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# Time optimal control for a nonholonomic system with state constraint

Jérôme Lohéac<sup>1,2,3</sup> and Jean-François Scheid<sup>1,2,3</sup>

<sup>1</sup>Université de Lorraine, Institut Élie Cartan de Nancy, UMR 7502, BP239,  
F-54506 Vandœuvre-lès-Nancy Cedex, France

<sup>2</sup>CNRS, Institut Élie Cartan de Nancy, UMR 7502, BP239,  
F-54506 Vandœuvre-lès-Nancy Cedex, France

<sup>3</sup>Inria, Villers-lès-Nancy, F-54600, France

loheac@iecn.u-nancy.fr, scheid@iecn.u-nancy.fr

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**Abstract.** The aim of this paper is to tackle the time optimal controllability of an  $(n + 1)$ -dimensional nonholonomic integrator. In the optimal control problem we consider, the state variables are subject to a bound constraint. We give a full description of the optimal control and optimal trajectories are explicitly obtained. The optimal trajectories we construct, lie in a 2-dimensional plane and they are composed of arcs of circle.

**Keywords.** Nonholonomic systems, Brockett integrator, time optimal control, Pontryagin's maximum principle, state constraints.

**AMS subject classifications (2000).** 49J15, 37J60, 93C15, 34H05.

## 1 Introduction

This paper is devoted to the study of the time optimal controllability of an  $(n + 1)$ -dimensional nonholonomic differential system with constraint on the state variables. The optimal control problem we deal with is bilinear with respect to the state and control variables. Nonholonomic systems have been intensively studied in numerous works and we only refer to Bloch [4] where a comprehensive survey is given in connection with control theory. Minimum time control problems for nonholonomic systems have also been considered in the literature and explicit optimal solutions have been computed for the case  $n = 2$  when no constraint is imposed on the state variables. We refer to Bloch [4] where the Lagrangian approach is used in order to compute optimal controls when no state constraint is imposed. We also refer to Beals, Gaveau and Greiner [3] (see also Prieur and Trélat [16]) where the minimal time control problem is solved through a complete description of the geodesics in a sub-Riemannian manifold. As far as we know, there is no explicit optimal solutions when constraints are imposed on the state variables. The aim of this paper is to describe the optimal solutions to the minimal time control problem for a general nonholonomic system with a state constraint. This will be achieved with the use of Pontryagin's maximum principle extended to the state constraints cases (see Bonnans and Hermant [5], Harlt, Sethi and Vickson [12] or Ioffe and Tihomirov [14]).

The general nonholonomic system we consider is a multi-dimensional generalization of the Brockett integrator, see [6]. We mention that our generalization is different from the one originally given in [6]

but follows the one given by Beals, Gaveau and Greiner [3] for  $(2n + 1)$ -dimensional Heissenberg systems.

Before introducing in details the general nonholonomic system we are interested in this work, we give some notations used throughout this paper. Let  $n \geq 2$  be an integer and let  $M$  denotes a  $n \times n$  real non-symmetric matrix (i.e.  $M^T \neq M$ ). By  $\langle \cdot, \cdot \rangle$  we denote the inner product in  $\mathbb{R}^n$  and  $|\cdot|$  is the corresponding Euclidean norm. The null vector in  $\mathbb{R}^n$  is denoted by  $\mathbf{0}_n$ . We fix two values  $c \in (0, +\infty]$  (possibly  $c = +\infty$ ) and  $\bar{y} \in \mathbb{R}$ ,  $\bar{y} \neq 0$  which will stand respectively for the bound of the state constraint and the nontrivial target to be reached for one of the variable components. Then, the optimal control problem we consider reads as follows:

**Time optimal control problem.** *Find the minimal time  $T^* > 0$ , such that there exist functions  $(y, \mathbf{x}) : [0, T^*] \rightarrow \mathbb{R} \times \mathbb{R}^n$  and a control variable  $\mathbf{u} : [0, T^*] \rightarrow \mathbb{R}^n$  satisfying the differential system*

$$\dot{y} = \langle M\mathbf{x}, \mathbf{u} \rangle \tag{1.1}$$

$$\dot{\mathbf{x}} = \mathbf{u} \tag{1.2}$$

with the given initial and final states

$$(y(0), \mathbf{x}(0)) = (0, \mathbf{0}_n), \tag{1.3}$$

$$(y(T^*), \mathbf{x}(T^*)) = (\bar{y}, \mathbf{0}_n), \tag{1.4}$$

and subject to the constraints

$$|\mathbf{u}(t)| \leq 1 \quad (t \in [0, T^*]), \tag{1.5}$$

$$|\mathbf{x}(t)| \leq c \quad (t \in [0, T^*]). \tag{1.6}$$

In the optimal control problem (1.1)–(1.6), functions  $(y, \mathbf{x}) : \mathbb{R}_+ \rightarrow \mathbb{R} \times \mathbb{R}^n$  are the state variables of the system whereas  $\mathbf{u} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is the control variable. Equation (1.3) is the initial condition for the state variables and (1.4) is the final state to be reached in a minimal time. The constraint (1.5) on the control variables is necessary to make system (1.1)–(1.4) relevant. Without any constraint on the boundedness of the control variables, the minimum time control problem does not make sense. Indeed, the control variable can be seen as a velocity variable and if we do not require the control  $\mathbf{u}$  to be bounded, the minimal time tends to zero. The constraint (1.6) on the state variable  $\mathbf{x}$  is mainly motivated by the work of Lohéac, Scheid, Tucsnaak [15]. In this paper, the authors tackle a time optimal problem arising from fluid dynamics and self-propulsion of a deformable body immersed into a Stokes fluid. The body is able to move into the fluid by changing its shape. In this context, the state variable  $y$  stands for the position of the mass center of the body in the fluid, whereas  $\mathbf{x}$  corresponds to the magnitudes of radial deformations. In order to deal with small deformations, a state constraint of type (1.6) has to be imposed on the magnitudes  $\mathbf{x}$ .

The paper is organized as follows. In Section 2, after giving some basic results on the controllability of the nonholonomic differential system (1.1)–(1.6), we state the main result of this paper. Optimal trajectories are fully described and optimal solutions to the constrained nonholonomic problem (1.1)–(1.6) are explicitly given. In Section 3, general properties on the optimal control problem are obtained. In particular, a "bang-bang" property is established for the optimal control. Next, in Section 4 we apply an Hamiltonian approach taking into account the state constraint and we make use of Pontryagin's maximum principle in order to get fine properties on the optimal control. Section 5 is devoted to the 2-dimensional control case ( $n = 2$ ) where explicit optimal solutions are obtained. Finally, in Section 6 we prove the main result in the  $n$ -dimensional control case by using the explicit solutions builds in Section 5.

## 2 Controllability results and statement of the main result

In this section we start to establish some controllability results on the nonholonomic system (1.1)–(1.6). The main result of this paper is given at the end of the section.

For a given control variable  $\mathbf{u}$ , the Cauchy problem (1.1)–(1.3) admits a unique solution. More precisely, one can easily checked that the following existence result holds.

**Theorem 2.1.** *Let  $\mathbf{u} \in L^\infty(\mathbb{R})^n$  be given. There exists a unique solution  $(y, \mathbf{x}) \in W^{1,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})^n$  to the differential system*

$$\begin{aligned} \dot{y} &= \langle M\mathbf{x}, \mathbf{u} \rangle \\ \dot{\mathbf{x}} &= \mathbf{u} \end{aligned}$$

with initial conditions

$$(y(0), \mathbf{x}(0)) = (0, \mathbf{0}_n) \in \mathbb{R} \times \mathbb{R}^n.$$

For every  $\mathbf{u} \in L^\infty(\mathbb{R}_+)^n$ , according to Theorem 2.1, we can defined for  $t \geq 0$ ,

$$\begin{aligned} \mathcal{X}_{\mathbf{u}}(t) &= \mathbf{x}(t), \\ \mathcal{Y}_{\mathbf{u}}(t) &= y(t), \end{aligned}$$

where  $(y, \mathbf{x}) \in W^{1,\infty}(\mathbb{R}_+) \times W^{1,\infty}(\mathbb{R}_+)^n$  is the solution of (1.1)–(1.3) with the control variable  $\mathbf{u}$ .

We now address the controllability problem associated to the differential system (1.1)–(1.3).

**Controllability problem.** *Find a time  $T > 0$  and a control variable  $\mathbf{u} \in L^\infty(0, T)^n$  such that*

$$(\mathcal{Y}_{\mathbf{u}}(T), \mathcal{X}_{\mathbf{u}}(T)) = (\bar{y}, \mathbf{0}_n) \tag{2.7}$$

and which satisfy the control constraint

$$|\mathbf{u}(t)| \leq 1 \quad (\text{for a.e. } t \in (0, T)) \tag{2.8}$$

together with the state constraint

$$|\mathcal{X}_{\mathbf{u}}(t)| \leq c \quad (t \in [0, T]). \tag{2.9}$$

This problem can be solved by using tools coming from the geometric control theory. The controllability result for problem (2.7)–(2.9) reads as follows.

**Proposition 2.2.** *There exist a time  $T > 0$  and a control variable  $\mathbf{u} \in L^\infty(0, T)^n$  such that (2.7) is satisfied with the constraints (2.8) and (2.9).*

Proposition 2.2 can be proved by using the Chow theorem based on the computation of Lie brackets (see for instance [1, 10]). This result can be obtained by a slight modification of the controllability result proved in [15, Theorem 4.1] (see also [10, Example 3.20] and [4]), so we do not give the proof of Proposition 2.2. We just want to point out that the assumption that the matrix  $M$  is not symmetric i.e.  $M \neq M^\top$ , is crucial in order to obtain non-null Lie bracket.

The controllability result of Proposition 2.2 allows to define the set  $\mathcal{U}^T(M)$  of controls  $\mathbf{u}$  solution of (2.7)–(2.9) in time  $T$  by

$$\mathcal{U}^T(M) = \{ \mathbf{u} \in L^\infty(0, T)^n, |\mathbf{u}| \leq 1 \text{ and } |\mathcal{X}_{\mathbf{u}}| \leq c \text{ a.e. in } (0, T), (\mathcal{Y}_{\mathbf{u}}(T), \mathcal{X}_{\mathbf{u}}(T)) = (\bar{y}, \mathbf{0}_n) \}. \tag{2.10}$$

When no confusion occurs, we will simply write  $\mathcal{U}^T$  instead of  $\mathcal{U}^T(M)$ .

Applying the classical Filippov Theorem (see for instance [1, 8, 12]) to the time optimal control problem, we can easily obtain the following result. For the proof, we refer to [15, Proposition 4.5] where a similar time optimal control result is proved.

**Proposition 2.3.** *The set of times  $T > 0$  such that there exists a control variable  $\mathbf{u} \in L^\infty(0, T)^n$  satisfying (2.7)–(2.9), admits a minimum value  $T^*(M) > 0$ .*

We will say that  $T^*(M)$  defined in Proposition 2.3 is the *optimal time* and a corresponding control  $\mathbf{u} \in \mathcal{U}^{T^*(M)}$  is a *time optimal control*. The optimal time  $T^*(M)$  depends on the matrix  $M$  but when there is no possible confusion, we do not mention this dependency by simply writing  $T^*$ . The 2-tuple  $(T^*, \mathbf{u})$  where  $\mathbf{u} \in \mathcal{U}^{T^*}$  is called an *optimal solution* of (2.7)–(2.9).

We are now in position to give the main result of this paper.

**Theorem 2.4.** *Let  $\lambda^*$  be the largest modulus of the eigenvalues of the skew part of the matrix  $M$ , i.e.*

$$\lambda^* = \max \left\{ |\lambda|, \lambda \in \text{sp} \left( \frac{1}{2}(M - M^\top) \right) \right\}.$$

Let us also define the two real positive values,

$$d^* = \sqrt{\frac{2|\bar{y}|}{\pi\lambda^*}} \quad \text{and} \quad \tau = \frac{\pi c}{2}.$$

Then, the optimal time  $T^*$  of the problem (2.7)–(2.9) is given by:

$$T^* = \begin{cases} \pi d^* & \text{if } c \geq d^*, \\ \frac{\pi d^{*2}}{2c} + \tau & \text{otherwise.} \end{cases} \quad (2.11)$$

Moreover, the problem (2.7)–(2.9) admits a time optimal control  $\mathbf{u} \in C^0([0, T^*])^n$  given by :

$$\mathbf{u}(t) = \gamma_1(t)\mathbf{w}_1 + \gamma_2(t)\mathbf{w}_2 \quad (t \in [0, T^*]), \quad (2.12)$$

where  $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^n$  are two orthonormal vectors satisfying

$$\frac{1}{2}(M - M^\top)\mathbf{w}_1 = \lambda^*\mathbf{w}_2 \quad \text{and} \quad \frac{1}{2}(M - M^\top)\mathbf{w}_2 = -\lambda^*\mathbf{w}_1.$$

The function  $\mathbf{v} = (\gamma_1, \gamma_2) \in C^0([0, T^*])^2$  is defined as follows.

- if  $c \geq d^*$ , we have

$$\mathbf{v}(t) = R \left( \text{sign}(\bar{y}) \frac{2\pi}{T^*} t \right) \mathbf{v}_0 \quad (t \in [0, T^*]). \quad (2.13)$$

- if  $c < d^*$ , we have

$$\mathbf{v}(t) = \begin{cases} R \left( \frac{2 \text{sign}(\bar{y})}{c} t \right) \mathbf{v}_0 & \text{if } t \in [0, \tau), \\ -R \left( \frac{\text{sign}(\bar{y})}{c} (t - \tau) \right) \mathbf{v}_0 & \text{if } t \in [\tau, T^* - \tau), \\ R \left( \frac{\text{sign}(\bar{y})}{c} (2t - 3(T^* - \tau)) \right) \mathbf{v}_0 & \text{if } t \in [T^* - \tau, T^*]. \end{cases} \quad (2.14)$$

In the above,  $R$  stands for the rotation matrix,  $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  for  $\theta \in \mathbb{R}$ . The vector  $\mathbf{v}_0 \in \mathbb{R}^2$  is chosen such that  $|\mathbf{v}_0| = 1$ .

From Theorem 2.4, one can give a complete description of the optimal trajectory  $t \mapsto \mathcal{X}_{\mathbf{u}}(t)$  associated to the optimal control  $\mathbf{u}$  given by (2.12). Indeed, integrating (2.12) yields  $\mathcal{X}_{\mathbf{u}}(t) = \alpha_1(t)\mathbf{w}_1 + \alpha_2(t)\mathbf{w}_2$  for all  $t \in [0, T^*]$  with  $\alpha_1(t) = \int_0^t \gamma_1(s) ds$  and  $\alpha_2(t) = \int_0^t \gamma_2(s) ds$ . As a result, we obtain that the optimal trajectory  $t \mapsto \mathcal{X}_{\mathbf{u}}(t)$  associated to  $\mathbf{u}$  lies on the plane spanned by  $\{\mathbf{w}_1, \mathbf{w}_2\}$ . In the case where  $c \geq d^*$  i.e. when the bound of the state constraint is large enough, the optimal trajectory  $t \mapsto \mathcal{X}_{\mathbf{u}}(t)$  describes a circle of diameter  $d^* \leq c$  passing through the origin  $\mathbf{0}_n$ . When  $c < d^*$ , the optimal trajectory  $t \mapsto \mathcal{X}_{\mathbf{u}}(t)$  is a  $C^1$ -curve formed by arcs of circles. In particular, in that case one can easily check that  $|\mathcal{X}_{\mathbf{u}}(t)| = c$  for all  $t \in [\tau, T^* - \tau]$  and  $|\mathcal{X}_{\mathbf{u}}(t)| < c$  for all  $t \in [0, \tau] \cup (T^* - \tau, T^*]$ . An example of an optimal trajectory is shown in Figure 1 in the case where  $c < d^*$ . In Figure 1, the trajectory  $t \mapsto \mathcal{X}_{\mathbf{u}}(t)$  is displayed in the plane spanned by the two orthonormal vectors  $\mathbf{w}_1$  and  $\mathbf{w}_2$ .

**Remark 2.5.** We point out that the minimal time is a decreasing function of the largest modulus  $\lambda^*$ . More precisely, for any couple  $(M, \tilde{M})$  of non-symmetric square matrices, we have:

$$\max\{|\lambda|, \lambda \in \text{sp}(M - M^\top)\} < \max\{|\lambda|, \lambda \in \text{sp}(\tilde{M} - \tilde{M}^\top)\} \Rightarrow T^*(M) < T^*(\tilde{M}),$$

whatever the size of the matrices are.

**Remark 2.6.** In the case where the bound of the state constraint is large enough, i.e. when  $c \geq d^*$ , the state constraint does not play any role since the optimal trajectory will never reach the constraint. In the case of time optimal control for the Brockett integrator, explicit optimal solutions have been already obtained in [16] for the 2-dimensional control case  $n = 2$ , without constraint on the state variables. We also refer to [3] where a complete description of the geodesics of  $(2n + 1)$ -dimensional Heisenberg groups are given. For the latter, no state constraint are considered in their study.

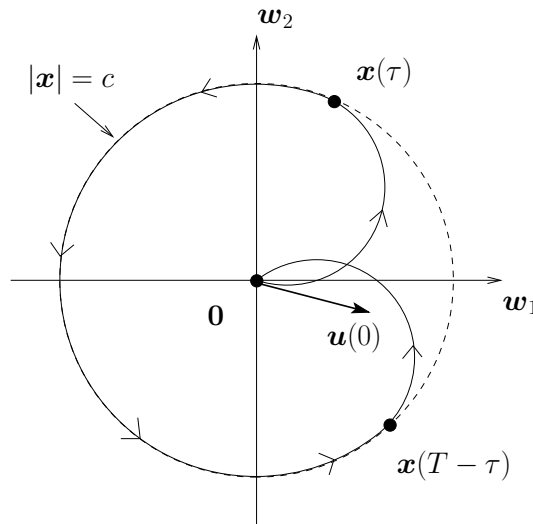


Figure 1: Optimal trajectory  $t \mapsto \mathcal{X}_{\mathbf{u}}(t)$  in the case where  $c < d^*$ .

### 3 Preliminary properties of the optimal solution

In this section we establish some properties of the time optimal solutions of the control problem (2.7)–(2.9). These properties will be very useful when integrating the maximum principle in Section 4 in order to obtain fine properties of the optimal trajectories. We first mention a result about the reversibility property of the control problem.

**Proposition 3.1.** *Let  $T > 0$  and  $\mathbf{u} \in \mathcal{U}^T$  be solution of the control problem (2.7)–(2.9). We define the function  $\tilde{\mathbf{u}} \in L^\infty(0, T)^n$  by  $\tilde{\mathbf{u}}(t) = \mathbf{u}(T - t)$  for almost every  $t \in (0, T)$ . Then, we have*

$$(\mathcal{Y}_{\tilde{\mathbf{u}}}(T), \mathcal{X}_{\tilde{\mathbf{u}}}(T)) = (-\bar{y}, \mathbf{0}_n)$$

with  $|\tilde{\mathbf{u}}(t)| \leq 1$  for a.e.  $t \in (0, T)$  and  $|\mathcal{X}_{\tilde{\mathbf{u}}}(t)| \leq c$  for all  $t \in (0, T)$ .

In other words, if  $\mathbf{u}$  is a control steering the state trajectory from  $(0, \mathbf{0}_n)$  to  $(\bar{y}, \mathbf{0}_n)$  in time  $T$ , then  $\tilde{\mathbf{u}} = \mathbf{u}(T - \cdot)$  is a control which steers the state trajectory from  $(0, \mathbf{0}_n)$  to  $(-\bar{y}, \mathbf{0}_n)$  in the same time  $T$ .

The next result is concerned with the non-symmetric matrix  $M$  and it asserts that only the skew part of  $M$  plays a role in the control problem (2.7)–(2.9). More precisely, we have the following result.

**Proposition 3.2.** *For every  $T > 0$ , we have  $\mathcal{U}^T(M) = \mathcal{U}^T(\frac{1}{2}(M - M^\top))$ . Moreover,  $T^*(M) = T^*(\frac{1}{2}(M - M^\top))$ .*

This means that a function  $\mathbf{u} \in \mathcal{U}^T(M)$  is solution of the control problem (2.7)–(2.9) if and only if it is a solution of the control problem obtained from (2.7)–(2.9) by replacing the matrix  $M$  with its skew part, that is to say by considering the equation

$$\dot{y} = \left\langle \frac{1}{2} (M - M^\top) \mathbf{x}, \mathbf{u} \right\rangle,$$

in place of equation (1.1).

*Proof of Proposition 3.2.* We first prove that  $\mathcal{U}^T(M) \subset \mathcal{U}^T(\frac{1}{2}(M - M^\top))$ . Let us choose  $\mathbf{u} \in \mathcal{U}^T(M)$  and we set  $\mathbf{x} = \mathcal{X}_{\mathbf{u}}$ ,  $y = \mathcal{Y}_{\mathbf{u}}$ . Equation (1.1) can be written in the form:

$$\dot{y} = \left\langle \frac{1}{2} (M + M^\top) \mathbf{x}, \dot{\mathbf{x}} \right\rangle + \left\langle \frac{1}{2} (M - M^\top) \mathbf{x}, \dot{\mathbf{x}} \right\rangle.$$

Since the matrix  $\frac{1}{2} (M + M^\top)$  is symmetric, we have

$$\left\langle \frac{1}{2} (M + M^\top) \mathbf{x}, \dot{\mathbf{x}} \right\rangle = \frac{1}{2} \frac{d}{dt} \left\langle \frac{1}{2} (M + M^\top) \mathbf{x}, \mathbf{x} \right\rangle.$$

Integrating  $\left\langle \frac{1}{2} (M + M^\top) \mathbf{x}, \dot{\mathbf{x}} \right\rangle$  over  $t \in [0, T]$  yields  $\int_0^T \left\langle \frac{1}{2} (M + M^\top) \mathbf{x}, \dot{\mathbf{x}} \right\rangle dt = 0$  due to the initial and final conditions  $\mathbf{x}(0) = \mathbf{x}(T) = \mathbf{0}_n$ . Then, we have

$$y(T) - y(0) = \int_0^T \left\langle \frac{1}{2} (M - M^\top) \mathbf{x}, \dot{\mathbf{x}} \right\rangle dt.$$

As a result, the symmetric part of  $M$  does not play any role in the control system. This fact clearly implies that  $\mathcal{U}^T(M) \subset \mathcal{U}^T(\frac{1}{2}(M - M^\top))$ . In a very similar way, one can prove that  $\mathcal{U}^T(\frac{1}{2}(M - M^\top)) \subset \mathcal{U}^T(M)$ . The Proposition is then proved.  $\square$

The next Proposition shows that the time optimal control is "bang-bang", that is to say the bound of the constraint (2.8) is reached by an optimal control for almost every time.

**Proposition 3.3.** *Let  $T^* > 0$  and  $\mathbf{u} \in \mathcal{U}^{T^*}$  be an optimal solution of the control problem (2.7)–(2.9). Then,  $\mathbf{u}$  satisfies for almost every  $t \in (0, T^*)$ ,*

$$|\mathbf{u}(t)| = 1 \quad (3.1)$$

and

$$\mathbf{u}(t) \in \text{Ker}(M - M^\top)^\perp. \quad (3.2)$$

*Proof.* To prove (3.1), we argue by contradiction. Let us assume that there exists a time optimal control  $\mathbf{u}$  which does not satisfies (3.1). We define the function  $s \in W^{1,\infty}(0, T^*)$  by

$$s(t) = \int_0^t |\mathbf{u}(r)| \, dr \quad (t \in [0, T^*]). \quad (3.3)$$

Let us denote  $\tilde{T} = s(T^*)$ . Since  $\bar{y} \neq 0$ , the optimal control  $\mathbf{u}$  is necessarily a non-null function and then the function  $s$  is non-constant on  $[0, T^*]$ . As a result, we have that  $\tilde{T} > 0$ . In addition, from the fact that  $|\mathbf{u}(t)| \leq 1$  for almost every  $t \in (0, T^*)$  and since we assume that  $\mathbf{u}$  does not satisfy (3.1), we deduce that  $\tilde{T} = s(T^*) < T^*$ . Moreover, since  $s$  is an nondecreasing function, there exists a right inverse function  $s_r : [0, \tilde{T}] \rightarrow [0, T^*]$  such that  $s(s_r(\sigma)) = \sigma$  for every  $\sigma \in [0, \tilde{T}]$ . In addition,  $s_r$  is a nondecreasing function. This fact implies that  $s_r$  is almost everywhere differentiable in  $[0, \tilde{T}]$  and we have that  $\dot{s}(s_r) \dot{s}_r = 1$  a.e. in  $[0, \tilde{T}]$ . Using (3.3), we deduce that  $|\mathbf{u}(s_r)| \dot{s}_r = 1$  a.e. in  $[0, \tilde{T}]$ . Therefore, we obtain

$$|\mathbf{u}(s_r)| \neq 0 \quad \text{a.e. in } [0, \tilde{T}]$$

Now, we introduce the following functions defined for almost every  $\sigma \in [0, \tilde{T}]$  by

$$\tilde{\mathbf{u}}(\sigma) = \frac{\mathbf{u}(s_r(\sigma))}{|\mathbf{u}(s_r(\sigma))|}, \quad (3.4)$$

$$\tilde{y}(\sigma) = y(s_r(\sigma)), \quad (3.5)$$

$$\tilde{\mathbf{x}}(\sigma) = \mathbf{x}(s_r(\sigma)), \quad (3.6)$$

where  $y = \mathcal{Y}_{\mathbf{u}} \in W^{1,\infty}(0, T^*)$  and  $\mathbf{x} = \mathcal{X}_{\mathbf{u}} \in W^{1,\infty}(0, T^*)^n$ . We are going to prove that  $\tilde{\mathbf{u}}$  is a control function providing a solution to (2.7)–(2.9) in the smaller time  $\tilde{T} < T^*$ . This will lead to a contradiction to the fact that  $T^*$  is the minimal time.

We first notice that  $\tilde{\mathbf{u}} \in L^\infty(0, \tilde{T})^n$  and  $(\tilde{y}, \tilde{\mathbf{x}}) \in W^{1,\infty}(0, \tilde{T}) \times W^{1,\infty}(0, \tilde{T})^n$ . For almost every  $\sigma \in [0, \tilde{T}]$ , we have

$$\begin{aligned} \frac{d\tilde{y}}{d\sigma}(\sigma) &= \frac{1}{|\mathbf{u}(s_r(\sigma))|} \dot{y}(s_r(\sigma)) = \langle M\tilde{\mathbf{x}}(\sigma), \tilde{\mathbf{u}}(\sigma) \rangle, \\ \frac{d\tilde{\mathbf{x}}}{d\sigma}(\sigma) &= \frac{1}{|\mathbf{u}(s_r(\sigma))|} \dot{\mathbf{x}}(s_r(\sigma)) = \tilde{\mathbf{u}}(\sigma). \end{aligned}$$

Thus, we deduce that  $\tilde{y} = \mathcal{Y}_{\tilde{\mathbf{u}}}$  and  $\tilde{\mathbf{x}} = \mathcal{X}_{\tilde{\mathbf{u}}}$ . The constraints (2.8) and (2.9) are fulfilled by the control  $\tilde{\mathbf{u}}$ . In addition, we have

$$\begin{aligned} \tilde{y}(0) &= 0 & \text{and} & \quad \tilde{\mathbf{x}}(0) = \mathbf{0}_n, \\ \tilde{y}(\tilde{T}) &= y(T^*) = \bar{y} & \text{and} & \quad \tilde{\mathbf{x}}(\tilde{T}) = \mathbf{x}(T^*) = \mathbf{0}_n. \end{aligned}$$



Then  $(\tilde{T}, \tilde{\mathbf{u}})$  is a solution to the control system (2.7)–(2.9) and since  $\tilde{T} < T^*$ , we obtain a contradiction. The *bang-bang* property (3.1) is then proved.

Now, we turn to the proof of (3.2). Using the space decomposition

$$\mathbb{R}^n = \text{Ker} \left( \frac{1}{2}(M - M^\top) \right) \oplus \text{Ker} \left( \frac{1}{2}(M - M^\top) \right)^\perp, \quad (3.7)$$

we split  $\mathbf{u}(t) = \mathbf{u}_0(t) + \mathbf{u}_1(t)$  with  $\mathbf{u}_0(t) \in \text{Ker}(\frac{1}{2}(M - M^\top))$  and  $\mathbf{u}_1 \in \text{Ker}(\frac{1}{2}(M - M^\top))^\perp$ , for almost every  $t \in [0, T^*]$ . Let us define  $\mathbf{x}_i = \mathcal{X}_{\mathbf{u}_i}$  for  $i \in \{0, 1\}$ . We clearly have that  $\mathcal{X}_{\mathbf{u}} = \mathbf{x}_0 + \mathbf{x}_1$  with  $\mathbf{x}_0(t) \in \text{Ker}(\frac{1}{2}(M - M^\top))$  and  $\mathbf{x}_1(t) \in \text{Ker}(\frac{1}{2}(M - M^\top))^\perp$ , for every  $t \in [0, T]$ . Expanding the scalar product in (1.1) with the splitting (3.7) yields

$$\mathcal{Y}_{\mathbf{u}}(T^*) = \int_0^{T^*} \left\langle \frac{1}{2} (M + M^\top) \mathcal{X}_{\mathbf{u}}(t), \mathbf{u}(t) \right\rangle dt + \int_0^{T^*} \left\langle \frac{1}{2} (M - M^\top) \mathcal{X}_{\mathbf{u}}(t), \mathbf{u}(t) \right\rangle dt.$$

We have that  $\mathbf{u} = \dot{\mathcal{X}}_{\mathbf{u}}$  and  $(M + M^\top)$  is a symmetric matrix, thus the first integral term in the above relation is null (see the proof of Proposition 3.2). Then we obtain

$$\begin{aligned} \mathcal{Y}_{\mathbf{u}}(T^*) &= \int_0^{T^*} \left( \left\langle \frac{1}{2} (M - M^\top) \mathbf{x}_0(t), \mathbf{u}(t) \right\rangle + \left\langle \frac{1}{2} (M - M^\top) \mathbf{x}_1(t), \mathbf{u}_0(t) \right\rangle + \right. \\ &\quad \left. + \left\langle \frac{1}{2} (M - M^\top) \mathbf{x}_1(t), \mathbf{u}_1(t) \right\rangle \right) dt \\ &= \int_0^{T^*} \left\langle \frac{1}{2} (M - M^\top) \mathbf{x}_1(t), \mathbf{u}_1(t) \right\rangle dt = \mathcal{Y}_{\mathbf{u}_1}(T^*). \end{aligned}$$

Thus, we conclude that  $(T^*, \mathbf{u}_1)$  is an optimal solution, that is  $\mathbf{u}_1 \in \mathcal{U}^{T^*}$ . Since  $\mathbf{u}$  and  $\mathbf{u}_1$  are time optimal controls, they both satisfy the *bang-bang* property (3.1) and consequently we have that  $\mathbf{u}_0 = \mathbf{0}_n$  and  $\mathbf{u} = \mathbf{u}_1$ . This ends the proof.  $\square$

## 4 Maximum principle

Before considering the Hamiltonian approach of the optimal problem, we recall some basic facts about the Radon measures and bounded variations functions. We say that  $\eta$  is a Radon measure on  $[0, T]$  if it is a regular Borel measure which is finite on the compacts. For more details on Radon measure, we refer to [11]. The space  $BV(0, T)$  of functions of bounded variations is defined by

$$BV(0, T) = \left\{ \mu : [0, T] \rightarrow \mathbb{R} \text{ measurable, } \sup_{n \in \mathbb{N}^*} \left( \sum_{k=0}^n |\mu(t_{k+1}) - \mu(t_k)| \right) < +\infty \right\},$$

where the supremum is taken over all finite partitions  $0 = t_0 < t_1 < \dots < t_n = T$ . For a complete description and properties of functions of bounded variations, we refer to [2] (see also [11]). We only recall the basic fact that for every function  $\mu \in BV(0, T)$  and  $t \in (0, T)$ , one can define the left and right limits

$$\mu(t^-) = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \mu(t - \varepsilon), \quad \mu(t^+) = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \mu(t + \varepsilon).$$

and the jump value is denoted by

$$[\mu](t) = \mu(t^+) - \mu(t^-).$$

Now, we consider the Hamiltonian approach so as to obtain necessary optimality conditions for the time optimal problem. In order to take in consideration the state constraint, we follow a classical procedure, as described for instance in [14, 5, 9, 7]. The Hamiltonian of the system (2.7)–(2.9) for the minimal time problem is the function  $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  defined by (see [15, Section 5]):

$$\mathcal{H}(\mathbf{x}, \mathbf{u}, p_0, \mathbf{p}, s_0) = \langle p_0 M \mathbf{x} + \mathbf{p}, \mathbf{u} \rangle + s_0. \quad (4.1)$$

In order to take into account the state constraint (2.9), we introduce the function  $g_c \in C^\infty(\mathbb{R}^n, \mathbb{R})$  defined, for every  $\boldsymbol{\xi} \in \mathbb{R}^n$ , by

$$g_c(\boldsymbol{\xi}) = \begin{cases} \frac{1}{2}(|\boldsymbol{\xi}|^2 - c^2) & \text{if } c \in \mathbb{R}_+^*, \\ -1 & \text{if } c = +\infty. \end{cases} \quad (4.2)$$

Let  $T > 0$  be the minimal time corresponding to an optimal control  $\mathbf{u} \in L^\infty(0, T)^n$  with the associated state trajectory  $(y, \mathbf{x}) = (\mathcal{Y}_{\mathbf{u}}, \mathcal{X}_{\mathbf{u}}) \in W^{1, \infty}(0, T)^{n+1}$ . We define the set

$$E_c = \{t \in [0, T], |\mathcal{X}_{\mathbf{u}}(t)| = c\}.$$

The Pontryagin maximum principle (see [14, Section 5.2.1, Theorem 1] or [5, Theorem 2.2]) asserts that there exist  $s_0 \leq 0$ ,  $\mathbf{q}_0 \in \mathbb{R}^n$ ,  $\mathbf{q}_1 \in \mathbb{R}^n$ ,  $p_0 \in \mathbb{R}$ ,  $\mathbf{p} \in BV(0, T)^n$  and a non-negative Radon measure  $\eta$  with support in  $E_c$ , not all null, such that (2.7)–(2.9) hold together with the co-state equations

$$\mathbf{p}(t) = \mathbf{q}_1 + \int_t^T p_0 M^\top \mathbf{u} dt - \int_{[t, T]} \nabla g_c(\mathbf{x}) d\eta, \quad (4.3)$$

$$\mathbf{p}(0) = \mathbf{q}_0 \quad (4.4)$$

and in addition

$$\mathcal{H}(\mathbf{x}, \mathbf{u}, p_0, \mathbf{p}, s_0) = \max_{\substack{\mathbf{v} \in \mathbb{R}^n \\ |\mathbf{v}| \leq 1}} \mathcal{H}(\mathbf{x}, \mathbf{v}, p_0, \mathbf{p}, s_0). \quad (4.5)$$

Since the Hamiltonian  $\mathcal{H}$  and the state constraint function  $g_c$  defined by (4.2) do not depend on the time variable  $t$ , we deduce that (see [14, Section 5.2.2]):

$$\mathcal{H}(\mathbf{x}, \mathbf{u}, p_0, \mathbf{p}, s_0) = 0. \quad (4.6)$$

We emphasize that in the presence of state constraints, the function  $\mathbf{p}$  can be discontinuous because the co-state equation (4.3) involves an integral with respect to the measure  $\eta$ .

In the next Proposition, we give some property of the optimal control  $\mathbf{u}$  satisfying (2.7)–(2.9) together with the adjoint variables satisfying (4.3)–(4.6).

**Proposition 4.1.** *Let us consider  $T > 0$ ,  $\mathbf{u} \in L^\infty(0, T)^n$  and  $(y, \mathbf{x}) = (\mathcal{Y}_{\mathbf{u}}, \mathcal{X}_{\mathbf{u}}) \in W^{1, \infty}(0, T)^{n+1}$  satisfying the control problem (2.7)–(2.9). We also consider  $s_0 \leq 0$ ,  $\mathbf{q}_0 \in \mathbb{R}^n$ ,  $\mathbf{q}_1 \in \mathbb{R}^n$ ,  $p_0 \in \mathbb{R}$ ,  $\mathbf{p} : [0, T] \rightarrow \mathbb{R}^n$  and a non-negative regular measure  $\eta$  with support in  $E_c = \{t \in [0, T], |\mathcal{X}_{\mathbf{u}}(t)| = c\}$ , not all null, satisfying (4.3)–(4.6). Then, we necessarily have that*

$$s_0 < 0, \quad \mathbf{q}_0 \neq \mathbf{0}_n, \quad \mathbf{q}_1 \neq \mathbf{0}_n, \quad p_0 \neq 0,$$

and  $\mathbf{u}, \mathbf{p} \in C^0([0, T]^n) \cap C^\infty([0, T] \setminus E_c)^n$  satisfy

$$\mathbf{u} = \frac{p_0 M \mathbf{x} + \mathbf{p}}{-s_0} \quad \text{in } [0, T]. \quad (4.7)$$

In addition, we have that

$$\langle \mathbf{x}, \mathbf{u} \rangle = 0 \quad \text{in } E_c, \quad (4.8)$$

and the measure  $\eta$  satisfies the property

$$\eta(\{t\}) = 0 \quad \text{for every } t \in [0, T]. \quad (4.9)$$

**Remark 4.2.** A time  $\tau \in (0, T)$  such that  $|\mathbf{x}(\tau)| = c$  and with the property that there exists  $\varepsilon_0 > 0$  such that  $|\mathbf{x}(\tau - \varepsilon)| < c$  for every  $\varepsilon \in (0, \varepsilon_0]$  will be called a reaching time. The property (4.8) implies that the state constraint (2.9) can only be reached tangentially at any reaching time  $\tau$ .

*Proof.* To begin with, we give some properties of the set  $E_c$ . Since  $\eta$  is a regular measure, we obtain from (4.3) that  $\mathbf{p} \in BV(0, T)^n$  and  $\mathbf{p}$  is continuous from the left. Notice that since  $\mathbf{x} \in C^0([0, T])^n$  and  $\mathbf{x}(0) = \mathbf{x}(T) = 0$ , we have that  $E_c$  is a closed set of  $(0, T)$  and  $[0, T] \setminus E_c$  is a set of positive measure.

Since the support of  $\eta$  is included in  $E_c$ , we obtain from (4.3) that  $\mathbf{p}$  is almost everywhere differentiable on the open set  $[0, T] \setminus E_c$  and we have,

$$\dot{\mathbf{p}}(t) = -p_0 M^\top \mathbf{u}(t) \quad \text{for a.e. } t \in (0, T) \setminus E_c. \quad (4.10)$$

• We start to prove that  $s_0 < 0$ . To this end, we assume by contradiction that  $s_0 = 0$ . Then  $\mathbf{q}_0$ ,  $\mathbf{q}_1$ ,  $p_0$ ,  $\mathbf{p}$  and  $\eta$  are not all zero and due to (4.6), we have

$$0 = \langle p_0 M \mathbf{x} + \mathbf{p}, \mathbf{u} \rangle \quad \text{a.e. in } [0, T]. \quad (4.11)$$

We introduce the set  $\mathcal{A} = \{t \in [0, T], p_0 M \mathbf{x}(t) + \mathbf{p}(t) \neq \mathbf{0}_n\}$ . Using the maximum principle (4.5) we deduce that, for almost every  $t \in \mathcal{A}$ , we have  $\mathbf{u}(t) = \frac{p_0 M \mathbf{x}(t) + \mathbf{p}(t)}{|p_0 M \mathbf{x}(t) + \mathbf{p}(t)|}$  and then  $\langle p_0 M \mathbf{x}(t) + \mathbf{p}(t), \mathbf{u}(t) \rangle \neq 0$ . Due to (4.11), we obtain that  $\mathcal{A}$  is a negligible set and thus

$$\mathbf{p} = -p_0 M \mathbf{x} \quad \text{a.e. in } [0, T]. \quad (4.12)$$

The above equation yields  $\mathbf{p} \in W^{1, \infty}(0, T)^n$ . Differentiating (4.12) and comparing the result with the equation (4.10), we obtain

$$\mathbf{0}_n = p_0 (M - M^\top) \mathbf{u}(t) \quad \text{for a.e. } t \in [0, T] \setminus E_c. \quad (4.13)$$

Now, under the assumption  $s_0 = 0$ , we are going to prove that  $p_0 \neq 0$ . Arguing by contradiction, we assume that  $p_0 = 0$ . Using (4.12), we obtain that  $\mathbf{p} = \mathbf{0}_n$  in  $[0, T]$ , hence  $\mathbf{q}_0 = \mathbf{0}_n$  and for every  $t \in [0, T]$ , we have

$$\mathbf{q}_1 = \int_t^T \nabla g_c(\mathbf{x}) d\eta.$$

Since  $T \notin E_c$  then  $\eta(\{T\}) = 0$  and then  $\mathbf{q}_1 = \mathbf{0}_n$  and for all  $t \in [0, T]$ , we obtain

$$\int_{[t, T]} \nabla g_c(\mathbf{x}) d\eta = \mathbf{0}_n. \quad (4.14)$$

If  $c = +\infty$ , we clearly have that  $\eta = 0$  and hence  $s_0$ ,  $p_0$ ,  $\mathbf{q}_0$ ,  $\mathbf{q}_1$ ,  $\mathbf{p}$  and  $\eta$  are all zero, leading to a contradiction. Thus, in the case where  $c = +\infty$ , we have  $p_0 \neq 0$ . For the case  $0 < c < +\infty$ , we will show that  $\eta = 0$  and the same contradiction as above will hold. Since  $\nabla g_c(\mathbf{x}) = \mathbf{x}$ , the equation (4.14) implies that for every  $t_0$  and  $t_1$  with  $0 \leq t_0 \leq t_1 \leq T$ , we have

$$\int_{[t_0, t_1]} \mathbf{x} d\eta = \mathbf{0}_n.$$

Let  $t_0$  be chosen in the support of  $\eta$ . Then, for every  $\varepsilon > 0$ , the measure of the interval  $[t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, T]$  with respect to  $\eta$  is positive. Since the support of  $\eta$  is contained in  $E_c$ , we have  $t_0 \in E_c$  and hence  $|\mathbf{x}(t_0)| = c$ . Using the fact that  $\mathbf{x}$  is continuous, we obtain that there exists  $\varepsilon > 0$  such that  $\langle \mathbf{x}(t_0), \mathbf{x}(t) \rangle \geq \frac{c^2}{2}$  for every  $t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, T]$ . On the other hand, due to (4.14) we have,

$$0 = \int_{[t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, T]} \langle \mathbf{x}(t_0), \mathbf{x}(t) \rangle d\eta \geq \frac{c^2}{2} \eta([t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, T]).$$

We deduce that  $\eta([t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, T]) = 0$  which contradicts the fact that  $t_0$  is in the support of  $\eta$ . Consequently, the support of  $\eta$  is empty and thus  $\eta = 0$ . As a result,  $s_0, p_0, \mathbf{q}_0, \mathbf{q}_1, \mathbf{p}$  and  $\eta$  are all zero which leads to a contradiction.

So, under the assumption  $s_0 = 0$ , we have proved that  $p_0 \neq 0$ . Now, we deduce from (4.13) that  $\mathbf{u} \in \text{Ker}(M - M^\top)$  in the set  $[0, T] \setminus E_c$  of positive measure. This is in contradiction to the property (3.2) in Proposition 3.3. Therefore we have proved that  $s_0 < 0$ .

• Now, we turn to the proof of (4.7). Since  $s_0 < 0$ , we deduce from (4.6) that  $p_0 M \mathbf{x}(t) + \mathbf{p}(t) \neq \mathbf{0}_n$  for almost every  $t \in [0, T]$ . Hence, using (4.5), we obtain

$$\mathbf{u}(t) = \frac{p_0 M \mathbf{x}(t) + \mathbf{p}(t)}{|p_0 M \mathbf{x}(t) + \mathbf{p}(t)|} \quad \text{for a.e. } t \in [0, T].$$

Inserting the above expression in (4.6), we obtain (4.7) and as a result we have that  $\mathbf{u} \in BV(0, T)^n$ .

• Let us prove that  $\mathbf{u}$  is a  $C^\infty$  function on the open set  $[0, T] \setminus E_c$ . From the co-state equation (4.3), we infer that  $\mathbf{p}$  has the  $W^{1, \infty}$ -regularity on the set  $[0, T] \setminus E_c$ . Therefore, according to (4.7) we obtain that  $\mathbf{u}$  has also the  $W^{1, \infty}$ -regularity on  $[0, T] \setminus E_c$ . Differentiating (4.7) and (4.3), we obtain that  $\mathbf{u}$  satisfies the differential equation

$$\dot{\mathbf{u}}(t) = \frac{p_0}{-s_0} (M - M^\top) \mathbf{u}(t) \quad \text{for a.e. } t \in [0, T] \setminus E_c.$$

This ensures that  $\mathbf{u}$  is a  $C^\infty$  function on  $[0, T] \setminus E_c$ . Using (4.7) we clearly obtain that  $\mathbf{p}$  also possesses the  $C^\infty$ -regularity on  $[0, T] \setminus E_c$ .

• Now, we prove that  $\mathbf{u}, \mathbf{p} \in C^0([0, T])$  and that the property (4.9) on the measure  $\eta$  holds. These results are obvious when  $E_c = \emptyset$ . So we only consider the case where  $0 < c < +\infty$  and  $E_c \neq \emptyset$ . We start to prove that  $\mathbf{p}$  is continuous on  $[0, T]$ . According to the co-state equation (4.3), it is sufficient to prove that the function  $m : t \mapsto \int_t^T \mathbf{x} d\eta$  is continuous on  $[0, T]$ . We argue by contradiction. Let us assume that the function  $m$  is not continuous on  $[0, T]$ . Hence, there exists a time  $\tau \in [0, T]$  such that  $[m](\tau) = \mathbf{x}(\tau) \alpha$  with  $\alpha = \eta(\{\tau\}) > 0$ . Therefore, using (4.7) and (4.3), we obtain

$$-s_0 [\mathbf{u}](\tau) = [\mathbf{p}](\tau) = \alpha \mathbf{x}(\tau). \quad (4.15)$$

Notice that since we already know that  $\mathbf{u}$  is a  $C^\infty$  function in  $[0, T] \setminus E_c$ , we necessarily have that  $\tau \in E_c$ . In particular, we know that  $\tau \neq 0$  and  $\tau \neq T$ . Using (4.6) with the expression (4.7) of  $\mathbf{u}$ , we obtain

$$|p_0 M \mathbf{x} + \mathbf{p}|^2 = s_0^2 \quad \text{in } BV(0, T),$$

which implies

$$|p_0 M \mathbf{x}(\tau) + \mathbf{p}(\tau^+)|^2 = |p_0 M \mathbf{x}(\tau) + \mathbf{p}(\tau^-)|^2.$$

It follows that

$$2 \langle p_0 M \mathbf{x}(\tau), [\mathbf{p}](\tau) \rangle + [|\mathbf{p}|^2](\tau) = 0. \quad (4.16)$$

Using (4.7), (4.15) and the fact that  $|\mathbf{x}(\tau)| = c$ , equation (4.16) yields after straightforward calculations,

$$-2s_0\alpha\langle\mathbf{u}(\tau^-), \mathbf{x}(\tau)\rangle + c^2\alpha^2 = 0. \quad (4.17)$$

Since  $\alpha > 0$ , we obtain  $\alpha = \frac{2s_0}{c^2}\langle\mathbf{u}(\tau^-), \mathbf{x}(\tau)\rangle$ . We know that  $\tau \in E_c$ , i.e.  $|\mathbf{x}(\tau)| = c$ , thus we necessarily have  $\frac{d|\mathbf{x}|^2}{dt}(\tau^-) \geq 0$ . This leads to  $\langle\mathbf{u}(\tau^-), \mathbf{x}(\tau)\rangle \geq 0$  and we deduce that  $\alpha = \frac{2s_0}{c^2}\langle\mathbf{u}(\tau^-), \mathbf{x}(\tau)\rangle \leq 0$ , which is in contradiction to the fact that  $\alpha > 0$ . So,  $\mathbf{p}$  cannot have discontinuity points and in view of (4.7), we have also shown that  $\mathbf{u} \in C^0([0, T])^n$ . From (4.3), we have  $[\mathbf{p}](t) = \eta(\{t\})\mathbf{x}(t)$  for every  $t \in [0, T]$  then the continuity of  $\mathbf{p}$  implies (4.9).

• We are now in position to prove (4.8). In fact, we are going to prove that  $\langle\mathbf{u}(\tau), \mathbf{x}(\tau)\rangle = 0$  for every  $\tau \in E_c$  or equivalently  $\frac{d|\mathbf{x}|^2}{dt}(\tau) = 0$  for every  $\tau \in E_c$ . Let us choose  $\tau \in E_c$ . Using the Taylor formula, we have

$$|\mathbf{x}(\tau + \theta)|^2 = |\mathbf{x}(\tau)|^2 + \theta \frac{d|\mathbf{x}|^2}{dt}(\tau) + o(\theta) = c^2 + \theta \frac{d|\mathbf{x}|^2}{dt}(\tau) + o(\theta) \quad (\theta \rightarrow 0).$$

Then, in order to satisfy the constraint  $|\mathbf{x}(\tau + \theta)|^2 \leq c^2$  for every  $\theta \in \mathbb{R}$  small enough, we necessarily have  $\frac{d|\mathbf{x}|^2}{dt}(\tau) = 0$ .

• The proof of the properties  $\mathbf{q}_0 \neq \mathbf{0}_n$ ,  $\mathbf{q}_1 \neq \mathbf{0}_n$  is a consequence of the continuity of  $\mathbf{u}$ . Indeed, we have  $\mathbf{q}_0 = \mathbf{p}(0)$  and  $\mathbf{q}_1 = \mathbf{p}(T)$  and due to (4.7) and the fact that  $\mathbf{u} \in C^0([0, T])^n$ , we deduce that  $\mathbf{u}(0) = \frac{1}{-s_0}\mathbf{q}_0$  and  $\mathbf{u}(T) = \frac{1}{-s_0}\mathbf{q}_1$ . Since  $\mathbf{u}$  is continuous in  $[0, T]$  and satisfies the bang-bang property (3.1), we necessarily have  $\mathbf{q}_0 \neq \mathbf{0}_n$ ,  $\mathbf{q}_1 \neq \mathbf{0}_n$ .

• Finally, we prove that  $p_0 \neq 0$ . We argue by contradiction. Let us assume that  $p_0 = 0$ . We define  $\tau = \sup\{t \in [0, T], |\mathbf{x}(\sigma)| < c \text{ for every } \sigma \in [0, t]\}$  and we see that  $\tau \in (0, T]$ . Using the co-state equation (4.3) with  $p_0 = 0$ , we obtain that  $\mathbf{p}$  is constant on  $[0, \tau]$  and hence using (4.4) and (4.7), we have  $\mathbf{p}(t) = \mathbf{q}_0$ ,  $\mathbf{u}(t) = \frac{\mathbf{q}_0}{-s_0}$  for every  $t \in [0, \tau]$ . Thus  $\mathbf{x}(t) = \frac{\mathbf{q}_0}{-s_0}t$  for every  $t \in [0, \tau]$ . If  $\tau = T$ , we have  $\mathbf{x}(T) = \frac{\mathbf{q}_0}{-s_0}T \neq 0$  which is in contradiction to the final condition (2.7). If  $0 < \tau < T$ , then we necessarily obtain  $\tau \in E_c$ . Moreover, we have  $\frac{d|\mathbf{x}|^2}{dt}(\tau) = 2\langle\mathbf{x}(\tau), \mathbf{u}(\tau)\rangle = \frac{|\mathbf{q}_0|^2}{s_0^2}\tau \neq 0$  which is in contradiction with (4.8). Thus, we have proved that  $p_0 \neq 0$ .

The proof of the Proposition is completed.  $\square$

We conclude this section by giving the explicit form of the optimal controls and trajectories in region where the state constraint is not reached.

**Lemma 4.3.** *Let  $\mathbf{u} \in L^\infty(0, T^*)^n$  be a time optimal control. Then, for every  $t_0, t_1 > 0$  with  $t_0 < t_1$  such that  $|\mathcal{X}_{\mathbf{u}}| < c$  in  $(t_0, t_1)$ , we have*

$$\mathbf{u}(t) = \exp\left(\delta(t - t_0)\frac{M - M^\top}{2}\right)\mathbf{u}(t_0) \quad (t \in [t_0, t_1]), \quad (4.18)$$

with  $\delta = \frac{2p_0}{-s_0} \neq 0$ .

*Proof.* The Pontryagin maximum principle ensures that there exist  $s_0 \leq 0$ ,  $\mathbf{q}_0 \in \mathbb{R}^n$ ,  $\mathbf{q}_1 \in \mathbb{R}^n$ ,  $p_0 \in \mathbb{R}$ ,  $\mathbf{p} : [0, T] \rightarrow \mathbb{R}^n$  and a non-negative regular measure  $\eta$  with support included in  $E_c$ , not all zero such that such (2.7)–(2.9) and (4.3)–(4.6) hold. Since the support of  $\eta$  does not intersect the interval  $[t_0, t_1]$ , we deduce from the co-state equation (4.3) and the  $C^\infty$ -regularity of  $\mathbf{p}$  in  $(t_0, t_1)$  that the following differential equation holds:

$$\dot{\mathbf{p}} = -p_0 M^\top \mathbf{u} \quad \text{in } (t_0, t_1).$$

Hence differentiating (4.7), we obtain,

$$\dot{\mathbf{u}} = \frac{1}{-s_0}(p_0 M \mathbf{u} - p_0 M^\top \mathbf{u}) = \frac{2p_0}{-s_0} \left( \frac{M - M^\top}{2} \right) \mathbf{u} \quad \text{in } (t_0, t_1).$$

The general solution of the above differential equation for  $\mathbf{u}$  is given by (4.18).  $\square$

## 5 Optimal solution for the 2-dimensional control case

In this section, we study the case of a 2-dimensional control, i.e. the case  $n = 2$ . We shall give the explicit expression of a 2-dimensional optimal control together with the associated optimal trajectory for the state variables. We first remark that any  $2 \times 2$  non-symmetric real matrix  $M$  possesses a skew part given by

$$\frac{1}{2}(M - M^\top) = \lambda J,$$

where  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\lambda$  is a non-zero real number. Introducing the rotation matrix  $R$  defined by  $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  for every  $\theta \in \mathbb{R}$ , we have that  $J = R(\frac{\pi}{2})$  and  $R(\theta) = \exp(\theta J)$  for every  $\theta \in \mathbb{R}$ .

The first result is devoted to the description of the optimal trajectory for  $\mathbf{x}$  into a region where the state constraint is not reached.

**Proposition 5.1.** *Let  $T^* > 0$  and  $\mathbf{u} \in L^\infty(0, T^*)^2$  be an optimal solution to the control problem (2.7)–(2.9) with  $n = 2$ . We denote by  $y = \mathcal{Y}_{\mathbf{u}}$  and  $\mathbf{x} = \mathcal{X}_{\mathbf{u}}$  the corresponding optimal state variables. Then there exists  $\delta \neq 0$  such that for every  $t_0, t_1 > 0$  with  $t_0 < t_1$  such that  $|\mathcal{X}_{\mathbf{u}}| < c$  in  $(t_0, t_1)$ , we have, for every  $t \in [t_0, t_1]$ ,*

$$\mathbf{u}(t) = R(\delta\lambda(t - t_0))\mathbf{u}(t_0), \quad (5.1)$$

$$\mathbf{x}(t) = \mathbf{x}(t_0) - \frac{1}{\lambda\delta} J (R(\delta\lambda(t - t_0)) - I_2) \mathbf{u}(t_0), \quad (5.2)$$

$$y(t) = y(t_0) + S(t_0, t_1) - \frac{1}{\delta} \langle \mathbf{u}(t_0), \mathbf{x}(t) - \mathbf{x}(t_0) \rangle + \lambda \langle J\mathbf{x}(t_0), \mathbf{x}(t) \rangle + \frac{t - t_0}{\delta}, \quad (5.3)$$

where  $S(t_0, t_1) = \frac{1}{2}(\langle M_S \mathbf{x}(t_1), \mathbf{x}(t_1) \rangle - \langle M_S \mathbf{x}(t_0), \mathbf{x}(t_0) \rangle)$  with  $M_S = \frac{1}{2}(M + M^\top)$ . The real value  $\lambda \neq 0$  is uniquely determined by the skew part of the  $2 \times 2$  non-symmetric matrix  $M$  written in the form  $\frac{1}{2}(M - M^\top) = \lambda J$ .

In addition, the optimal state trajectory  $t \in [t_0, t_1] \mapsto \mathbf{x}(t) \in \mathbb{R}^2$  is a parametrization of an arc of circle with radius  $\frac{1}{|\delta\lambda|}$  and center  $\mathbf{x}(t_0) - \frac{1}{\delta\lambda} J\mathbf{u}(t_0) \in \mathbb{R}^2$ .

*Proof.* The expression (5.1) for  $\mathbf{u}$  is nothing more than the expression (4.18) in Lemma 4.3 for the 2-dimensional control case. Moreover, integrating (5.1) between  $t_0$  and  $t \in (t_0, t_1)$  yields the expression (5.2) for  $\mathbf{x}$ . Let us turn to the proof of the expression (5.3) for  $y$ . We integrate (1.1) between  $t_0$  and

$t \in (t_0, t_1)$  :

$$\begin{aligned}
y(t) - y(t_0) &= \int_{t_0}^t \langle M\mathbf{x}(s), \mathbf{u}(s) \rangle ds \\
&= \int_{t_0}^t \langle M_S\mathbf{x}(s), \mathbf{u}(s) \rangle ds + \int_{t_0}^t \langle \lambda J\mathbf{x}(s), \mathbf{u}(s) \rangle ds \\
&= \frac{1}{2}(\langle M_S\mathbf{x}(t_1), \mathbf{x}(t_1) \rangle - \langle M_S\mathbf{x}(t_0), \mathbf{x}(t_0) \rangle) + \int_{t_0}^t \langle \lambda J\mathbf{x}(s), \mathbf{u}(s) \rangle ds \quad (5.4)
\end{aligned}$$

Using (5.1) and (5.2) into the last integral term in (5.4), we obtain

$$\begin{aligned}
\int_{t_0}^t \langle \lambda J\mathbf{x}(s), \mathbf{u}(s) \rangle ds &= \int_{t_0}^t \left\langle \lambda J \left( \frac{-1}{\lambda\delta} J(\mathbf{u}(s) - \mathbf{u}(t_0)) + \mathbf{x}(t_0) \right), \mathbf{u}(s) \right\rangle ds \\
&= \int_{t_0}^t \left\langle \frac{1}{\delta} (\mathbf{u}(s) - \mathbf{u}(t_0)) + \lambda J\mathbf{x}(t_0), \mathbf{u}(s) \right\rangle ds \\
&= \frac{1}{\delta} \int_{t_0}^t |\mathbf{u}(s)|^2 ds - \frac{1}{\delta} \left\langle \mathbf{u}(t_0), \int_{t_0}^t \mathbf{u}(s) ds \right\rangle + \lambda \left\langle J\mathbf{x}(t_0), \int_{t_0}^t \mathbf{u}(s) ds \right\rangle \\
&= \frac{t - t_0}{\delta} - \frac{1}{\delta} \langle \mathbf{u}(t_0), \mathbf{x}(t) - \mathbf{x}(t_0) \rangle + \lambda \langle J\mathbf{x}(t_0), \mathbf{x}(t) \rangle \quad (5.5)
\end{aligned}$$

Combining (5.4) with (5.5) yields (5.3).

Now, we prove that the optimal trajectory  $t \in (t_0, t_1) \mapsto \mathbf{x}(t)$  describes an arc of circle. To see this, we define  $\mathbf{c}_0 = \mathbf{x}(t_0) - \frac{1}{\delta\lambda} J\mathbf{u}(t_0) \in \mathbb{R}^2$  and we have, for every  $t \in (t_0, t_1)$ ,

$$|\mathbf{x}(t) - \mathbf{c}_0|^2 = \left| \frac{-1}{\delta\lambda} JR(\delta\lambda(t - t_0)) \mathbf{u}(t_0) \right|^2 = \frac{|\mathbf{u}(t_0)|^2}{|\delta\lambda|^2} = \frac{1}{|\delta\lambda|^2}.$$

This completes the proof of Proposition 5.1.  $\square$

In the next Lemma, we give a property on the optimal state trajectory which asserts that the trajectory does not pass through the origin except at the initial and final times. This result will be useful for the complete description of the optimal trajectory.

**Lemma 5.2.** *Let  $n = 2$ . We suppose that the  $2 \times 2$  matrix  $M$  in problem (1.1)–(1.6) is given by  $M = \lambda J$  with  $\lambda \in \mathbb{R}^*$ . Let  $T^*(M)$  be the corresponding optimal time and  $\mathbf{u} \in \mathcal{U}^{T^*}(M)$  a time optimal control. Then  $\{t \in [0, T^*], \mathcal{X}_{\mathbf{u}}(t) = \mathbf{0}_2\} = \{0, T^*\}$ .*

*Proof.* As usual, we denote  $(y, \mathbf{x}) = (\mathcal{Y}_{\mathbf{u}}, \mathcal{X}_{\mathbf{u}})$  the optimal state trajectory associated to  $\mathbf{u}$ . It is clear that we have  $\{0, T^*\} \subset \{t \in [0, T^*], \mathbf{x}(t) = \mathbf{0}_2\}$ . We assume by contradiction that there exists  $\sigma \in (0, T^*)$  such that  $\mathbf{x}(\sigma) = \mathbf{0}_2$ .

It is easy to see that the control  $\tilde{\mathbf{u}} = J\mathbf{u}$  is also a time optimal control, i.e.  $\tilde{\mathbf{u}} \in \mathcal{U}^{T^*}$ . Indeed, let us introduce  $(\tilde{y}, \tilde{\mathbf{x}}) = (y, J\mathbf{x})$ . Then we have

$$\dot{\tilde{\mathbf{x}}} = J\dot{\mathbf{x}} = J\mathbf{u} = \tilde{\mathbf{u}} \quad \text{and hence} \quad \dot{\tilde{y}} = \dot{y} = \langle \lambda J\mathbf{x}, \mathbf{u} \rangle = \langle \lambda J(J\mathbf{x}), J\mathbf{u} \rangle = \langle \lambda J\tilde{\mathbf{x}}, \tilde{\mathbf{u}} \rangle.$$

This implies that  $(\mathcal{Y}_{\tilde{\mathbf{u}}}, \mathcal{X}_{\tilde{\mathbf{u}}}) = (\tilde{y}, \tilde{\mathbf{x}})$ . In particular, we have  $\tilde{\mathbf{x}}(\sigma) = \mathbf{x}(\sigma) = \mathbf{0}_2$  and  $\tilde{y}(\sigma) = y(\sigma)$ . Then, due to the Bellman principle, the controls  $\hat{\mathbf{u}}$  defined by

$$\begin{aligned}
\hat{\mathbf{u}} : [0, T^*] &\rightarrow \mathbb{R}^2 \\
t &\mapsto \begin{cases} \mathbf{u}(t) & \text{if } t \leq \sigma \\ \tilde{\mathbf{u}}(t) & \text{if } \sigma < t \leq T^* \end{cases}
\end{aligned}$$

is a time optimal control for the control problem (2.7)–(2.9). However,  $\hat{\mathbf{u}}$  is not smooth at time  $\sigma$  and  $\sigma \notin E_c = \{t \in [0, T], |\mathcal{X}_{\mathbf{u}}(t)| = c\}$  since  $\mathcal{X}_{\hat{\mathbf{u}}}(\sigma) = \mathbf{0}_2$  (see Figure 2).

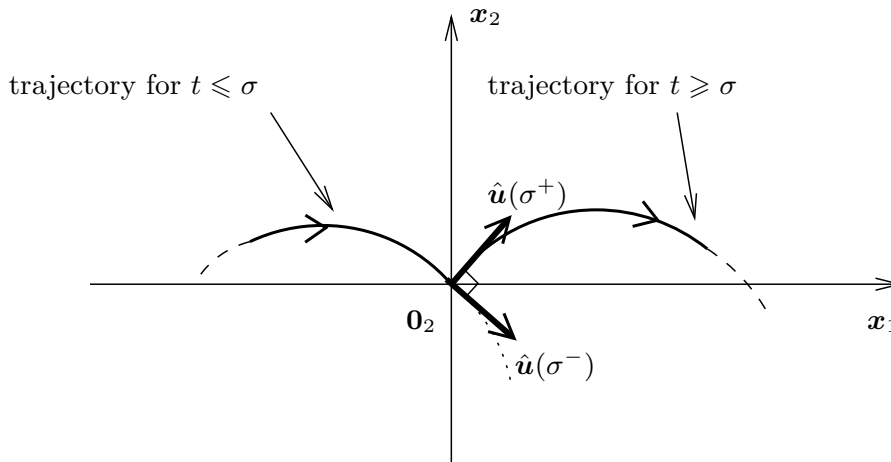


Figure 2: Counter example for proving that the optimal trajectory for  $\mathbf{x}$  does not pass through  $\mathbf{0}_2$ , except at initial and final times.

This is in contradiction with the regularity of the optimal controls proved in Proposition 4.1. Thus, Lemma 5.2 is proved.  $\square$

We are now in position to give the full description of the optimal trajectory when the state constraints are imposed. The following result is a particular version of Theorem 2.4 when  $n = 2$ , i.e. when considering the 2-dimensional control case.

**Proposition 5.3.** *Let  $M$  be a  $2 \times 2$  real non-symmetric matrix. We define the two real positive values,*

$$d^* = \sqrt{\frac{2|\bar{y}|}{\pi|\lambda|}} \quad \text{and} \quad \tau = \frac{\pi c}{2}, \quad (5.6)$$

where the real value  $\lambda \neq 0$  is uniquely determined by the skew part of the  $2 \times 2$  matrix  $M$  written in the form  $\frac{1}{2}(M - M^\top) = \lambda J$ . Then, the optimal time  $T^*$  for the problem (2.7)–(2.9) with the  $2 \times 2$  matrix  $M$  is given by:

$$T^* = \begin{cases} \pi d^* & \text{if } c \geq d^*, \\ \frac{\pi d^{*2}}{2c} + \tau & \text{otherwise.} \end{cases} \quad (5.7)$$

Moreover, the problem (2.7)–(2.9) admits an time optimal control  $\mathbf{u} \in C^0([0, T^*])^n$  defined as follows:

- if  $c \geq d^*$ , we have

$$\mathbf{u}(t) = R\left(\text{sign}(\bar{y}) \frac{2\pi}{T^*} t\right) \mathbf{u}_0 \quad (t \in [0, T^*]). \quad (5.8)$$

In this case, the optimal trajectory  $t \mapsto \mathcal{X}_{\mathbf{u}}(t)$  for  $t \in [0, T^*]$  describes a circle with diameter  $d^* \leq c$ .



- if  $c < d^*$ , we have

$$\mathbf{u}(t) = \begin{cases} R\left(\frac{2\operatorname{sign}(\bar{y})}{c}t\right)\mathbf{u}_0 & \text{if } t \in [0, \tau), \\ -R\left(\frac{\operatorname{sign}(\bar{y})}{c}(t-\tau)\right)\mathbf{u}_0 & \text{if } t \in [\tau, T^* - \tau), \\ R\left(\frac{\operatorname{sign}(\bar{y})}{c}(2t - 3(T^* - \tau))\right)\mathbf{u}_0 & \text{if } t \in [T^* - \tau, T^*]. \end{cases} \quad (5.9)$$

In this case, the optimal trajectory  $t \mapsto \mathcal{X}_{\mathbf{u}}(t)$  for  $t \in [0, T^*]$  is composed by three arcs of circle. The first arc of circle corresponds to the trajectory starting from  $\mathbf{x}(0) = 0$  until the state constraint is reached tangentially at time  $\tau = \frac{c\pi}{2}$  with  $|\mathbf{x}(\tau)| = c$ . Then, the trajectory stays on the constraint, that is  $|\mathbf{x}(t)| = c$  for times  $t \in [\tau, T^* - \tau]$ . Finally, at time  $t = T^* - \tau$ , the trajectory leaves tangentially the state constraint and describes an arc of circle in order to reach the final state  $\mathbf{x}(T^*) = 0$ .

In the above, the vector  $\mathbf{u}_0 \in \mathbb{R}^2$  is chosen such that  $|\mathbf{u}_0| = 1$ .

*Proof.* This proof is organized as follows. Firstly, we will prove the result for the case  $c \geq d^*$ . Afterwards, the result will be shown for  $c < d^*$ .

According to Proposition 3.2, we can assume without loss of generality that  $M = \lambda J$ . Let  $T^* = T^*(M) > 0$  be the minimal time for the problem (2.7)–(2.9) and let  $\mathbf{u} \in \mathcal{U}^{T^*}(M)$  be a time optimal control. We denote the corresponding state variables  $y = \mathcal{Y}_{\mathbf{u}}$  and  $\mathbf{x} = \mathcal{X}_{\mathbf{u}}$ .

- In order to prove the result for the case  $c \geq d^*$ , we first pay attention to the case  $c = +\infty$ . In this case, the state constraint (2.9) is never reached and according to Proposition 5.1 an optimal control has the form

$$\mathbf{u}(t) = R(\delta\lambda t)\mathbf{u}_0 \quad (t \in [0, T^*]),$$

with  $\mathbf{u}_0 \in \mathbb{R}^2$ ,  $|\mathbf{u}_0| = 1$  and  $\delta \in \mathbb{R}^*$ . The state trajectory  $(y, \mathbf{x})$  has the following representation for every  $t \in [0, T^*]$ :

$$\begin{aligned} \mathbf{x}(t) &= \frac{-1}{\lambda\delta} J (R(\delta\lambda t) - I_2) \mathbf{u}_0, \\ y(t) &= \frac{t}{\delta} - \frac{1}{\delta} \langle \mathbf{u}_0, \mathbf{x}(t) \rangle. \end{aligned}$$

In order to reach the final state (2.7), we must have

$$\bar{y} = \frac{T^*}{\delta} \quad \text{and} \quad |\delta\lambda|T^* \in 2\pi\mathbb{N}^*.$$

Hence, there exists  $k \in \mathbb{N}^*$  such that

$$T^* = \sqrt{\frac{2k\pi|\bar{y}|}{|\lambda|}} \quad \text{and} \quad \delta = \frac{T^*}{\bar{y}} = \operatorname{sign}(\bar{y}) \sqrt{\frac{2k\pi}{|\lambda\bar{y}|}}. \quad (5.10)$$

Moreover, Proposition 5.1 implies that

$$\sup_{t \in [0, T^*]} |\mathbf{x}(t)| = \frac{1}{|\delta\lambda|} = \sqrt{\frac{2|\bar{y}|}{k\pi|\lambda|}} = \frac{d^*}{\sqrt{k}}. \quad (5.11)$$

Since we deal with the case  $c = +\infty$ , the minimal time  $T^*$  is necessarily obtained with  $k = 1$  and we obtain  $T^* = \pi d^*$  with the corresponding optimal control  $\mathbf{u}$  given by (5.8).

One can easily check that the optimal control  $\mathbf{u}$  previously obtained in the case  $c = +\infty$  still remains an optimal control in the constrained case  $d^* \leq c < +\infty$  with the same minimal time  $T^*$ .

From Proposition 5.1 we deduce that the associated trajectory  $t \in [0, T^*] \mapsto \mathbf{x}(t)$  is a circle of diameter  $d^*$ .

• Now we investigate the case where  $c < d^*$ . We will first show that the state constraint for  $\mathbf{x}$  is necessarily reached in a time  $\tau \in (0, T^*)$ . Then, we will compute an optimal control.

1. Let us assume by contradiction that the state constraint (2.9) is never reached. In that case, the optimal time and the time optimal control have the same expressions as the ones already obtained in the case  $c = +\infty$ . In particular, the expression (5.10) for the minimal time  $T^*$  and the estimate (5.11) for  $\mathbf{x}$  hold true for  $k \geq 1$ . Since  $c < d^*$ , estimate (5.11) implies that  $k \geq 2$ . Hence at time  $\sigma = \frac{T^*}{k} < T^*$ , we have  $\mathbf{x}(\sigma) = \mathbf{0}_2$ . This is in contradiction with Lemma 5.2. Consequently, the state constraint (2.9) is necessarily reached and according to Proposition 4.1 (see also Remark 4.2) the constraint is reached tangentially.

2. We now turn to the computation of the optimal control.

• Firstly, we prove the following characterization for the optimal solution:

- (a) the state constraint is reached tangentially at time  $\tau = \frac{\pi c}{2}$ , i.e.  $|\mathbf{x}(\tau)| = c$  and  $\langle \mathbf{x}(\tau), \mathbf{u}(\tau^-) \rangle = 0$ ;
- (b) the trajectory for  $\mathbf{x}$  stays on the constraint between times  $\tau$  and  $T^* - \tau$ ;
- (c) finally the state trajectory leaves (tangentially) the state constraint at time  $T^* - \tau$  in order to reach the final state  $(y(T^*), \mathbf{x}(T^*)) = (\bar{y}, \mathbf{0}_2)$ .

We recall that due to proposition 5.1, we know that the trajectory for  $\mathbf{x}$  is composed by arcs inside the region where the state constraint is not reached. Obviously, when the trajectory lies on the constraint,  $\mathbf{x}$  also describes an arc of circles. In addition, from (5.1)–(5.2), one can see that the trajectory can change of arc of circle only when reaching or leaving the state constraint.

We start by proving the description of the trajectory announced in (a).

Let  $\tilde{\tau} = \inf \{t \in [0, T^*], |\mathbf{x}(t)| = c, \forall s \in [0, t), |\mathbf{x}(s)| < c\}$  be the first reaching time. According to the first item, we know that  $\tilde{\tau}$  exists. Since  $\mathbf{x}$  reaches the constraint in a tangential manner (see Proposition 4.1 and Remark 4.2) then  $t \in [0, \tilde{\tau}] \mapsto \mathbf{x}(t)$  is a parametrization of a semicircle of radius  $c/2$ . In addition, due to the fact that  $|\mathbf{u}(t)| = 1$  for every  $t \in [0, \tilde{\tau}]$  and that  $\mathbf{u}$  is continuous, we deduce that  $\tilde{\tau} = \frac{c\pi}{2} = \tau$ .

Using Proposition 5.1, we obtain that the radius of this semi-circle is  $\frac{1}{|\delta\lambda|} = \frac{c}{2}$ , and then

$$\delta = \epsilon_0 \frac{2}{\lambda c} \quad \text{with } \epsilon_0 = \pm 1. \quad (5.12)$$

We now prove that the last part of the trajectory, described in (c), is also an arc of circle. We first prove that if the trajectory leaves the constraint at time  $\sigma$  then we necessarily have that  $\sigma = T^* - \tau$  and the state trajectory is never reached again. Using (5.1)–(5.2), with  $\delta$  given by (5.12), we obtain that

$$\mathbf{x}(\sigma + \tau) = \mathbf{0}_2 \quad \text{and} \quad |\mathbf{x}(t)| \neq 0 \quad (t \in [\sigma, \sigma + \tau)).$$

Consequently, due to Lemme 5.2, we obtain that  $\sigma + \tau = T^*$ .

Then we have proved that the optimal trajectory  $t \mapsto \mathbf{x}(t)$  for  $t \in [0, T]$ , has the characterization given by (a), (b) and (c).

• We now turn to the calculation of an optimal control.

Using (5.3), we obtain

$$y(\tau) = y(T^*) - y(T^* - \tau) = \frac{\tau}{\delta} = \epsilon_0 \frac{\lambda c^2 \pi}{4}. \quad (5.13)$$

For every  $t \in [\tau, T^* - \tau]$ , we have  $|\mathbf{x}(t)| = c$  and  $\mathbf{x}$  must have the form:

$$\mathbf{x}(t) = R \left( \frac{\nu(t)}{c} \right) \mathbf{x}(\tau) \quad (t \in [\tau, T^* - \tau]), \quad (5.14)$$

with  $\nu \in W^{1,\infty}(\tau, T^* - \tau)$  being an unknown function. We have

$$\mathbf{u}(t) = \dot{\mathbf{x}}(t) = \dot{\nu}(t) J R \left( \frac{\nu(t)}{c} \right) \frac{\mathbf{x}(\tau)}{c} \quad \text{for a.e. } t \in (\tau, T^* - \tau). \quad (5.15)$$

Due to the fact that  $|\mathbf{u}(t)| = 1$  for every  $t \in (0, T^*)$ , we have that  $|\dot{\nu}| = 1$  everywhere in  $(\tau, T^* - \tau)$  and since  $\mathbf{u}$  is continuous, we have that  $\dot{\nu}$  is a constant and we denote  $\epsilon_1 = \dot{\nu} = \pm 1$ .

Using (5.14) and (5.15), we obtain  $\langle \lambda J \mathbf{x}(t), \mathbf{u}(t) \rangle = \lambda c \epsilon_1$  for every  $t \in (\tau, T^* - \tau)$ , then

$$y(T^* - \tau) - y(\tau) = \int_{\tau}^{T^* - \tau} \dot{y}(t) dt = \int_{\tau}^{T^* - \tau} \langle \lambda J \mathbf{x}(t), \mathbf{u}(t) \rangle dt = \lambda c \epsilon_1 (T^* - 2\tau). \quad (5.16)$$

Then, using (5.13) and (5.16), we obtain

$$\begin{aligned} y(T^*) &= y(\tau) + (y(T^* - \tau) - y(\tau)) + (y(T^*) - y(T^* - \tau)) \\ &= \epsilon_0 \frac{\lambda c^2 \pi}{4} + \lambda c \epsilon_1 (T^* - 2\tau) + \epsilon_0 \frac{\lambda c^2 \pi}{4} = \epsilon_0 \frac{\lambda c^2 \pi}{2} + \lambda c \epsilon_1 (T^* - c\pi). \end{aligned}$$

Then

$$\bar{y} - \epsilon_0 \frac{\lambda c^2 \pi}{2} = \lambda c \epsilon_1 (T^* - c\pi). \quad (5.17)$$

Since the trajectory of  $\mathbf{x}$  is composed by arcs of circle and since the state constraint is reached and leaved tangentially, we deduce that the length of the trajectory  $t \mapsto \mathbf{x}(t)$  for  $t \in [0, T^*]$  is necessarily larger than the perimeter of a circle of radius  $c/2$ . So, we have

$$\int_0^{T^*} |\dot{\mathbf{x}}(t)| dt > \pi c.$$

Since  $|\dot{\mathbf{x}}| = |\mathbf{u}| = 1$  on  $[0, T^*]$ , we deduce that  $T^* > c\pi$ . Since we assumed that  $c < d^* = \sqrt{\frac{2|\bar{y}|}{|\lambda|\pi}}$ , we obtain from (5.17) that  $\epsilon_1 = \text{sign}(\lambda \bar{y})$  and then using (5.17), the minimum for  $T^*$  is obtain for  $\epsilon_0 = \epsilon_1$ . Therefore, the minimal time  $T^*$  is given by

$$T^* = \frac{|\bar{y}|}{|\lambda|c} + \frac{\pi c}{2}.$$

A straightforward calculation shows that the control given by (5.9) is optimal.

The proof of Proposition 5.3 is completed.  $\square$

We conclude this section by giving a numerical example of a fluid-structure interaction control problem. We consider a swimming control problem for a deformable body immersed into a Stokes fluid. The body is able to self-propell into the viscous fluid by changing its shape. As already mentioned in the introductory section at the beginning of the paper, the state variables  $\boldsymbol{x}$  correspond to the magnitudes of the deformations and a state constraint is imposed in order to ensure in particular that the deformation map is one-to-one. We refer to [15] for more details. In this context, the minimal time control problem involves the Shapere-Wilczek matrix (see [17, 15]) given by  $M = \frac{3}{35} \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$ . An optimal trajectory  $t \mapsto \boldsymbol{x}(t)$  with the state constraint  $|\boldsymbol{x}| \leq 1$  is depicted in Figure 3.

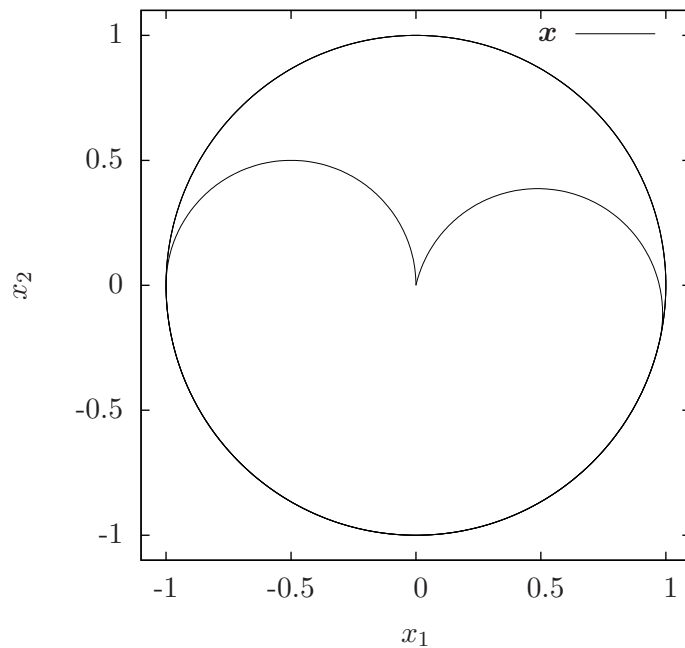


Figure 3: Case  $n = 2$ . Optimal trajectory  $t \mapsto \boldsymbol{x}(t)$  with the Shapere-Wilczek matrix. The state constraint is  $|\boldsymbol{x}(t)| \leq 1$ .

The state variable  $y$  stands for the vertical position of the mass center of the body and we choose the target  $\bar{y} = 1$ . In Figure 4, we plot the optimal trajectory  $t \mapsto y(t)$  obtained with the full matrix  $M$ . We also display in Figure 4, the optimal trajectory obtained by only considering the skew-symmetric part  $\frac{1}{2}(M - M^\top)$  instead of the full matrix  $M$ . For these two matrices, we emphasize that the dynamics of the system are different although the optimal controls are the same as proved in Proposition 3.2.

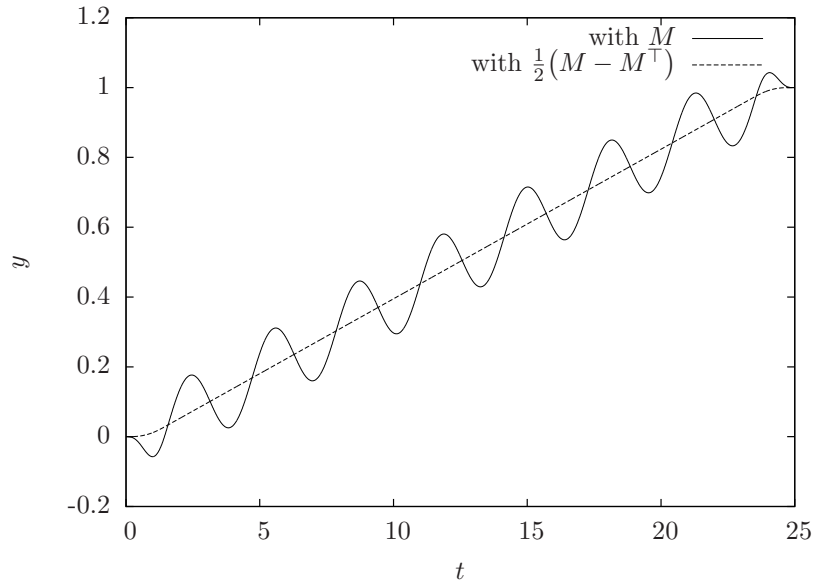


Figure 4: Case  $n = 2$ . Optimal trajectories  $t \mapsto y(t)$  with the full Shapere-Wilczek matrix and with its skew-symmetric part. The state constraint is  $|\boldsymbol{x}(t)| \leq 1$  and the objective is  $\bar{y} = 1$ .

## 6 Proof of the main result for the $n$ -dimensional control case

In this section, we consider the general  $n$ -dimensional control case with  $n \geq 2$  and with an  $n \times n$  real non-symmetric matrix  $M$ . We will construct a  $n$ -dimensional optimal control from the 2-dimensional control computed in the previous section. To begin with, we make some reductions on the  $n \times n$  matrix  $M$  of the control problem (1.1)–(1.6). Let us first notice that due to Proposition 3.2, we can assume without any loss of generality that  $M$  is a skew-symmetric matrix. Moreover, due to the property (3.2) in Proposition 3.3, the null space of  $M$  does not play any role in the control problem (2.7)–(2.9). Consequently, we can also assume that  $M$  is an invertible matrix. This means in particular that the dimension number  $n$  of the matrix  $M$  is even. The eigenvalues of the skew-symmetric invertible matrix  $M$  are pure imaginary numbers  $i\lambda_1, -i\lambda_1, i\lambda_2, -i\lambda_2, \dots, i\lambda_l, -i\lambda_l$  with  $2l = n$  and with  $\lambda_1 \geq \dots \geq \lambda_l > 0$ . In addition, the skew-symmetric invertible matrix  $M$  can be diagonalised in an orthogonal basis to a block diagonal real matrix. More precisely, there exists a real orthogonal matrix  $P$  such that (see [13, Corollary 2.5.14])

$$P^\top M P = \mathbf{J}(\Lambda) \quad \text{with} \quad \mathbf{J}(\Lambda) = \begin{pmatrix} \lambda_1 J & & \\ & \ddots & \\ & & \lambda_l J \end{pmatrix}, \quad (6.1)$$

where  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\Lambda = (\lambda_1, \dots, \lambda_l)$ . Moreover, the columns of the orthogonal matrix  $P$  are composed by the real vectors  $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{2l}$  which satisfy, for  $j = 1, \dots, l$ ,

$$\begin{aligned} M \boldsymbol{w}_{2j-1} &= \lambda_j \boldsymbol{w}_{2j}, \\ M \boldsymbol{w}_{2j} &= -\lambda_j \boldsymbol{w}_{2j-1}. \end{aligned} \quad (6.2)$$

One can easily check that, for every  $T > 0$ , we have

$$\boldsymbol{u} \in \mathcal{U}^T(\mathbf{J}(\Lambda)) \Leftrightarrow P \boldsymbol{u} \in \mathcal{U}^T(M) \quad (6.3)$$

Consequently, the minimal time for the matrix  $M$  is equal to the minimal time for the matrix  $\mathbf{J}(\Lambda)$  i.e.,

$$T^*(M) = T^*(\mathbf{J}(\Lambda)).$$

For the rest of the section, we will denote the minimal time  $T^*(\Lambda)$  instead of  $T^*(M)$  and when no confusion occurs, we will simply write  $T^*$ .

According to (6.3), it is sufficient to prove Theorem 2.4 for the  $n$ -dimensional control problem with the block diagonal matrix  $\mathbf{J}(\Lambda)$  defined by (6.1) instead of the full matrix  $M$ . Moreover, the vector  $\Lambda \in \mathbb{R}^l$  is composed by the positive imaginary part of the eigenvalues of  $\frac{1}{2}(M - M^\top)$  taken with their multiplicities and arranged in decreasing order. We will obtain a  $n$ -dimensional optimal control with the  $n \times n$  matrix  $\mathbf{J}(\Lambda)$  starting from a 2-dimensional optimal control for the  $2 \times 2$  matrix  $\lambda^* J$  where  $\lambda^* = \max \Lambda$ . This will be made possible thanks to a monotonicity argument and a zero-invariance property of the optimal solution.

Let us introduce some notations that will allow us to link the  $n$ -dimensional control case with the 2-dimensional one. For every  $l \in \mathbb{N}^*$ , we define the maps:

$$\begin{aligned} \Pi_j^l : \mathbb{R}^{2l} &\rightarrow \mathbb{R}^2 \\ \begin{pmatrix} v_1 \\ \vdots \\ v_{2j-1} \\ v_{2j} \end{pmatrix} &\mapsto \begin{pmatrix} v_{2j-1} \\ v_{2j} \end{pmatrix} \quad (j \in \{1, \dots, l\}). \end{aligned}$$

The following result is concerned with a *monotonicity* property for the map  $\Lambda \mapsto T^*(\Lambda)$ .

**Lemma 6.1.** *Let  $\Lambda = (\lambda_j)_{j \in \{1, \dots, l\}} \in \mathbb{R}_+^{*l}$  and  $\tilde{\Lambda} = (\tilde{\lambda}_j)_{j \in \{1, \dots, \tilde{l}\}} \in \mathbb{R}_+^{*\tilde{l}}$  be two non-increasing and positive finite sequences such that  $\tilde{l} \geq l \geq 1$  and  $\tilde{\lambda}_j \geq \lambda_j$  for every  $j \in \{1, \dots, l\}$ . Then we have*

$$T^*(\tilde{\Lambda}) \leq T^*(\Lambda).$$

*Proof.* Let  $\mathbf{u} \in C^0(0, T^*(\Lambda))^{2l}$  be an optimal control associated to the matrix  $\mathbf{J}(\Lambda)$ . Then, we define  $\tilde{\mathbf{u}} \in C^0(0, T^*(\Lambda))^{2\tilde{l}}$  by

$$\Pi_j^{\tilde{l}}(\tilde{\mathbf{u}}(t)) = \begin{cases} \sqrt{\frac{\lambda_j}{\tilde{\lambda}_j}} \Pi_j^l(\mathbf{u}(t)) & \text{if } j \leq l \\ \mathbf{0}_2 & \text{otherwise,} \end{cases}$$

for every  $t \in [0, T^*(\Lambda)]$  and for all  $j \in \{1, \dots, \tilde{l}\}$ . Let  $(y, \mathbf{x}) = (\mathcal{Y}_{\mathbf{u}}, \mathcal{X}_{\mathbf{u}}) \in W^{1,\infty}(0, T^*(\Lambda)) \times W^{1,\infty}(0, T^*(\Lambda))^{2l}$  and  $(\tilde{y}, \tilde{\mathbf{x}}) = (\mathcal{Y}_{\tilde{\mathbf{u}}}, \mathcal{X}_{\tilde{\mathbf{u}}}) \in W^{1,\infty}(0, T^*(\Lambda)) \times W^{1,\infty}(0, T^*(\Lambda))^{2\tilde{l}}$  be the state variables associated with the matrices  $\mathbf{J}(\Lambda)$  and  $\mathbf{J}(\tilde{\Lambda})$  respectively. Then, for every  $t \in [0, T^*(\Lambda)]$ , we clearly have

$$\Pi_j^{\tilde{l}}(\tilde{\mathbf{x}}(t)) = \begin{cases} \sqrt{\frac{\lambda_j}{\tilde{\lambda}_j}} \Pi_j^l(\mathbf{x}(t)) & \text{if } j \leq l \\ \mathbf{0}_2 & \text{otherwise,} \end{cases}$$

for all  $j \in \{1, \dots, \tilde{l}\}$ . This implies that  $|\tilde{\mathbf{x}}(t)| \leq c$  for every  $t \in [0, T^*(\Lambda)]$  and  $\tilde{\mathbf{x}}(T^*(\Lambda)) = \mathbf{0}_{2\tilde{l}}$ . Moreover, we have for every  $t \in [0, T^*(\Lambda)]$ ,

$$\dot{\tilde{y}}(t) = \sum_{i=1}^{\tilde{l}} \left\langle \tilde{\lambda}_i J \Pi_i^{\tilde{l}}(\tilde{\mathbf{x}}(t)), \Pi_i^{\tilde{l}}(\tilde{\mathbf{u}}(t)) \right\rangle = \sum_{i=1}^l \left\langle \lambda_i J \Pi_i^l(\mathbf{x}(t)), \Pi_i^l(\mathbf{u}(t)) \right\rangle = \dot{y}(t).$$

Hence,  $\tilde{y}(t) = y(t)$  for every  $t \in [0, T^*(\Lambda)]$  and  $\tilde{y}(T^*(\Lambda)) = \bar{y}$ . This ends the proof.  $\square$

Now, we turn to a zero-invariance property satisfied by any optimal control.

**Lemma 6.2.** *Let  $l \in \mathbb{N}^*$ ,  $\lambda \in \mathbb{R}_+^*$  and  $\Lambda = (\lambda, \dots, \lambda) \in \mathbb{R}_+^{*l}$ . We consider  $\mathbf{u} \in \mathcal{U}^{T^*}(\mathbf{J}(\Lambda))$  a time optimal control with the matrix  $\mathbf{J}(\Lambda)$  and satisfying  $\Pi_k^l(\mathbf{u}(0)) = \mathbf{0}_2$  for some  $k \in \{1, \dots, l\}$ . Then  $\Pi_k^l \mathbf{u}(t) = \mathbf{0}_2$  for every  $t \in [0, T^*]$ .*

*Proof.* As usual, we define  $\mathbf{x} = \mathcal{X}_{\mathbf{u}}$  and  $y = \mathcal{Y}_{\mathbf{u}}$  the state variables associated to the optimal control  $\mathbf{u} \in \mathcal{U}^{T^*}(\mathbf{J}(\Lambda))$ . We recall that the Pontryagin maximum principle ensures that there exist  $s_0, p_0, \mathbf{p}, \mathbf{q}_0, \mathbf{q}_1$  and  $\eta$  not all null such that (2.7)–(2.9) and (4.3)–(4.6) hold and Proposition 4.1 ensures that

$$s_0 < 0, p_0 \neq 0, \mathbf{q}_0 \neq \mathbf{0}_n, \mathbf{q}_1 \neq \mathbf{0}_n.$$

In addition, according to Proposition 4.1, we know that  $\mathbf{u}, \mathbf{p} \in C^0([0, T])^{2l}$ . Finally, taking into account (4.7) at time  $t = 0$ , we have  $\Pi_k^l \mathbf{p}(0) = \mathbf{0}_2$  and in view of (4.4), we obtain  $\Pi_k^l \mathbf{q}_0 = \mathbf{0}_2$ . In addition, since  $\mathbf{p}$  is continuous, we deduce from (4.3)–(4.4) that

$$\mathbf{q}_0 = \mathbf{q}_1 - \int_0^{T^*} p_0 \mathbf{J}(\Lambda) \mathbf{u} dt - \int_0^{T^*} \mathbf{x} d\eta. \quad (6.4)$$

Notice that due to (4.9), we have  $\int_{[t_0, t_1]} f d\eta = \int_{(t_0, t_1)} f d\eta$  for every  $f \in C([0, T^*])$  and for all  $0 \leq t_0 \leq t_1 \leq T^*$ . Hence, we write  $\int_{t_0}^{t_1} f d\eta$  for these integrals with respect to the measure  $\eta$ .

Combining (6.4) together with (4.3), we obtain

$$\mathbf{p}(t) = \mathbf{q}_0 + \int_0^t p_0 \mathbf{J}(\Lambda) \mathbf{u} dt + \int_0^t \mathbf{x} d\eta \quad (t \in [0, T^*]). \quad (6.5)$$

In addition, since  $\mathbf{u}$  is a time optimal control with the matrix  $\mathbf{J}(\Lambda)$ , we know from (4.7) that

$$-s_0 \mathbf{u} = p_0 \mathbf{J}(\Lambda) \mathbf{x} + \mathbf{p}.$$

Hence, we deduce

$$-s_0 \Pi_k^l \mathbf{u} = p_0 \Pi_k^l (\mathbf{J}(\Lambda) \mathbf{x}) + \Pi_k^l \mathbf{p} = p_0 \lambda J \Pi_k^l \mathbf{x} + \Pi_k^l \mathbf{p}. \quad (6.6)$$

Using (6.5) with (6.6), we obtain for every  $t \in [0, T^*]$ ,

$$\begin{aligned} -s_0 |\Pi_k^l \mathbf{u}(t)|^2 &= \left\langle p_0 \lambda J \Pi_k^l \mathbf{x}(t) + \Pi_k^l \mathbf{p}(t), \Pi_k^l \mathbf{u}(t) \right\rangle \\ &= \left\langle p_0 \lambda J \Pi_k^l \mathbf{x}(t), \Pi_k^l \mathbf{u}(t) \right\rangle + \left\langle \Pi_k^l \mathbf{q}_0, \Pi_k^l \mathbf{u}(t) \right\rangle + \left\langle \Pi_k^l \int_0^t p_0 \mathbf{J}(\Lambda) \mathbf{u} dt, \Pi_k^l \mathbf{u}(t) \right\rangle + \\ &\quad + \left\langle \Pi_k^l \int_0^t \mathbf{x} d\eta, \Pi_k^l \mathbf{u}(t) \right\rangle. \end{aligned}$$

Using the fact that  $\Pi_k^l \mathbf{q}_0 = \mathbf{0}_2$  and  $\Pi_k^l \mathbf{x}(t) = \int_0^t \Pi_k^l \mathbf{u}(\sigma) d\sigma$ , we obtain

$$\begin{aligned} -s_0 |\Pi_k^l \mathbf{u}(t)|^2 &= \left\langle \int_0^t 2p_0 \lambda J \Pi_k^l \mathbf{u} dt, \Pi_k^l \mathbf{u} \right\rangle + \left\langle \int_0^t \Pi_k^l \mathbf{x} d\eta, \Pi_k^l \mathbf{u}(t) \right\rangle \\ &= \int_0^t \left\langle 2p_0 \lambda J \Pi_k^l \mathbf{u}(s), \Pi_k^l \mathbf{u}(t) \right\rangle ds + \int_0^t \int_0^s \left\langle \Pi_k^l \mathbf{u}(\sigma), \Pi_k^l \mathbf{u}(t) \right\rangle d\sigma d\eta_s. \end{aligned}$$

The Cauchy-Schwartz inequality yields

$$\begin{aligned} -s_0 |\Pi_k^l \mathbf{u}(t)|^2 &\leq \int_0^t 2|p_0 \lambda| |\Pi_k^l \mathbf{u}(s)| |\Pi_k^l \mathbf{u}(t)| ds + \int_0^t \int_0^s |\Pi_k^l \mathbf{u}(\sigma)| |\Pi_k^l \mathbf{u}(t)| d\sigma d\eta_s \\ &\leq |\Pi_k^l \mathbf{u}(t)| \left( \int_0^t 2|p_0 \lambda| |\Pi_k^l \mathbf{u}(s)| ds + \int_0^t \int_0^s |\Pi_k^l \mathbf{u}(\sigma)| d\sigma d\eta_s \right). \end{aligned}$$

Finally, since we have  $\eta([0, t]) < +\infty$ , one can apply the Fubini Theorem (see [11, Theorem 1, Section 1.4]) to get

$$\begin{aligned} -s_0 |\Pi_k^l \mathbf{u}(t)| &\leq \int_0^t 2|p_0 \lambda| |\Pi_k^l \mathbf{u}(s)| ds + \int_0^t \int_0^s \chi_{[0, s]}(\sigma) |\Pi_k^l \mathbf{u}(\sigma)| d\sigma d\eta_s \\ &\leq \int_0^t 2|p_0 \lambda| |\Pi_k^l \mathbf{u}(s)| ds + \int_0^t \eta([\sigma, t]) |\Pi_k^l \mathbf{u}(\sigma)| d\sigma \\ &\leq \int_0^t (2|p_0 \lambda| + \eta([0, T^*])) |\Pi_k^l \mathbf{u}(s)| ds. \end{aligned}$$

Hence, we obtain, for every  $t \in [0, T^*]$ ,

$$|\Pi_k^l \mathbf{u}(t)| \leq \int_0^t \frac{2|p_0 \lambda| + \eta([0, T^*])}{-s_0} |\Pi_k^l \mathbf{u}(s)| ds.$$

Applying Gronwall's Lemma, we conclude that  $\Pi_k^l \mathbf{u}(t) = \mathbf{0}_2$  for every  $t \in [0, T^*]$ .  $\square$

The following result deals with an invariance property of the optimal solutions under suitable rotations. We denote by  $O_n$  the set of the  $n \times n$  real orthogonal matrices.

**Lemma 6.3.** *Let  $l \in \mathbb{N}^*$ ,  $\lambda \in \mathbb{R}_+^*$  and  $\Lambda = (\lambda, \dots, \lambda) \in \mathbb{R}_+^{*l}$ . We consider an orthogonal matrix  $\mathbf{R} \in O_{2l}$  such that  $\mathbf{R}$  and  $\mathbf{J}(\Lambda)$  commute. If  $\mathbf{u} \in \mathcal{U}^{T^*}(\mathbf{J}(\Lambda))$  is a time optimal control with the matrix  $\mathbf{J}(\Lambda)$  then  $\mathbf{R}\mathbf{u} \in \mathcal{U}^{T^*}(\mathbf{J}(\Lambda))$  is also a time optimal control.*

*Proof.* As usual, we denote by  $\mathbf{x} = \mathcal{X}_{\mathbf{u}}$  the optimal state trajectory associated to the time optimal control  $\mathbf{u} \in \mathcal{U}^{T^*}(\mathbf{J}(\Lambda))$ . Let us define  $\tilde{\mathbf{u}} = \mathbf{R}\mathbf{u}$  and  $\tilde{\mathbf{x}} = \mathbf{R}\mathbf{x}$  where  $\mathbf{R} \in O_{2l}$  commutes with the matrix  $\mathbf{J}(\Lambda)$ . We clearly have that  $\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{u}}$  and  $\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}(T^*) = \mathbf{0}_{2l}$  and  $|\tilde{\mathbf{x}}| = |\mathbf{x}|$ . Since  $\mathbf{R}$  is an orthogonal matrix which commutes with  $\mathbf{J}(\Lambda)$ , we obtain that  $\langle \mathbf{J}(\Lambda)\tilde{\mathbf{x}}, \tilde{\mathbf{u}} \rangle = \langle \mathbf{J}(\Lambda)\mathbf{x}, \mathbf{u} \rangle$  and hence  $\mathcal{Y}_{\tilde{\mathbf{u}}} = \mathcal{Y}_{\mathbf{u}}$ . Therefore, we have  $\tilde{\mathbf{u}} \in \mathcal{U}^{T^*}(\mathbf{J}(\Lambda))$  i.e.  $\tilde{\mathbf{u}} = \mathbf{R}\mathbf{u}$  is also a time optimal control.  $\square$

In the next Lemma we give a particular matrix  $\mathbf{R} \in O_{2l}$  which satisfies Lemma 6.3.

**Lemma 6.4.** *Let  $l \in \mathbb{N}^*$ ,  $\lambda \in \mathbb{R}_+^*$ ,  $\Lambda = (\lambda, \dots, \lambda) \in \mathbb{R}_+^{*l}$  and  $\mathbf{v} \in \mathbb{R}^{2l}$ . There exists  $\mathbf{R} \in O_{2l}$  such that  $\mathbf{R}$  and  $\mathbf{J}(\Lambda)$  commute and  $\Pi_j^l(\mathbf{R}\mathbf{v}) = \mathbf{0}_2$  for every  $j \in \{2, \dots, l\}$ .*

*Proof.* We can assume without loss of generality that  $\lambda = 1$ . Using an iterative procedure, it is enough to show the result for  $l = 2$ . This is done by considering the map  $\mathcal{R}$  defined for every  $\mathbf{v} \in \mathbb{R}^4$  with  $|\mathbf{v}| = 1$  by,

$$\mathcal{R}(\mathbf{v}) = \begin{cases} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} & \text{if } \mathbf{v}^1 = \mathbf{0}_2, \\ \begin{pmatrix} |v^1| I_2 & \frac{1}{|v^1|} (\langle \mathbf{v}^1, \mathbf{v}^2 \rangle I_2 - \langle J\mathbf{v}^1, \mathbf{v}^2 \rangle J) \\ \frac{-1}{|v^1|} (\langle \mathbf{v}^1, \mathbf{v}^2 \rangle I_2 + \langle J\mathbf{v}^1, \mathbf{v}^2 \rangle J) & |v^1| I_2 \end{pmatrix} & \text{otherwise,} \end{cases}$$

where  $\mathbf{v}^i = \Pi_i^2(\mathbf{v})$  for every  $i \in \{1, 2\}$ .



Using relation  $|\mathbf{v}^1|^2 \mathbf{v}^2 = \langle \mathbf{v}^1, \mathbf{v}^2 \rangle \mathbf{v}^1 + \langle J\mathbf{v}^1, \mathbf{v}^2 \rangle J\mathbf{v}^1$ , one can check that for every  $\mathbf{v} \in \mathbb{R}^4$ , with  $|\mathbf{v}| = 1$  we have  $\mathcal{R}(\mathbf{v}) \in O_4$ . Moreover  $\mathcal{R}(\mathbf{v})$  and  $\mathbf{J}(1, 1)$  commute and  $\Pi_2^2(\mathcal{R}(\mathbf{v})\mathbf{v}) = \mathbf{0}_2$ .  $\square$

We are now in position to prove Theorem 2.4 for the  $n$ -dimensional control case, starting from a 2-dimensional optimal control associated to the matrix  $\lambda^*J$ . More precisely, for  $\mathbf{u} \in \mathcal{U}^{T^*(\lambda^*)}(\lambda^*J)$ , we define  $\mathbf{u}^* \in L^\infty(0, T^*(\lambda^*))^{2l}$  by

$$\Pi_k^l \mathbf{u}^*(t) = \begin{cases} \mathbf{u}(t) & \text{if } k = 1 \\ \mathbf{0}_2 & \text{if } 2 \leq k \leq l \end{cases} \quad (t \in [0, T^*(\lambda^*)]). \quad (6.7)$$

We clearly have that  $\mathbf{u}^* \in \mathcal{U}^{T^*(\lambda^*)}(\mathbf{J}(\Lambda))$ . We now want to show that  $\mathbf{u}^* \in \mathcal{U}^{T^*(\mathbf{J}(\Lambda))}(\mathbf{J}(\Lambda))$  i.e.  $\mathbf{u}^*$  is an optimal control for the problem (2.7)–(2.9) with matrix  $\mathbf{J}(\Lambda)$ . This is equivalent to prove that  $T^*(\Lambda) = T^*(\lambda^*)$ .

From Lemma 6.1, we have

$$T^*(\lambda^*) \geq T^*(\Lambda) \geq T^*(\Lambda^*),$$

where  $\Lambda^* = (\lambda^*, \dots, \lambda^*) \in \mathbb{R}^l$ . Now, we prove that

$$T^*(\Lambda^*) \geq T^*(\lambda^*).$$

To this end, we consider  $\tilde{\mathbf{u}} \in \mathcal{U}^{T^*(\Lambda^*)}(\mathbf{J}(\Lambda^*))$ . Lemmas 6.3 and 6.4 imply that there exists an orthogonal matrix  $\mathbf{R} \in O_{2l}$  such that  $\mathbf{R}\tilde{\mathbf{u}} \in \mathcal{U}^{T^*(\Lambda^*)}(\mathbf{J}(\Lambda^*))$  with  $\Pi_k^l(\mathbf{R}\tilde{\mathbf{u}}(0)) = \mathbf{0}_2$  for every  $k \in \{2, \dots, l\}$ . Hence the zero-invariance property of Lemma 6.2 implies that  $\Pi_k^l(\mathbf{R}\tilde{\mathbf{u}}(t)) = \mathbf{0}_2$  for every  $k \in \{2, \dots, l\}$  and  $t \in [0, T^*(\Lambda^*)]$ . Finally, it remains clear that  $\mathbf{u} = \Pi_1^l(\mathbf{R}\tilde{\mathbf{u}}) \in \mathcal{U}^{T^*(\Lambda^*)}(\lambda^*J)$ . Then we have  $T^*(\lambda^*) \leq T^*(\Lambda^*)$ .

Hence we have proved that  $T^*(\lambda^*) = T^*(\Lambda) = T^*(\Lambda^*)$  and this proves that the function  $\mathbf{u}^*$  defined by (6.7) is an optimal control for the control problem (2.7)–(2.9) with the matrix  $\mathbf{J}(\Lambda)$ . According to (6.3), an optimal control for Problem (2.7)–(2.9) with the skew-symmetric invertible matrix  $M$  is given by  $P\mathbf{u}^*$ . Writing down  $\mathbf{u}^* = (\gamma_1, \gamma_2, 0, \dots, 0)^\top$  with  $\gamma_1, \gamma_2 \in C([0, T^*])$  and since the columns of the matrix  $P$  are composed by the vectors  $\mathbf{w}_1, \dots, \mathbf{w}_n$ , we obtain

$$P\mathbf{u}^* = \gamma_1 \mathbf{w}_1 + \gamma_2 \mathbf{w}_2.$$

The proof of Theorem 2.4 is completed.

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