-line contained in $\sigma^\vee$ then the ring*

$$A_L := \bigoplus_{m\in L\cap M} A_m\chi^m$$

is normal and noetherian.

Proof. Let

$$S = \{m \in \sigma_M^\vee,\, A_m \neq \{0\}\}$$

be the weight semigroup of A. Assume that $S \neq \sigma_M^\vee$. Then there exist $e \in \mathbb{Z}_{>0}$ and $m \in M$ such that $m \notin S$ and $e\cdot m \in S$. Since A is a noetherian ring, by [GY, Lemma 2.2] the A_0-module A_{em} is a fractional ideal of A_0. By Theorem 1.4 we obtain

$$A_{em} = H^0(Y, \mathcal{O}_Y(D_{em}))$$

for some integral divisor $D_{em} \in \mathrm{Div}_{\mathbb{Z}}(Y)$. Let f be a nonzero section of

$$H^0\left(Y, \mathcal{O}_Y\left(\left\lfloor \frac{D_{em}}{e} \right\rfloor\right)\right).$$

This element exists by virtue of Theorem 1.4. We have the inequalities

$$\mathrm{div}\, f^e \geq -e\left\lfloor \frac{D_{em}}{e} \right\rfloor \geq -D_{em}.$$

The normality of A implies $f \in A_m$. This contradicts our assumption and gives (i).

For the second assertion we notice by 2.7 and by the argument of [AH, Lemma 4.1] that A_L is noetherian.

It remains to show that A_L is normal. Let $\alpha \in \mathrm{Frac}\, A_L$ be an integral element over A_L. By normality of A and $K_0[\chi^m]$ we obtain $\alpha \in A \cap K_0[\chi^m] = A_L$ and so A_L is normal. □

In the sequel, we introduce some useful notation of convex geometry.

Notation 2.10. Let

$$(m_i, e_i),\ i = 1, \ldots, r$$

be elements of $M \times \mathbb{Z}$ such that the vectors $m_1, \ldots, m_r$ are nonzero and generate the lattice M. Then the cone $\omega = \mathrm{Cone}(m_1, \ldots, m_r)$ is full dimensional in $M_{\mathbb{Q}}$. Consider the $\omega^\vee$-polyhedron

$$\Delta = \{\, v \in N_{\mathbb{Q}},\ \langle m_i, v\rangle \geq -e_i,\ i = 1, 2, \ldots, r \,\}.$$

Let $L = \mathbb{Q}_{\geq 0} \cdot m$ be a half-line contained in ω with primitive vector m. In other words, the element m generates the semigroup $L \cap M$. Denote by $\mathscr{H}_L$ the Hilbert basis in the lattice $\mathbb{Z}^r$ of the nonzero cone

$$p^{-1}(L) \cap (\mathbb{Q}_{\geq 0})^r, \quad \text{where} \quad p : \mathbb{Q}^r \to M_{\mathbb{Q}}$$

is the $\mathbb{Q}$-linear map sending the canonical basis onto $(m_1, \ldots, m_r)$. We let

$$\mathscr{H}_L^\star = \left\{ (s_1, \ldots, s_r) \in \mathscr{H}_L\,,\ \sum_{i=1}^r s_i \cdot m_i \neq 0 \right\}.$$

For any vector $(s_1, \ldots, s_r) \in \mathscr{H}_L^\star$ there exists a unique $\lambda(s_1, \ldots, s_r) \in \mathbb{Z}_{>0}$ such that

$$\sum_{i=1}^r s_i \cdot m_i = \lambda(s_1, \ldots, s_r) \cdot m.$$

The argument of the proof of the following lemma uses only elementary facts of commutative algebra and of convex geometry. This is the key idea in order to obtain the Altmann-Hausen's presentation of Theorem 2.5 (ii).

Lemma 2.11. *Let* $\min\langle m, \Delta\rangle = \min_{v\in\Delta}\langle m, v\rangle$. *Under the assumptions of* 2.10 *we have*

$$\min\langle m, \Delta\rangle = -\min_{(s_1,\ldots,s_r)\,\in\,\mathscr{H}_L^\star} \frac{\sum_{i=1}^r s_i \cdot e_i}{\lambda(s_1, \ldots, s_r)}.$$

Proof. Let t be a variable over the field of complex numbers $\mathbb{C}$. Consider the M-graded subalgebra

$$A = \mathbb{C}[t][t^{e_1}\chi^{m_1}, \ldots, t^{e_r}\chi^{m_r}] \subset \mathbb{C}(t)[M].$$

The field of fractions of A is the same as that of $\mathbb{C}(t)[M]$. Using results of [Ho] the normalization of the algebra A is

$$\bar{A} = \mathbb{C}[\omega_0 \cap (M \times \mathbb{Z})], \text{ where } \omega_0 \subset M_{\mathbb{Q}} \times \mathbb{Q}$$

is the rational cone generated by $(0,1), (m_1, e_1), \ldots, (m_r, e_r)$. Actually a routine calculation shows that

$$\omega_0 = \{(w, -\min\langle w, \Delta\rangle + e) \,|\, w \in \omega,\, e \in \mathbb{Q}_{\geq 0}\}$$

and so

$$\bar{A} = \bigoplus_{m \in \omega \cap M} H^0(\mathbb{A}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(\lfloor \min\langle m, \Delta\rangle \rfloor \cdot (0)))\chi^m,$$

where $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$.

The sublattice $G \subset M$ generated by $p(\mathscr{H}^\star_L)$ is a subgroup of $\mathbb{Z} \cdot m$. Therefore there exists a unique integer $d \in \mathbb{Z}_{>0}$ such that $G = d\,\mathbb{Z} \cdot m$. For an element $m' \in \omega \cap M$, we denote by $A_{m'}$ (resp. $\bar{A}_{m'}$) the graded piece of A (resp. $\bar{A}$) corresponding to m'. Then the normalization $\bar{A}^{(d)}_L$ of the algebra

$$A^{(d)}_L := \bigoplus_{s \geq 0} A_{sdm}\chi^{sdm} \text{ is } B_L := \bigoplus_{s \geq 0} \bar{A}_{sdm}\chi^{sdm}.$$

Futhermore

$$A_L = \bigoplus_{s \geq 0} A_{sm}\chi^{sm}$$

is generated over $\mathbb{C}[t]$ by the elements

$$f_{(s_1,\ldots,s_r)} := \prod_{i=1}^{r} (t^{e_i}\chi^{m_i})^{s_i} = t^{\sum_{i=1}^r s_i e_i}\chi^{\lambda(s_1,\ldots,s_r)m},$$

where $(s_1, \ldots, s_r)$ runs $\mathscr{H}^\star_L$. By the choice of the integer d we have $A^{(d)}_L = A_L$. Considering the G-graduation of $A^{(d)}_L$ for any $(s_1, \ldots, s_r) \in \mathscr{H}^\star_L$ the element $f_{(s_1,\ldots,s_r)}$ of the graded ring $A^{(d)}_L$ has degree

$$\deg f_{(s_1,\ldots,s_r)} := \frac{\lambda(s_1, \ldots, s_r)}{d}\ .$$

Letting

$$D = -\min_{(s_1,\ldots,s_r) \in \mathscr{H}^\star_L} \frac{\operatorname{div} f_{(s_1,\ldots,s_r)}}{\deg f_{(s_1,\ldots,s_r)}} = -\min_{(s_1,\ldots,s_r) \in \mathscr{H}^\star_L} d \cdot \frac{\sum_{i=1}^r s_i e_i}{\lambda(s_1, \ldots, s_r)} \cdot (0)$$

by Corollary 1.7 we obtain

$$\bar{A}^{(d)}_L = \bigoplus_{s \geq 0} H^0(\mathbb{A}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(\lfloor sD \rfloor))\chi^{sdm}.$$

The equality $\bar{A}^{(d)}_L = B_L$ implies that for any integer $s \geq 0$

$$H^0(\mathbb{A}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(\lfloor \min\langle sd \cdot m, \Delta\rangle \rfloor \cdot (0))) = H^0(\mathbb{A}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(\lfloor sD \rfloor)).$$

Hence by Lemma 1.6 we have

$$D = \min \langle d \cdot m, \Delta \rangle \cdot (0).$$

Dividing by d, we obtain the desired formula. □

Let A be an M-graded algebra satisfying the assumptions of 2.5 (ii). Using the D.P.D. presentation on each half line of the weight cone $\sigma^\vee$ (see Lemma 1.6), we can build a map

$$\sigma^\vee \to \mathrm{Div}_{\mathbb{Q}}(Y), \quad m \mapsto D_m.$$

It is upper convex, positively homogeneous, and verifies for any $m \in \sigma_M^\vee$,

$$A_m = H^0(C, \mathcal{O}_C(\lfloor D_m \rfloor)).$$

By Lemma 2.11, this map is piecewise linear (see [AH, 2.11]) or equivalently $m \mapsto D_m$ is the evaluation map of a polyhedral divisor. The following proof precises this idea.

Proof of 2.5 (ii). By 2.7 we may consider

$$f = (f_1 \chi^{m_1}, \ldots, f_r \chi^{m_r})$$

a system of homogeneous generators of A with nonzero vectors $m_1, \ldots, m_r \in M$. We use the same notation as in 2.4. Denote by $\mathfrak{D}$ the σ-polyhedral divisor $\mathfrak{D}[f]$. Let us show that $A = A_0[\mathfrak{D}]$. Let $L = \mathbb{Q}_{\geq 0} \cdot m$ be a half-line contained in $\omega = \sigma^\vee$ with m being the primitive vector of L. By Lemma 2.9, the graded subalgebra

$$A_L := \bigoplus_{m' \in L \cap M} A_{m'} \chi^{m'} \subset K_0[\chi^m]$$

is normal, noetherian, and has the same field of fractions as that of $K_0[\chi^m]$. Futhermore with the same notation as in 2.10, the algebra A_L is generated by the set

$$\left\{ \prod_{i=1}^r (f_i \chi^{m_i})^{s_i}, \ (s_1, \ldots, s_r) \in \mathscr{H}_L^\star \right\}.$$

By Corollary 1.7, if

$$D_m := - \min_{(s_1, \ldots, s_r) \in \mathscr{H}_L^\star} \frac{\sum_{i=1}^r s_i \operatorname{div} f_i}{\lambda(s_1, \ldots, s_r)}$$

then $A_L = A_0[D_m]$ with respect to the variable χ^m. By Lemma 2.11 for any closed point $z \in Z$ we have

$$h_z[f](m) = \min \langle m, \Delta_z[f] \rangle = - \min_{(s_1, \ldots, s_r) \in \mathscr{H}_L^\star} \frac{\sum_{i=1}^r s_i \operatorname{ord}_z f_i}{\lambda(s_1, \ldots, s_r)}.$$

Hence $\mathfrak{D}(m) = D_m$. Since this equality holds for all primitive vectors belonging to $\sigma^\vee$ one concludes that $A = A_0[\mathfrak{D}]$. The uniqueness of $\mathfrak{D}$ is straightforward (see Theorem 1.4 and [La, 2.2]). □

Using well known facts on Dedekind domains we obtain the following result.

Proposition 2.12. *Let A_0 be a Dedekind domain and let B_0 be the integral closure of A_0 in a finite separable extension L_0/K_0, where $K_0 = \operatorname{Frac} A_0$. Let $\mathfrak{D} = \sum_{z\in Z} \Delta_z \cdot z$ be a polyhedral divisor over A_0, where $Z \subset Y = \operatorname{Spec} A_0$ is the subset of closed points. Letting $Y' = \operatorname{Spec} B_0$ and considering the natural projection $p : Y' \to Y$, B_0 is a Dedekind domain and we have the formula*

$$A_0[\mathfrak{D}] \otimes_{A_0} B_0 = B_0[p^\star\mathfrak{D}] \text{ with } p^\star\mathfrak{D} = \sum_{z\in Z} \Delta_z \cdot p^\star(z).$$

Exemple 2.13. Consider the polyhedral divisor

$$\mathfrak{D} = \Delta_{(t)} \cdot (t) + \Delta_{(t^2+1)} \cdot (t^2+1)$$

over the Dedekind ring $A_0 = \mathbb{R}[t]$, where the coefficients are

$$\Delta_{(t)} = (-1, 0) + \sigma, \;\; \Delta_{(t^2+1)} = [(0, 0), (1, 0)] + \sigma,$$

and $\sigma \subset \mathbb{Q}^2$ is the rational cone generated by $(1, 0)$ and $(1, 1)$. An easy computation shows that

$$A_0[\mathfrak{D}] = \mathbb{R}\left[t, -t\chi^{(1,0)}, \chi^{(0,1)}, t(t^2+1)\chi^{(1,-1)}\right] \simeq \frac{\mathbb{R}[x_1, x_2, x_3, x_4]}{((1+x_1^2)x_2 + x_3x_4)},$$

where x_1, x_2, x_3, x_4 are independent variables over $\mathbb{R}$. Let $\zeta = \sqrt{-1}$. Considering the natural projection $p : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ we obtain

$$p^\star\mathfrak{D} = \Delta_0 \cdot 0 + \Delta_{(t^2+1)} \cdot \zeta + \Delta_{(t^2+1)} \cdot (-\zeta).$$

Letting $B_0 = \mathbb{C}[t]$ one concludes that $A_0[\mathfrak{D}] \otimes_{\mathbb{R}} \mathbb{C} = B_0[p^\star\mathfrak{D}]$.

3. Multigraded algebras and algebraic function fields

In this section, we study another type of multigraded algebras. They are described by a proper polyhedral divisor over an algebraic function field in one variable. Fix an arbitrary field $\mathbf{k}$. Let us recall a classical definition.

3.1. An *algebraic function field* (in one variable) over $\mathbf{k}$ is a field extension $K_0/\mathbf{k}$ verifying the following conditions.

(i) The transcendence degree of K_0 over $\mathbf{k}$ is equal to one.
(ii) Every element of K_0 that is algebraic over $\mathbf{k}$ belongs to $\mathbf{k}$.

Actually, by virtue of our convention, a smooth projective curve C over $\mathbf{k}$ gives naturally an algebraic function field $K_0/\mathbf{k}$ by letting $K_0 = \mathbf{k}(C)$. As an application of the valuative criterion of properness (see [EGA II, Section 7.4]), every algebraic function field $K_0/\mathbf{k}$ is the field of rational functions of a unique (up to isomorphism) smooth projective curve C over $\mathbf{k}$. In the next paragraph, we recall the construction of the curve C starting from an algebraic function field K_0.

3.2. A *valuation ring* of K_0 is a proper subring $\mathcal{O} \subset K_0$ strictly containing $\mathbf{k}$ and such that for any nonzero element $f \in K_0$, either $f \in \mathcal{O}$ or $\frac{1}{f} \in \mathcal{O}$. By [St, 1.1.6] every valuation ring of K_0 is the ring associated to a discrete valuation of $K_0/\mathbf{k}$. A subset $P \subset K_0$ is called a *place* of K_0 if there is some valuation ring $\mathcal{O}$ of K_0 such that P is the maximal ideal of $\mathcal{O}$. We denote by $\mathscr{R}_{\mathbf{k}} K_0$ the set of places of K_0. The latter is called the *Riemann surface* of K_0. By [EGA II, 7.4.18] the set $\mathscr{R}_{\mathbf{k}} K_0$ can be identified with a smooth projective curve C over the field $\mathbf{k}$ such that $K_0 = \mathbf{k}(C)$.

In the sequel we consider $C = \mathscr{R}_{\mathbf{k}} K_0$ as a geometrical object with its structure of scheme. By convention an element z belonging to C is a closed point. We write P_z the associated place to a point $z \in C$. Note that we keep the notation of convex geometry introduced in 2.1.

3.3. Let M, N be dual lattices and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. A *σ-polyhedral divisor* over K_0 (or over C) is a formal sum $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ with $\Delta_z \in \mathrm{Pol}_\sigma(N_{\mathbb{Q}})$ and $\Delta_z = \sigma$ for all but finitely many $z \in C$. Again we let

$$\mathfrak{D}(m) = \sum_{z \in C} \min_{v \in \Delta_z} \langle m, v \rangle \cdot z$$

be the *evaluation* in $m \in \sigma^\vee$; that is, a $\mathbb{Q}$-divisor over the curve C. We let $\kappa(P) = \mathcal{O}/P$, where $\mathcal{O}$ is the valuation ring of a place P. The field $\kappa(P)$ is a finite extension of $\mathbf{k}$ [St, 1.1.15] and we call it the *residual field* of P. The *degree* of $\mathfrak{D}$ is the Minkowski sum

$$\deg \mathfrak{D} = \sum_{z \in C} [\kappa(P_z) : \mathbf{k}] \cdot \Delta_z,$$

where $[\kappa(P) : \mathbf{k}]$ is the dimension of the $\mathbf{k}$-vector space $\kappa(P)$. The number $[\kappa(P) : \mathbf{k}]$ is also called the degree of the place P. Given $m \in \sigma^\vee$ we have naturally the relation $(\deg \mathfrak{D})(m) = \deg \mathfrak{D}(m)$.

We can now introduce the notion of properness for polyhedral divisors. See [AH, 2.7, 2.12] for other particular cases.

Definition 3.4. A σ-polyedral divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is called *proper* if it satifies the following conditions.

(i) The polyhedron $\deg \mathfrak{D}$ is strictly contained in the cone σ.
(ii) If $\deg \mathfrak{D}(m) = 0$ then m belongs to the boundary of $\sigma^\vee$ and a multiple of $\mathfrak{D}(m)$ is principal.

Our next main result gives a description similar to that in 2.5 for algebraic function fields. For a proof of 3.5 (iii) we refer to the argument of [La, 2.4].

Theorem 3.5. *Let $\mathbf{k}$ be a field and let $C = \mathscr{R}_{\mathbf{k}} K_0$ be the Riemann surface of an algebraic function field $K_0/\mathbf{k}$. Then the following statements hold.*

(i) *Let*

$$A = \bigoplus_{m \in \sigma^\vee_M} A_m \chi^m$$

be an M-graded normal noetherian $\mathbf{k}$-subalgebra of $K_0[M]$ with weight cone $\sigma^\vee$ and $A_0 = \mathbf{k}$. If A and $K_0[M]$ have the same field of fractions then there exists a unique proper σ-polyhedral divisor $\mathfrak{D}$ over C such that $A = A[C, \mathfrak{D}]$, where

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma^\vee_M} H^0(C, \mathcal{O}_C(\lfloor \mathfrak{D}(m) \rfloor)) \chi^m.$$

(ii) *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor over C. Then the algebra $A[C, \mathfrak{D}]$ is M-graded, normal, and finitely generated with weight cone $\sigma^\vee$. Futhermore it has the same field of fractions as that of $K_0[M]$.*

(iii) *Let*

$$A = \mathbf{k}[f_1\chi^{m_1}, \ldots, f_r\chi^{m_r}]$$

be an M-graded subalgebra of $K_0[M]$ with the $f_i\chi^{m_i}$ homogeneous of nonzero degree m_i. Let $f = (f_1\chi^{m_1}, \ldots, f_r\chi^{m_r})$. Assume that A and $K_0[M]$ have the same field of fractions. Then $\mathfrak{D}[f]$ is the proper σ-polyhedral divisor such that the normalization of A is $A[C, \mathfrak{D}[f]]$ (see 2.4).

For the proof of 3.5 we need some preliminary results. We begin by collecting some properties from a M-graded algebra A as in 3.5 (i) to some graded subring A_L.

Lemma 3.6. *Let A be an M-graded algebra satisfying the assumptions of* 3.5 (i)*. Given a half-line $L = \mathbb{Q}_{\geq 0} \cdot m \subset \sigma^\vee$ with a primitive vector m consider the subalgebra*

$$A_L = \bigoplus_{m' \in L \cap M} A_{m'}\chi^{m'}.$$

Let

$$Q(A_L)_0 = \left\{ \frac{a}{b} \mid \ a \in A_{sm},\, b \in A_{sm},\, b \neq 0,\, s \geq 0 \right\}.$$

Then the following assertions hold.

(i) *The algebra A_L is finitely generated and normal.*
(ii) *Either $Q(A_L)_0 = \mathbf{k}$ or $Q(A_L)_0 = K_0$.*
(iii) *If $Q(A_L)_0 = \mathbf{k}$ then $A_L = \mathbf{k}[\beta\chi^{dm}]$ for some $\beta \in K_0^\star$ and some $d \in \mathbb{Z}_{>0}$.*

Proof. The proof of (i) is similar to that of 2.9 (ii) and so we omitted it.

The field $Q(A_L)_0$ is an extension of $\mathbf{k}$ contained in K_0. If the transcendence degree of $Q(A_L)_0$ over $\mathbf{k}$ is zero then by normality of A_L we have $Q(A_L)_0 = \mathbf{k}$. Otherwise the extension $K_0/Q(A_L)_0$ is algebraic. Let α be an element of K_0. Then there exist $a_1, \ldots, a_d \in Q(A_L)_0$ with $a_d \neq 0$ such that

$$\alpha^d = \sum_{j=1}^{d} a_j \alpha^{d-j}.$$

Let

$$I = \{i \in \{1, \ldots, d\},\ a_i \neq 0\}.$$

For any $i \in I$ we write $a_i = \frac{p_i}{q_i}$ with $p_i, q_i \in A_L$ being homogeneous of the same degree. Considering $q = \prod_{i \in I} q_i$ we obtain the equality

$$(\alpha q)^d = \sum_{j=1}^{d} a_j q^j (\alpha q)^{d-j}.$$

The normality of A_L gives $\alpha q \in A_L$, proving that $\alpha = \alpha q/q \in Q(A_L)_0$.

To show (iii) we let $S \subset \mathbb{Z} \cdot m$ be the weight semigroup of the graded algebra A_L. Since L is contained in the weight cone $\sigma^\vee$, S is nonzero. Therefore if G is the subgroup generated by S then there exists $d \in \mathbb{Z}_{>0}$ such that $G = \mathbb{Z}\, d \cdot m$. Letting $u = \chi^{dm}$ we can write

$$A_L = \bigoplus_{s \geq 0} A_{sdm} u^s.$$

Thus for all homogeneous elements $a_1u^l, a_2u^l \in A_L$ of the same degree we have $\frac{a_1}{a_2} \in Q(A_L)_0^\star = \mathbf{k}^\star$ so that

$$A_L = \bigoplus_{s\in S'} kf_s\, u^s,$$

where $S' := \frac{1}{d}\, S$ and $f_s \in \mathbf{k}(C)^\star$. Let us fix homogeneous generators $f_{s_1}u^{s_1}, \ldots, f_{s_r}u^{s_r}$ of the G-graded algebra A_L. Consider $d' := \text{g.c.d}(s_1, \ldots, s_r)$. If $d' > 1$ then the inclusion $S \subset dd'\,\mathbb{Z} \cdot m$ yields a contradiction. So $d' = 1$ and there are some integers $l_1, \ldots, l_r$ such that $1 = \sum_{i=1}^r l_i s_i$. The element

$$\beta u = \prod_{i=1}^r (f_{s_i}u^{s_i})^{l_i}$$

verifies

$$\frac{(\beta u)^{s_1}}{f_{s_1}u^{s_1}} \in Q(A_L)_0^\star = \mathbf{k}^\star.$$

By normality of A_L, $\beta u \in A_L$ and so $A_L = \mathbf{k}[\beta u] = \mathbf{k}[\beta\chi^{dm}]$, establishing (iii). □

The following lemma is well known. For the main argument we refer the reader to [De, Section 3], [AH, 9.1].

Lemma 3.7. *Let D_1, D_2, D be $\mathbb{Q}$-divisors on C. Then the following hold.*

(i) *If D has positive degree then there exists $d \in \mathbb{Z}_{>0}$ such that the invertible sheaf $\mathcal{O}_C(\lfloor dD \rfloor)$ of $\mathcal{O}_C$-modules is very ample. Futhermore, the graded algebra*

$$B = \bigoplus_{l\geq 0} H^0(C, \mathcal{O}_C(\lfloor lD \rfloor))t^l,$$

where t is a variable over $\mathbf{k}(C)$, is finitely generated. The field of fractions of B is $\mathbf{k}(C)(t)$.

(ii) *Assume that for $i = 1, 2$ we have either $\deg D_i > 0$ or rD_i is principal for some $r \in \mathbb{Z}_{>0}$. If for any $s \in \mathbb{N}$ the inclusion*

$$H^0(C, \mathcal{O}_C(\lfloor sD_1 \rfloor)) \subset H^0(C, \mathcal{O}_C(\lfloor sD_2 \rfloor))$$

holds then we have $D_1 \leq D_2$.

In the next corollary, we keep the notation of Lemma 3.6. Using Demazure's Theorem for normal graded algebras, we show that each A_L admits a D.P.D. presentation given on the same smooth projective curve.

Corollary 3.8. *There exists a unique $\mathbb{Q}$-divisor D on C such that*

$$A_L = \bigoplus_{s\geq 0} H^0(C, \mathcal{O}_C(\lfloor sD \rfloor))\chi^{sm}$$

and the following hold.

(i) *If $Q(A_L)_0 = \mathbf{k}$ then $D = \frac{\operatorname{div} f}{d}$ for some $f \in K_0^\star$ and some $d \in \mathbb{Z}_{>0}$.*
(ii) *If $Q(A_L)_0 = K_0$ then $\deg D > 0$.*
(iii) *If $f_1\chi^{s_1 m}, \ldots, f_r\chi^{s_r m}$ are homogeneous generators of the algebra A_L then*

$$D = -\min_{1\leq i\leq r} \frac{\operatorname{div} f_i}{s_i}\,.$$

Proof. (i) Assume that $Q(A_L)_0 = \mathbf{k}$. By Lemma 3.6, $A_L = \mathbf{k}[\beta\,\chi^{dm}]$ for some $\beta \in K_0^\star$ and some $d \in \mathbb{Z}_{>0}$. Thus, we can take $D = \frac{\operatorname{div}\beta^{-1}}{d}$. The uniqueness in this case is easy. This gives assertion (i).

(ii) The field of rational functions of the normal variety $\operatorname{Proj} A_L$ is $K_0 = Q(A_L)_0$. Since $\operatorname{Proj} A_L$ is a smooth projective curve over $A_0 = \mathbf{k}$, we may identify its points with the places of K_0. Therefore the existence and the uniqueness of D follow from Demazure's Theorem (see [De, Theorem 3.5]). Futhermore $Q(A_L)_0 \neq \mathbf{k}$ implies that $\dim_{\mathbf{k}} A_{sm} \geq 2$, for some $s \in \mathbb{Z}_{>0}$. Hence by [St, 1.4.12] we obtain $\deg D > 0$.

The proof of (iii) follows from 3.7 and from the argument in [FZ, 3.9]. □

As a consequence of Corollary 3.8, again we can apply the formula of convex geometry of 2.11 to obtain the existence of the polyhedral divisor $\mathfrak{D}$ in the statement of 3.5 (i).

Proof of 3.5 (i). Let us adopt the notation introduced in 2.4 and 2.10. Let

$$f = (f_1\chi^{m_1}, \ldots, f_r\chi^{m_r})$$

be a system of homogeneous generators of A. Consider a half-line

$$L = \mathbb{Q}_{\geq 0} \cdot m \subset \sigma^\vee$$

with primitive vector $m \in M$. By Corollary 3.8

$$A_L = \bigoplus_{s\geq 0} H^0(C, \mathcal{O}_C(\lfloor sD_m \rfloor))\chi^{sm}$$

for a unique $\mathbb{Q}$-divisor D_m on C. By the proof of [AH, Lemma 4.1] the algebra A_L is generated by

$$\prod_{i=1}^{r} (f_i\chi^{m_i})^{s_i}, \text{ where } (s_1, \ldots s_r) \in \mathscr{H}_L^\star.$$

By Corollary 3.8 (iii) and Lemma 2.11 we have $\mathfrak{D}[f](m) = D_m$ and so $A = A[C, \mathfrak{D}[f]]$.

It remains to show that $\mathfrak{D} = \mathfrak{D}[f]$ is proper; the uniqueness of $\mathfrak{D}$ will be given by Lemma 3.7 (ii). Denote by $S \subset C$ the union of the supports of divisors $\operatorname{div} f_i$, for $i = 1, \ldots, r$. Let $v \in \deg \mathfrak{D}$. We can write

$$v = \sum_{z\in S} [\kappa(P_z) : \mathbf{k}] \cdot v_z$$

for some $v_z \in \Delta_z[f]$. Therefore for any i we have

$$\langle m_i, \sum_{z\in S} [\kappa(P_z) : \mathbf{k}] \cdot v_z \rangle \geq -\sum_{s\in S} [\kappa(P_z) : \mathbf{k}] \cdot \operatorname{ord}_z f_i = -\deg \operatorname{div} f_i = 0$$

and so $\deg \mathfrak{D} \subset \sigma$. If $\deg \mathfrak{D} = \sigma$ then one concludes that $\operatorname{Frac} A$ is different from $\operatorname{Frac} K_0[M]$, contradicting our assumption. Hence $\deg \mathfrak{D} \neq \sigma$. Let $m \in \sigma_M^\vee$ be such that $\deg \mathfrak{D}(m) = 0$. Then m belongs to the boundary of $\sigma^\vee$. Consider the half-line L generated by m. Applying Corollary 3.8 (i) for the algebra A_L, we deduce that a multiple of $\mathfrak{D}(m)$ is principal, proving that $\mathfrak{D}$ is proper. □

Proof of 3.5 (ii). Let us show that $A = A[C, \mathfrak{D}]$ and $K_0[M]$ have the same field of fractions. Let $L = \mathbb{Q}_{\geq 0} \cdot m$ be a half-line intersecting $\sigma^\vee$ with its relative interior and having m for primitive vector. Since $\deg \mathfrak{D}(m) > 0$ by Lemma 3.7 (i) we have

Frac $A_L = K_0(\chi^m)$, yielding our first claim. As a consequence, $\sigma^\vee$ is the weight cone of the M-graded algebra A. The proof of the normality is similar to that of 2.5 (i).

Let us show further that A is finitely generated. First we may consider a subdivision of $\sigma^\vee$ by regular strongly convex polyhedral cones $\omega_1, \ldots, \omega_s$ such that for any i we have $\omega_i \cap \operatorname{relint} \sigma^\vee \neq \emptyset$, ω_i is full dimensional, and $\mathfrak{D}$ is linear on ω_i. Fix $1 \leq i \leq s$ and a positive integer k. Let $(e_1, \ldots, e_n)$ be a basis of M generating the cone ω_i and such that $e_1 \in \operatorname{relint} \sigma^\vee$. By properness there exists $d \in \mathbb{Z}_{>0}$ such that every $\mathfrak{D}(de_j)$ is a globally generated integral divisor. Letting

$$A_{\omega_i,k} = \bigoplus_{(a_1,\ldots,a_n)\in\mathbb{Z}^n} H^0\left(C, \mathcal{O}\left(\left\lfloor \sum_{i=1}^n a_i k e_i \right\rfloor\right)\right) \chi^{\sum_i a_i k e_i}$$

we consider homogeneous elements $f_1\chi^{m_1}, \ldots, f_r\chi^{m_r} \in A_{\omega_i,d}$ obtained by taking generators of the space of global sections of every $\mathcal{O}(\mathfrak{D}(de_j))$ and homogeneous generators of the graded algebra

$$B = \bigoplus_{l\geq 0} H^0(C, \mathcal{O}_C(\mathfrak{D}(dle_1)))\chi^{lde_1},$$

see Lemma 3.7 (i). Using Theorem 3.5 (iii) the normalization of $\mathbf{k}[f_1\chi^{m_1}, \ldots, f_r\chi^{m_r}]$ is $A_{\omega_i,d}$ and so by Theorem 2 in [Bou, V3.2] the algebra $A_{\omega_i} = A_{\omega_i,1}$ is finitely generated. One concludes by taking the surjection $A_{\omega_1} \otimes \ldots \otimes A_{\omega_s} \to A$. □

In the next assertion, we study how the algebra associated to a polyhedral divisor over a smooth projective curve changes when we extend the scalars passing to the algebraic closure of the ground field $\mathbf{k}$. Assertions (i), (ii) are classical for the theory of algebraic function fields and the proofs are omitted.

Proposition 3.9. *Assume that* $\mathbf{k}$ *is a perfect field and let* $\bar{\mathbf{k}}$ *be an algebraic closure of* $\mathbf{k}$. *Denote by* $\mathfrak{S}_{\bar{\mathbf{k}}/\mathbf{k}}$ *the absolute Galois group of* $\mathbf{k}$. *For a smooth projective curve* C *over* $\mathbf{k}$ *associated to an algebraic function field* $K_0/\mathbf{k}$ *the following hold.*

(i) *The field* $\bar{K_0} = \bar{\mathbf{k}} \cdot K_0$ *is an algebraic function field over* $\bar{\mathbf{k}}$.

(ii) *The group* $\mathfrak{S}_{\bar{\mathbf{k}}/\mathbf{k}}$ *acts naturally on* $C\left(\bar{\mathbf{k}}\right) = \mathscr{R}_{\bar{\mathbf{k}}}\bar{K_0}$ *by*

$$g \cdot (\lambda f) = g(\lambda) f \quad \text{and} \quad g \star P = \{\, g \cdot F \mid F \in P\},$$

where $g \in \mathfrak{S}_{\bar{\mathbf{k}}/\mathbf{k}}$, $\lambda \in \bar{\mathbf{k}}$, $f \in K_0$, *and* $P \in C\left(\bar{\mathbf{k}}\right)$. *Any* $\mathfrak{S}_{\bar{\mathbf{k}}/\mathbf{k}}$*-orbit of* $C\left(\bar{\mathbf{k}}\right)$ *is a finite set and corresponds to a fiber of the surjective map*

$$S : C\left(\bar{\mathbf{k}}\right) \to C, \quad P \mapsto P \cap K_0.$$

In other words, there is a bijection between the set of $\mathfrak{S}_{\bar{\mathbf{k}}/\mathbf{k}}$*-orbits of* $C\left(\bar{\mathbf{k}}\right)$ *and the curve* C.

(iii) *If* $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ *is a proper* σ*-polyedral divisors over* C *then*

$$A[C, \mathfrak{D}] \otimes_{\mathbf{k}} \bar{\mathbf{k}} = A\left[C\left(\bar{\mathbf{k}}\right), \mathfrak{D}_{\bar{\mathbf{k}}}\right],$$

where $\mathfrak{D}_{\bar{\mathbf{k}}}$ *is the proper* σ*-polyhedral divisor over* $C\left(\bar{\mathbf{k}}\right)$ *defined by*

$$\mathfrak{D}_{\bar{\mathbf{k}}} = \sum_{z\in C} \Delta_z \cdot S^\star(z) \text{ with } S^\star(z) = \sum_{z'\in S^{-1}(z)} z'.$$

Proof. (iii) Given a Weil $\mathbb{Q}$-divisor D over C, by [St, Theorem 3.6.3] we obtain

$$H^0(C, \mathcal{O}_C(\lfloor D \rfloor)) \otimes_{\mathbf{k}} \bar{\mathbf{k}} = H^0(C\left(\bar{\mathbf{k}}\right), \mathcal{O}_{C\left(\bar{\mathbf{k}}\right)}(\lfloor S^\star D \rfloor)).$$

The proof of (iii) follows from the computation of $A[C, \mathfrak{D}] \otimes_{\mathbf{k}} \bar{\mathbf{k}}$. The properness of $\mathfrak{D}_{\bar{\mathbf{k}}}$ is given for instance by 3.5 (i). □

Remark 3.10. It is well known that every finitely generated extension of a perfect field is separable. However for non-perfect case, we may consider the inseparable algebraic function field of one variable

$$K_0 = \operatorname{Frac} \frac{\mathbf{k}[X,\, Y]}{(tX^2 + s + \, Y^2)}\,,$$

where $\mathbf{k} = \mathbb{F}_2(s, t)$ is the rational function field in two variables. Consequently, for any proper polyhedral divisor $\mathfrak{D}$ over $C = \mathscr{R}_{\mathbf{k}}\, K_0$, the ring $A[C, \mathfrak{D}] \otimes_{\mathbf{k}} \bar{\mathbf{k}}$ contains a nonzero nilpotent element.

4. Split affine $\mathbb{T}$-varieties of complexity one

As an application of the results in the previous sections, we can give now a combinatorial description of split affine $\mathbb{T}$-varieties of complexity one over any field $\mathbf{k}$.

4.1. Let $\mathbb{T}$ be a split algebraic torus over $\mathbf{k}$. Denote by M and N its dual lattices of characters and of one parameter subgroups. Let $X = \operatorname{Spec} A$ be an affine variety over $\mathbf{k}$. Assume that $\mathbb{T}$ acts on X. Then the associated morphism $A \to A \otimes_{\mathbf{k}} \mathbf{k}[\mathbb{T}]$ endows A with an M-grading. Conversely, an M-grading on the algebra A yields naturally a $\mathbb{T}$-action on X. Consider the subextension $Q(A)_0 \subset \mathbf{k}(X)$ of $\mathbf{k}$ generated by the quotients a/b, where $a, b \in A$ are homogeneous of the same degree. The *complexity* of the $\mathbb{T}$-action is the transcendence degree of $Q(A)_0$ over $\mathbf{k}$.

We say that X is a $\mathbb{T}$-*variety* if X is normal and if the $\mathbb{T}$-action on X is effective[2]. This is equivalent to say that A is normal and the set of its weights generates M.

Definition 4.2. Let C be a smooth curve over $\mathbf{k}$ and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. A σ-polyhedral divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is called *proper* if it satifies one of the following statement.

(i) C is affine. In particular, $\mathfrak{D}$ is a polyhedral divisor over the Dedekind ring $A_0 = \mathbf{k}[C]$.
(ii) C is projective and $\mathfrak{D}$ is a proper polyhedral divisor in the sense of 3.4.

We denote by $A[C, \mathfrak{D}]$ the associated M-graded algebra.

Combining 2.5 and 3.5 one can describe a split affine $\mathbb{T}$-variety of complexity one by a proper polyhedral divisor.

Theorem 4.3.

(i) *To any split affine $\mathbb{T}$-variety $X = \operatorname{Spec} A$ over $\mathbf{k}$ of complexity one there is some proper σ-polyhedral divisor $\mathfrak{D}$ on a smooth curve C over $\mathbf{k}$ such that $A \simeq A[C, \mathfrak{D}]$ as M-graded algebras.*

[2]Seeing $\mathbb{T}$ as a representable group functor, this means that the kernel of the natural transformation of group functors $\mathbb{T} \to \operatorname{Aut} X$ is trivial.

(ii) *Conversely, if $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth curve C then $X = \operatorname{Spec} A$, where $A = A[C, \mathfrak{D}]$, defines a split affine $\mathbb{T}$-variety of complexity one.*

Proof. (i) Let $\sigma \subset N_{\mathbb{Q}}$ be the dual of the weight cone of A. Remark that we can choose some weight vectors $\chi^m \in \operatorname{Frac} A$ such that $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ and such that we have an embedding

$$A \subset \bigoplus_{m \in M} Q(A)_0 \, \chi^m = Q(A)_0[M],$$

making A an M-graded subalgebra. Futhermore A and $Q(A)_0[M]$ have the same field of fractions. The graded piece A_0 is the algebra of $\mathbb{T}$-invariants. Denote by K_0 the field of fractions of A_0. Assume that $A_0 \neq \mathbf{k}$. Then we have $K_0 = Q(A)_0$. Indeed, by assumption every algebraic element of K_0 over $\mathbf{k}$ belongs to $\mathbf{k}$. Therefore the trancendence degree of $K_0/\mathbf{k}$ is equal to 1 so that $Q(A)_0/K_0$ is algebraic. Using the normality of A_0 one concludes that $K_0 = Q(A)_0$. Remark further that the ring A_0 is a Dedekind Domain. By Theorem 2.5 (ii) we obtain $A = A[C, \mathfrak{D}]$ for some σ-polyhedral divisor $\mathfrak{D}$ over A_0. If $A_0 = \mathbf{k}$ then one concludes by Theorem 3.5 (i). Assertion (ii) follows immediately from 2.5 (i) and 3.5 (ii). □

4.4. By a *principal σ-polyhedral divisor* $\mathfrak{F}$ over C we mean a pair $(\varphi, \mathfrak{D})$ with a morphism of semigroup $\varphi : \sigma_M^\vee \to \mathbf{k}(C)^\star$ and a σ-polyhedral divisor $\mathfrak{D}$ over C such that for any $m \in \sigma_M^\vee$ we have

$$\mathfrak{D}(m) = \operatorname{div}_C \mathfrak{F}(m).$$

Actually starting from $\mathfrak{F}$ and choosing a finite generating set of $\sigma_M^\vee$ one can easily construct $\mathfrak{D}$ satisfying the equalities as before. Usually we write $\mathfrak{F}$ and $\mathfrak{D}$ by the same letter.

The following result provides a description of equivariant isomorphisms between two affine $\mathbb{T}$-varieties of complexity one over the same base curve. See [AH, Section 8, 9] for higher complexity when the ground field is algebraically closed of characteric zero.

Proposition 4.5. *Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two proper σ-polyhedral divisors over a smooth curve C. Then $A[C, \mathfrak{D}]$ and $A[C, \mathfrak{D}']$ are equivariantly isomorphic if and only if the following assertion holds. There exist a principal σ-polyhedral divisor $\mathfrak{F}$, a linear automorphism F of $M_{\mathbb{Q}}$ preserving $\sigma_M^\vee$ and, $\phi \in \operatorname{Aut} C$ such that for any $m \in \sigma_M^\vee$ we have*

$$\phi^\star \mathfrak{D}(m) = \mathfrak{D}'(F(m)) + \mathfrak{F}(m).$$

Proof. Let $K_0 = \mathbf{k}(C)$. Let $\psi : A[C, \mathfrak{D}] \to A[C, \mathfrak{D}']$ be an isomorphism of M-graded algebras. Since each homogeneous element is sent into a homogeneous element, the morphism ψ extends to an automorphism of $K_0[M]$. We have also automorphisms of $\mathbf{k}[C]$ and $\mathbf{k}(C)$ coming from an element $\phi \in \operatorname{Aut} C$. Composing by $(\phi^\star)^{-1}$ we may suppose that ψ is the identity map on K_0. Futhermore there exists a linear automorphism F of $M_{\mathbb{Q}}$ preseving $\sigma_M^\vee$ such that for any $m \in M$ we have $\psi(\chi^m) = f_m \chi^{F(m)}$ for some $f_m \in K_0^\star$. Again we may suppose that F is the identity. One observes that $m \mapsto f_m$, $\sigma_M^\vee \to K_0^\star$ defined a principal σ-polyhedral divisor $\mathfrak{F}$. Using Theorem 1.4 and Lemma 3.7 (ii) for any $m \in \sigma_M^\vee$ we have the equality $\mathfrak{D}(m) = \mathfrak{D}'(m) + \mathfrak{F}(m)$. The converse is straightforward and left to the reader. □

5. Non-split case via Galois descent

In view of the result of the above section, we provide a combinatorial description of affine normal varieties endowed with a (non-nescessary split) torus action of complexity one (see 5.4 for a precise definition). This can be compared with well known descriptions for toric and spherical varieties, see [Bry], [Vo], [ELST], [Hu].

5.1. For a field extension $F/\mathbf{k}$ and an algebraic scheme X over $\mathbf{k}$ we let

$$X(F) = X \times_{\operatorname{Spec}\mathbf{k}} \operatorname{Spec} F.$$

This is an algebraic scheme over F. An algebraic *torus* of dimension n is an algebraic group $\mathbf{G}$ over $\mathbf{k}$ such that there exists a finite Galois extension $E/\mathbf{k}$ yielding an isomorphism of algebraic groups $\mathbf{G}(E) \simeq \mathbb{G}_m^n(E)$ $(\star)$, where $\mathbb{G}_m$ is the multiplicative group scheme over $\mathbf{k}$. We say that the torus $\mathbf{G}$ *splits in the extension* $E/\mathbf{k}$ if we have an isomorphism similar to $(\star)$. For more details concerning the theory of non-split reductive group the reader may consult [BoTi].

Below $\mathbf{G}$ is a torus over $\mathbf{k}$ that splits in a finite Galois extension $E/\mathbf{k}$. Denote by $\mathfrak{S}_{E/\mathbf{k}}$ the Galois group of $E/\mathbf{k}$. Consider also M and N the dual lattices of characters and of one parameter subgroups of the split torus $\mathbf{G}(E)$. Notice that in the sequel most of our varieties are defined over the field E. We start by precising the following classical notion.

Definition 5.2.

(i) A $\mathfrak{S}_{E/\mathbf{k}}$-action on a variety V over E is called *semi-linear* if $\mathfrak{S}_{E/\mathbf{k}}$ acts by scheme automorphisms over $\mathbf{k}$ and if for any $g \in \mathfrak{S}_{E/\mathbf{k}}$ the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ g\ } & V \\ \downarrow & & \downarrow \\ \operatorname{Spec} E & \xrightarrow[\ g\ ]{} & \operatorname{Spec} E \end{array}$$

is commutative.

(ii) Let B be an algebra over E. A *semi-linear* $\mathfrak{S}_{E/\mathbf{k}}$-action on B is an action by automorphisms of algebras over $\mathbf{k}$ such that for all $a \in B$, $\lambda \in E$, and $g \in \mathfrak{S}_{E/\mathbf{k}}$

$$g \cdot (\lambda a) = g(\lambda) g \cdot a.$$

If V is affine then having a semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on V is equivalent to having a semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on the algebra $E[V]$.

Next, we recall a well known description of algebraic tori related to finite groups actions on lattices.

5.3. The Galois group $\mathfrak{S}_{E/\mathbf{k}}$ acts naturally on the torus

$$\mathbf{G}(E) = \mathbf{G} \times_{\operatorname{Spec}\mathbf{k}} \operatorname{Spec} E$$

by the second factor. The corresponding action on $E[M]$ is determinated by a linear $\mathfrak{S}_{E/\mathbf{k}}$-action on M (see e.g. [ELST, Proposition 2.5], [Vo, Section 1]) permuting the Laurent monomials.

Conversely, given a linear $\mathfrak{S}_{E/\mathbf{k}}$-action on M we have a semi-linear action on $E[M]$ defined by

$$g \cdot (\lambda\chi^m) = g(\lambda)\chi^{g\cdot m},$$

where $g \in \mathfrak{S}_{E/\mathbf{k}}$, $\lambda \in E$ and $m \in M$, respects the Hopf algebra structure. As a consequence of the Speiser's Lemma, we obtain a torus $\mathbf{G}$ over $\mathbf{k}$ that splits in $E/\mathbf{k}$. In addition, the semi-linear action that we have built on $\mathbf{G}(E) = \operatorname{Spec} E[M]$ is exacty the natural semi-linear action on the second factor.

The following definition introduces the category of $\mathbf{G}$-varieties.

Definition 5.4. A $\mathbf{G}$-*variety of complexity* d (splitting in $E/\mathbf{k}$) is a normal variety over $\mathbf{k}$ with a $\mathbf{G}$-action and such that $X(E)$ is a $\mathbf{G}(E)$-variety of complexity d in sense of Section 4. A $\mathbf{G}$-*morphism* between $\mathbf{G}$-varieties X, Y over $\mathbf{k}$ is a morphism $f : X \to Y$ of varieties over $\mathbf{k}$ such that

$$\begin{array}{ccc} \mathbf{G} \times X & \xrightarrow{\mathrm{id}\times f} & \mathbf{G} \times Y \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

is commutative.

An important class of semi-linear actions is provided by those respecting a split torus action. The $\mathfrak{S}_{E/\mathbf{k}}$-action on $\mathbf{G}(E)$ is given as in paragraph 5.3.

Definition 5.5.

(i) Let B be an M-graded algebra over E. A semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on B is called *homogeneous* whether it sends homogeneous elements into homogeneous elements.

(ii) A semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on a $\mathbf{G}(E)$-variety V *respects the* $\mathbf{G}(E)$-*action* if the following diagram

$$\begin{array}{ccc} \mathbf{G}(E) \times V & \xrightarrow{g\times g} & \mathbf{G}(E) \times V \\ \downarrow & & \downarrow \\ V & \xrightarrow[g]{} & V \end{array}$$

commutes, where g runs $\mathfrak{S}_{E/\mathbf{k}}$.

With the assumption that V is affine, a semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on the variety V respecting the $\mathbf{G}(E)$-action corresponds to a homogeneous semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on the algebra $E[V]$.

The following result is classically stated for the category of quasi-projective varieties (see the proof of [Hu2, 1.10]). In the setting of affine $\mathbf{G}$-varieties we include a short argument.

Lemma 5.6. *Let V be an affine $\mathbf{G}(E)$-variety of complexity d over E with a semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action. Then the quotient $X = V/\mathfrak{S}_{E/\mathbf{k}}$ is an affine $\mathbf{G}$- variety of complexity d. We have a natural isomorphism of $\mathbf{G}(E)$-varieties $X(E) \simeq V$ respecting the $\mathfrak{S}_{E/\mathbf{k}}$-actions.*

Proof. It is known that $R = B^{\mathfrak{S}_{E/\mathbf{k}}}$ is finitely generated. Let us show that R is normal. Letting L be the field of fractions of R and considering $f \in L$ an integral element over R, by normality of B, we have $f \in B \cap L = R$. This proves the normality of R. Using the above definition, the variety X is endowed with a $\mathbf{G}$-action. The rest of the proof follows from Speiser's Lemma. □

Fixing an affine $\mathbf{G}$-variety X of complexity d over E, an $E/\mathbf{k}$-*form* of X is an affine $\mathbf{G}$-variety Y over $\mathbf{k}$ such that we have a $\mathbf{G}(E)$-isomorphism $X(E) \simeq Y(E)$. Our aim is to give a combinatorial description of $E/\mathbf{k}$-forms of X. Let us recall first in this context some notion of non-abelian Galois cohomology (see e.g. [ELST, Section 2.5] for the category of varieties).

5.7. Let Y, Y' be $E/\mathbf{k}$-forms of the fixed affine $\mathbf{G}$-variety X. The Galois group $\mathfrak{S}_{E/\mathbf{k}}$ acts on the set of $\mathbf{G}(E)$-isomorphisms between $Y(E)$ and $Y'(E)$. Consequently, it acts also by group automorphisms on the group of $\mathbf{G}(E)$-automorphims $\mathrm{Aut}_{\mathbf{G}(E)}(X(E))$ of $X(E)$. More precisely, recall that for any $g \in \mathfrak{S}_{E/\mathbf{k}}$ and any $\mathbf{G}(E)$-isomorphism $\varphi : Y(E) \to Y'(E)$ one defines $g(\varphi)$ by the following commutative diagram

$$\begin{array}{ccc} Y(E) & \xrightarrow{g(\varphi)} & Y'(E) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ Y(E) & \xrightarrow[\varphi]{} & Y'(E) \end{array}.$$

Note that this $\mathfrak{S}_{E/\mathbf{k}}$-action depends on the data of the $E/\mathbf{k}$-forms Y, Y'. Now given a $\mathbf{G}(E)$-isomorphism $\psi : X(E) \to Y(E)$ the map

$$a : \mathfrak{S}_{E/\mathbf{k}} \to \mathrm{Aut}_{\mathbf{G}(E)}(X(E)), \;\; g \mapsto a_g = \psi^{-1} \circ g(\psi)$$

is a 1-*cocycle*. This means that for all $g, g' \in \mathfrak{S}_{E/\mathbf{k}}$ we have

$$a_g \circ g(a_{g'}) = \psi^{-1} \circ g(\psi) \circ g\left(\psi^{-1} \circ g'(\psi)\right) = a_{gg'}.$$

Let $\phi : Y \to Y'$ be a $\mathbf{G}$-isomorphism and take a $\mathbf{G}(E)$-isomorphism $\varphi : X(E) \to Y'(E)$ giving a 1-cocycle b as above. The diagram

$$\begin{array}{ccc} X(E) & \xrightarrow{\psi} & Y(E) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \phi' = \phi \times \mathrm{id}} \\ X(E) & \xrightarrow[\varphi]{} & Y'(E) \end{array}$$

is commutative, where $\alpha \in \mathrm{Aut}_{\mathbf{G}(E)}(X(E))$ and ϕ' is the extension ϕ. Since for any $g \in \mathfrak{S}_{E/\mathbf{k}}$ we have $g(\phi') = \phi'$, it follows that

$$b_g = \alpha \circ a_g \circ g\left(\alpha^{-1}\right).$$

In this case, we say that the cocycles a and b are *cohomologous*. We obtain as well a map Φ between the pointed set of isomorphism classes of $E/\mathbf{k}$-forms of X and the pointed set

$$H^1(E/\mathbf{k}, \mathrm{Aut}_{\mathbf{G}(E)}(X(E)))$$

of cohomology classes of 1-cocycles $a : \mathfrak{S}_{E/\mathbf{k}} \to \mathrm{Aut}_{\mathbf{G}(E)}(X(E))$.

Conversely, starting with a cocycle a the map

$$\mathfrak{S}_{E/\mathbf{k}} \to \mathrm{Aut}_{\mathbf{G}(E)}(X(E)), \;\; g \mapsto a_g \circ g$$

is a semi-linear action on $X(E)$ respecting the $\mathbf{G}(E)$-action. According to Lemma 5.6 one can associate an $E/\mathbf{k}$-form W of X by taking the quotient $X(E)/\mathfrak{S}_{E/\mathbf{k}}$. Changing a by a cohomologous 1-cocycle gives an $E/\mathbf{k}$-form of X isomorphic to W. Thus we deduce that the map Φ is bijective.

Moreover, let γ be a semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on $X(E)$. Remark that

$$\begin{array}{ccc} X(E) & \xrightarrow{\gamma(g')} & X(E) \\ {\scriptstyle g^{-1}}\downarrow & & \downarrow{\scriptstyle g^{-1}} \\ X(E) & \xrightarrow[g(\gamma(g'))]{} & X(E) \end{array}$$

commutes for all $g, g' \in \mathfrak{S}_{E/\mathbf{k}}$. Hence the equality $a_g = \gamma(g) \circ g^{-1}$ defines a 1-cocycle a. A straightforward verification shows that $H^1(E/\mathbf{k}, \mathrm{Aut}_{\mathbf{G}(E)}(X(E)))$ is also in bijection with the pointed set of conjugacy classes of semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-actions on $X(E)$ respecting the $\mathbf{G}(E)$-action.

As explained in the above paragraph, classifying the pointed set of $E/\mathbf{k}$-forms of X is equivalent to classifying all possible semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-actions on $X(E)$. Thus generalizing the notion of proper polyhedral divisors, we consider the combinatorial counterpart of this classification.

Definition 5.8. Let C be a smooth curve over E and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex cone. A $\mathfrak{S}_{E/\mathbf{k}}$-*invariant* σ-*polyhedral divisor* over C is a 4-uplet $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ verifying the following conditions.

(i) $\mathfrak{D}$ (resp. $\mathfrak{F}$) is a proper (resp. principal) σ-polyhedral divisor over C.

(ii) The curve C is endowed with a semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action

$$\mathfrak{S}_{E/\mathbf{k}} \times C \to C, \;\; (g, z) \mapsto g \star z.$$

This yields naturally an action on the space of Weil $\mathbb{Q}$-divisors over C. More precisely, given $g \in \mathfrak{S}_{E/\mathbf{k}}$ and a $\mathbb{Q}$-divisor D over C we let

$$g \star D = \sum_{z \in C} a_{g^{-1} \star z} \cdot z, \;\; \text{where} \;\; D = \sum_{z \in C} a_z \cdot z.$$

(iii) The lattice M is endowed with a linear $\mathfrak{S}_{E/\mathbf{k}}$-action

$$\mathfrak{S}_{E/\mathbf{k}} \times M \to M, \;\; (g, m) \mapsto g \cdot m$$

preserving the subset $\sigma_M^\vee$.

The 4-uplet $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ satisfies in addition the equality

$$g \star (\mathfrak{D}(m) + \mathfrak{F}(m)) = \mathfrak{D}(g \cdot m) + \mathfrak{F}(g \cdot m),$$

where $m \in \sigma_M^\vee$ and $g \in \mathfrak{S}_{E/\mathbf{k}}$.

The following result is a direct consequence of Hilbert's Theorem 90. For the convenience of the reader we give a short argument.

Lemma 5.9. *Let E_0/K_0 be a finite Galois extension with Galois group $\mathfrak{S}_{E_0/K_0}$. Assume that $\mathfrak{S}_{E_0/K_0}$ acts linearly on M. For any $g \in \mathfrak{S}_{E_0/K_0}$ consider a morphism of groups $f_g : M \to E_0^\star$ satisfying the equalities*

$$f_{gh}(m) = g\,(f_h(m))\, f_g(h \cdot m),$$

where $g, h \in \mathfrak{S}_{E_0/K_0}$ and $m \in M$. Then there exists a morphism of groups $b : M \to E_0^\star$ such that for all $g \in \mathfrak{S}_{E_0/K_0}$, $m \in M$ we have

$$f_g(m) = b(g \cdot m) g(b(m))^{-1}\,.$$

Proof. The opposite of $\mathfrak{S}_{E_0/K_0}$ is the group H with underlying set $\mathfrak{S}_{E_0/K_0}$ and the multiplication law defined by $g \star h = hg$, where $g, h \in H$. For $g \in H$ we denote by $a_g : M \to E_0^\star$ the morphism of groups defined by

$$a_g(m) = g^{-1}(f_g(m)),$$

where $m \in M$. We can also define an H-action by group automorphisms on the split torus

$$T = \mathrm{Hom}(M, E_0^\star)$$

over E_0 by letting $(g \cdot \alpha)(m) = g^{-1}(\alpha(g \cdot m))$, where $\alpha \in T$, $g \in H$, and $m \in M$. Considering $g, h \in H$ we obtain

$$a_{h\star g}(m) = (gh)^{-1}(f_{gh}(m)) = (gh)^{-1}(g(f_h(m)) f_g(h \cdot m)) = a_h(m)(h \cdot a_g)(m)$$

so that $g \mapsto a_g$ is a 1-cocycle. By the Hilbert Theorem 90 one has

$$H^1(H, T) \simeq H^1(E_0/K_0, T) = 1.$$

Hence there exists $b \in T$ such that for any $g \in H$ we have $a_g = b \cdot (g \cdot b^{-1})$. These latter equalities provide our result. □

The next theorem yields a classification of affine $\mathbf{G}$-varieties of complexity one in terms of invariant polyhedral divisors.

Theorem 5.10. *Let $\mathbf{G}$ be a torus over $\mathbf{k}$ splitting in a finite Galois extension $E/\mathbf{k}$. Denote by $\mathfrak{S}_{E/\mathbf{k}}$ the Galois group of $E/\mathbf{k}$.*

(i) *Every affine $\mathbf{G}$-variety of complexity one splitting in $E/\mathbf{k}$ is described by a $\mathfrak{S}_{E/\mathbf{k}}$-invariant proper polyhedral divisor over a smooth curve.*

(ii) *Conversely, let C be a smooth curve over E. For a $\mathfrak{S}_{E/\mathbf{k}}$-invariant proper σ-polyhedral divisor $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ over C one can endow the algebra $A[C, \mathfrak{D}]$ with a homogeneous semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action and associate an affine $\mathbf{G}$-variety of complexity one over $\mathbf{k}$ splitting in $E/\mathbf{k}$ by taking $X = \mathrm{Spec}\, A$, where*

$$A = A[C, \mathfrak{D}]^{\mathfrak{S}_{E/\mathbf{k}}}.$$

Proof. (i) Let X be a $\mathbf{G}$-variety of complexity one over $\mathbf{k}$. According to Theorem 4.3 we may suppose that $B = A[C, \mathfrak{D}]$ is the coordinate ring of $X(E)$ for some proper σ-polyhedral divisor $\mathfrak{D}$ over a smooth curve C. The algebra B is endowed with a homogeneous semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action. Let $E_0 = E(C)$. Extending this action on $E_0[M]$ we remark that E_0 and $E[C]$ are preserved. We obtain a semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on C. If C is projective then one defines the $\mathfrak{S}_{E/\mathbf{k}}$-action on C by the following way ; given a place $P \subset E_0$ we let

$$g \star P = \{g \star f \mid f \in P\}.$$

In the case where C is arbitrary the Speiser Lemma gives the equality

$$E_0 = E \cdot K_0, \text{ where } K_0 = E_0^{\mathfrak{S}_{E/\mathbf{k}}}.$$

The finite extension E_0/K_0 is Galois. We have a natural identification $\mathfrak{S}_{E/\mathbf{k}} \simeq \mathfrak{S}_{E_0/K_0}$ with the Galois group of E_0/K_0. For all $m \in M$, $g \in \mathfrak{S}_{E/\mathbf{k}}$ we have

$$g \cdot (f\chi^m) = g(f) f_g(m) \chi^{\Gamma(g,m)} \tag{1}$$

for some some element f_g in the split torus $T = \operatorname{Hom}(M, E_0^\star)$ and some $\Gamma(g,m) \in M$. We observe that Γ is a linear action on M. Denote by $g \cdot m$ the lattice vector $\Gamma(g,m)$. For all $g, h \in \mathfrak{S}_{E/\mathbf{k}}$ we have

$$f_{gh}(m)\chi^m = gh \cdot \chi^m = g \cdot (h \cdot \chi^m) = g(f_h(m)) f_g(h \cdot m) \chi^{gh \cdot m}.$$

Using Lemma 5.9 there exists $b \in T$ such that for all $m \in M$, $g \in \mathfrak{S}_{E/\mathbf{k}}$ we have $f_g(m) = b(g \cdot m)/g(b(m))$. We let $\mathfrak{F}$ be the principal σ-polyhedral divisor associated to b.

It remains to show the equalities

$$g \star (\mathfrak{D}(m) + \mathfrak{F}(m)) = \mathfrak{D}(g \cdot m) + \mathfrak{F}(g \cdot m), \ \ \forall m \in \sigma_M^\vee, \forall g \in \mathfrak{S}_{E/\mathbf{k}}. \tag{2}$$

First of all, we remark that if $f \in E_0^\star$ and $g \in \mathfrak{S}_{E/\mathbf{k}}$ then $g \star \operatorname{div} f = \operatorname{div} g(f)$. Let $f\chi^m \in B$ be homogeneous of degree m. The transformation of $f\chi^m$ by g is an element of B of degree $g \cdot m$ and so

$$\operatorname{div} g(f) f_g(m) + \mathfrak{D}(g \cdot m) \geq 0.$$

This implies that

$$g \star (-\operatorname{div} f + \mathfrak{F}(m)) \leq \mathfrak{F}(g \cdot m) + \mathfrak{D}(g \cdot m).$$

According to Corollary 1.7 and Corollary 3.8 (iii) we obtain

$$g \star (\mathfrak{D}(m) + \mathfrak{F}(m)) \leq \mathfrak{D}(g \cdot m) + \mathfrak{F}(g \cdot m).$$

The converse inequality uses a similar argument. One concludes that $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ is an invariant polyhedral divisor.

(ii) Again if $b \in T$ corresponds to $\mathfrak{F}$ then by virtue of (2) one defines a homogeneous semi-linear $\mathfrak{S}_{E/\mathbf{k}}$-action on $A[C, \mathfrak{D}]$ by letting $f_g(m) = b(g \cdot m)/g(b(m))$ and by the equality (1). The rest of the proof is a consequence of Lemma 5.6. □

Let us provide an elementary example.

Exemple 5.11. Consider the σ-polyhedral divisor $\mathfrak{D}$ over $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$ defined by

$$((1,0) + \sigma) \cdot \zeta + ((0,1) + \sigma) \cdot (-\zeta) + ((1,-1) + \sigma) \cdot 0,$$

where σ is the first quadrant $\mathbb{Q}^2_{\geq 0}$ and $\zeta = \sqrt{-1}$. We endow $\mathfrak{D}$ with a structure of $\mathfrak{S}_{\mathbb{C}/\mathbb{R}}$-invariant polyhedral divisors by considering first $\mathfrak{F}$ given by the morphism $(m_1, m_2) \mapsto t^{m_2 - m_1}$. We have a $\mathfrak{S}_{\mathbb{C}/\mathbb{R}}$-action

$$\mathfrak{S}_{\mathbb{C}/\mathbb{R}} \to \mathrm{GL}_2(\mathbb{Z}), \ \ g \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on the lattice $\mathbb{Z}^2$, where g is the generator of $\mathfrak{S}_{\mathbb{C}/\mathbb{R}}$. The algebra $\mathbb{C}[t]$ has the natural complex conjugaison action $\star$ of $\mathfrak{S}_{\mathbb{C}/\mathbb{R}}$. A direct computation shows that

$$A = \mathbb{C}\left[t, \frac{1}{t(t-\zeta)}\chi^{(1,0)}, \frac{t}{t+\zeta}\chi^{(0,1)}\right]$$

and so $X = \operatorname{Spec} A$ is the affine space $\mathbb{A}^3_{\mathbb{C}}$. More concretely, the $\mathfrak{S}_{\mathbb{C}/\mathbb{R}}$-action on the algebra A is obtained by

$$g \cdot (f(t)\chi^{(m_1,m_2)}) = f(\bar{t})t^{2(m_1-m_2)}\chi^{(m_2,m_1)}.$$

Letting $x = t^{-1}(1-\zeta)^{-1}\chi^{(1,0)}$ and $y = t(1+\zeta)^{-1}\chi^{(0,1)}$ we observe that $A^{\mathfrak{S}_{\mathbb{C}/\mathbb{R}}} = \mathbb{R}[t, x+y, \zeta(x-y)]$. Hence $X/\mathfrak{S}_{\mathbb{C}/\mathbb{R}} \simeq \mathbb{A}^3_{\mathbb{R}}$.

Next we describe the pointed set of $E/\mathbf{k}$-forms of an affine $\mathbf{G}$-varieties of complexity one in terms of polyhedral divisors.

Definition 5.12. The invariant σ-polyhedral divisors $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ and $(\mathfrak{D}, \mathfrak{F}', \star', \cdot')$ over C are *conjugated* if they verify the following. There exist $\varphi \in \operatorname{Aut}(C)$, a principal σ-polyhedral divisor $\mathfrak{E}$ over C, and a linear automorphism $F \in \operatorname{Aut}(M)$ giving an automorphism of the E-algebra $A[C, \mathfrak{D}]$ (see 4.5) such that for any $g \in \mathfrak{S}_{E/\mathbf{k}}$ the diagrams

$$\begin{array}{ccc} C & \xrightarrow{g\star} & C \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ C & \xrightarrow[g\star']{} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{g\cdot} & M \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ M & \xrightarrow[g\cdot']{} & M \end{array}$$

commute and for any $m \in M$ we have

$$\frac{\mathfrak{F}(g\cdot m)}{g\star\mathfrak{F}(m)} = \frac{g\star(\varphi^{-1})^\star\mathfrak{E}(m)\cdot(\varphi^{-1})^\star\mathfrak{F}'(F(g\cdot m))}{\mathfrak{E}(g\cdot m)\cdot g\star(\varphi^{-1})^\star\mathfrak{F}'(F(m))}\,.$$

Consider X an affine $\mathbf{G}$-variety of complexity one described by the invariant polyhedral divisor $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$. We denote by $\mathscr{E}_X(E/\mathbf{k})$ the pointed set of conjugacy classes of $\mathfrak{S}_{E/\mathbf{k}}$-invariant σ-polyhedral divisors over C of the form $(\mathfrak{D}, \mathfrak{F}', \star', \cdot')$.

As a direct consequence of the discusion of 5.7 we obtain the following.

Corollary 5.13. *Let C be a smooth curve over E. Given an affine $\mathbf{G}$-variety X of complexity one associated to a $\mathfrak{S}_{E/\mathbf{k}}$-invariant polyhedral divisor $(\mathfrak{D}, \mathfrak{F}, \star, \cdot)$ over C, we have a bijection of pointed sets*

$$\mathscr{E}_X(E/\mathbf{k}) \simeq H^1(E/\mathbf{k}, \operatorname{Aut}_{\mathbf{G}(E)}(X(E))).$$

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