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# POLYHEDRAL DIVISORS, DEDEKIND DOMAINS AND ALGEBRAIC FUNCTION FIELDS

KEVIN LANGLOIS

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Abstract. In this paper, we show that the presentation of affine  $\mathbb{T}$ -varieties of complexity one in terms of polyhedral divisors holds over an arbitrary field. We describe also a class of multigraded algebras over Dedekind domains.

## INTRODUCTION

There exist several descriptions of  $\mathbb{T}$ -varieties, see for instance [KKMS], [De], [FZ], [AH], [Ti], [AHS]. In this paper, we show that the presentation of affine  $\mathbb{T}$ -varieties of complexity one in terms of polyhedral divisors holds over an arbitrary field. We describe also a class of multigraded algebras over Dedekind domains, see 2.6, 3.12.

Before reformulating our result, let us recall some notation. In the following all algebraic structures are defined over a field  $\mathbf{k}$ .

A split algebraic torus  $\mathbb{T}$  of dimension  $n$  is an algebraic group scheme isomorphic to  $\mathbb{G}_m^n$ . Let  $M$  be the character lattice of a torus  $\mathbb{T}$ . Then defining a  $\mathbb{T}$ -action on an affine variety  $X$  is equivalent to having an  $M$ -grading on the algebra  $\mathbf{k}[X]$  [SGA III, 4.7.3]. An affine  $\mathbb{T}$ -variety of complexity one is a normal variety endowed with an effective  $\mathbb{T}$ -action (i.e the weights of  $A = \mathbf{k}[X]$  generate  $M$ ) such that the field

$$K_0 = \left\{ \frac{a}{b}, a, b \in A \text{ homogeneous of same degree} \right\} \cup \{0\}$$

has transcendence degree over  $\mathbf{k}$  equal to one.

In order to describe affine  $\mathbb{T}$ -varieties of complexity one, we consider a smooth curve  $C$ . A point  $z \in C$  is assumed to be closed. Denote by  $N$  the one parameter lattice of  $\mathbb{T}$ . Fix a strongly convex polyhedral cone  $\sigma \subset N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$ . We define as in [AH] a divisor  $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$  with  $\sigma$ -polyhedral coefficients in  $N_{\mathbb{Q}}$ . The degree of  $\mathfrak{D}$  is

$$\deg \mathfrak{D} = \sum_{z \in C} [\kappa_z : \mathbf{k}] \cdot \Delta_z,$$

where  $\kappa_z$  is the residual field at the point  $z$ . The properness of  $\mathfrak{D}$  and the corresponding  $M$ -graded algebra  $A[C, \mathfrak{D}]$  are given by the usual definitions, see [AH, 2.12]. Using some ideas developed in [La], we obtain the following result.

**Theorem 0.1.** *Let  $X$  be an affine  $\mathbb{T}$ -variety of complexity one over a field  $\mathbf{k}$ . If  $\sigma^\vee \subset M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$  is the weight cone of  $A = \mathbf{k}[X]$  then there exists a proper  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  on a smooth curve  $C$  such that  $A \simeq A[C, \mathfrak{D}]$  as multigraded algebras.*

Let us give a brief summary of the contents of each section. In the first section, we extend the D.P.D. presentation to the context of Dedekind domains. This fact has been mentioned by Flenner and Zaidenberg, see the introduction of [FZ]. In the second section, we study a class of multigraded algebras over Dedekind domains. In the last section, we classify affine  $\mathbb{T}$ -varieties of complexity one over  $\mathbf{k}$ .

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## 1. GRADED ALGEBRAS AND DEDEKIND DOMAINS

In this section we generalize the Dolgachev-Pinkham-Demazure (D.P.D.) presentation to the context of Dedekind domains. See [FZ, §3] for the case of normal affine surfaces with parabolic  $\mathbb{C}^*$ -actions.

**1.1.** An integral domain  $A_0$  is called a *Dedekind domain* (or Dedekind ring) if it is not a field and if it satisfies the following conditions.

- (i) The ring  $A_0$  is noetherian;
- (ii) The ring  $A_0$  is integrally closed in its field of fractions;
- (iii) Every nonzero prime ideal is a maximal ideal.

Let us mention several classical examples of Dedekind domains.

**Example 1.2.** Let  $K$  be a number field (i.e.  $K$  is a finite extension of  $\mathbb{Q}$ ). If  $\mathbb{Z}_K$  denotes the ring of elements of  $K$  which are integral over  $\mathbb{Z}$  then  $\mathbb{Z}_K$  is a Dedekind ring. For instance the ring of integers  $\mathbb{Z}$  and the ring of Gaussian integers  $\mathbb{Z}[\sqrt{-1}]$  are Dedekind rings. If  $p$  is a prime number and if  $\zeta \in \mathbb{C}$  is a primitive  $p$ th root of unity then  $\mathbb{Z}[\zeta]$  is a Dedekind domain. However there exist quadratic extensions of  $\mathbb{Z}$  that are not Dedekind. Indeed if  $d$  is a positive integer then the ring  $\mathbb{Z}[\sqrt{d}]$  is Dedekind if and only if  $d \equiv 2, 3 \pmod{4}$ .

Let  $k$  be a field and let  $A$  be a finitely generated normal algebra over  $k$  of dimension one. This means that the affine scheme  $C = \text{Spec } A$  is an affine smooth algebraic curve over  $k$ . Then the coordinate ring  $A = k[C]$  is Dedekind. The algebra of power series  $k[[t]]$  in one variable over the field  $k$  is a Dedekind domain. More generally every principal ideal domain (and so every discrete valuation ring) that is not a field is a Dedekind domain. The example of  $\mathbb{Z}[\sqrt{-5}]$  shows that the converse is false, in general.

If  $A_0$  is a Dedekind domain then the localisation of  $A_0$  by a multiplicative subset  $S$  of  $A_0$  is either a Dedekind Domain or a field [Mi, 3.4]. In the case where  $S$  is the complement of a nonzero prime ideal  $\mathfrak{p}$  the localisation is a discrete valuation ring. Furthermore if  $K_0$  is the field of fractions of  $A_0$  and  $L$  is a finite separable extension of  $K_0$  then the integral closure of  $A_0$  in  $L$  is also a Dedekind ring [Mi, 2.29].

**1.3.** Let  $A_0$  be an integral domain and let  $K_0$  be the field of fractions of  $A_0$ . Recall that a *fractional ideal*  $\mathfrak{b}$  is a finitely generated nonzero  $A_0$ -submodule of  $K_0$ . We remark that every fractional ideal is of the form  $\frac{1}{f} \cdot \mathfrak{a}$  where  $f \in A_0$  is nonzero and  $\mathfrak{a}$  is a

nonzero ideal of  $A_0$ . If  $\mathfrak{b}$  is equal to  $u \cdot A_0$  for some nonzero element  $u$  belonging to  $K_0 := \text{Frac } A_0$  then we say that  $\mathfrak{b}$  is a *principal* fractional ideal.

The following gives a description of fractional ideals of  $A_0$  in terms of Weil divisors on  $Y = \text{Spec } A_0$  when  $A_0$  is a Dedekind domain. This assertion is well known [Ke, §14]. For convenience of the reader we include a short proof.

**Theorem 1.4.** *Let  $A_0$  be a Dedekind ring with field of fractions  $K_0$ . Let  $Y := \text{Spec } A_0$ . Then the map*

$$\text{Div}_{\mathbb{Z}}(Y) \rightarrow \text{Id}(A_0), \quad D \mapsto H^0(Y, \mathcal{O}_Y(D))$$

*is a bijection between the set of integral Weil divisors on  $Y$  and the set of fractional ideals of  $A_0$ . Every fractional ideal is locally free of rank 1 as  $A_0$ -module and the natural map*

$$H^0(Y, \mathcal{O}_Y(D)) \otimes H^0(Y, \mathcal{O}_Y(D')) \rightarrow H^0(Y, \mathcal{O}_Y(D + D'))$$

*is surjective. The divisor  $D$  is principal (resp. effective) if and only if the corresponding fractional ideal is principal (resp. contains  $A_0$ ).*

*Proof.* By [Ha, Proposition II.6.11], the group of Weil divisors on  $Y$  coincides with the group of Cartier divisors. In particular, every  $A_0$ -module  $H^0(Y, \mathcal{O}_Y(D))$  is of finite type [Ha, Corollary II.5.5], locally free of rank one and so has a nonzero global section. Therefore the map

$$\text{Div}_{\mathbb{Z}}(Y) \rightarrow \text{Id}(A_0)$$

is well defined.

Let  $D, D'$  be divisors of  $\text{Div}_{\mathbb{Z}}(Y)$ . Then by the previous observation the  $\mathcal{O}_Y$ -sheaves  $\mathcal{O}_Y(D) \otimes \mathcal{O}_Y(D')$  and  $\mathcal{O}_Y(D + D')$  are isomorphic. By standard arguments this induces an isomorphism on the level of global sections.

Combining Proposition 14.6 and Theorem 14.8 in [Ke], every nonzero prime ideal of  $A_0$  is the module of global sections of an invertible sheaf over  $\mathcal{O}_Y$ . Thus by the primary decomposition [Ke, Theorem 14.11], the map  $\text{Div}_{\mathbb{Z}}(Y) \rightarrow \text{Id}(A_0)$  is surjective.

Assume that

$$H^0(Y, \mathcal{O}_Y(D)) = H^0(Y, \mathcal{O}_Y(D'))$$

for some  $D, D' \in \text{Div}_{\mathbb{Z}}(Y)$ . Then we can write  $D = D_+ - D_-$  and  $D' = D'_+ - D'_-$  where  $D_+, D'_+, D_-, D'_+$  are integral effective divisors. By tensoring we obtain the equality

$$H^0(Y, \mathcal{O}_Y(-D_- - D'_+)) = H^0(Y, \mathcal{O}_Y(-D'_- - D_+))$$

between ideals of  $A_0$ . Again using the decomposition in prime ideals we get  $-D_- - D'_+ = -D'_- - D_+$  and so  $D = D'$ . One concludes that the map is injective.

Assume that  $H^0(Y, \mathcal{O}_Y(D))$  contains  $A_0$ . Write  $D = D_+ - D_-$  with  $D_+, D_-$  effective and having disjoint supports. Then by our assumption

$$H^0(Y, \mathcal{O}_Y(0)) = A_0 = A_0 \cap H^0(Y, \mathcal{O}_Y(D)) = H^0(Y, \mathcal{O}_Y(-D_-)).$$

This yields  $D_- = 0$  and so  $D$  is effective. The rest of the proof is straightforward.  $\square$

**Notation 1.5.** Let  $A_0$  be a Dedekind domain. As usual, for a  $\mathbb{Q}$ -divisor  $D$  on  $Y := \text{Spec } A_0$  we denote by  $A_0[D]$  the graded algebra

$$\bigoplus_{i \in \mathbb{N}} H^0(Y, \mathcal{O}_Y([iD])) t^i$$

where  $t$  is a variable over the field  $K_0 := \text{Frac } A_0$ . Note that  $A_0[D]$  is normal as intersection of discrete valuation rings with field of fractions  $K_0(t)$  (see [De, §2.7]).

The next lemma provides a D.P.D. presentation for a class of graded subrings of  $K_0[t]$ . It will be useful later on. See [FZ, Theorem 3.2(a)] for a geometric proof concerning affine surfaces with a parabolic  $\mathbb{C}^*$ -action.

**Lemma 1.6.** *Let  $A_0$  be a Dedekind ring with the field of fractions  $K_0$ , and let*

$$A = \bigoplus_{i \in \mathbb{N}} A_i t^i \subset K_0[t]$$

*be a normal graded subalgebra of finite type over  $A_0$  such that for every  $i$ ,  $A_i \subset K_0$  is an  $A_0$ -submodule. Assume that the field of fractions of  $A$  is  $K_0(t)$ . Then there exists a unique  $\mathbb{Q}$ -divisor  $D$  on  $Y = \text{Spec } A_0$  such that for any  $i \in \mathbb{N}$ ,*

$$A_i = H^0(Y, \mathcal{O}_Y([iD])).$$

*Futhermore we have  $Y = \text{Proj } A$ .*

*Proof.* By Hilbert's Basis Theorem the ring  $A$  is noetherian. So each  $A_i$  is either zero or a fractional ideal of  $A_0$  (see [G-Y, Lemma 2.2]). Using Theorem 1.4 for every  $i \in \mathbb{N}$  such that  $A_i \neq \{0\}$  we can write

$$A_i = H^0(Y, \mathcal{O}_Y(D_i))$$

for some  $D_i \in \text{Div}_{\mathbb{Z}}(Y)$ . By [Bou, §III.3, Proposition 3] we can find a positive integer  $d$  such that the  $d$ th Veronese subring

$$A^{(d)} := \bigoplus_{i \geq 0} A_{di} t^{di}$$

is generated as  $A_0$ -algebra by  $A_d t^d$ . One observes that for each  $i \in \mathbb{N}$ , the graded piece  $A_{di}$  is nonzero. Proceeding by induction, for any  $i \in \mathbb{N}$  the equality  $D_{di} = iD_d$  holds. Put  $D = \frac{D_d}{d}$ . Then using the normality of  $A$  and  $A_0[D]$ , we have for any homogenous element  $f \in K_0[t]$  the equivalences

$$f \in A_0[D] \Leftrightarrow f^d \in A_0[D] \Leftrightarrow f^d \in A \Leftrightarrow f \in A.$$

Hence  $A = A_0[D]$ .

Let  $D'$  be another  $\mathbb{Q}$ -divisor on  $Y$  such that  $A = A_0[D']$ . Comparing the graded pieces of  $A_0[D]$  and of  $A_0[D']$ , by Theorem 1.4 we obtain

$$[iD] = [iD']$$

for any  $i \in \mathbb{N}$ . Hence

$$D' = \lim_{i \rightarrow \infty} \frac{[iD']}{i} = \lim_{i \rightarrow \infty} \frac{[iD]}{i} = D$$

and so the decomposition is unique.

It remains to show the equality  $Y = \text{Proj } A$ . Let  $V := \text{Proj } A$ . By [Ha, §II Exercices 5.13] and [Bou, §III.1, Proposition 3] we may assume that  $A = A_0[D]$  is generated by

$A_1$ . Since the sheaf  $\mathcal{O}_Y(D)$  is locally free of rank one over  $\mathcal{O}_Y$  there are  $g_1, \dots, g_s \in A_0$  such that

$$Y = Y_{g_1} \cup \dots \cup Y_{g_s},$$

where  $Y_{g_j} = \text{Spec}(A_0)_{g_j}$  (i.e. the localization by  $g_j$ ) and such that for  $e = 1, \dots, s$ ,

$$A_1 \otimes_{A_0} (A_0)_{g_e} = \mathcal{O}_Y(D)(Y_{g_e}) = h_e \cdot A_0$$

for some  $h_e \in K_0^*$ . Let  $\pi : V \rightarrow Y$  be the natural morphism induced by the inclusion  $A_0 \subset A$ . The preimage of the open subset  $Y_{g_e}$  under  $\pi$  is

$$\text{Proj } A \otimes_{A_0} (A_0)_{g_e} = \text{Proj}(A_0)_{g_e}[A_1 \otimes_{A_0} (A_0)_{g_e} t] = \text{Proj}(A_0)_{g_e}[h_e t] = Y_{g_e}.$$

Hence  $\pi$  is the identity map and so  $Y = V$ , as required.  $\square$

Using the same argument as in [FZ, Proposition 3.9] we deduce the following corollary.

**Corollary 1.7.** *Let  $A_0$  be a Dedekind ring with field of fractions  $K_0$  and let  $t$  be a variable over  $K_0$ . Consider the subalgebra*

$$A = A_0[f_1 t^{m_1}, \dots, f_r t^{m_r}] \subset K_0[t],$$

where  $m_1, \dots, m_r$  are positive integers and  $f_1, \dots, f_r \in K_0^*$  are such that the field of fractions of  $A$  is the same as that of  $K_0[t]$ . Then the normalization of  $A$  is equal to  $A_0[D]$  where  $D$  is the  $\mathbb{Q}$ -divisor

$$D = - \min_{1 \leq i \leq r} \text{div } f_i.$$

*Proof.* Since for  $i = 1, \dots, r$ , we have  $f_i \in H^0(Y, \mathcal{O}_Y(\lfloor m_i D \rfloor))$ , the ring  $A$  is contained in  $A_0[D]$ . By normality of  $A_0[D]$  we obtain  $\bar{A} \subset A_0[D]$ . Notice that  $\bar{A}$  satisfies the assumptions of Lemma 1.6. Thus there exists a  $\mathbb{Q}$ -divisor  $D'$  on  $Y$  such that  $A = A_0[D']$  and so  $D' \leq D$ . Now the inequalities

$$\text{div } f_i + \lfloor m_i D' \rfloor \geq 0, \quad i = 1, \dots, r,$$

imply that  $D = D'$ .  $\square$

## 2. MULTIGRADED ALGEBRAS AND DEDEKIND DOMAINS

Let  $A_0$  be a Dedekind ring and let  $K_0$  be its field of fractions. We denote by  $K_0[M]$  the semigroup algebra of a lattice  $M$ .

The purpose of this section is to study normal noetherian effectively  $M$ -graded  $A_0$ -subalgebras of  $K_0[M]$ . We show below that these subalgebras admit a description in terms of polyhedral divisors introduced by Altmann-Hausen [A-H]. We start by recalling some necessary notation from convex geometry.

**2.1.** Let  $N$  be a lattice and let  $M = \text{Hom}(N, \mathbb{Z})$  be its dual. We denote by  $N_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} N$  and  $M_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} M$  the associated dual  $\mathbb{Q}$ -linear spaces. To any linear form  $m$  of  $M_{\mathbb{Q}}$  and to any vector  $v$  of  $N_{\mathbb{Q}}$ , we let

$$\langle m, v \rangle = m(v).$$

A polyhedral cone  $\sigma \subset N_{\mathbb{Q}}$  is called *strongly convex* if it admits a vertex. This is equivalent to say that the dual cone

$$\sigma^{\vee} := \{ m \in M_{\mathbb{Q}}, \forall v \in \sigma, \langle m, v \rangle \geq 0 \}$$

is full dimensional or that  $\sigma$  does not contain any affine line.

We fix now a strongly convex polyhedral cone  $\sigma \subset N_{\mathbb{Q}}$ . A subset  $Q \subset N_{\mathbb{Q}}$  is a *polytope* if  $Q$  is non-empty and is the convex hull of a finite number of vectors. We define  $\text{Pol}_{\sigma}(N_{\mathbb{Q}})$  to be the set of polyhedra which can be written as the Minkowski sum  $P = Q + \sigma$  with  $Q$  a polytope of  $N_{\mathbb{Q}}$ .

**Definition 2.2.** Let  $A_0$  be a Dedekind domain. Consider the subset  $Z$  of closed points of the affine scheme  $Y := \text{Spec } A_0$ . A  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  over  $A_0$  is a formal sum

$$\mathfrak{D} = \sum_{z \in Z} \Delta_z \cdot z,$$

where  $\Delta_z$  belongs to  $\text{Pol}_{\sigma}(N_{\mathbb{Q}})$  and  $\Delta_z = \sigma$  for all but finitely many  $z$  in  $Z$ . Let  $z_1, \dots, z_r$  be elements of  $Z$  such that for any  $z \in Z$  and for  $i = 1, \dots, r$ ,  $z \neq z_i$  implies  $\Delta_z = \sigma$ . If the meaning of  $A_0$  is clear from the context, then we write

$$\mathfrak{D} = \sum_{i=1}^r \Delta_{z_i} \cdot z_i.$$

**Example 2.3.** If  $A_0 = k[[t]]$  is the algebra of power series in one variable over a field  $k$  then  $Z$  has a unique element given by the ideal  $t k[[t]]$ . Thus defining a  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  over  $A_0$  is equivalent to considering an element of  $\text{Pol}_{\sigma}(N_{\mathbb{Q}})$ .

If  $A_0 = \mathbb{Z}$  is the ring of integers then  $Z$  can be seen as the set of prime numbers.

Starting from a  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  we can build an  $M$ -graded algebra over  $A_0$  with weight cone  $\sigma^{\vee}$  in the same way as in [A-H, §3].

**2.4.** Let  $m$  be an element of  $\sigma^{\vee}$ . Then for any  $z \in Z$  the expression

$$h_z(m) := \min \langle m, \Delta_z \rangle$$

is well defined. The function  $h_z$  on  $\sigma^{\vee}$  is upper convex and positively homogeneous. It is identically zero if and only if  $\Delta_z = \sigma$ . The *evaluation* of  $\mathfrak{D}$  in a vector  $m \in \sigma^{\vee}$  is the  $\mathbb{Q}$ -divisor on  $Y = \text{Spec } A_0$  given by

$$\mathfrak{D}(m) := \sum_{z \in Z} h_z(m) \cdot z.$$

Letting  $K_0$  be the field of fractions of  $A_0$  we consider the semigroup algebra

$$K_0[M] := \bigoplus_{m \in M} K_0 \cdot \chi^m,$$

where the  $\chi^m$ 's satisfy the relations  $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$  for all  $m, m' \in M$ . To simplify notation for a polyhedral cone  $\omega \subset M_{\mathbb{Q}}$  we let  $\omega_M := \omega \cap M$ . We denote by  $A_0[\mathfrak{D}]$  the  $M$ -graded subring

$$\bigoplus_{m \in \sigma_M^{\vee}} A_m \chi^m \subset K_0[M] \quad \text{with } A_m = H^0(Y, \mathcal{O}_Y([\mathfrak{D}(m)]))$$

for any  $m \in \sigma_M^{\vee}$ . In particular, every  $A_m$  is a fractional ideal of the ring  $A_0$ .

**Notation 2.5.** For a closed point  $z \in Z$  of the scheme  $Y = \text{Spec } A_0$  the function

$$\nu_z : K_0^* \rightarrow \mathbb{Z}$$

is the valuation given by the local ring  $\mathcal{O}_{Y,z}$ . Let

$$f = (f_1 \chi^{m_1}, \dots, f_r \chi^{m_r})$$

be an  $r$ -uplet of homogeneous elements of  $K_0[M]$ . We assume that the vectors  $m_1, \dots, m_r$  generate the cone  $\sigma^\vee$ . We denote by  $\mathfrak{D}[f]$  the  $\sigma$ -polyhedral divisor

$$\sum_{z \in \mathbb{Z}} \Delta_z[f] \cdot z, \quad \text{where } \Delta_z[f] := \{v \in N_{\mathbb{Q}}, \langle m_i, v \rangle \geq -\nu_z(f_i), i = 1, 2, \dots, r\}.$$

The main result of this section is the following theorem.

**Theorem 2.6.** *Let  $A_0$  be a Dedekind domain with field of fractions  $K_0$  and let  $\sigma \subset N_{\mathbb{Q}}$  be a strongly convex polyhedral cone. Then the following hold.*

- (i) *If  $\mathfrak{D}$  is a  $\sigma$ -polyhedral divisor over  $A_0$  then the algebra  $A_0[\mathfrak{D}]$  is normal, noetherian, and has the same field of fractions as that of  $K_0[M]$ ;*
- (ii) *Conversely, let*

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m$$

*be a normal noetherian  $M$ -graded subalgebra of  $K_0[M]$  with weight cone  $\sigma^\vee$ . Assume that the rings  $A$  and  $K_0[M]$  have the same field of fractions. Then there exists a unique  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  over  $A_0$  such that  $A = A_0[\mathfrak{D}]$ ;*

- (iii) *More explicitly, if*

$$f = (f_1 \chi^{m_1}, \dots, f_r \chi^{m_r})$$

*is an  $r$ -uplet of homogeneous elements of  $K_0[M]$  with the vectors  $m_1, \dots, m_r$  generating the lattice  $M$  then the normalization of the ring*

$$A = A_0[f_1 \chi^{m_1}, \dots, f_r \chi^{m_r}]$$

*is equal to  $A_0[\mathfrak{D}[f]]$ .*

Let us give some examples.

**Example 2.7.** Let  $A_0 := \mathbb{Z}$ . Let  $x, y$  be variables over  $\mathbb{Q}$  and consider the  $\mathbb{Z}^2$ -graded subring

$$A = \mathbb{Z} \left[ \frac{2}{3} xy^2, \frac{1}{9} x, \frac{4}{3} x^2 y \right] \subset \mathbb{Q} \left[ x, y, \frac{1}{x}, \frac{1}{y} \right].$$

We will compute the normalization  $\bar{A}$  of  $A$ . Note that  $N_{\mathbb{Q}}$  is identified with  $\mathbb{Q}^2$ . By Theorem 2.6 we have  $\bar{A} = A_0[\mathfrak{D}]$ , where  $\mathfrak{D} = \Delta_2 \cdot (2) + \Delta_3 \cdot (3)$  with

$$\Delta_2 = \{(v_1, v_2) \in \mathbb{Q}^2 \mid v_1 + 2v_2 \geq -1, v_1 \geq 0, 2v_1 + v_2 \geq -2\}$$

and

$$\Delta_3 = \{(v_1, v_2) \in \mathbb{Q}^2 \mid v_1 + 2v_2 \geq 1, v_1 \geq 2, 2v_1 + v_2 \geq 1\}.$$

More precisely, the weight cone of  $A$  is

$$\omega = \mathbb{Q}_{\geq 0}(1, 2) + \mathbb{Q}_{\geq 0}(1, 0).$$

By [La, Proposition 1.10], for any

$$(m_1, m_2) \in \omega_{\mathbb{Z}^2} := \omega \cap \mathbb{Z}^2$$

we have the equality

$$\mathfrak{D}(m_1, m_2) = -\frac{m_2}{2} \cdot (2) + \left(2m_1 - \frac{1}{2}m_2\right) \cdot (3).$$



The graded pieces are given by

$$A_0[\mathfrak{D}] = \bigoplus_{(m_1, m_2) \in \omega_{\mathbb{Z}^2}} H^0(Y, \mathcal{O}_Y([\mathfrak{D}(m_1, m_2)])) x^{m_1} y^{m_2},$$

where  $Y := \text{Spec } \mathbb{Z}$ . Actually

$$(1) \quad A_0[\mathfrak{D}] = \mathbb{Z} \left[ \frac{1}{9} x, \frac{2}{3} xy, \frac{2}{3} xy^2 \right].$$

Indeed let  $(m_1, m_2)$  be a vector of  $\omega_{\mathbb{Z}^2}$  and assume that  $m_2 = 2r$  is even. Then the integer  $m_1 - r$  is non-negative. The graded piece  $A_{(m_1, m_2)}$  of  $A_0[\mathfrak{D}]$  corresponding to the pair  $(m_1, m_2)$  is

$$A_{(m_1, m_2)} = \mathbb{Z} \frac{2^r}{3^{2m_1-r}} x^{m_1} y^{m_2} = \mathbb{Z} \left( \frac{1}{9} x \right)^{m_1-r} \cdot \left( \frac{2}{3} xy^2 \right)^r.$$

Assume that  $m_2 = 2r + 1$  is odd. Then  $m_1 - (r + 1) \geq 0$  and

$$A_{(m_1, m_2)} = \mathbb{Z} \frac{2^{r+1}}{3^{2m_1-(r+1)}} x^{m_1} y^{m_2} = \mathbb{Z} \frac{2}{3} xy \cdot \left( \frac{1}{9} x \right)^{m_1-(r+1)} \cdot \left( \frac{2}{3} xy^2 \right)^r.$$

Thus all graded pieces of  $A_0[\mathfrak{D}]$  are generated by the elements  $\frac{1}{9} x, \frac{2}{3} xy, \frac{2}{3} xy^2$ . Hence (1) holds.

**Example 2.8.** Let  $k$  be a field and let  $A_0 := k[[t]]$  be the algebra of power series in one variable. Let  $z$  be the unique element of  $Z$  corresponding to the maximal ideal  $t k[[t]]$ . Choose an element  $\Delta = \Delta_z$  of  $\text{Pol}_\sigma(N_{\mathbb{Q}})$ . If  $\mathfrak{D} = \Delta_z \cdot z$  then

$$A_0[\mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} t^{-[h_z(m)]} k[[t]] \chi^m = \bigoplus_{m \in \sigma_M^\vee} t^{-[h_z(m)]} k[t] \otimes_{k[t]} k[[t]] \chi^m.$$

Letting

$$\omega(\Delta) := \{(m, e) \in M_{\mathbb{Q}} \times \mathbb{Q}, e \geq -h_z(m)\} \quad \text{and denoting } \widehat{M} := M \times \mathbb{Z}$$

we consider the algebra over  $k$

$$k[\omega(\Delta)_{\widehat{M}}] = \bigoplus_{(m, e) \in \omega(\Delta)_{\widehat{M}}} k \cdot \chi^{(m, e)}$$

of the affine semigroup  $\omega(\Delta)_{\widehat{M}}$ . The variable  $t$  is identified with  $\chi^{(0,1)}$ . Therefore we have

$$A_0[\mathfrak{D}] = k[\omega(\Delta)_{\widehat{M}}] \otimes_{k[t]} k[[t]].$$

In the next example we treat the case where  $A_0$  is not a principal ideal domain.

**Example 2.9.** For a number field  $K$ , the group of classes  $\text{Cl } K$  is the quotient of the abelian group of fractional ideals of  $K$  by the subgroup of principal fractional ideals. In other words,  $\text{Cl } K = \text{Pic } Y$ , where  $Y := \text{Spec } \mathbb{Z}_K$  is the affine scheme associated to the ring of integers of  $K$ . It is known that the group  $\text{Cl } K$  is finite [Mi, Theorem 4.4]. Furthermore  $\mathbb{Z}_K$  is a principal ideal domain if and only if  $\text{Cl } K$  is trivial.

Let  $K = \mathbb{Q}(\sqrt{-5})$ . Then  $\mathbb{Z}_K = \mathbb{Z}[\sqrt{-5}]$  and the group  $\text{Cl } K$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  [Mi, Example 4.6]. A set of representatives in  $\text{Cl } K$  is given by the fractional ideals  $\mathfrak{a} = (2, 1 + \sqrt{-5})$  and  $\mathbb{Z}_K$ . Given  $x, y$  variables over  $K$ , consider the  $\mathbb{Z}^2$ -graded subring

$$A = \mathbb{Z}_K [3x^2y, 2y, 6x] \subset K \left[ x, y, \frac{1}{x}, \frac{1}{y} \right].$$

We will describe the normalization  $\bar{A}$  of  $A$ . Denoting respectively by  $\mathfrak{b}, \mathfrak{c}$  the prime ideals  $(3, 1 + \sqrt{-5})$  and  $(3, 1 - \sqrt{-5})$ , we have the decompositions

$$(2) = \mathfrak{a}^2, \quad (3) = \mathfrak{b} \cdot \mathfrak{c}.$$

Observe that the ideals  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$  are distincts. Thus we have

$$\text{div } 2 = 2 \cdot \mathfrak{a} \quad \text{and} \quad \text{div } 3 = \mathfrak{b} + \mathfrak{c},$$

where  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$  is seen as closed points of  $Y = \text{Spec } \mathbb{Z}_K$ . Let  $\mathfrak{D}$  be the polyhedral divisor over  $\mathbb{Z}_K$  given by  $\Delta_{\mathfrak{a}} \cdot \mathfrak{a} + \Delta_{\mathfrak{b}} \cdot \mathfrak{b} + \Delta_{\mathfrak{c}} \cdot \mathfrak{c}$  with the polyhedra

$$\Delta_{\mathfrak{a}} = \{(v_1, v_2) \in \mathbb{Q}^2 \mid 2v_1 + v_2 \geq 0, v_2 \geq -2, v_1 \geq -2\} \quad \text{and}$$

$$\Delta_{\mathfrak{b}} = \Delta_{\mathfrak{c}} = \{(v_1, v_2) \in \mathbb{Q}^2 \mid 2v_1 + v_2 \geq -1, v_2 \geq 0, v_1 \geq -1\}.$$

By Theorem 2.6  $\bar{A} = A_0[\mathfrak{D}]$ , where  $A_0 = \mathbb{Z}_K$ . The weight cone of  $A$  is the first quadrant  $\omega = (\mathbb{Q}_{\geq 0})^2$ . An easy computation (cf, [La, Proposition 1.10]) shows that for all  $m_1, m_2 \in \mathbb{N}$ ,

$$\mathfrak{D}(m_1, m_2) = \min(m_1 - 2m_2, -2m_1 + 4m_2) \cdot \mathfrak{a} + \min\left(-\frac{m_1}{2}, -m_1 + m_2\right) \cdot (\mathfrak{b} + \mathfrak{c}).$$

Letting

$$\omega_1 = \mathbb{Q}_{\geq 0}(0, 1) + \mathbb{Q}_{\geq 0}(2, 1) \quad \text{and} \quad \omega_2 = \mathbb{Q}_{\geq 0}(2, 1) + \mathbb{Q}_{\geq 0}(1, 0),$$

on the cone  $\omega_1$ , we have

$$\mathfrak{D}(m_1, m_2) = (m_1 - 2m_2) \cdot \mathfrak{a} - \frac{m_1}{2} \cdot (\mathfrak{b} + \mathfrak{c}),$$

and on  $\omega_2$ ,

$$\mathfrak{D}(m_1, m_2) = (-2m_1 + 4m_2) \cdot \mathfrak{a} + (-m_1 + m_2) \cdot (\mathfrak{b} + \mathfrak{c}).$$

With the same notation as in 2.7, for  $i = 1, 2$  we let

$$A_{\omega_i} = \bigoplus_{(m_1, m_2) \in \omega_i \cap \mathbb{Z}^2} A_{(m_1, m_2)} x^{m_1} y^{m_2}$$

be the direct sum of graded pieces of  $A_0[\mathfrak{D}]$  corresponding to  $\omega_i \cap \mathbb{Z}^2$ . Then  $A_{\omega_2}$  is generated over  $\mathbb{Z}_K$  by the elements  $6x$  and  $3x^2y$ . Fix an element  $(m_1, m_2)$  in  $\omega_1 \cap \mathbb{Z}^2$ . If  $m_1 = 2r$  is even then  $r - m_1$  is non-positive. It follows that

$$A_{(m_1, m_2)} = \mathbb{Z}_K (3xy^2)^r \cdot (2y)^{m_2 - r}.$$

Otherwise,  $m_1 = 2r + 1$  is odd,  $m_2 - r - 1 \geq 0$ , and  $A_{(m_1, m_2)}$  is the ideal of  $\mathbb{Z}_K$  generated by the elements

$$(3xy^2)^r \cdot (2y)^{m_2 - r - 1} \cdot (3(1 + \sqrt{-5})xy), \quad (3xy^2)^r \cdot (2y)^{m_2 - r - 1} \cdot 6xy.$$

We conclude that

$$\bar{A} = A_0[\mathfrak{D}] = \mathbb{Z}_K [2y, 6xy, 3(1 + \sqrt{-5})xy, 3x^2y, 6x].$$

The proof of Theorem 2.6 needs some preparations. We start by a well known result giving an equivalence between noetherian and finitely generated properties of multigraded algebras.

**Theorem 2.10** (G-Y, Theorem 1.1). *Let  $G$  be a finitely generated abelian group and let  $A$  be a  $G$ -graded ring. Then the following statements are equivalent.*

- (i) *The ring  $A$  is noetherian.*
- (ii) *The graded piece  $A_e$  corresponding to the neutral element is a noetherian ring and the  $A_e$ -algebra  $A$  is finitely generated.*

The next lemma will enable us to show that the ring  $A_0[\mathfrak{D}]$  is noetherian.

**Lemma 2.11.** *Let  $D_1, \dots, D_r$  be  $\mathbb{Q}$ -divisors on  $Y = \text{Spec } A_0$ . Then the  $A_0$ -algebra*

$$B = \bigoplus_{(m_1, \dots, m_r) \in \mathbb{N}^r} H^0 \left( Y, \mathcal{O}_Y \left( \left[ \sum_{i=1}^r m_i D_i \right] \right) \right)$$

*is finitely generated.*

*Proof.* Let  $d$  be a positive integer such that for  $i = 1, \dots, r$ , the divisor  $dD_i$  is integral. Consider the lattice polytope

$$Q := \{ (m_1, \dots, m_r) \in \mathbb{Q}^r \mid 0 \leq m_i \leq d, i = 1, \dots, r \}.$$

The subset  $Q \cap \mathbb{N}^r$  being finite, the  $A_0$ -module

$$E := \bigoplus_{(m_1, \dots, m_r) \in \mathbb{N}^r \cap Q} H^0 \left( Y, \mathcal{O}_Y \left( \left[ \sum_{i=1}^r m_i D_i \right] \right) \right)$$

is finitely generated (see Theorem 1.4). Let  $(m_1, \dots, m_r)$  be an element of  $\mathbb{N}^r$ . Write  $m_i = dq_i + r_i$  with  $q_i, r_i \in \mathbb{N}$  such that  $0 \leq r_i < d$ . The equality

$$\left[ \sum_{i=1}^r m_i D_i \right] = \sum_{i=1}^r q_i [dD_i] + \left[ \sum_{i=1}^r r_i D_i \right]$$

implies that every homogeneous element of  $B$  of degree  $(m_1, \dots, m_r)$  can be written as a polynomial in  $E$ . Therefore if  $f_1, \dots, f_s$  generate the  $A_0$ -module  $E$  then we

$$B = A_0[E] = A_0[f_1, \dots, f_s],$$

proving our statement. □

Next we give a proof of the first part of Theorem 2.6.

*Proof of Theorem 2.6 (i).* Let  $A = A_0[\mathfrak{D}]$ . Let us show that

$$(2) \quad \text{Frac } A = \text{Frac } K_0[M].$$

Indeed since  $A_0 \subset A$  we have  $K_0 \subset \text{Frac } A$ . Let  $m$  be a vector of  $M$ . Then the cone  $\sigma^\vee$  is full dimensional and so there is  $m_1, m_2 \in \sigma_M^\vee$  such that  $m = m_1 - m_2$ . By Theorem 1.4 there exist nonzero elements  $a \in A_{m_1}$ ,  $b \in A_{m_2}$ . Hence

$$\chi^m = \frac{b}{a} \cdot \frac{a\chi^{m_1}}{b\chi^{m_2}} \in \text{Frac } A,$$

proving (2).

Let us show further that  $A$  is a normal ring. Given a closed point  $z \in Z$  and an element of  $v \in \Delta_z$ , consider the map

$$\nu_{z,v} : K_0[M] - \{0\} \rightarrow \mathbb{Z}$$

defined as follows. Let  $\alpha$  be a nonzero element of  $K_0[M]$  with decomposition in homogeneous elements

$$\alpha = f_1\chi^{m_1} + \dots + f_r\chi^{m_r}.$$

This means that  $f_1, \dots, f_r$  belong to  $K_0^*$  and the  $m_i$ 's are distinct. Then let

$$\nu_{z,v}(\alpha) := \min_{1 \leq i \leq r} \{\nu_z(f_i) + \langle m_i, v \rangle\}.$$

The map  $\nu_{z,v}$  defines a discrete valuation on  $\text{Frac } A$ . Denote by  $\mathcal{O}_{v,z}$  the associated local ring. By the definition of the algebra  $A_0[\mathfrak{D}]$  we have

$$A = K_0[M] \cap \bigcap_{z \in Z} \bigcap_{v \in \Delta_z} \mathcal{O}_{v,z}.$$

Hence  $A$  is normal as intersection of normal subrings of the field of fractions  $\text{Frac } A$ .

It remains to show that  $A$  is noetherian. By Hilbert's Basis Theorem, it suffices to show that  $A$  is finitely generated. Let  $\lambda_1, \dots, \lambda_e$  be full dimensional regular subcones of  $\sigma^\vee$  giving a subdivision<sup>1</sup> such that for any  $i$ , the evaluation map

$$\sigma^\vee \rightarrow \text{Div}_{\mathbb{Q}}(Y), \quad m \mapsto \mathfrak{D}(m)$$

is linear on  $\lambda_i$ . Fix  $i \in \mathbb{N}$  such that  $1 \leq i \leq e$ . Consider the distinct elements  $v_1, \dots, v_n$  of the Hilbert basis of  $\lambda_i$ . Denote by  $A_{\lambda_i}$  the algebra

$$\bigoplus_{m \in \lambda_i \cap M} H^0(Y, \mathcal{O}_Y([\mathfrak{D}(m)])\chi^m).$$

Then the vectors  $v_1, \dots, v_n$  form a basis of the lattice  $M$  and so

$$A_{\lambda_i} \simeq \bigoplus_{(m_1, \dots, m_n) \in \mathbb{N}^n} H^0 \left( Y, \mathcal{O}_Y \left( \left[ \sum_{i=1}^n m_i \mathfrak{D}(v_i) \right] \right) \right).$$

By Lemma 2.11,  $A_{\lambda_i}$  is finitely generated over  $A_0$ . The natural surjective map

$$A_{\lambda_1} \otimes \dots \otimes A_{\lambda_e} \rightarrow A$$

shows that  $A$  is also finitely generated. This completes the proof of Theorem 2.6 (i).  $\square$

For the second part of Theorem 2.6, we need the following lemma.

**Lemma 2.12.** *Assume that  $A$  verifies the assumptions of Theorem 2.6 (ii). Then the following hold.*

- (i) *For any  $m \in \sigma_M^\vee$  we have  $A_m \neq \{0\}$ . In other words, the weight semigroup of the  $M$ -graded algebra  $A$  is  $\sigma_M^\vee$ ;*
- (ii) *If  $L = \mathbb{Q}_{\geq 0} \cdot m'$  is a half-line contained in  $\sigma^\vee$  then the ring*

$$A_L := \bigoplus_{m \in L \cap M} A_m$$

*is normal and noetherian.*

<sup>1</sup>see for instance [CLS, Theorem 11.1.9].

*Proof.* Let

$$S = \{m \in \sigma_M^\vee, A_m \neq \{0\}\}$$

be the weight semigroup of  $A$ . Assume that  $S \neq \sigma_M^\vee$ . Then  $S$  is not saturated in  $M$ . So there exists  $e \in \mathbb{Z}_{>0}$  and  $m \in M$  such that  $m \notin S$  and  $e \cdot m \in S$ . Since  $A$  is a noetherian ring, by [G-Y, Lemma 2.2] the  $A_0$ -module  $A_{em}$  is a fractional ideal of  $A_0$ . By Theorem 1.4

$$A_{em} = H^0(Y, \mathcal{O}_Y(D_{em}))$$

for some integral divisor  $D_{em} \in \text{Div}_{\mathbb{Z}}(Y)$ . Let  $g$  be a nonzero element of

$$H^0\left(Y, \mathcal{O}_Y\left(\left\lfloor \frac{D_{em}}{e} \right\rfloor\right)\right).$$

This element exists by virtue of Theorem 1.4. We have the inequalities

$$\text{div } g^e \geq -e \left\lfloor \frac{D_{em}}{e} \right\rfloor \geq -D_{em}.$$

Since

$$g^e \chi^{em} \in \text{Frac } K_0[M] = \text{Frac } A$$

is integral over  $A$  the normality of  $A$  implies  $g \chi^m \in A_m$ . This contradicts our assumption  $S \neq \sigma_M^\vee$  and gives (i).

For the second assertion we notice by Theorem 2.10 and by arguments of [A-H, Lemma 4.1] that the ring  $A_L$  is noetherian.

It remains to show that  $A_L$  is normal. Let  $\alpha \in \text{Frac } A_L$  be an integral element over  $A_L$ . By normality of  $A$  and  $K_0[\chi^m]$  we obtain  $\alpha \in A \cap K_0[\chi^m] = A_L$  and so  $A_L$  is normal, proving our lemma.  $\square$

In the next paragraphs we provide a proof of Theorem 2.6 (ii). We apply the Dolgachev-Pinkham-Demazure construction given in section 1. We start by fixing some notation.

**Notation 2.13.** Let

$$(m_i, e_i), \quad i = 1, \dots, r$$

be elements of  $M \times \mathbb{Z}$  such that the vectors  $m_1, \dots, m_r$  generate the lattice  $M$ . Then the cone  $\omega = \text{Cone}(m_1, \dots, m_r)$  is full dimensional in  $M_{\mathbb{Q}}$ . Consider the  $\omega^\vee$ -polyhedron

$$\Delta = \{v \in N_{\mathbb{Q}}, \langle m_i, v \rangle \geq -e_i, \quad i = 1, 2, \dots, r\}.$$

Let  $L = \mathbb{Q}_{\geq 0} \cdot m$  be a half-line contained in  $\omega$  with primitive vector  $m$ . In other words, the element  $m$  generates the semigroup  $L \cap M$ . Denote by  $\mathcal{H}_L$  the Hilbert basis of the cone

$$p^{-1}(L) \cap (\mathbb{Q}_{\geq 0})^r, \quad \text{where } p : \mathbb{Q}^r \rightarrow M_{\mathbb{Q}}$$

is the  $\mathbb{Q}$ -linear map sending the canonical basis onto  $(m_1, \dots, m_r)$ . We put

$$\mathcal{H}_L^* := \{(s_1, \dots, s_r) \in \mathcal{H}_L, \sum_{i=1}^r s_i \cdot m_i \neq 0\}.$$

For any vector  $(s_1, \dots, s_r) \in \mathcal{H}_L^*$ , there exists a unique  $\lambda(s_1, \dots, s_r) \in \mathbb{Z}_{>0}$  such that

$$\sum_{i=1}^r s_i \cdot m_i = \lambda(s_1, \dots, s_r) \cdot m.$$

**Lemma 2.14.** *Under the assumptions of 2.13, we have*

$$\min \langle m, \Delta \rangle = - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^*} \frac{\sum_{i=1}^r s_i \cdot e_i}{\lambda(s_1, \dots, s_r)}.$$

*Proof.* Let  $t$  be a variable over  $\mathbb{C}$  and consider the  $M$ -graded algebra

$$A = \mathbb{C}[t][t^{e_1} \chi^{m_1}, \dots, t^{e_r} \chi^{m_r}] \subset \mathbb{C}(t)[M].$$

The field of fractions of  $A$  is the same as that of  $\mathbb{C}(t)[M]$ . By [La, Theorem 2.4], the normalization of the algebra  $A$  is

$$\bar{A} = \bigoplus_{m \in \omega \cap M} H^0(\mathbb{A}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1}([\min \langle m, \Delta \rangle] \cdot (0))) \chi^m,$$

where  $\mathbb{A}_{\mathbb{C}}^1 := \text{Spec } \mathbb{C}[t]$  is the complex affine line. The sublattice  $G \subset M$  generated by  $p(\mathcal{H}_L^*)$  is a subgroup of  $\mathbb{Z} \cdot m$ . There exists a unique integer  $d \in \mathbb{Z}_{>0}$  such that  $G = d\mathbb{Z} \cdot m$ . For an element  $m' \in \omega \cap M$ , we denote by  $A_{m'}$  (resp.  $\bar{A}_{m'}$ ) the graded piece of  $A$  (resp.  $\bar{A}$ ) corresponding to  $m'$ . Then the normalization  $\bar{A}_L^{(d)}$  of the algebra

$$A_L^{(d)} := \bigoplus_{s \geq 0} A_{sdm} \chi^{sdm} \text{ is } B_L := \bigoplus_{s \geq 0} \bar{A}_{sdm} \chi^{sdm}.$$

Indeed, since  $A_L^{(d)} \subset A$  we have  $\bar{A}_L^{(d)} \subset \bar{A}$ . Similarly,  $\bar{A}_L^{(d)} \subset \mathbb{C}(t)[\chi^{dm}]$  and so

$$\bar{A}_L^{(d)} \subset \mathbb{C}(t)[\chi^{dm}] \cap \bar{A} = B_L.$$

Conversely, let  $s$  be a natural integer and consider  $\alpha \in \bar{A}_{sdm} \chi^{sdm}$ . Then  $\alpha$  satisfies an equation of integral dependence

$$\alpha^e = a_1 \alpha^{e-1} + \dots + a_e, \text{ where } a_e \neq 0,$$

with  $a_i \in A$  for every  $i = 1, \dots, e$ . We may assume that  $a_i = 0$  or  $a_i$  homogenous of degree  $isd \cdot m$ . Thus  $\alpha$  is integral over  $A_L^{(d)}$ . Since  $G = d\mathbb{Z} \cdot m$ , the element  $\alpha$  belongs to

$$\text{Frac } B_L = \mathbb{C}(t)(\chi^{dm}) = \text{Frac } A_L^{(d)}$$

and so  $\alpha \in \bar{A}_L^{(d)}$ . Thus  $B_L = \bar{A}_L^{(d)}$ . Moreover the algebra

$$A_L := \bigoplus_{s \geq 0} A_{sm} \chi^{sm}$$

is generated over  $\mathbb{C}[t]$  by the elements

$$f_{(s_1, \dots, s_r)} := \prod_{i=1}^r (t^{e_i} \chi^{m_i})^{s_i} = t^{\sum_{i=1}^r s_i e_i} \chi^{\lambda(s_1, \dots, s_r) m}, \quad (s_1, \dots, s_r) \in \mathcal{H}_L^*.$$

By the choice of the integer  $d$  we have  $A_L^{(d)} = A_L$ . Considering the  $G$ -graduation of  $A_L^{(d)}$ , for any  $(s_1, \dots, s_r) \in \mathcal{H}_L^*$  the element  $f_{(s_1, \dots, s_r)}$  of the graded ring  $A_L^{(d)}$  has degree

$$\deg f_{(s_1, \dots, s_r)} := \frac{\lambda(s_1, \dots, s_r)}{d}.$$

Letting

$$D := - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^*} \frac{\operatorname{div} f_{(s_1, \dots, s_r)}}{\operatorname{deg} f_{(s_1, \dots, s_r)}} = - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^*} d \cdot \frac{\sum_{i=1}^r s_i e_i}{\lambda(s_1, \dots, s_r)} \cdot (0),$$

by Corollary 1.7 we obtain

$$\bar{A}_L^{(d)} = \bigoplus_{s \geq 0} H^0(\mathbb{A}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1}(\lfloor sD \rfloor)) \chi^{sdm}.$$

The equality  $\bar{A}_L^{(d)} = B_L$  implies

$$H^0(\mathbb{A}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1}(\lfloor \min \langle sd \cdot m, \Delta \rangle \rfloor \cdot (0))) = H^0(\mathbb{A}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1}(\lfloor sD \rfloor))$$

for all integer  $s \geq 0$ . Hence

$$D = \min \langle d \cdot m, \Delta \rangle \cdot (0).$$

Dividing by  $d$ , we obtain the desired formula.  $\square$

Let  $A$  be an  $M$ -graded algebra satisfying the assumptions of Theorem 2.6 (ii). Using the D.P.D. presentation on each half line of the weight cone  $\sigma^\vee$ , we can build a map

$$\sigma^\vee \rightarrow \operatorname{Div}_{\mathbb{Q}}(Y), \quad m \mapsto D(m).$$

It is upper convex, positively homogeneous, and verifies for any  $m \in \sigma_M^\vee$ ,

$$A_m = H^0(C, \mathcal{O}_C(\lfloor D(m) \rfloor)).$$

By Lemma 2.14, this map is piecewise linear (see [AH, Proposition 2.11]) or equivalently  $m \mapsto D(m)$  is the evaluation map of a polyhedral divisor. The following precises this idea.

*Proof of Theorem 2.6 (ii).* Let

$$f = (f_1 \chi^{m_1}, \dots, f_r \chi^{m_r})$$

be a system of homogeneous generators of  $A_0$  with nonzero vectors  $m_1, \dots, m_r \in M$ . Denote by  $\mathfrak{D}$  the  $\sigma$ -polyhedral divisor  $\mathfrak{D}[f]$  (see Notations 2.5). Let us show that  $A = A_0[\mathfrak{D}]$ . Let  $L = \mathbb{Q}_{\geq 0} \cdot m$  be a half-line contained in  $\omega := \sigma^\vee$  with  $m$  being the primitive vector of  $L$ . By Lemma 2.12, the graded subalgebra

$$A_L := \bigoplus_{m' \in L \cap M} A_{m'} \chi^{m'} \subset K_0[\chi^m]$$

is normal, noetherian, and has the same field of fractions as that of  $K_0[\chi^m]$ . Moreover, with the same notations as in 2.13, the algebra  $A_L$  is generated by the set

$$\left\{ \prod_{i=1}^r (f_i \chi^{m_i})^{s_i}, (s_1, \dots, s_r) \in \mathcal{H}_L^* \right\}.$$

By Corollary 1.7, if

$$D(m) := - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^*} \frac{\sum_{i=1}^r s_i \operatorname{div} f_i}{\lambda(s_1, \dots, s_r)}$$

then  $A_L = A_0[D(m)]$  with respect to the variable  $\chi^m$ . By Lemma 2.14, for any closed point  $z \in Z$  we have

$$h_z[f](m) = \min\langle m, \Delta_z[f] \rangle = - \min_{(s_1, \dots, s_r) \in \mathcal{H}_L^*} \frac{\sum_{i=1}^r s_i \nu_z(f_i)}{\lambda(s_1, \dots, s_r)}.$$

Hence  $D(m) = \mathfrak{D}(m)$ . Since this equality is true for all primitive vectors belonging to the cone  $\omega$ , we conclude that  $A = A_0[\mathfrak{D}]$ . The uniqueness of  $\mathfrak{D}$  is straightforward (see Theorem 1.4 and [La, Lemma 2.2]).  $\square$

The following proof is essentially the same as that of Theorem 2.4 in [La]. For the convenience of the reader we recall the main argument.

*Proof of Theorem 2.6 (iii).* By Theorem 2.6 (ii), we can write  $\bar{A} = A_0[\mathfrak{D}]$  for some  $\sigma$ -polyhedral divisor  $\mathfrak{D} = \sum_{z \in Z} \Delta_z \cdot z$  over  $A_0$ . Since each homogeneous element  $f_i \chi^{m_i}$  belongs to  $A_0[\mathfrak{D}[f]]$ , by normality we have the inclusion

$$A_0[\mathfrak{D}] = \bar{A} \subset A_0[\mathfrak{D}[f]].$$

As follow from Theorem 1.4, this implies that for any  $m \in \sigma^\vee$ ,  $\mathfrak{D}(m) \leq \mathfrak{D}[f](m)$ . We have  $\mathfrak{D}(m_i) + \text{div } f_i \geq 0$ , for every  $i$ . This yields that for any closed point  $z \in Z$ , the inclusion  $\Delta_z \subset \Delta_z[f]$ . Thus for  $m \in \sigma^\vee$ ,  $\mathfrak{D}(m) \geq \mathfrak{D}[f](m)$  and so  $\mathfrak{D} = \mathfrak{D}[f]$  holds, proving our theorem.  $\square$

### 3. MULTIGRADED ALGEBRAS AND ALGEBRAIC FUNCTION FIELDS

In this section, we show that the description of affine  $\mathbb{T}$ -varieties of complexity one in terms of polyhedral divisors ([KKMS], [AH], [Ti]) holds over an arbitrary field. Our approach uses the ideas of the proof of Theorem 2.6. Let  $\mathbf{k}$  be a field.

**3.1.** An *algebraic function field* (in one variable) over  $\mathbf{k}$  is a field extension  $K_0/\mathbf{k}$  verifying the following conditions.

- (i) The transcendence degree of  $K_0$  over  $\mathbf{k}$  is equal to 1.
- (ii) Every algebraic element of  $K_0$  over  $\mathbf{k}$  belongs to  $\mathbf{k}$ .

**3.2.** Actually every algebraic function fields  $K_0/\mathbf{k}$  is the field of rational functions of a unique (up to isomorphism) smooth algebraic projective curve  $C$  over  $\mathbf{k}$  [EGA II, §7.4]. For instance, the projective line  $\mathbb{P}_{\mathbf{k}}^1$  yields the field  $\mathbf{k}(t)$  where  $t$  is a variable.

In the next paragraph, we recall the construction of the curve  $C$  starting from an algebraic function fields  $K_0$ .

**3.3.** A *valuation ring* of  $K_0$  is a proper subring  $\mathcal{O} \subset K_0$  strictly containing  $\mathbf{k}$  and such that for any nonzero element  $f \in K_0$ , either  $f \in \mathcal{O}$  or  $\frac{1}{f} \in \mathcal{O}$ . By [St, Theorem 1.1.6] every valuation ring of  $K_0$  is the ring associated to a discrete valuation of  $K_0/\mathbf{k}$ . A subset  $P \subset K_0$  is called a *place* of  $K_0$  if there is some valuation ring  $\mathcal{O}$  of  $K_0$  such that  $P$  is the maximal ideal of  $\mathcal{O}$ . We denote by  $\mathfrak{R}_{\mathbf{k}} K_0$  the set of places of  $K_0$ . The latter is called the *Riemann surface* of  $K_0$ . By [EGA II, 7.4.18] the set  $\mathfrak{R}_{\mathbf{k}} K_0$  can be identified with a smooth projective curve  $C$  such that  $K_0 = \mathbf{k}(C)$ .

In the sequel we consider  $C = \mathfrak{R}_{\mathbf{k}} K_0$  as a geometrical object with its structure of scheme. An element  $z$  belonging to  $C$  is a closed point. For  $z \in C$ ,  $P_z$  is the associated place. We will use the classical Weil divisors theory available e.g. in [St, §1.4].



**3.4.** Let  $\sigma \subset N_{\mathbb{Q}}$  be a strongly polyhedral convex cone. We define as well a  $\sigma$ -polyhedral divisor  $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$  and its evaluation, see [A-H, §2]. For a place  $P$ , we let  $\kappa(P) := \mathcal{O}/P$  where  $\mathcal{O}$  is the valuation ring of  $K_0$  containing  $P$  as maximal ideal. The field  $\kappa(P)$  is a finite extension of  $\mathbf{k}$  [St, Proposition 1.1.15]. The *degree* of  $\mathfrak{D}$  is the Minkowski sum

$$\deg \mathfrak{D} = \sum_{z \in C} [\kappa(P_z) : \mathbf{k}] \cdot \Delta_z.$$

The number  $[\kappa(P) : \mathbf{k}]$  is the dimension of the  $\mathbf{k}$ -vector space  $\kappa(P)$ . It is also called the degree of the place  $P$ . For  $m \in \sigma^\vee$ , we have the relation  $(\deg \mathfrak{D})(m) = \deg \mathfrak{D}(m)$ . When  $\mathbf{k}$  is algebraically closed, we have for any  $z \in C$ ,  $[\kappa(P_z) : \mathbf{k}] = 1$  and we recover the usual notion of degree for polyhedral divisors.

**Definition 3.5.** A  $\sigma$ -polyhedral divisor  $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$  is called *proper* if it satisfies the following conditions.

- (i) The polyhedron  $\deg \mathfrak{D}$  is strictly contained in the cone  $\sigma$ .
- (ii) If  $\deg \mathfrak{D}(m) = 0$  then  $m$  belongs to the boundary of  $\sigma$  and a multiple of  $\mathfrak{D}(m)$  is principal.

In our next result, we give a description similar to that in 2.6 for algebraic function fields.

**Theorem 3.6.** *Let  $\mathbf{k}$  be a field and let  $C := \mathfrak{R}_{\mathbf{k}} K_0$  be the Riemann surface of an algebraic function field  $K_0/\mathbf{k}$ . Then the following hold.*

- (i) *Let*

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m$$

*be an  $M$ -graded normal noetherian subalgebra of  $K_0[M]$  with weight cone  $\sigma^\vee$  such that for any  $m \in \sigma_M^\vee$ ,  $A_m \subset K_0$  is a  $\mathbf{k}$ -vector subspace. If  $A_0 = \mathbf{k}$  and  $\text{Frac } A = \text{Frac } K_0[M]$  then there exists a unique proper  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  on the curve  $C$  such that  $A = A[C, \mathfrak{D}]$ , where*

$$A[C, \mathfrak{D}] := \bigoplus_{m \in \sigma_M^\vee} H^0(C, \mathcal{O}_C([\mathfrak{D}(m)]) \chi^m.$$

- (ii) *Let  $\mathfrak{D}$  be a proper  $\sigma$ -polyhedral divisor over  $C$ . Then the algebra  $A[C, \mathfrak{D}]$  is  $M$ -graded, normal, and finitely generated with weight cone  $\sigma^\vee$ . Furthermore it has the same field of fractions as that of  $K_0[M]$ .*
- (iii) *Let*

$$A = \mathbf{k}[f_1 \chi^{m_1}, \dots, f_r \chi^{m_r}]$$

*be an  $M$ -graded subalgebra of  $K_0[M]$  with the  $f_i \chi^{m_i}$ 's homogeneous and let  $f := (f_1 \chi^{m_1}, \dots, f_r \chi^{m_r})$ . Assume that  $\text{Frac } A = \text{Frac } K_0[M]$ . Then  $\mathfrak{D}[f]$  is the proper  $\sigma$ -polyhedral divisor such that the normalization of  $A$  in  $K_0[M]$  is  $A[C, \mathfrak{D}[f]]$ .*

For the proof of Theorem 3.6 we need some preliminary results.

**Lemma 3.7.** *Let*

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m$$

be an  $M$ -graded algebra satisfying the assumptions of Theorem 3.6 (i). Given a half-line  $L = \mathbb{Q}_{\geq 0} \cdot m \subset \sigma^\vee$  with a primitive vector  $m$ , consider the subalgebra

$$A_L := \bigoplus_{m' \in L \cap M} A_{m'} \chi^{m'}.$$

Let

$$Q(A_L)_0 := \left\{ \frac{a}{b}, a \in A_{sm}, b \in A_{sm}, b \neq 0, s \geq 0 \right\}.$$

Then the following hold.

- (i) The algebra  $A_L$  is finitely generated and normal.
- (ii) Either  $Q(A_L)_0 = \mathbf{k}$  or  $Q(A_L)_0 = K_0$ .
- (iii) If  $Q(A_L)_0 = \mathbf{k}$  then  $A_L = \mathbf{k}[\beta \chi^{dm}]$  for some  $\beta \in K_0^*$  and some  $d \in \mathbb{Z}_{>0}$ .

*Proof.* The proof of (i) is similar to that of Lemma 2.12 (ii) and so we omit it.

The field  $Q(A_L)_0$  is an extension of  $\mathbf{k}$  contained in  $K_0$ . If the transcendence degree of  $Q(A_L)_0$  over  $\mathbf{k}$  is zero then by normality of  $A_L$  we have  $Q(A_L)_0 = \mathbf{k}$ . Otherwise the extension  $K_0/Q(A_L)_0$  is algebraic. Let  $\alpha$  be an element of  $K_0$ . Then there exist  $a_1, \dots, a_d \in Q(A_L)_0$  with  $a_d \neq 0$  such that

$$\alpha^d = a_1 \alpha^{d-1} + a_2 \alpha^{d-2} + \dots + a_d.$$

Letting

$$I := \{i \in \{1, \dots, d\}, a_i \neq 0\},$$

for any  $i \in I$  we write  $a_i = \frac{p_i}{q_i}$  with  $p_i, q_i \in A_L$  being homogeneous of the same degree. Considering  $q := \prod_{i \in I} q_i$  we obtain the equality

$$(\alpha q)^d = a_1 q (\alpha q)^{d-1} + a_2 q^2 (\alpha q)^{d-2} + \dots + q^d a_d,$$

where  $\alpha q \in A_L$ . Write the decompositions in homogeneous components

$$\alpha q = \sum_{i=1}^r f_i \chi^{s_i m} \quad \text{and} \quad q = a \chi^{sm}.$$

Then

$$\alpha = \frac{\alpha q}{q} = \sum_{i=1}^r \frac{f_i}{a} \chi^{(s_i - s)m}$$

has degree zero in  $K_0[M]$ . This implies that  $r = 1$  and  $s_1 = s$ . Hence  $\alpha \in Q(A_L)_0$ . This establishes (ii).

To prove (iii), we let  $S \subset \mathbb{Z} \cdot m$  be the weight semigroup of the graded algebra  $A_L$ . Since  $L$  is contained in the weight cone  $\sigma^\vee$ ,  $S$  is nonzero. Therefore if  $G$  is the subgroup generated by  $S$  then there exists  $d \in \mathbb{Z}_{>0}$  such that  $G = \mathbb{Z}d \cdot m$ . Putting  $u := \chi^{dm}$  we can write

$$A_L = \bigoplus_{s \geq 0} A_{sdm} u^s.$$

Thus for all homogeneous elements  $a_1 u^{r_1}, a_2 u^{r_2} \in A_L$  of the same degree we have  $\frac{a_1}{a_2} \in Q(A_L)_0^* = \mathbf{k}^*$ , so that

$$A_L = \bigoplus_{s \in S'} k f_s u^s,$$

where  $S' := \frac{1}{d}S$  and  $f_s \in \mathbf{k}(C)^*$ . Let us fix homogeneous generators  $f_{s_1}u^{s_1}, \dots, f_{s_r}u^{s_r}$  of the  $G$ -graded algebra  $A_L$ . Consider  $d' := \text{g.c.d.}(s_1, \dots, s_r)$ . If  $d' > 1$  then the inclusion  $S \subset dd'\mathbb{Z} \cdot m$  yields a contradiction. So  $d' = 1$  and there are some nonzero integers  $l_1, \dots, l_r$  such that  $1 = \sum_{i=1}^r l_i s_i$ . The element

$$\beta u = \prod_{i=1}^r (f_{s_i} u^{s_i})^{l_i}$$

verifies

$$\frac{(\beta u)^{s_1}}{f_{s_1} u^{s_1}} \in Q(A_L)_0^* = \mathbf{k}^*.$$

By normality of  $A_L$ ,  $\beta u \in A_L$  and so  $A_L = \mathbf{k}[\beta u] = \mathbf{k}[\beta \chi^{dm}]$ , proving (iii).  $\square$

The following lemma is well known. For the convenience of the reader we provide a short proof using the Riemann-Roch Theorem [St, §1.5].

**Lemma 3.8.** *Let  $D_1, D_2$  be  $\mathbb{Q}$ -divisors on  $C$ . Assume that for  $i = 1, 2$ , we have either  $\deg D_i > 0$  or  $rD_i$  is principal for some  $r \in \mathbb{Z}_{>0}$ . If for any  $s \in \mathbb{N}$ , the inclusion*

$$H^0(C, \mathcal{O}_C(\lfloor sD_1 \rfloor)) \subset H^0(C, \mathcal{O}_C(\lfloor sD_2 \rfloor))$$

holds, then  $D_1 \leq D_2$ .

*Proof.* First of all, if  $rD_1 = \text{div } \alpha$  for some  $\alpha \in K_0^*$  and for some  $r \in \mathbb{Z}_{>0}$ , then by our assumption, for any  $s \in \mathbb{N}$ ,

$$-s \text{div } \alpha + \lfloor rsD_2 \rfloor \geq 0.$$

This yields  $D_2 \geq D_1$ .

Assume that  $\deg D_1 > 0$ . For a  $\mathbb{Q}$ -divisor  $D$ , we denote by  $D_{z'}$  the coefficient of  $D$  corresponding to  $z' \in C$ . Fix  $z \in C$  and denote by  $g$  the genus of  $C$ . Since the degree of  $D_1$  is positive, we may consider  $s \in \mathbb{Z}_{>0}$ , where  $sD_1$  is integral and such that the inequality

$$\deg sD_1 - [\kappa(P_z) : \mathbf{k}] \geq 2g - 1$$

holds. Then the integral divisor

$$D_{(s,z)} := \sum_{z' \in C - \{z\}} (sD_1)_{z'} \cdot z' + ((sD_1)_z - 1) \cdot z$$

has degree greater than  $2g - 1$ . By the Riemann-Roch Theorem [St, Theorem 1.5.17] we have

$$\begin{aligned} h^0(C, \mathcal{O}_C(D_{(s,z)})) &= \deg D_{(s,z)} + 1 - g \\ &= \deg sD_1 - [\kappa(P_z) : \mathbf{k}] + 1 - g < \deg sD_1 + 1 - g = h^0(C, \mathcal{O}_C(sD_1)). \end{aligned}$$

This implies that there exists

$$\gamma \in H^0(C, \mathcal{O}_C(sD_1)) - H^0(C, \mathcal{O}_C(D_{(s,z)})).$$

Thus  $\gamma$  verifies the equality  $\nu_z(\gamma) + s(D_1)_z = 0$ , where  $\nu_z$  is the valuation corresponding to the place  $P_z$ . By assumption  $\gamma$  belongs to  $H^0(C, \mathcal{O}_C(\lfloor sD_2 \rfloor))$ . We obtain  $\nu_z(\gamma) + \lfloor s(D_2)_z \rfloor \geq 0$  and so  $(D_1)_z \leq (D_2)_z$ . This is true for any  $z \in C$ . Hence  $D_1 \leq D_2$ , proving our lemma.  $\square$

In the next corollary, we keep the notation of Lemma 3.7. Using the Demazure's Theorem for graded algebras, we show that each  $A_L$  admits a D.P.D. presentation given on the same smooth projective curve.

**Corollary 3.9.** *There exists a unique  $\mathbb{Q}$ -divisor  $D$  on  $C$  such that*

$$A_L = \bigoplus_{s \geq 0} H^0(C, \mathcal{O}_C(\lfloor sD \rfloor)) \chi^{sm}$$

and the following hold.

- (i) If  $Q(A_L)_0 = \mathbf{k}$  then  $D = \frac{\operatorname{div} f}{d}$  for some  $f \in K_0^*$  and some  $d \in \mathbb{Z}_{>0}$ .
- (ii) If  $Q(A_L)_0 = K_0$  then  $\deg D > 0$ .
- (iii) If  $f_1 \chi^{s_1 m}, \dots, f_r \chi^{s_r m}$  are homogeneous generators of the algebra  $A_L$  then

$$D = - \min_{1 \leq i \leq r} \frac{\operatorname{div} f_i}{s_i}.$$

*Proof.* Assume that  $Q(A_L)_0 = \mathbf{k}$ . By Lemma 3.7,  $A_L = \mathbf{k}[\beta \chi^{dm}]$  for some  $\beta \in K_0^*$  and some  $d \in \mathbb{Z}_{>0}$ . If  $D = \frac{\operatorname{div} \beta^{-1}}{d}$  then  $A_L = A[C, D]$ . The uniqueness in this case is easy.

Otherwise, the field of rational functions of the normal variety  $\operatorname{Proj} A_L$  is  $K_0 = Q(A_L)_0$ . Since  $\operatorname{Proj} A_L$  is a smooth projective curve over  $A_0 = \mathbf{k}$ , we may identify its points with the places of  $K_0$ . Therefore the existence and the uniqueness of  $D$  follow from Demazure's Theorem (see [De, Theorem 3.5]). Furthermore  $Q(A_L)_0 \neq \mathbf{k}$  implies that  $\dim_{\mathbf{k}} A_{sm} \geq 2$ , for some  $s \in \mathbb{Z}_{>0}$ . Hence by [St, Corollary 1.4.12] we obtain  $\deg D > 0$ .

For the last assertion, we fix homogenous generators  $f_1 \chi^{s_1 m}, \dots, f_r \chi^{s_r m}$  of the  $\mathbf{k}$ -algebra  $A_L$ . Letting

$$D' := - \min_{1 \leq i \leq r} \frac{\operatorname{div} f_i}{s_i},$$

by [De, §2.7],  $A_L$  is contained in

$$A[C, D'] := \bigoplus_{s \geq 0} H^0(C, \mathcal{O}_C(\lfloor sD' \rfloor)) \chi^{sm}$$

as a graded subalgebra. By Lemma 3.8,  $D \leq D'$ . To show the inequality  $D' \leq D$  one can use the same argument as that in Corollary 1.7.  $\square$

In the next paragraph, we keep the notation introduced in 2.5 and 2.13. For the proof of 3.6 (iii), we refer the reader to [La, 2.4].

*Proof of Theorem 3.6 (i).* Let  $f = (f_1 \chi^{m_1}, \dots, f_r \chi^{m_r})$  be a system of homogeneous generators of  $A$ . Consider a half-line  $L = \mathbb{Q}_{\geq 0} \cdot m \subset \sigma^\vee$  with primitive vector  $m \in M$ . By Corollary 3.9

$$A_L = \bigoplus_{s \geq 0} H^0(C, \mathcal{O}_C(\lfloor sD(m) \rfloor)) \chi^{sm}$$

for a unique  $\mathbb{Q}$ -divisor  $D(m)$  on the curve  $C$ . The algebra  $A_L$  is generated by

$$(f_1 \chi^{m_1})^{s_1} \dots (f_r \chi^{m_r})^{s_r}$$

where  $(s_1, \dots, s_r) \in \mathcal{H}_L^*$ . By Lemma 2.14 we have  $\mathfrak{D}[f](m) = D(m)$  and so  $A = A[C, \mathfrak{D}[f]]$ .

It remains to show that  $\mathfrak{D} := \mathfrak{D}[f]$  is proper. Denote by  $S \subset C$  the union of the supports of divisors  $\operatorname{div} f_i$ , for  $i = 1, \dots, r$ . Let  $v \in \operatorname{deg} \mathfrak{D}$ . We can write

$$v = \sum_{z \in S} [\kappa(P_z) : \mathbf{k}] \cdot v_z$$

for some  $v_z \in \Delta_z := \Delta_z[f]$ . Therefore for any  $i$  we have

$$\langle m_i, \sum_{z \in S} [\kappa(P_z) : \mathbf{k}] \cdot v_z \rangle \geq - \sum_{s \in S} [\kappa(P_s) : \mathbf{k}] \cdot \nu_s(f_i) = -\operatorname{deg} \operatorname{div} f_i = 0,$$

and so  $\operatorname{deg} \mathfrak{D} \subset \sigma$ . If  $\operatorname{deg} \mathfrak{D} = \sigma$  then by Corollary 3.9 one concludes that  $\operatorname{Frac} A$  is different from  $\operatorname{Frac} K_0[M]$ , contradicting our assumption. Hence  $\operatorname{deg} \mathfrak{D} \neq \sigma$ . Let  $m' \in \sigma_M^\vee$  be such that  $\operatorname{deg} \mathfrak{D}(m') = 0$ . Then  $m'$  belongs to the boundary of  $\sigma^\vee$  (see [CLS, §1.2]). Consider the half-line  $L'$  generated by  $m'$ . Applying Corollary 3.9 for the algebra  $A_{L'}$ , we deduce that a multiple of  $\mathfrak{D}(m')$  is principal.  $\square$

*Proof of Theorem 3.6 (ii).* Let us show that the field of fractions of  $A := A[C, \mathfrak{D}]$  is  $\operatorname{Frac} K_0[M]$ . For any half-line  $L = \mathbb{Q}_{\geq 0} \cdot m$ , intersecting  $\sigma^\vee$  with its relative interior, the weight semigroup of  $A_L$  generates the sublattice  $L \cap M$ . Furthermore,  $Q(A_L)_0 = K_0$  and so  $K_0 \subset \operatorname{Frac} A$ . Since the relative interior of  $\sigma^\vee$  contains a basis of  $M$ , we obtain our first claim. This argument also proves that  $\sigma^\vee$  is the weight cone of  $A$ . The proof of the normality of  $A[C, \mathfrak{D}]$  is similar to that of Theorem 2.6 (i) and is left to the reader.

Let us prove that  $A[C, \mathfrak{D}]$  is finitely generated. Passing to a subdivision we may assume that  $\sigma^\vee$  is a strongly convex regular cone. Let  $(e_1, \dots, e_n)$  be a basis of  $M$  such that  $e_1, \dots, e_n$  generate  $\sigma^\vee$ . Let  $d \in \mathbb{Z}_{>0}$  be such that  $\mathfrak{D}(d \cdot e_i)$  is an integral divisor for  $i = 1, \dots, n$ . The line bundles  $\mathcal{O}_C(\mathfrak{D}(d \cdot e_i))$  are globally generated. Therefore by [Ha2], the  $\mathbf{k}$ -algebra

$$A_d = \bigoplus_{(s_1, \dots, s_n) \in \mathbb{N}^n} H^0 \left( Y, \mathcal{O}_Y \left( \sum_{i=1}^n s_i \mathfrak{D}(d \cdot e_i) \right) \right) \chi^{\sum_{i=1}^n s_i e_i}.$$

is finitely generated. Using the same argument as before, we obtain

$$L := \operatorname{Frac} A_d = \operatorname{Frac} K_0[M_d],$$

where  $M_d$  is the sublattice generated by  $d \cdot e_1, \dots, d \cdot e_n$ . The extension  $\operatorname{Frac} K_0[M]/L$  is finite. Since the integral closure of  $A_d$  in  $\operatorname{Frac} K_0[M]$  is  $A[C, \mathfrak{D}]$ , by [Bou, §V.3.2, Theorem 2], the algebra  $A[C, \mathfrak{D}]$  is finitely generated.  $\square$

In order to have a geometrical interpretation of the previous results 2.6 and 3.6, we can regard an affine variety over  $\mathbf{k}$  as a representable functor in the category of  $k$ -algebras.

**3.10.** Recall that  $\mathbb{G}_m = \mathbb{G}_{m, \mathbf{k}}$  denotes the multiplicative group scheme of the field  $\mathbf{k}$ . A (split) *algebraic torus*  $\mathbb{T}$  over  $\mathbf{k}$  is an algebraic group isomorphic to  $\mathbb{G}_m^n$ , for some integer  $n \geq 1$ . Let  $X$  be an affine variety represented by an integral finitely generated  $\mathbf{k}$ -algebra  $A = \mathbf{k}[X]$ . Assume that a torus  $\mathbb{T}$  acts on  $X$ . This means that there is a natural transformation

$$\Phi : \mathbb{T} \times X \rightarrow X$$

such that for every  $\mathbf{k}$ -algebra  $B$ , the map  $\Phi(B)$  is a  $\mathbb{T}(B)$ -action on the set  $X(B)$ . The coaction gives a morphism of  $\mathbf{k}$ -algebras

$$A \rightarrow A \otimes_{\mathbf{k}} \mathbf{k}[M],$$

where  $M$  is the character lattice of  $\mathbb{T}$  and  $\mathbf{k}[M]$  is the semigroup algebra of  $M$ . Using this morphism, we endow  $A$  with an  $M$ -grading. Conversely an  $M$ -grading on the algebra  $A$  yields naturally a  $\mathbb{T}$ -action on  $X$  (see [SGA III, 4.7.3]).

We say that  $X$  is an affine  $\mathbb{T}$ -variety if  $A$  is normal and if the weight semigroup of  $A$  generates  $M$ . We define as well the *complexity* of a  $\mathbb{T}$ -action as the transcendence degree over  $\mathbf{k}$  of the field

$$Q(A)_0 = \left\{ \frac{a}{b}, a, b \in A \text{ homogeneous of same degree} \right\} \cup \{0\}.$$

**3.11.** More generally, if  $C$  is an arbitrary smooth algebraic curve over  $\mathbf{k}$  then we define a proper  $\sigma$ -polyhedral divisors  $\mathfrak{D}$  over  $C$  in the following way. There is two cases. If  $C$  is affine then  $\mathfrak{D}$  is a polyhedral divisor over the Dedekind Domain  $A_0 = \mathbf{k}[C]$ . In this case, we denote by  $A[C, \mathfrak{D}]$  the algebra  $A_0[\mathfrak{D}]$  given in 2.4. Otherwise  $C$  is projective and the polyhedral divisor  $\mathfrak{D}$  verifies Definition 3.5.

Combining 2.6 and 3.6, one can describes an affine  $\mathbb{T}$ -variety of complexity one over the field  $\mathbf{k}$  by a polyhedral divisors on a smooth curve.

**Corollary 3.12.** *Let  $\mathbb{T}$  be a split algebraic torus over  $\mathbf{k}$ . Denote by  $N$  its one parameter lattice and by  $M$  its character lattice with the natural duality. Then the following hold.*

- (i) *For every affine  $\mathbb{T}$ -variety  $X$  of complexity one represented by an  $M$ -graded algebra  $A$  with weight cone  $\sigma^\vee \subset M_{\mathbb{Q}}$  there is a proper  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  on a smooth curve  $C$  over  $\mathbf{k}$  such that  $A \simeq A[C, \mathfrak{D}]$ .*
- (ii) *Conversely, if  $\sigma \subset N_{\mathbb{Q}}$  is a strongly convex cone and if  $\mathfrak{D}$  is a proper  $\sigma$ -polyhedral divisor on a smooth curve  $C$  over  $\mathbf{k}$ , then*

$$X[C, \mathfrak{D}] := \text{Hom}(A[C, \mathfrak{D}], -)$$

*is an affine  $\mathbb{T}$ -variety of complexity one.*

*Proof.* The assertion (ii) follows immediately from 2.6 (i) and 3.6 (ii).

Given an affine  $\mathbb{T}$ -variety of complexity one, consider the corresponding  $M$ -graded algebra  $A$ . For any  $m \in M$ , let

$$Q(A)_m = \left\{ \frac{a}{b}, a \in A_{m+m'}, b \in A_{m'}, b \neq 0 \right\}.$$

For a vector  $m \in M$ , one can find a nonzero element  $a_m$  such that for all  $m, m' \in M$ ,  $a_{m+m'} = a_m \cdot a_{m'}$ . Hence  $A$  is an  $M$ -graded subring of

$$R := \bigoplus_{m \in M} Q(A)_m = \bigoplus_{m \in M} Q(A)_0 \cdot a_m.$$

Assume that  $A_0 \neq \mathbf{k}$ . Let us show that in this case  $K_0 = Q(A)_0$ . By normality of  $A$ , every algebraic element of  $K_0$  over  $\mathbf{k}$  belongs to  $\mathbf{k}$ . Therefore the transcendence degree of  $K_0/\mathbf{k}$  is greater than 1 and so  $Q(A)_0/K_0$  is algebraic. Using the same argument as that in 3.7, one concludes that  $K_0 = Q(A)_0$ . Since  $A$  is normal and finitely generated,  $A_0$  is too. The ring  $A_0$  is a Dedekind Domain. Identifying  $R$  with the semigroup algebra  $K_0[M]$ ,  $A$  verifies the assumption of 2.6 (ii) and so  $A \simeq A[C, \mathfrak{D}]$  for some polyhedral divisor  $\mathfrak{D}$  over the curve  $C = \text{Spec } A_0$ .

In the case where  $A_0 = \mathbf{k}$  the proof is similar by using Theorem 3.6 (i).  $\square$

#### REFERENCES

- [AH] K. Altmann, J. Hausen. *Polyhedral divisors and algebraic torus actions*. Math. Ann. **334**. (2006). 557-607.
- [AHS] K. Altmann, J. Hausen, H. Süß. *Gluing affine torus actions via divisorial fans*. Transform. Groups. **13**. (2008). 215-242.
- [Bou] N. Bourbaki. *Elements of Mathematics*. Commutative Algebra. Vol 8. Hermann. (1972).
- [CLS] D. Cox, J. Little, H. Schenck. *Toric Varieties*. Graduate Studies in Mathematics. AMS. **124**. (2011).
- [De] M. Demazure. *Anneaux gradués normaux*. Trav. Cours 37. (1988). 35-68.
- [SGA III] M. Demazure. A. Grothendieck. *Séminaires de Géométrie Algébrique III*. Propriétés Générales des Schémas en Groupes. Lecture Notes in Mathematics. **151, 152, 153**. (1970).
- [FZ] H. Flenner, M. Zaidenberg. *Normal affine surfaces with  $\mathbb{C}^*$ -actions* Osaka J.Math. **40**. (2003). 981-1009.
- [GY] S. Goto, K. Yamagishi. *Finite Generation of Noetherian Graded Rings*. Proc. Amer. **89**. (1983).
- [EGA II] A. Grothendieck. *Éléments de Géométrie Algébrique II*. Etude globale élémentaire de quelques classes de morphismes (en collaboration avec J. Dieudonné). Pub. Math. I.H.É.S. tome 8. (1961). 5-222.
- [Ha2] R. Hartshorne. *Ample Vector Bundles*. Pub. Math. IHÉS. **29**. (1966). 63-94.
- [Ha] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer-Verlag. **52**. (1977).
- [KKMS] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I*. Lecture Notes in Mathematics, Springer-Verlag, Berlin. **339**. (1973).
- [Ke] G. Kemper. *A course in Commutative Algebra*. Graduate Texts in Mathematics, Springer-Verlag, Berlin. **256**. (2011).
- [La] K. Langlois. *Clôture intégrale et opérations de tores algébriques de complexité un dans les variétés affines*. arxiv: 1202.1261.v2 (2012). 25p.
- [Li] A. Liendo. *Affine  $\mathbb{T}$ -varieties of complexity one and locally nilpotent derivations*. Transform. Groups. **15**. (2010). 389-425.
- [Mi] J.S. Milne. *Algebraic Number Theory*. <http://www.jmilne.org/math/CourseNotes/ant.html>.
- [St] H. Stichtenoth *Algebraic Function Fields and Codes*. Universitext, Springer-Verlag. (1993).
- [Ti] D. Timashev. *Torus actions of complexity one*. In : Toric topology. Contemp. Math. AMS. **60**. (2008). 349-364.

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