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On the equivariant cohomology of Hilbert schemes of points in the plane

Pierre-Emmanuel Chaput, Laurent Evain

October 31, 2018

Abstract

Let $S = \mathbb{A}^2$ be the affine plane regarded as a toric variety with an action of the 2-dimensional torus T . We study the equivariant Chow ring $A_K^*(S^{[n]})$ of the punctual Hilbert scheme $S^{[n]}$ with equivariant coefficients inverted. We compute base change formulas in $A_K^*(S^{[n]})$ between the natural bases introduced by Nakajima, Ellingsrud and Strømme, and the classical basis associated with the fixed points. We compute the equivariant commutation relations between creation/annihilation operators. We express the class of the small diagonal in $S^{[n]}$ in terms of the equivariant Chern classes of the tautological bundle. We prove that the nested Hilbert scheme $S_0^{[n, n+1]}$ parametrizing nested punctual subschemes of degree n and $n + 1$ is irreducible.

Introduction

If S is a quasi-projective smooth surface, let $S^{[n]}$ be the Hilbert scheme parameterizing the zero dimensional subschemes of degree n in S . Following Nakajima and Grojnowski, a first tool to study the Chow ring $A^*(S^{[n]}, \mathbb{Q})$ is to consider the direct sum $\bigoplus_{n \in \mathbb{N}} A^*(S^{[n]}, \mathbb{Q})$ and operators acting linearly on this direct sum. Then, a lot of structure and information lies in the commutation relations of the various operators. In the case $S = \mathbb{A}^2$, this approach yields a basis of $A^*(S^{[n]}, \mathbb{Q})$ that we call Nakajima's basis and a description of the ring structure on it [Na99, Le99].

When $S = \mathbb{A}^2$, another approach is the use of the equivariant Chow rings. The 2-dimensional torus T acts on $S^{[n]}$. The equivariant Chow ring with respect to the action of the full torus T has been computed in [Ev07] in the case $S = \mathbb{P}^2$, but this is a purely equivariant approach independent of Nakajima's framework. Similarly Bialynicki-Birula's theorem [BB76] yields a basis of the classical and equivariant Chow rings which has been studied in [ES87] and which we call Ellingsrud-Strømme's basis.

There are equivariant analogues of the operators introduced by Nakajima et al which act on the equivariant Chow ring. Following Vasserot [Va01], it is natural to compute these equivariant operators. In his paper, Vasserot does not consider the full action of the torus T , but the action of a non-generic one-dimensional subtorus $T' \subset T$. He computes several operators in T' -equivariant Chow rings and their commutators. As a consequence, he obtains a description of the T' -equivariant and of the classical Chow ring of the Hilbert scheme of \mathbb{A}^2 .

On the other hand, Schiffmann and Vasserot study an algebra of operators acting on the equivariant K -theory of Hilbert schemes [SV09]. Since the correspondences defining q_i for $i > 1$ are singular, they do not define operators on the K -theory, this is the reason why the authors only consider the operator algebra generated by q_1, q_{-1} and multiplication by some tautological bundles.

In this work we consider the action of the operators q_i for all i on the T -equivariant Chow rings. The apparent difficulty coming from the non projectivity of \mathbb{A}^2 is not severe : we have all the standard constructions and properties of intersection theory that we need (pushforward, correspondences, composition of correspondences...) provided that we work in the tensored equivariant Chow ring $A_T^*(S^{[n]}) \otimes_{A_T^*(pt)} K$, instead of $A_T^*(S^{[n]})$, where K is the fraction field of $A_T^*(pt)$, the equivariant Chow ring of a point (Section 1).

However this construction also has its drawback: the pushforward of a contractant non proper morphism does not need to vanish (see Lemma 40 for an example) and some key arguments of the classical situation are not valid in our equivariant context. Let $S_0^{[n]}$ denote the set of subschemes z_n of length n supported as the origin, and let $S_0^{[p,q]}$ denote the similar set of couples of nested subschemes ($z_p \subset z_q$). When one wants to compute the composition of two correspondences, the ubiquitous local situation that one has to understand are the classes $\pi_*[C]$, where C is some subvariety in $S^{[p,q]}$ and π is the projection to $S^{[p]}$ or to $S^{[q]}$. The geometry is under control when both z_p and z_q are curvilinear for the generic pair $(z_p, z_q) \in C$. In the other cases, the restriction of π to C is contractant and therefore $\pi_*[C]$ is zero in the classical Chow ring [Na99, Le99]. However $\pi_*[C]$ need not vanish in the equivariant Chow ring. Our remedy is to prove that $S_0^{[n,n+1]}$ is irreducible and well understood (Theorem 19, Section 2) (whereas $S_0^{[p,q]}$ is not in general, Proposition 18). Then follows our construction to compute the commutators: we use algebraic arguments to reduce to the case when one of the operators adds only one point.

In Section 4, we consider the classical operators acting on $A_K := \bigoplus_{n \in \mathbb{N}} A_T^*(S^{[n]}) \otimes_{A_T^*(pt)} K$, namely the creation/destruction operators q_i , the boundary operator ∂ , and an auxiliary operator ρ . All these operators are defined by a correspondence. Provided that the correspondence is smooth, the computation is easily done with the Bott formula. This is the strategy to compute q_1 and q_{-1} in the fixed point basis (Proposition 27 and 28). All the other correspondences are singular at some points and a turnaround is needed to compute the corresponding operators.

Computing restriction to fixed points, we prove the formula $\partial = -2c_1(\mathcal{O}_X[n])$, where $\mathcal{O}_X[n]$ denotes the tautological bundle. Following Lehn and Schiffmann-Vasserot's ideas, we consider various commutators starting with q_1, q_{-1} and ∂ . We end up with recursion formulas for the q_i 's, $|i| > 1$ (Theorem 34). In particular this yields base change formulas between the fixed point basis and Nakajima's basis (Example 36).

To compute the commutation relations between the q_i 's (Theorem 52), using once again the same general idea as in [SV09], we use algebraic computations to reduce to the case of operators of conformal degree one. The algebraic reduction leads to Proposition 43, a formula apparently new even in the non equivariant context. We use geometric arguments to get rid of the excess intersection components which appear in this case.

The class $\delta_n \in A^{n-1}(S^{[n]})$ of the small diagonal $\Delta_n \subset S^{[n]}$ parameterizing the subschemes supported on a single point has an expression in terms of the equivariant Chern classes of the tautological bundle: $\delta_n = (-1)^{n-1} n c_{n-1}(\mathcal{O}_X[n])$. The original proof by Lehn [Le99, Theorem

4.6] remains true in our context. We give a new proof which relies on an algebraic expression for the operator q_n (Theorem 55).

Finally, we give an application of our equivariant computations at the level of classical Chow rings. We investigate the base change between Nakajima's basis and Ellingsrud and Strømme's basis. Our theorem asserts that these bases are equal up to sign (and a normalisation constant with our conventions) in classical Chow rings (Theorem 64). Our strategy of proof is to interpret the Bialynicki-Birula cells in terms of operators: we introduce new creation operators $q_{i,X}$ such that the basis introduced by Ellingsrud and Strømme is obtained applying these operators on the vacuum. We express the $q_{i,X}$ in terms of the creation operators q_i and we get a base change formula in the equivariant Chow ring. Projecting this relation in the usual Chow ring gives the asserted formula.

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1 Pushforward with non proper morphisms

We work over an algebraically closed field k of any characteristic. Let T be a 2-dimensional torus. The T -equivariant Chow ring $A_T^*(pt)$ of a point is isomorphic to a polynomial ring in two variables U, V . We denote by $K = \mathbb{Q}(U, V)$ the field of fractions of $A_T^*(pt)$. Moreover, if X is any T -variety, we denote by $A_K^*(X)$ the tensor product $A_T^*(X) \otimes_{\mathbb{Z}[U, V]} K$. We denote the product of two classes x, y in a Chow ring indifferently by $x \cdot y$ or by $x \cup y$.

In this section $f : X \rightarrow Y$ is an equivariant morphism between smooth varieties. Moreover, we assume that X and Y are *filtrable*, in the sense of Definition 3.2 in [Bri97].

When $f : X \rightarrow Y$ is a proper equivariant morphism, there is a well defined pushforward $f_* : A_T^*(X) \rightarrow A_T^*(Y)$. Since we shall work with the affine plane, we are in a non projective setting and we have to deal with non proper morphisms.

The goal of this section is to explain that a good notion of pushforward f_*^K exists, when f is a non proper morphism, provided that the restriction to fixed points $f^T : X^T \rightarrow Y^T$ is proper. This notion is applied to define correspondences. We show that these correspondences defined using non proper morphisms satisfy formal properties similar to the correspondences defined in the usual setting when f is proper.

Definition 1 *If f is as above, $f^T : X^T \rightarrow Y^T$ will denote the restriction of f to T -fixed points. The morphisms $f_*^K : A_K^*(X) \rightarrow A_K^{*+\dim Y - \dim X}(Y)$ (when f is proper) and $f_K^* : A_K^*(Y) \rightarrow A_K^*(X)$ are derived from the standard morphisms $f_*^T : A_T^*(X) \rightarrow A_T^{*+\dim Y - \dim X}(Y)$ and $f_T^* : A_T^*(Y) \rightarrow A_T^*(X)$ after tensorisation over $\mathbb{Z}[U, V]$ by K .*

Let $f : X \rightarrow Y$ be any T -equivariant morphism, and consider the following commutative diagram:

$$\begin{array}{ccc} X^T & \xrightarrow{i} & X \\ \downarrow f^T & & \downarrow f \\ Y^T & \xrightarrow{j} & Y \end{array} \quad . \quad (1)$$

Since i_*^K is an isomorphism by [EG98, Theorem 1], the following definition is meaningful:

Definition 2 *If f^T is proper, define*

$$f_*^{T,K} = j_*^K (f^T)_*^K (i_*^K)^{-1} : A_K^*(X) \rightarrow A_K^{*+\dim Y - \dim X}(Y) .$$

If f is proper, then $f_*^{T,K} = f_*^K$, by the functoriality of the pushforward in the proper case. Since there is therefore no possible confusion, we will denote $f_*^{T,K}$ simply by f_*^K .

Theorem 3 *The morphism f_*^K satisfies the following properties:*

1. *Functoriality: if we have T -equivariant morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that f^T and g^T are proper, then $(g \circ f)_*^K = g_*^K \circ f_*^K$.*
2. *Projection formula: assume here that X and Y are smooth, so that $A_K^*(X)$ and $A_K^*(Y)$ are rings. For any $\alpha \in A_K^*(X)$ and $\beta \in A_K^*(Y)$, we have the equality $f_*^K(\alpha) \cdot \beta = f_*^K(\alpha \cdot f_*^K(\beta))$.*

3. We have the equality $g_*^K f_K^* = l_K^* h_*^K$ if f, g, h, l are as in the following diagram:

$$\begin{array}{ccc} X \times Y \times Z & \xrightarrow{g} & Y \times Z \\ \downarrow f & & l \downarrow \\ X \times Y & \xrightarrow{h} & Y \end{array}$$

Proof.

Functoriality. Consider the following diagram:

$$\begin{array}{ccc} X^T & \xrightarrow{i} & X \\ \downarrow f^T & & \downarrow f \\ Y^T & \xrightarrow{j} & Y \\ \downarrow g^T & & \downarrow g \\ Z^T & \xrightarrow{k} & Z \quad . \end{array}$$

Then we have

$$\begin{aligned} (g \circ f)_*^K &= k_*^K ((g \circ f)^T)_*^K (i_*^K)^{-1} \\ &= k_*^K (g^T)_*^K (f^T)_*^K (i_*^K)^{-1} \\ &= k_*^K (g^T)_*^K (j_*^K)^{-1} \circ j_*^K (f^T)_*^K (i_*^K)^{-1} \\ &= g_*^K \circ f_*^K \quad . \end{aligned}$$

Projection formula. Let $\alpha \in A_K^*(X)$ and $\beta \in A_K^*(Y)$. By commutativity of the diagram (1), we have:

$$\begin{aligned} \alpha \cdot f_K^* \beta &= \alpha \cdot (i_K^*)^{-1} (f^T)_K^* j_K^* \beta \\ &= i_K^* (i_K^*)^{-1} \alpha \cdot (i_K^*)^{-1} (f^T)_K^* j_K^* \beta \\ &= i_K^* ((i_K^*)^{-1} \alpha \cdot i_K^* (i_K^*)^{-1} (f^T)_K^* j_K^* \beta), \end{aligned}$$

where the last equality is due to the usual projection formula for i_K^* . Applying $(j_K^*)^{-1}$ to both sides of the equality we get: $(i_K^*)^{-1} (\alpha \cdot f_K^* \beta) = (i_K^*)^{-1} \alpha \cdot (f^T)_K^* j_K^* \beta$. Therefore,

$$\begin{aligned} f_*^K \alpha \cdot \beta &= j_*^K (f^T)_*^K (i_*^K)^{-1} \alpha \cdot \beta \\ &= j_*^K (f^T)_*^K ((i_*^K)^{-1} \alpha \cdot (f^T)_K^* j_K^* \beta) \\ &= j_*^K (f^T)_*^K ((i_*^K)^{-1} (\alpha \cdot f_K^* \beta)) \\ &= f_*^K (\alpha \cdot f_K^* \beta) \end{aligned}$$

(the second equality follows from the projection formula for $j \circ f^T$.)

Property 3. Consider the following diagram:

$$\begin{array}{ccccccc} X^T \times Y^T \times Z^T & \xrightarrow{b} & X \times Y \times Z & \xrightarrow{g} & Y \times Z & \xleftarrow{d} & Y^T \times Z^T \\ f^T \downarrow & & \downarrow f & & l \downarrow & & \downarrow l^T \\ X^T \times Y^T & \xrightarrow{a} & X \times Y & \xrightarrow{h} & Y & \xleftarrow{c} & Y^T \quad . \end{array}$$

The property that we want to prove holds when the pushforward maps are taken with proper morphisms, so that for example $b_*^K(f^T)_K^* = f_K^* a_*^K$. The second following equality is a consequence of this remark, the third and the fourth equalities are similar:

$$\begin{aligned}
g_*^K f_K^* &= d_*^K (g^T)_*^K (b_*^K)^{-1} f_K^* \\
&= d_*^K (g^T)_*^K (f^T)_K^* (a_*^K)^{-1} \\
&= d_*^K (l^T)_K^* (h^T)_*^K (a_*^K)^{-1} \\
&= l_K^* c_*^K (h^T)_*^K (a_*^K)^{-1} \\
&= l_K^* h_*^K .
\end{aligned}$$

□

In practice, f_*^K can be computed by a ‘‘Bott formula’’, as in the proper case. Assume that X is smooth. Since X^T is smooth, $A_T^*(X^T) = \bigoplus A_T^*(X_i)$ where the sum runs through the irreducible components X_i of X^T . We denote by $c_{top}(N_{X^T, X})$ the operator which acts on $A_T^*(X^T)$ through multiplication by the equivariant Chern class c_{d_i} of the normal bundle $N_{X_i, X}$ on the component $A_T^*(X_i)$, where d_i is the codimension of X_i in X . Similarly, there is a class $c_{top}(N_{Y^T, Y})$. In $A_K^*(X)$ (or $A_K^*(Y)$), the Chern class c_{d_i} is equal to the sum of an invertible element and a nilpotent element, according to the proof of [Bri97, Proposition 3.2(i)]. Therefore it is invertible and $c_{top}(N_{X^T, X})$ is an invertible operator.

Lemma 4 *Assume that X is smooth. Let $i : X^T \rightarrow X$ be the natural inclusion. The pullback i_K^* is invertible with inverse $i_*^K \frac{1}{c_{top}(N_{X^T, X})}$.*

Proof. For $\alpha \in A_T^*(X^T)$, the self-intersection formula reads

$$i_T^* i_*^T \alpha = c_{top}(N_{X^T, X}) \alpha .$$

Thus the lemma follows from the fact that i_*^K is invertible by the localization theorem [EG98, Theorem 1]. □

Theorem 5 (Bott Formula) *Recall the diagram (1) and assume that X and Y are smooth. Let $\alpha \in A_T^*(X)$. Then*

$$j_K^* f_*^K(\alpha) = c_{top}(N_{Y^T, Y}) (f^T)_*^K \left(\frac{1}{c_{top}(N_{X^T, X})} i_K^*(\alpha) \right) .$$

In particular, when both X and Y have a finite number of fixed points $x_1, \dots, x_n, y_1, \dots, y_p$, the formula expresses f_^K in terms of the localization at these fixed points:*

$$(f_*^K \alpha)(y_k) = \sum_{f(x_i)=y_k} \frac{c_{top}(T_{y_k, Y})}{c_{top}(T_{x_i, X})} \alpha_{x_i} ,$$

where $T_{y_k, Y}$ and $T_{x_i, X}$ are the tangent T -representations.

Proof. By Lemma 4, the inverse of i_*^K is $\frac{1}{c_{top}(N_{X^T, X})}i_K^*$. Then,

$$\begin{aligned} j_K^* f_*^K \alpha &= j_K^* j_*^K (f^T)_*^K (i_*^K)^{-1} \alpha \\ &= c_{top}(N_{Y^T, Y}) (f^T)_*^K (i_*^K)^{-1} \alpha \\ &= c_{top}(N_{Y^T, Y}) (f^T)_*^K \left(\frac{1}{c_{top}(N_{X^T, X})} i_K^*(\alpha) \right). \end{aligned}$$

□

Definition 6 *An equivariant correspondence is a closed T -stable subvariety $C \subset X \times Y$ such that $C^T \rightarrow Y^T$ is a proper morphism. Let π_X and π_Y be the projections from C to X and Y respectively. The classes of such varieties C generate a subspace in $A_K^*(X \times Y)$ and we still call equivariant correspondence a class in this subspace. An equivariant correspondence C yields a morphism $f : A_K^*(X) \rightarrow A_K^*(Y)$ defined by $f(\alpha) = (\pi_Y)_*^K (\pi_X)_*^K(\alpha)$.*

Proposition 7 *Assume that X, Y, Z are smooth varieties. Let $C \subset X \times Y$ and $D \subset Y \times Z$ be two equivariant correspondences, and f and g the associated morphisms. Let $\pi_{12}, \pi_{13}, \pi_{23}$ the projections from $X \times Y \times Z$ to $X \times Y, X \times Z$ and $Y \times Z$ respectively. Then the $(\pi_{13})_*^K ((\pi_{12})_*^K [C] \cup (\pi_{23})_*^K [D])$ is an equivariant correspondence with associated morphism $g \circ f$.*

Proof. When $f : C \rightarrow Y$ and $g : D \rightarrow Z$ are proper, the proof is classical and relies on functoriality of the pushforward, on the projection formula, and on the third property of Theorem 3. Thus the proof is mutatis mutandis the same as in the proper case if we check that $E = (\pi_{13})_*^K ((\pi_{12})_*^K [C] \cup (\pi_{23})_*^K [D]) \in A_K^*(X \times Z)$ is well defined and if we can associate to E a morphism $f_E : A_K^*(X) \rightarrow A_K^*(Z)$ with the help of the standard formulas of intersection theory.

Let $L = (\pi_{12}^{-1} C \cap \pi_{23}^{-1} D)$. Since $D^T \rightarrow Z^T$ is proper, so is $X^T \times D^T \rightarrow X^T \times Z^T$. It follows that $L^T = (C^T \times Z^T) \cap (X^T \times D^T) \rightarrow X^T \times Z^T$ is proper and that $(\pi_{13})_*^K(L)$ is well defined.

We now prove that $h : L^T \rightarrow Z^T$ is proper. In particular, there is a well defined morphism $(\pi_3^L)_*^K : A_K^*(L) \rightarrow A_K^*(Z)$. We consider the factorization of $h : L^T \rightarrow Z^T$ as $g \circ f$ with $f : L^T \rightarrow D^T$ and $g : D^T \rightarrow Z^T$. The morphism g is proper since D is an equivariant correspondence. The morphism $f : L^T \rightarrow D^T$ is the extension of the proper morphism $C^T \rightarrow Y^T$ by the morphism $D^T \rightarrow Y^T$. Then h is proper as a composition of proper morphisms.

When the intersection L is proper, then $E = (\pi_{13})_*^K(L)$ and f_E is defined to be the composite morphism $(\pi_3^L)_*^K \circ (\pi_1^L)_*^K : A_K^*(X) \rightarrow A_K^*(L) \rightarrow A_K^*(Z)$.

When the intersection L is not proper, then $(\pi_{12})_*^K [C] \cup (\pi_{23})_*^K [D]$ is representable by a refined intersection I , ie. by a linear combination of classes $I = \sum \alpha_i [L_i]$, with $L_i \subset L$ a closed subvariety. In particular, $L_i^T \rightarrow Z^T$ is proper as a composition of the proper morphisms $L_i^T \rightarrow L^T \rightarrow Z^T$. Similarly $L_i^T \rightarrow X^T \times Z^T$ is proper since it factorizes through L^T . Then the pushforward $E = (\pi_{13})_*^K (\sum \alpha_i L_i)$ is well defined. The associated morphism is $f_E = \sum \alpha_i f_{L_i}$ with $f_{L_i} = (\pi_3^{L_i})_*^K \circ (\pi_1^{L_i})_*^K$. □

Definition 8 *Suppose that $\pi : X^T \rightarrow \text{Spec } k$ is proper. Then there is a well-defined pushforward map $\pi_*^K : A_K^*(X) \rightarrow K$. This yields a K -bilinear product on $A_K^*(X)$ defined by $\langle \alpha, \beta \rangle_X = \pi_*^K(\alpha \cup \beta)$.*

When $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ are both non degenerate (for instance when X and Y have a finite number of fixed points), then every map $f : A_K^*(X) \rightarrow A_K^*(Y)$ admits a dual map $f^\vee : A_K^*(Y) \rightarrow A_K^*(X)$.

Definition 9 *If $C \subset X \times Y$ is a correspondence, the dual correspondence C^\vee is the correspondence in $Y \times X$ which is canonically identified with C under the natural isomorphism $X \times Y \simeq Y \times X$. In particular, if C^\vee is an equivariant correspondence, it yields a map $A_K^*(Y) \rightarrow A_K^*(X)$.*

Proposition 10 *Assume that X and Y are smooth. Let $C \subset X \times Y$ be an equivariant correspondence and $f : A_K^*(X) \rightarrow A_K^*(Y)$ the associated morphism. Suppose that $C^\vee \subset Y \times X$ is an equivariant correspondence and that $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ are non degenerate. Then the dual map f^\vee is defined by the dual correspondence C^\vee .*

Proof. In the classical setting, the proof relies on the functoriality of the pushforward and on the projection formula. Both arguments remain valid in our context. \square

Restriction to fixed points does not commute with the bilinear product (see Lemma 26). This remark is important when one wants to compute f^\vee on fixed points.

2 Tangent space to $S^{[n,n+1]}$

We denote by S the affine plane \mathbb{A}^2 . Let $n \geq 0$ be an integer, we denote by $S^{[n]}$ the Hilbert scheme parameterizing length n subschemes of S [Gro61] ($S^{[0]} = \text{Spec } k$). Given $z \in S^{[n]}$, we denote by $I_z \subset k[X, Y]$ the corresponding ideal, of codimension n . Given p, q integers with $0 \leq p < q$, we denote by $S^{[p,q]}$ the “nested” Hilbert scheme, namely the subscheme of $S^{[p]} \times S^{[q]}$ consisting of pairs (s, b) such that $I_s \supset I_b$. The torus $(k^*)^2$ will be denoted by T . It acts on the plane S : we use the convention that an element $(u, v) \in T$ acts on a monomial $X^a Y^b \in k[X, Y]$ by $(u, v) \cdot X^a Y^b = (uX)^a (vY)^b$. This induces an action of T on each Hilbert scheme $S^{[n]}$ and $S^{[p,q]}$. We will denote by $aU + bV$ the weight on T defined by $(aU + bV)(u, v) = u^a v^b$. Given a monomial $m = X^a Y^b$, we denote by $wt(m) = aU + bV$ its weight. Any character of T defines naturally an element in $A^1(pt)$, thus our notation is compatible with the notation $A^*(pt) = \mathbb{Z}[U, V]$ in Section 1.

Several arguments in the present paper rely on a tangent space argument. In fact, at a T -fixed point $z \in S^{[n]}$, the tangent space $T_z S^{[n]}$ has several combinatorial descriptions. One [Ev04] is in terms of significant cleft pairs and another [Na99] in terms of boxes of the corresponding staircase. We recall in this section the necessary material to be comfortable with these two notions. We give two applications. First, we compute the tangent space at a toric point in $S^{[n,n+1]}$ as a representation of T . Then Theorem 19 proves the irreducibility of $S_0^{[n,n+1]} \subset S^{[n,n+1]}$ parameterizing the pairs of subschemes $s \subset b$ with the support of b equal to the origin.

2.1 Tangent space to the Hilbert schemes

First, observe that a T -fixed point z in $S^{[n]}$ is defined by an ideal

$$I_z = \bigoplus_{(a,b) \notin E} k \cdot X^a Y^b \ ,$$

where $E \subset \mathbb{N}^2$ satisfies $(\mathbb{N}^2 \setminus E) + \mathbb{N}^2 \subset (\mathbb{N}^2 \setminus E)$. Such (finite) subsets $E \subset \mathbb{N}^2$ will be called *staircases*. A partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_l > 0)$ is by definition a finite sequence of non increasing positive natural numbers, l is called the length of λ , and $|\lambda| = \sum_{i=1}^l \lambda_i$ is the weight of λ . We denote by \mathcal{P}_n the set of partitions of weight n . If E is a finite staircase associated with a T -fixed point $z \in S^{[n]}$, there exists a unique partition λ with weight n such that $(a, b) \in E \Leftrightarrow a + 1 \leq l, b < \lambda_{a+1}$.

We begin by recalling the description given in [Ev04] of $T_z S^{[n]}$ when $z \in S^{[n]}$ is a T -fixed point.

- A monomial $c \in I_z$ is called a *cleft* whenever $X^{-1} \cdot c \notin I_z$ and $Y^{-1} \cdot c \notin I_z$.
- A Laurent monomial is called *positive* (resp., *negative*) if it belongs to $Y^{-1}k[X, Y^{-1}]$ (resp., $X^{-1}k[X^{-1}, Y]$).
- A weight $aU + bV$ with $a \geq 0$ and $b < 0$ resp. $a < 0$ and $b \geq 0$ will be called *positive* resp. *negative*.
- A *cleft pair* is a pair (c, m) such that c is a cleft, m is a monomial not belonging to I_z , and m/c is either positive or negative (in which case, we say that (c, m) is positive or negative, respectively).

Now let $\mathcal{C} := \{c_1, \dots, c_l\}$ denote the set of clefts, which we order following the convention that c_{i+1}/c_i be positive for $1 \leq i \leq l-1$.

For each positive (resp., negative) cleft pair (c_i, m) , let $s := s_i$ denote the least common multiple of c_i and c_{i+1} (resp., c_i and c_{i-1}). We say that (c_i, m) is *significant* if $ms/c_i \in I_z$. To (c_i, m) we associate the vector $\varphi = \varphi_{(c_i, m)}$ in $T_z S^{[n]} \simeq \text{Hom}_{k[X, Y]}(I_z, k[X, Y]/I_z)$ defined by

$$\begin{cases} \varphi(c_j) = mc_j/c_i \text{ if } j \leq i \\ \varphi(c_j) = 0 \text{ if } j > i \end{cases} \quad \text{resp.} \quad \begin{cases} \varphi(c_j) = 0 \text{ if } j < i \\ \varphi(c_j) = mc_j/c_i \text{ if } j \geq i \end{cases}$$

According to [Ev04, Theorem 3], the set of elements $\varphi_{(c, m)}$ for all significant cleft pairs (c, m) is a basis of $T_z S^{[n]}$.

On the other hand, Nakajima gives a combinatorial description of the weights occurring in $T_z S^{[n]}$ [Na99, Proposition 5.8]. Given a staircase E and $e \in E$, let $a(e) := \max\{i \mid X^i \cdot e \in E\}$ and let $b(e) := \max\{j \mid Y^j \cdot e \in E\}$. The set of positive weights in $T_z S^{[n]}$, counted with multiplicities, is the set of weights of the form $w_+(e) := a(e)U - (b(e) + 1)V$, and the set of negative weights is the set of weights the form $w_-(e) := -(a(e) + 1)U + b(e)V$.

We now give a bijection h_+ resp. h_- between the staircase E and the set of positive resp. negative significant cleft couples, preserving the weights, meaning that $wt(\varphi_{(h_{\pm}(e))}) = w_{\pm}(e)$. The bijection h_+ resp. h_- is defined as follows: given $e \in E$, we denote $c_1 = Y^b \cdot e$ resp. $c_1 = X^a \cdot e$, where b resp. a is the minimal integer such that $Y^b \cdot e \notin E$ resp. $X^a \cdot e \notin E$. We denote $c_2 = X^a \cdot e$ resp. $c_2 = Y^b \cdot e$, where a resp. b is the maximal integer such that $X^a \cdot e \in E$ resp. $Y^b \cdot e \in E$. We let i resp. j be the maximal integer such that $X^{-i} \cdot c_1 \notin E$ resp. $Y^{-j} \cdot c_1 \notin E$ (thus $X^{-i} \cdot c_1$ resp. $Y^{-j} \cdot c_1$ is a cleft). Finally we set $h_+(e) = (X^{-i} \cdot c_1, X^{-i} \cdot c_2)$ resp. $h_-(e) = (Y^{-j} \cdot c_1, Y^{-j} \cdot c_2)$: by construction of (c_1, c_2) this is a significant cleft couple.

Proposition 11 *The map h_+ resp. h_- is a bijection between E and the set of positive resp. negative significant cleft couples, and we have $wt(\varphi_{(h_+(e))}) = w_+(e)$ and $wt(\varphi_{(h_-(e))}) = w_-(e)$.*

Proof. By symmetry, we give the proof only in the positive case. With the notations before the proposition, we have $wt(\varphi_{(h_+(e))}) = wt(c_2c_1^{-1})$, which is readily $w_+(e)$. Thus to prove the proposition it suffices to describe the inverse of h_+ . Given a positive significant cleft couple (c, m) , let i be the maximal integer such that $X^i \cdot m \in E$. The inverse of h_+ maps (c, m) to the greatest common divisor of $X^i \cdot c$ and $X^i \cdot m$. \square

2.2 Computation of the tangent space of $S^{[n,n+1]}$

Let $(s, b) \in (S^{[n,n+1]})^T$ (s and b stand for small and big). We denote by E_s, E_b their staircases. Let $q : S^{[n,n+1]} \rightarrow S^{[n+1]}$ denote the natural projection, and let dq denote its differential at (s, b) . There is a natural exact sequence

$$0 \rightarrow \ker dq \rightarrow T_{(s,b)}S^{[n,n+1]} \xrightarrow{dq} T_bS^{[n+1]} .$$

The following is immediate:

Lemma 12 *The tangent space $T_{(s,b)}S^{[p,q]}$ of $S^{[p,q]}$ at (s, b) is the set of couples of homomorphisms $(\varphi, \psi) \in Hom_{k[X,Y]}(I_s, k[X, Y]/I_s) \times Hom_{k[X,Y]}(I_b, k[X, Y]/I_b)$ such that for all $f \in I_b$, we have*

$$\varphi(f) = \psi(f) \text{ mod } I_s.$$

Proof. This tangent space is included in the tangent space $T_{(s,b)}(S^{[p]} \times S^{[q]})$, which is the direct sum $Hom_{k[X,Y]}(I_s, k[X, Y]/I_s) \oplus Hom_{k[X,Y]}(I_b, k[X, Y]/I_b)$. Consider the ring $k[\epsilon]$, where $\epsilon^2 = 0$. Identifying the tangent bundle of $S^{[p]}$ resp. $S^{[q]}$ with the $k[\epsilon]$ -points of $S^{[p]}$ resp. $S^{[q]}$, we see that $(\varphi, \psi) \in T_{(s,b)}S^{[p,q]}$ if and only if the restriction of φ to I_b is equal to the quotient of ψ modulo I_s . \square

The following propositions 13 and 14 describe the infinitesimal deformations of b that admit a lift to a deformation of (s, b) . There are two cases, depending on the geometry of the staircases involved.

If m is a monomial, we denote by $x(m)$ its exponent for the variable X . In other words $x(X^aY^b) = a$. Similarly $y(X^aY^b) = b$. We denote by c_1, \dots, c_l the clefts of s , and by k the index such that $c_k \in E_b$.

Proposition 13 *Assume that $y(c_{k-1}) > y(c_k) + 1$, resp. $x(c_{k+1}) > x(c_k) + 1$. Then the positive, resp. negative part of $\text{Im } dq$ is the subspace of $T_bS^{[n+1]}$ generated by those $\varphi_{(c,m)}$ with $(c, m) \neq (Yc_k, X^{-1}c_i), i > k$, resp. $(c, m) \neq (Xc_k, Y^{-1}c_i), i < k$.*

Proof. By symmetry, it suffices to describe the positive part of $\text{Im } dq$ when $y(c_{k-1}) > y(c_k) + 1$.

The fact that many cleft pairs of s are also cleft pairs of b is a potential source of confusion. Consequently, given a pair (c, m) of both s and b , we will use $\varphi_{(c,m)}^n$ resp. $\varphi_{(c,m)}^{n+1}$ to denote the corresponding tangent vector in $T_sS^{[n]}$ resp. $T_bS^{[n+1]}$. Moreover we will use the convention that if $m \notin E_s$, then $\varphi_{(c,m)}^n = 0$. Let (c, m) be a positive significant cleft pair of b .

First note that if $c = c_i$ with $i \neq k$, then $\varphi_{(c,m)}^{n+1} \in \text{Im } dq$. In fact in this case it is clear that $(\varphi_{(c,m)}^n, \varphi_{(c,m)}^{n+1})$ is a tangent vector of $S^{[n,n+1]}$ (recall our convention that $\varphi_{(c,m)}^n$ is 0 if $m = c_k$).

Next consider the case where $c = Xc_k$. We also see that $\varphi_{(c,m)}^{n+1} \in \text{Im } dq$ since the pair $(\varphi_{(c_k, X^{-1}m)}^n, \varphi_{(Xc_k, m)}^{n+1})$ is a tangent vector of $S^{[n,n+1]}$ by Lemma 12.

Now, if $c = Yc_k$, the fact that (c, m) is a significant cleft pair implies that $Xm \in I_b$. If $m \neq X^{-1}c_i$ for any $i > k$, then $XY^{-1}m \in I_b$ and thus $\varphi_{(c_k, Y^{-1}m)}^n(Xc_k) = 0$, so that $(\varphi_{(c_k, Y^{-1}m)}^n, \varphi_{(Yc_k, m)}^{n+1})$ is a tangent vector of $S^{[n,n+1]}$.

It remains to show that the $\varphi_{(Yc_k, X^{-1}c_i)}^{n+1}$ -coefficient of any vector in $\text{Im } dq$ vanishes, for all $i > k$. To this end, let $(\varphi^n, \varphi^{n+1})$ be a tangent vector to the incidence variety. Considering φ^n resp. φ^{n+1} as an element of $\text{Hom}_{k[X,Y]}(I_s, k[X, Y]/I_s)$ resp. $\text{Hom}_{k[X,Y]}(I_b, k[X, Y]/I_b)$, we see that the coefficient of $Y^{-1}c_i$ in $\varphi^n(Xc_k)$ is equal to the coefficient of $X^{-1}c_i$ in $\varphi^n(Yc_k)$ (namely, those coefficients equal the coefficient of $X^{-1}Y^{-1}c_i$ in $\varphi^n(c_k)$). It follows from Lemma 12 that φ^{n+1} has the same property. On the other hand, among all the basis vectors $(\varphi_{(c,m)}^{n+1})$, $\varphi_{(Yc_k, X^{-1}c_i)}^{n+1}$ is the only vector for which these coefficients are not equal. Thus the $\varphi_{(Yc_k, X^{-1}c_i)}^{n+1}$ -coefficient in φ^{n+1} vanishes. \square

Proposition 14 *Assume that $y(c_{k-1}) = y(c_k) + 1$ resp. $x(c_{k+1}) = x(c_k) + 1$. Then the positive resp. negative part of $\text{Im } dq$ is the subspace of $T_b S^{[n+1]}$ generated by those $\varphi_{(c,m)}$ with $(c, m) \neq (c_{k-1}, X^{x(c_{k-1})-x(c_k)-1} \cdot c_i), i > k$ resp. $(c, m) \neq (c_{k+1}, Y^{y(c_{k+1})-y(c_k)-1} \cdot c_i), i < k$.*

Proof. The argument is similar to that used in the proof of the preceding proposition. For any positive significant cleft pair (c_i, m) with $i > k$ or $i < k - 1$, the vector $(\varphi_{(c_i, m)}^n, \varphi_{(c_i, m)}^{n+1})$ belongs to $T_{(s,b)} S^{[n,n+1]}$; whence, $\varphi_{(c_i, m)}^{n+1}$ belongs to the image of dq .

Similarly, if (c, m) is a positive cleft pair and $c = Xc_k$, then the pair $(\varphi_{(c_k, X^{-1}m)}^n, \varphi_{(c, m)}^{n+1})$ belongs to $T_{(s,b)} S^{[n,n+1]}$; we deduce $\varphi_{(c, m)}^{n+1} \in \text{Im } dq$.

We now consider positive significant cleft pairs of the form (c_{k-1}, m) . Assume that $i \geq k$ is such that $y(c_{i-1}) > y(m) \geq y(c_i)$. Since (c_{k-1}, m) is significant, we have

$$x(m) \geq x(c_i) + x(c_{k-1}) - x(c_k) - 1.$$

When $x(m) \geq x(c_i) + x(c_{k-1}) - x(c_k)$, the pair $(\varphi_{(c_{k-1}, m)}^n, \varphi_{(c_{k-1}, m)}^{n+1})$ belongs to the tangent space $T_{(s,b)} S^{[n,n+1]}$, so $\varphi_{(c_{k-1}, m)}^{n+1} \in \text{Im } dq$. When $x(m) = x(c_i) + x(c_{k-1}) - x(c_k) - 1$ and $y(m) > y(c_i)$, the pair $(\varphi_{(c_k, X^{x(c_k)-x(c_{k-1})} Y^{-1}m)}^n, \varphi_{(c_{k-1}, m)}^{n+1})$ belongs to $T_{(s,b)} S^{[n,n+1]}$, so $\varphi_{(c_{k-1}, m)}^{n+1} \in \text{Im } dq$.

Finally it remains to show that the $\varphi_{(c_{k-1}, X^{x(c_{k-1})-x(c_k)-1} c_i)}^{n+1}$ -coordinate of any vector in $\text{Im } dq$ vanishes. To this end, note that $\varphi_{(c_{k-1}, X^{x(c_{k-1})-x(c_k)-1} c_i)}^{n+1}$ is the only vector φ in our basis of $T_b S^{[n+1]}$ for which

$$[Y^{-1}c_i]\varphi(Xc_k) \neq [X^{-1}c_i]\varphi(Yc_k),$$

while

$$[Y^{-1}c_i]\psi(Xc_k) = [X^{-1}c_i]\psi(Yc_k)$$

for any $\psi \in \text{Hom}(I_s, k[X, Y]/I_s)$, where $[m]\varphi(n)$ denotes the coefficient of m in $\varphi(n)$. \square

While the description of $\text{Im}(dq)$ given in Propositions 13 and 14 depends on the shape of the Young tableau corresponding to s , the weights of the T -representation $T_b S^{[n+1]}/\text{Im} dq$ have a more uniform description. This space measures the obstructions to lift a deformation of b to a deformation of (s, b) .

Proposition 15 *The weights of the T -representation $T_b S^{[n+1]}/\text{Im} dq$ have multiplicity one and are given by*

$$(x(c_i) - x(c_k) - 1)U + (y(c_i) - y(c_k) - 1)V, i \neq k.$$

These weights are the weights of $\frac{c_i}{XYc_k}$, $i \neq k$.

Proof. By symmetry, it suffices to consider only the positive part of the quotient $T_b S^{[n+1]}/\text{Im} dq$. Assume first that $y(c_{k-1}) > y(c_k) + 1$. By Proposition 13, the positive weights of $T_b S^{[n+1]}/\text{Im} dq$ are the weights of $\frac{X^{-1}c_i}{Yc_k}$. Thus the proposition is proved in this case. Assume now that $y(c_{k-1}) = y(c_k) + 1$. By Proposition 14, the positive weights of $T_b S^{[n+1]}/\text{Im} dq$ are the weights of $\frac{X^{x(c_{k-1})-x(c_k)-1}c_i}{c_{k-1}}$. Since $c_{k-1} = X^{x(c_{k-1})-x(c_k)}Yc_k$, the proposition follows in this case too. \square

Proposition 16 *The weights of $\ker dq$ have multiplicity one and are the following:*

$$\begin{aligned} & -(y(c_{i-1}) - y(c_k) + 1)V + (x(c_i) - x(c_k) - 1)U \quad i > k \\ & -(x(c_k) - x(c_{i+1}) + 1)U + (y(c_i) - y(c_k) - 1)V \quad i < k \end{aligned}$$

These weights are the weights of the arrows from c_k to the corners of the partition corresponding to s .

Proof. This kernel consists of those morphisms $\varphi \in \text{Hom}_{k[X, Y]}(I_s, k[X, Y]/I_s)$ for which $\varphi(I_b) = 0$, by Lemma 12. It follows that if (c, m) is a cleft pair with $c \neq c_k$, we have

$$[\varphi_{(c, m)}^n]\varphi = 0$$

for every $\varphi \in \ker dq$. On the other hand, if (c_k, m) is a cleft pair and φ has a non-vanishing $\varphi_{(c_k, m)}^n$ -coefficient, the fact that

$$\varphi(Xc_k) = X\varphi(c_k) = 0 \text{ and } \varphi(Yc_k) = Y\varphi(c_k) = 0 \text{ mod } I_s$$

implies that m is a corner of s . Conversely, for m a corner of s , it is clear that $\varphi_{(c_k, m)}^n \in \ker dq$. Thus $\ker dq$ is generated by those elements $\varphi_{(c_k, m)}^n$ for which m is a corner of s . The proposition now follows immediately. \square

The last two propositions and the exact sequence together describe the tangent space $T_{(s, b)} S^{[n, n+1]}$ as a linear T -representation. This will be useful later on to compute equivariant Chern classes. Considering only the dimensions of these spaces, we recover the following well-known result of Cheah [Che98, Theorem 3.2.2] about the smoothness of $S^{[n, n+1]}$:

Proposition 17 *The incidence $S^{[n, n+1]}$ is a smooth irreducible subvariety of $S^{[n]} \times S^{[n+1]}$.*

2.3 Application to the irreducibility of $S_0^{[n,n+1]}$

We consider the subvariety $S_0^{[p,q]}$ of $S^{[p,q]}$ parameterizing incident schemes ($s \subset b$) with respective length p and q both supported at the origin. Recall Briançon's theorem which asserts the irreducibility of the variety $S_0^{[n]} \subset S^{[n]}$ parameterizing the subschemes of length n and support the origin. The corresponding theorem for pairs of incident schemes is not true, as shown by the following example.

Proposition 18 *The scheme $S_0^{[2,4]}$ is not irreducible.*

Proof. As $S_0^{[2,4]}$ is a strict subscheme of the irreducible 4-dimensional product $S_0^{[4]} \times S_0^{[2]}$, any component has dimension at most 3 and any irreducible $F \subset S_0^{[2,4]}$ of dimension 3 is an irreducible component. One irreducible component of dimension 3 is birational to $S_0^{[4]}$: its generic point parameterizes the couple (s, b) with b the generic curvilinear subscheme of length 4 and s the unique subscheme of b with length 2. An other 3-dimensional family is constructed as follows. Let (f, g) be two distinct linear forms and b the subscheme of S with equation (f^2, g^2) . Let s be any scheme of length 2 supported at the origin. Since $s \subset b$, the set of such (s, b) describe a subvariety $F \subset S_0^{[2,4]}$ of dimension $1 + 2 = 3$. \square

Although the general incidence $S_0^{[p,q]}$ is wild and difficult to describe, the case $(p, q) = (n, n + 1)$ behaves nicely. The rest of this section is devoted to the proof of the following theorem:

Theorem 19 *The incidence $S_0^{[n,n+1]}$ is irreducible of dimension n . At the generic point (s, b) , the subschemes s and b are curvilinear.*

We start the proof with a weaker version of the theorem, in the next proposition.

Proposition 20 *We have $\dim S_0^{[n,n+1]} = n$ and there is only one irreducible component H_0 in $S_0^{[n,n+1]}$ of dimension n . At a generic point $(s, b) \in H_0$, the subschemes s and b are curvilinear.*

Proof. Once again this result is a consequence of the detailed study of $S_0^{[n,n+1]}$ performed by Cheah [Che98, Proposition 3.4.11]. We give a short proof.

To apply Bialynicki-Birula's decomposition theorem in $S^{[n,n+1]}$ which is not compact, we first compactify $S^{[n,n+1]}$. So consider the inclusion $S^{[n,n+1]} \subset (\mathbb{P}^2)^{[n,n+1]}$. Since $S^{[i,i+1]}$ is smooth for all values of i , $(\mathbb{P}^2)^{[n,n+1]}$ is smooth. Consider the action of k^* induced by the action on the affine plane defined by $t \cdot X = t^\alpha X$, $t \cdot Y = tY$, where α is any integer strictly greater than $n + 1$.

Let O denote the origin of $\mathbb{A}^2 \subset \mathbb{P}^2$. Let Z be a subscheme of \mathbb{P}^2 . If Z is not supported on O , then the limit at $t = 0$ of $t \cdot Z$ is also not supported on O . For (s, b) a k^* -fixed point in $(\mathbb{P}^2)^{[n,n+1]}$ let $C_{(s,b)}$ be the corresponding Bialynicki-Birula cell: we have

$$(\mathbb{P}^2)_0^{[n,n+1]} = \coprod_{(s,b) \in (S_0^{[n,n+1]})^T} C_{(s,b)} .$$

Thus to prove the proposition it is enough to show that all the cells $C_{(s,b)}$ with s and b supported at the origin have dimension at most n and that exactly one has dimension n .

Let $(s, b) \in S_0^{[n, n+1]}$ be a given k^* -fixed point. Let us say that a tangent vector $x \in T_{(s, b)}S^{[n, n+1]}$ is contractant if it is an eigenvector for the k^* -action of positive weight. It is well-known that the dimension of $C_{(s, b)}$ is the number of independent contractant tangent vectors.

Let $x \in T_{(s, b)}S^{[n, n+1]}$ be a T -weight vector which is contractant and let $w = aU + bV$ be its weight. Its weight for k^* is $a\alpha + b$ and this is a positive integer. Since $\alpha > n + 1$, we have $a \geq 0$ so w is a positive weight (recall the definition of positive weights in Subsection 2.1). In particular, the vector space W generated by such contractant tangent vectors x satisfies $\dim W \leq \dim T^+ = n + 1$, where $T^+ \subset T_{(s, b)}S^{[n, n+1]}$ is the vector space generated by tangent vectors with positive weight. Recall the description of $T_{(s, b)}S^{[n, n+1]}$ given in terms of the projection $q : S^{[n, n+1]} \rightarrow S^{[n+1]}$ and its differential dq . By Proposition 15, all the tangent vectors of $T_bS^{[n+1]}$ of positive weight $-V$ are in the image of dq but are not contractant. Since there is at least one such vector, we get $\dim W \leq \dim T^+ - 1 = n$. Moreover, if we have equality, there is exactly one vector in $T_bS^{[n+1]}$ of weight $-V$. This implies that the partition corresponding to b is a rectangle: $\lambda = (m, m, \dots, m)$. But, if $m > 1$, the eigenspace of weight $-V$ in $\text{Im } dq$ has dimension 1, as well as the eigenspace of weight $-(m-1)V$ in $\ker dq$, by Proposition 16. Since these vectors are not contractant, we get $\dim W \leq n - 1$ and a contradiction. Thus the only possibility is $\lambda = (1, 1, \dots, 1)$, and the proposition is proved. \square

The next proposition describes the Bialynicki-Birula cells of dimension $n - 1$ introduced in the proof of the previous proposition.

Proposition 21 *A Bialynicki-Birula cell $C_{(s, b)}$ has dimension $n - 1$ if and only if the height of the partition λ associated to b is 2.*

Proof. We keep the notations of the previous proposition. If the height h of λ is 1, then $\dim C_{(s, b)} = n$ by the above. If $h \geq 3$, since there are at least h independent vectors in $\text{Im}(dq)$ of weight a positive multiple of $-V$, there are at least three tangent vectors in $T_{(s, b)}S^{[n, n+1]}$ which are positive non contractant, hence $\dim C_{(s, b)} \leq \dim T^+ - h \leq n - 2$. If $h = 2$, then $\lambda = (2^\alpha, 1^\beta)$ and the partition μ of s is $\mu_0 = (2^{\alpha-1}, 1^{\beta+1})$ or $\mu_1 = (2^\alpha, 1^{\beta-1})$. Let us denote by T_{cont} the subspace of a vector space T generated by the contractant tangent vectors. Then $\dim T_bS_{cont}^{[n+1]} = n - 1$.

Our description of the kernel $\ker(dq)$ and the coimage $\text{coIm}(dq)$ of dq (Propositions 14 and 16) show that when $\mu = \mu_1$ or $\beta = 0$, then $\dim \text{coIm}(dq)_{cont} = \dim \ker(dq)_{cont} = 0$. When $\mu = \mu_0$ and $\beta \neq 0$, then $\dim \text{coIm}(dq)_{cont} = \dim \ker(dq)_{cont} = 1$. Summing up, $\dim C_{(s, b)} = \dim(T_b)_{cont} - \dim \text{coIm}(dq)_{cont} + \dim \ker(dq)_{cont} = n - 1$ as required. \square

Proposition 22 *Let $L_n \subset S^{[n, n+1]}$ the set of (s, b) such that s is punctual. Then every component of L_n has dimension at least $n + 3$.*

Proof.

Following Gaffney and Lazarsfeld, if $f : X \rightarrow Y$ is a finite morphism between irreducible varieties we define the ramification locus $R_l \subset X$ containing the points x for which $f^{-1}(f(x))$ is a scheme whose support on x has length at least $l + 1$. When X is normal, Y non-singular and f surjective, then the components of R_l have codimension at most l [GL80, p.58],[Laz80].

We apply this theorem with $X = U_n$ the universal family over $Y = S^{[n,n+1]}$ whose fiber over (s, b) is the scheme s . It suffices to prove that U_n is normal. We shall prove that U_n is Cohen-Macaulay and smooth in codimension one, which implies normality according to Serre's criteria.

U_n is Cohen-Macaulay as it is flat over the smooth base $S^{[n,n+1]}$.

For any $\lambda = (\lambda_1, \dots, \lambda_k)$ ordered k -tuple with $\sum \lambda_i = n + 1$, we denote by $\Delta_\lambda \subset S^{[n+1]}$ the stratum of subschemes z of type λ , ie. $z = z_1 \amalg \dots \amalg z_k$ with $\text{length}(z_k) = \lambda_k$ and z_k punctual. Since any punctual $z_i(p)$ supported by p is the translation of a subscheme $z_i(0)$ supported by the origin, $\dim \Delta_\lambda = \dim(S^k \times S_0^{[\lambda_1]} \times \dots \times S_0^{[\lambda_k]})$

For $i \leq k$, let $\mu_i(\lambda) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$. For $\mu \subset \lambda$ with $\sum \mu_i = n$ let $D_{\mu,\lambda} \subset S^{[n,n+1]}$ be the image of the generically well defined quasi-finite map

$$\begin{aligned} S^k \times S_0^{[\mu_1, \lambda_1]} \times \dots \times S_0^{[\mu_k, \lambda_k]} &\rightarrow S^{[n,n+1]} \\ ((p_1, \dots, p_k), (t_1, w_1), \dots, (t_k, w_k)) &\mapsto (\amalg t_i(p_i), \amalg w_i(p_i)). \end{aligned}$$

Let $D_\lambda \subset S^{[n,n+1]}$ denote the inverse image of Δ_λ by the natural projection $S^{[n,n+1]} \rightarrow S^{[n+1]}$. Then $D_\lambda = \cup_{i \leq k} D_{\mu_i(\lambda), \lambda}$. For $\lambda \neq (2, 1, \dots, 1)$ and $\lambda \neq (1, \dots, 1)$, the codimension of D_λ in $S^{[n,n+1]}$ is at least 2 according to Proposition 20. In particular, no smoothness condition is required for the universal family U_n over D_λ . When $\lambda = (1, \dots, 1)$, the smoothness of U_n is obvious.

We consider now the case $\lambda = (2, 1, \dots, 1)$. Let $(s, b) \in D_\lambda$.

If $(s, b) \in D_{\mu,\lambda}$ with $\mu = (1, \dots, 1)$, then locally around (s, b) , $S^{[n,n+1]}$ is isomorphic to $S^{[1,2]} \times S^{n-1}$. The universal family U_n over $D_{\mu,\lambda}$ is locally a disjoint union of sheets. The sheets coming from the universal families over S are obviously smooth. The last sheet Z coming from the factor $S^{[1,2]}$ is such that the projection $Z \rightarrow S^{[1,2]}$ is an isomorphism (the fiber is zero dimensional with length 1), so this last sheet is smooth too.

If $(s, b) \in D_{\mu,\lambda}$ with $\mu = (2, 1, \dots, 1, 0, 1, \dots, 1)$, then locally around (s, b) , $S^{[n,n+1]}$ is isomorphic to $S^{[2]} \times S^{n-1}$. The universal family U_n is smooth since it is the disjoint union of the pullback of the smooth universal families over $S^{[2]}$ and S .

□

Corollary 23 *Every irreducible component of $S_0^{[n,n+1]}$ has dimension n or $n - 1$.*

Proof. Moving the subschemes of $S_0^{[n,n+1]}$ with translations, the product $S_0^{[n,n+1]} \times S$ parameterizes the set L of pairs $(s, b) \in S^{[n,n+1]}$ with s, b punctual with any support $p \in S$. We need to prove that the components of L have dimension $n + 1$ or $n + 2$. Consider the residual morphism $Res : S^{[n,n+1]} \rightarrow S$ that sends a pair (s, b) to the point q defined by the ideal $(I_s : I_b)$. Let $\Delta_x : L_n \rightarrow k$, $(s, b) \rightarrow x(s) - x(Res(s, b))$, where $x(s)$ denotes the x coordinate of the punctual subscheme s . Define similarly Δ_y . The components of L_n have dimension at least $n + 3$ by Proposition 22. From the equality $L = L_n \cap \Delta_x^{-1}(0) \cap \Delta_y^{-1}(0)$, we conclude that any component of L has dimension at least $n + 1$. The components of L have dimension at most $n + 2$ by Proposition 20. □

We now conclude the proof of Theorem 19. For the proof, we need to produce some universal families over Bialynicki-Birula cells. They are constructed from the description of the tangent space by a procedure similar to the one used in [Ev04].

Proof. By Corollary 23, the components of $S_0^{[n,n+1]}$ have dimension n or $n - 1$. The Bialynicki-Birula decomposition of $S_0^{[n,n+1]}$ is a partition into irreducible sets. It follows that the irreducible components of $S_0^{[n,n+1]}$ are the maximal sets for the inclusion among the closure of the Bialynicki-Birula cells. Since we already proved that there is a unique maximal component of dimension n , it remains to prove that the closure of the cells of dimension $n - 1$ described by Proposition 21 are not irreducible components of $S_0^{[n,n+1]}$.

Let $C_{s_0,b_0} \subset S_0^{[n,n+1]}$ be a Bialynicki-Birula cell of dimension $n - 1$, $(s_0, b_0) \in S_0^{[n,n+1]}$ the corresponding fixed point, $\lambda = (2^\alpha, 1^\beta)$ and μ be the partitions of b_0 and s_0 . Let $\mu_0 = (2^{\alpha-1}, 1^{\beta+1})$ and $\mu_1 = (2^\alpha, 1^{\beta-1})$. We have $\mu = \mu_0$ or $\mu = \mu_1$.

First, we remark that the irreducible components of $S_0^{[n,n+1]}$ are invariant under GL_2 , the group of linear automorphisms of the plane. In particular, if we prove that the generic point of C_{s_0,b_0} is not invariant under GL_2 , it follows that the closure $\overline{C_{s_0,b_0}}$ is not a component of $S_0^{[n,n+1]}$. Moreover we will use the following notation: we denote by $k[X, Y]_d$ the space of homogeneous polynomials of degree d and by $\pi_d : k[X, Y] \rightarrow k[X, Y]_d$ the natural projection. Given an ideal I , I_d will denote the subspace $\pi_d(I) \subset k[X, Y]_d$. Let us finally say that an admissible cleft couple for I_b is liftable if the corresponding infinitesimal deformation of I_b can be lifted to a infinitesimal deformation of the pair (I_s, I_b) .

- If $\alpha > 1$ and $\beta = 0$, then consider the ideal $I_b = (X^\alpha, Y^2 + \sum_{0 < i < \alpha, j \leq 1} c_{ij} X^i Y^j)$ and $I_s = I_b + (YX^{\alpha-1})$. The variables c_{ij} are the $(n - 1)$ coordinates on the Bialynicki-Birula cell C and I_s, I_b are the corresponding universal ideals. If (I_s, I_b) is the generic element in this cell, we have $(I_b)_1 = k \cdot X$, thus this generic point is not GL_2 -invariant.

- If $\alpha > 1$, $\beta = 1$ and $\mu = \mu_1$, we again have $(I_b)_1 = k \cdot X$ for the generic pair (I_s, I_b) , and this is not GL_2 -invariant.

- If $\alpha = 1$ and $\beta = 0$, then $I_b = (X, Y^2)$ and $I_s = (X, Y)$, so this point is not GL_2 -invariant. Similarly, if $\alpha = 1$, $\beta = 1$ and $\mu = \mu_1$, then $I_s = (X, Y^2)$, and this is not GL_2 -invariant.

- If $\alpha > 1$ and $\beta > 2$ then the cleft couple $((2, 0), (0, 2))$ is not admissible for I_b . It follows that the generic element (b, s) , $(I_b)_2$ has dimension one and is generated by a polynomial divisible by Y . In particular, the generic element of the cell is not GL_2 -invariant.

- If $\alpha > 1$, $\beta = 2$ and $\mu = \mu_0$, then the cleft couple $((2, 0)(0, 2))$ is admissible, but not liftable, by Proposition 15. Thus for the same reason the cell is not GL_2 -invariant.

It remains to consider the cases $(\beta = 1$ and $\mu = \mu_0)$, $(\alpha = 1, \beta > 1, \mu = \mu_0)$, $(\alpha = 1, \beta > 1, \mu = \mu_1)$, and $(\alpha > 1, \beta = 2$ and $\mu = \mu_1)$. For these cases, we will see that the closure of the Bialynicki-Birula cells are invariant under GL_2 , and thus we cannot apply the same arguments as above. Instead, we will prove that the closure $\overline{C_{s_0,b_0}}$ of the cell under consideration is not an irreducible component of $S_0^{[n,n+1]}$ as it is included in the unique (“curvilinear”) component of dimension n . To this end, we apply a change of coordinates to obtain simple equations for the generic point (s, b) of the cell C_{s_0,b_0} and we express (s, b) as the limit of $(s(t), b(t)) \in S_0^{[n,n+1]}$ with $s(t)$ and $b(t)$ curvilinear.

- Consider the case $\beta = 1$ and $\mu = \mu_0$. The universal families over C_{s_0,b_0} are described by coordinates c_{ij}, d and universal ideals $I_b = (X^{\alpha+1}, YX^\alpha, Y^2 + \sum_{i+j \geq 2, (i,j) \in \lambda} c_{ij} X^i Y^j)$, $I_s =$

$I_b + (YX^{\alpha-1} + dX^\alpha)$. Note that I_b contains all the monomials of degree $\alpha + 1$. The element in I_b with initial term Y^2 vanishes on a curve locally reducible around 0 as a union of two distinct smooth curves when the coefficients are generic. Up to a change of coordinates, one may suppose that the two branches have equations $X = 0$ and $Y = 0$. Then I_b contains XY and all the monomials of degree $\alpha + 1$. Thus $I_b = (X^{\alpha+1}, Y^{\alpha+1}, XY)$ because of the inclusion, and both ideals have the same colength. The ideal I_s has codimension one in I_b thus the general element has the form $I_s = I_b + (X^\alpha + dY^\alpha)$. Up to a linear change of coordinates of the form $Y \mapsto c.Y, X \mapsto X$, one may suppose that $I_s = I_b + (X^\alpha + Y^\alpha)$. For a generic pair (s, b) , b is the union of two curvilinear schemes of length $\alpha + 1$ supported at the origin, and s a colength one subscheme of b : this cell is invariant under automorphisms.

For $t \neq 0$ let $I_b(t) = (XY + t(X + Y), X^{2\alpha+1}, Y^{2\alpha+1})$ and $I_s(t) = (XY + t(X + Y), X^{2\alpha}, Y^{2\alpha})$. Let $(I_s(0), I_b(0))$ be the limit at $t = 0$ of $(I_s(t), I_b(t))$. Obviously $XY = \lim_{t \rightarrow 0} XY + t(X + Y) \in I_b(0)$.

Since $b(t)$ (resp. $s(t)$) has length $2\alpha + 1$ (resp. 2α) with support the origin, every monomial of degree $2\alpha + 1$ (resp. 2α) is in $I_b(t)$ (resp. $I_s(t)$). Since $X + Y = \frac{-XY}{t}$ modulo $I_b(t)$, we obtain $X(X+Y)^\alpha \in I_b(t)$ and $Y(X+Y)^\alpha \in I_b(t)$. Summing up, $I_b(0) \supset (XY, X(X+Y)^\alpha, Y(X+Y)^\alpha) = (XY, X^{\alpha+1}, Y^{\alpha+1})$. This inclusion is an equality since the two ideals have colength $2\alpha + 1$. The same reasoning with the curvilinear $s(t)$ instead of $b(t)$ shows that $I_s(0) \supset (XY, X^{\alpha+1}, Y^{\alpha+1})$. Modulo $I_s(t)$, $(X + Y)^\alpha = (\frac{-XY}{t})^\alpha = 0$. Thus $I_s(0) \supset (XY, X^{\alpha+1}, Y^{\alpha+1}, (X + Y)^\alpha)$ and the equality follows by length considerations. We have proved $I_b = I_b(0)$ and $I_s = I_s(0)$, as expected.

- If $\alpha > 1, \beta = 2, \mu = \mu_1$, we can perform as above a change of coordinates in order to reduce to the case $I_s = (XY, X^{\alpha+1}, Y^{\alpha+1})$ and $I_b = ((XY, X^{\alpha+2}, Y^{\alpha+2}, X^{\alpha+1} + Y^{\alpha+1}))$. For a generic pair (b, s) , b is a colength one subscheme in the union $c_1 \cup c_2$ of two curvilinear subschemes of length $\alpha + 2$, and s is the union $c'_1 \cup c'_2$, where $c'_i \subset c_i$ is the unique colength one subscheme: this cell is invariant under automorphisms. The same computation as above now shows that $I_s = \lim I_s(t)$, $I_b = \lim I_b(t)$ with $I_s(t) = (XY + t(X + Y), X^{2\alpha+1}, Y^{2\alpha+1})$ and $I_b(t) = (XY + t(X + Y), X^{2\alpha+2}, Y^{2\alpha+2})$.

- If $\alpha = 1, \beta > 1, \mu = \mu_0$, then $I_b = (X^{1+\beta}, XY + \sum_{2 \leq j \leq \beta} a_j X^j, Y^2 + \sum_{2 \leq j \leq \beta} a_j Y X^{j-1})$ and $I_s = I_b + (Y + \sum_{2 \leq j \leq \beta} a_j X^{j-1} + dX^\beta)$. Up to the coordinate change $X \mapsto X, Y \mapsto Y + \sum_{2 \leq j \leq \beta} a_j X^{j-1} + dX^\beta$, one may suppose that $I_b = (X^{1+\beta}, XY, Y^2)$ and $I_s = (X^{1+\beta}, Y)$. It follows that for a generic pair (b, s) , s is a curvilinear scheme and b the union of s and the 2-fat point: this cell is invariant under automorphisms. Consider the curvilinear ideals $c = (X^{2+\beta}, Y)$, and the automorphism $\phi_t : X \mapsto X, Y \mapsto tY + X^{\beta+1}$. The ideals $I_b(t) = \phi_t(c)$ and $I_s(t) = \phi_t(I_s) = I_s$ are such that $\lim_{t \rightarrow 0} I_s(t) = I_s$ and $\lim_{t \rightarrow 0} I_b(t) = I_b$.

- If $\alpha = 1, \beta > 1, \mu = \mu_1$, then $I_b = (X^{1+\beta}, XY + \sum_{2 \leq j \leq \beta} a_j X^j, Y^2 + \sum_{2 \leq j \leq \beta} a_j Y X^{j-1} + dX^\beta)$ and $I_s = I_b + (X^\beta)$. Up to the two coordinate changes $X \mapsto X, Y \mapsto Y + \sum_{2 \leq j \leq \beta} a_j X^{j-1}$, and then $X \mapsto X, Y \mapsto \lambda Y$, one may suppose that $I_b = (X^{1+\beta}, XY, Y^2 + X^\beta)$ and $I_s = (X^\beta, XY, Y^2)$.

For a generic pair (b, s) , b is a colength one subscheme in the union $c_1 \cup c_2$ of two curvilinear subschemes of length 3 and $n - 1$, and s is the union $c'_1 \cup c'_2$, where $c'_i \subset c_i$ is the unique colength one subscheme: this cell is invariant under automorphisms.

For $t \neq 0$ let $I_b(t) = (XY - t^2 Y + t^\beta X, X^{\beta+2}, Y^{\beta+2})$ and $I_s(t) = (XY - t^2 Y + t^\beta X, X^{\beta+1}, Y^{\beta+1})$. Let $(I_s(0), I_b(0))$ be the limit at $t = 0$ of $(I_s(t), I_b(t))$. Obviously $XY = \lim_{t \rightarrow 0} XY - t^2 Y + t^\beta X \in I_b(0) \cap I_s(0)$.

Since $b(t)$ (resp. $s(t)$) has length $\beta+2$ (resp. $\beta+1$) with support the origin, all the monomials in $k[X, Y]$ of degree $\beta+2$ (resp. $\beta+1$) are in $I_b(t)$ (resp. $I_s(t)$). A straightforward induction shows that

$$\forall k \geq 1, Y = t^{\beta-2}X + t^{\beta-4}X^2 + \dots + t^{\beta-2k}X^k + t^{-2k}X^kY \text{ mod } I_b(t) \cap I_s(t).$$

In particular, $e(t) := Y - t^{\beta-2}X - \dots - t^{-\beta-2}X^{\beta+1} \in I_b(t)$, $f(t) := Y - t^{\beta-2}X - \dots - t^{-\beta}X^\beta \in I_s(t)$, and

$$\begin{aligned} X^{\beta+1} &= \lim_{t \rightarrow 0} t^{\beta+2} e(t) \in I_b(0), \\ X^\beta &= \lim_{t \rightarrow 0} t^\beta f(t) \in I_s(0). \end{aligned}$$

Since $Y^2 = (Y - e(t))^2 \text{ mod } I_b(t)$ and $Y^2 = (Y - f(t))^2 \text{ mod } I_s(t)$, we get

$$\begin{aligned} g(t) &:= Y^2 - t^{2\beta-4}X^2 - \dots - (\beta-1)X^\beta - \beta t^{-2}X^{\beta+1} \in I_b(t), \\ h(t) &:= Y^2 - t^{2\beta-4}X^2 - \dots - (\beta-1)X^\beta \in I_s(t). \end{aligned}$$

It follows that

$$\begin{aligned} Y^2 + X^\beta &= \lim_{t \rightarrow 0} g(t) - \beta t^\beta e(t) \in I_b(0) \\ Y^2 &= \lim_{t \rightarrow 0} h(t) - (\beta-1)t^\beta f(t) \in I_s(0) \end{aligned}$$

Summing up, these limits prove that $I_s(0) \supset I_s$ and $I_b(0) \supset I_b$. The equalities follows from the inclusions by length considerations. \square

3 Bases of the equivariant Chow ring

We now present three natural bases $\text{fix}(\lambda)$, $\text{nak}(\lambda)$, $\text{es}(\lambda)$ of the K -vector space $A_K^*(S^{[n]})$. Our three bases of $A_K^*(S^{[n]})$ are naturally parameterized by the set \mathcal{P}_n of partitions λ of weight n .

Let $n \geq 0$ and let $i > 0$ be integers. We define some correspondences following Nakajima [Na99]:

Definition 24 *Let $Q_i^n \subset S^{[n]} \times S^{[n+i]}$ be the closure of the set of pairs (z_n, z_{n+i}) where $z_n \in S^{[n]}$ is arbitrary and $z_{n+i} \in S^{[n+i]}$ is the disjoint union of z_n and a punctual scheme of length i .*

The T -invariant correspondence $\coprod_n Q_i^n$ induces an operator (called ‘‘creation operator’’) $q_i : \oplus_n A_T^*(S^{[n]}) \rightarrow \oplus_n A_T^{*+i-1}(S^{[n+i]})$ on Chow groups. Assume now that $i < 0$. The ‘‘destruction operator’’ q_i is defined either as the dual of q_{-i} or with the correspondence $Q_i^n \subset S^{[n]} \times S^{[n+i]}$ which is dual to the correspondence Q_{-i}^{n+i} , in the sense of Definition 9. By Proposition 10, both definitions lead to the same operator. For any i , q_i has conformal degree i and cohomological degree $i-1$. We make the convention that $q_0 = 0$.

Given a partition λ of length l , we will denote by $\text{nak}(\lambda)$ the equivariant class obtained applying $q_{\lambda_l} \circ \dots \circ q_{\lambda_1}$ to the vacuum ϕ , where the vacuum is the fundamental class on $S^{[0]}$.

Since by [Na99], the classes $\text{nak}(\lambda)$ for $\lambda \in \mathcal{P}_n$ restrict to a basis of the non equivariant Chow group of $S^{[n]}$, $\{\text{nak}(\lambda), \lambda \in \mathcal{P}_n\}$ is a basis of $A_K^*(S^{[n]})$ over K .

Recall the classes introduced by Ellingsrud and Strømme in [ES87]. These classes are introduced for \mathbb{P}^2 but we can consider the same classes for \mathbb{A}^2 . We choose an injection $k^* \rightarrow T, t \mapsto (t^{-1}, t^{-d})$ where d is large. The action of T on $S^{[n]}$ induces an action of k^* . With the assumption that d is large enough, the k^* -fixed points are the T -fixed points; in particular there is a finite number of them and they are parameterized by partitions. More precisely, if $\lambda = (\lambda_1, \dots, \lambda_l)$ is a partition, we denote by x_λ the subscheme with ideal I_{x_λ} generated by the $l+1$ polynomials $X^{k-1}Y^{\lambda_k}$, where k varies from 1 to $l+1$, with the convention that $\lambda_{l+1} = 0$.

To each partition λ of weight n corresponds a Bialynicki-Birula cell containing the points $p \in S^{[n]}$ such that $\lim_{t \rightarrow 0} t.p = x_\lambda$. We denote $ES_\lambda \subset S^{[n]}$ the closure of this cell. Let l be the length of the partition λ . Geometrically, the Bialynicki-Birula cell associated to λ parameterizes the subschemes $Z \subset S$ for which there exist $x_1, \dots, x_l \in k$ such that each intersection $Z \cap \{X = x_i\}$ has length λ_i . The equivariant class of ES_λ in the Chow ring will be denoted es_λ . Since by definition $S^{[n]}$ has the cellular decomposition $S^{[n]} = \coprod_{\lambda \in \mathcal{P}_n} ES_\lambda$, where $\lambda \in \mathcal{P}_n$, the classes es_λ for $\lambda \in \mathcal{P}_n$ form a basis of $A_T^*(S^{[n]})$.

Finally, the classes $\text{fix}(\lambda) \in A_K^*(S^{[n]})$ are defined using the localization theorem [EG98, Theorem 1]. The set $(S^{[n]})^T$ contains the points x_λ parameterized by $\lambda \in \mathcal{P}_n$. Let $1_\lambda \in A_T^*((S^{[n]})^T)$ be the class corresponding to x_λ . Let $i : (S^{[n]})^T \rightarrow S^{[n]}$ denote the inclusion. By Lemma 4, $i_K^* : A_K^*(S^{[n]})^T \rightarrow A_K^*(S^{[n]})$ is an isomorphism.

Definition 25 Let $\text{fix}(\lambda)$ be the unique element in $A_K^*(S^{[n]})$ such that $i_K^*(\text{fix}(\lambda)) = 1_\lambda$.

Let us denote by $\text{Tan}(\lambda) \in \mathbb{Z}[U, V]$ the product of the weights of the tangent space $T_{x_\lambda} S^{[n]}$. According to the self-intersection formula, we have

$$\text{fix}(\lambda) = i_*(1_\lambda) / \text{Tan}(\lambda). \quad (2)$$

Recall Definition 8. We deduce from (2) the following lemma:

Lemma 26 We have $\langle \text{fix}(\lambda), \text{fix}(\lambda) \rangle_{S^{[n]}} = 1 / \text{Tan}(\lambda)$.

Proof. By (2), we have

$$\langle \text{fix}(\lambda), \text{fix}(\lambda) \rangle_{S^{[n]}} = \frac{\langle i_*(1_\lambda), i_*(1_\lambda) \rangle_{S^{[n]}}}{\text{Tan}(\lambda)^2}.$$

By Definition 8, this is $\pi_*^K(i_*(1_\lambda) \cup i_*(1_\lambda)) / \text{Tan}(\lambda)^2$, if $\pi : S^{[n]} \rightarrow \text{Spec } k$ denotes the projection to a point. Since $i_*(1_\lambda) \cup i_*(1_\lambda) = \text{Tan}(\lambda) \cdot i_*(1_\lambda)$, the lemma follows. \square

4 Classical Operators

Let us denote by A the direct sum $\bigoplus_n A_T^*(S^{[n]})$ and $A_K := \bigoplus_n A_K^*(S^{[n]})$. In this section, we consider the classical operators acting on A_K , namely the creation/destruction operators q_i and the boundary operator ∂ , and an auxiliary operator ρ . We compute them in the basis $\text{fix}(\lambda)$. We also compute the commutators of these operators.

The operators ∂, ρ, q_i for $i > 0$ are naturally defined on A and they are naturally extended to A_K . We use freely the same notation for the operators on A and on A_K . On the contrary, the operators q_i for $i < 0$ are defined on A_K but not on A . This is because their definition involves non proper morphisms.

In Theorem 34 we give an explicit algorithm to compute all operators q_i in the basis $\text{fix}(\lambda)$. With the help of this result, we checked on a computer our formulas for commutators, such as Theorem 52. However the computations are very tricky and we are not able to give a purely algebraic proof of these formulas: instead we use geometric arguments.

4.1 The operators q_1 and q_{-1}

Given a partition λ , we denote by $\lambda[1]$ the set of partitions μ with $\lambda \subset \mu$ and $|\mu| = |\lambda| + 1$. Given two partitions λ, μ with $\mu \in \lambda[1]$, we denote by $\text{Coker}(\lambda, \mu) \in \mathbb{Z}[U, V]$ the product of the weights of Proposition 15 and by $\text{Ker}(\lambda, \mu) \in \mathbb{Z}[U, V]$ the product of the weights of Proposition 16. With these notations we have the following proposition:

Proposition 27 *We have the following formula:*

$$q_1(\text{fix}(\lambda)) = \sum_{\mu \in \lambda[1]} \frac{\text{Coker}(\lambda, \mu)}{\text{Ker}(\lambda, \mu)} \text{fix}(\mu) .$$

Proof. Consider the following diagram:

$$\begin{array}{ccc} & S^{[n]} \times S^{[n+1]} & \\ \pi_n \swarrow & & \searrow \pi_{n+1} \\ S^{[n]} & & S^{[n+1]} \end{array}$$

Let $[S^{[n, n+1]}] \in A_T^*(S^{[n]} \times S^{[n+1]})$ denote the equivariant class of the incidence. By definition, we have $q_1(\text{fix}(\lambda)) = \pi_{n+1, *}^K (\pi_{n, K}^*(\text{fix}(\lambda)) \cup [S^{[n, n+1]}])$. Now, by Definition 25 and the following commutative diagram

$$\begin{array}{ccc} S^{[n]} \times S^{[n+1]} & \longleftarrow & (S^{[n]})^T \times (S^{[n+1]})^T \\ \downarrow & & \downarrow \\ S^{[n]} & \longleftarrow & (S^{[n]})^T , \end{array}$$

we have $\pi_{n, K}^*(\text{fix}(\lambda)) = \sum_{\mu} \text{fix}(\lambda) \otimes \text{fix}(\mu)$. By Proposition 17, $S^{[n, n+1]}$ is smooth, so the restriction of its class to a fixed point is the product of the weights of the normal space at this point. By Propositions 15 and 16, we thus have

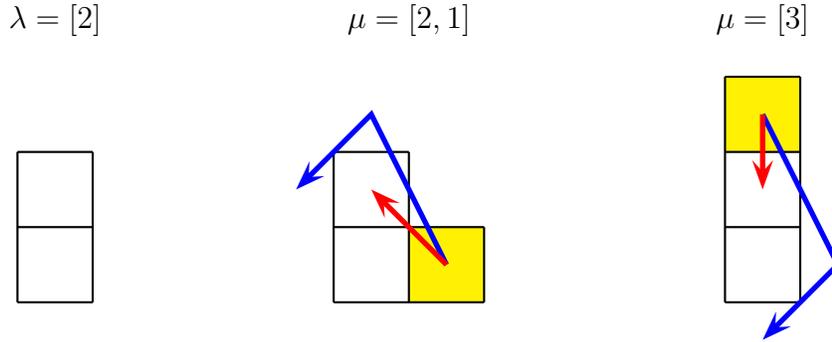
$$[S^{[n, n+1]}] = \sum_{\lambda, \mu: \mu \in \lambda[1]} \frac{\text{Tan}(\lambda) \text{Coker}(\lambda, \mu)}{\text{Ker}(\lambda, \mu)} \text{fix}(\lambda) \otimes \text{fix}(\mu).$$

Therefore,

$$\begin{aligned}
q_1(\text{fix}(\lambda)) &= \pi_{n+1,*}^K \left(\sum_{\mu \in \lambda[1]} \frac{\text{Tan}(\lambda) \text{Coker}(\lambda, \mu)}{\text{Ker}(\lambda, \mu)} \text{fix}(\lambda) \otimes \text{fix}(\mu) \right) \\
&= \sum_{\mu \in \lambda[1]} \frac{\text{Coker}(\lambda, \mu)}{\text{Ker}(\lambda, \mu)} \text{fix}(\mu),
\end{aligned}$$

using Theorem 5. This is what we wanted to prove. \square

For example, we have $q_1(\text{fix}([2])) = \frac{-2U+V}{-U+V} \text{fix}([2, 1]) + 3 \text{fix}([3])$. This is illustrated as follows, where the weights of the blue resp. red arrows are the numerators resp. denominators of the coefficients:



We deduce a formula for q_{-1} . Given a partition μ , let $\mu[-1]$ denote the set of partitions λ with $\lambda \subset \mu$ and $|\lambda| = |\mu| - 1$.

Proposition 28 *We have the following formula:*

$$q_{-1}(\text{fix}(\mu)) = \sum_{\lambda \in \mu[-1]} \frac{\text{Coker}(\lambda, \mu)}{\text{Ker}(\lambda, \mu)} \cdot \frac{\text{Tan}(\lambda)}{\text{Tan}(\mu)} \text{fix}(\lambda).$$

Proof. Let λ resp. μ be a partition of weight n resp. $n + 1$. Since q_{-1} is the adjoint of q_1 , we have the relation

$$\langle q_1(\text{fix}(\lambda)), \text{fix}(\mu) \rangle_{S^{[n+1]}} = \langle \text{fix}(\lambda), q_{-1}(\text{fix}(\mu)) \rangle_{S^{[n]}}.$$

Therefore the proposition follows from Lemma 26 and Proposition 27. \square

4.2 Class of the boundary and derivatives

We turn to the problem of determining the equivariant class of the divisor Δ_2 of non-reduced schemes. In the non equivariant setting on $S^{[n]}$, recall the following formula of Lehn which expresses the class $[\Delta_2]_{cla}$ of Δ_2 in terms of the classical Chern class c^{cla} of the tautological bundle: $[\Delta_2]_{cla} = -2c_1^{cla}(\mathcal{O}^{[n]})$. We prove an equivariant analog in the equivariant Chow ring: $[\Delta_2] = -2c_1(\mathcal{O}^{[n]})$, where the Chern class considered is the equivariant Chern class. Our method involves equivariant techniques and does not rely on Lehn's ideas.

Denote $\partial \in A_T^*(S^{[n]})$ the class of Δ_2 , and let $p : \text{Spec } k \rightarrow S^{[n]}$ be a T -fixed point. We'd like to compute $p^*\partial$. To this end, assume that p corresponds to the partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of weight n . We let $l(\lambda)$ denote the number of non-vanishing parts of λ , and $h(\lambda) = \lambda_1$. Let λ^\vee denote the partition dual to λ .

Proposition 29 *We have*

$$p^* \partial = -(\sum_{j=1}^{h(\lambda)} \lambda_j^\vee (\lambda_j^\vee - 1)) U - (\sum_{i=1}^{l(\lambda)} \lambda_i (\lambda_i - 1)) V.$$

Proof. We treat first the case $n = 2$. Then $S^{[2]}$ is the blow-up of $S \times S$ along the diagonal. Assume, moreover, that $\lambda = (2)$. Then $T_p S^{[2]}$ contains 4 eigenlines, of weight $-U, -U+V, -V, -2V$. In this case Δ is smooth, and the tangent space $T_p \Delta$ contains the three eigenlines of weight $-U, -V$ and $-U+V$: in fact, the first two lines are obtained by translating the double point p , and $-U+V$ is the weight of the deformation obtained with schemes supported at the origin. We deduce that $p^* \partial = -2V$.

We now consider the general case. Let $l = l(\lambda)$ and $h = h(\lambda)$. Given $\mathbf{x} = (x_1, \dots, x_l)$ and $\mathbf{y} = (y_1, \dots, y_h)$ tuples of elements in k , we let $I_{(\mathbf{x}, \mathbf{y})}$ denote the ideal generated by the $l+1$ polynomials $\prod_{i=1}^{m-1} (X - x_i) \cdot \prod_{j=1}^{\lambda_m} (Y - y_j)$, where m varies from 1 to $l+1$. When all the x_i and all the y_j are distinct, $k[X, Y]/I_{(\mathbf{x}, \mathbf{y})}$ is reduced and the corresponding set of points is the set of (x_i, y_j) where $i \leq l$ and $j \leq \lambda_i$. Thus $I_{(\mathbf{x}, \mathbf{y})}$ has length n . On the other hand, when $\mathbf{x} = (0, \dots, 0)$ and $\mathbf{y} = (0, \dots, 0)$, the ideal $I_{(\mathbf{x}, \mathbf{y})}$ is monomial and generated by the elements the $X^m Y^{\lambda_m}$, and thus also has length n . Since the length of this family of ideals is upper-semicontinuous, it follows that it is constant, and this family is flat.

In this way, we obtain a T -equivariant morphism $\varphi : k^{l+h} \rightarrow S^{[n]}$ with respect to the natural action on k^{l+h} . We now compute $\varphi^* \partial$. If $\{i_1, i_2\} \subset \{1, \dots, l\}$ is a subset with two elements, where we assume $i_1 < i_2$, we denote by $\Delta_{\{i_1, i_2\}} \subset k^{l+h}$ the class of the variety of tuples (\mathbf{x}, \mathbf{y}) with $x_{i_1} = x_{i_2}$ and $\partial_{\{i_1, i_2\}}$ its class in the equivariant Chow ring of k^{l+h} . Let $z : \text{Spec } k \rightarrow k^{h+l}$ be the origin of k^{h+l} ; since $\partial_{\{i_1, i_2\}}$ is defined by one equation of weight U , it follows that $z^* \partial_{\{i_1, i_2\}} = -U$. Similarly, if $j_1 < j_2$, let $\Delta_{\{j_1, j_2\}}$ be the divisor defined by $y_{j_1} = y_{j_2}$, and let $\partial_{\{j_1, j_2\}}$ denote its class. We have $z^* \partial_{\{i_1, i_2\}} = -V$.

We claim that

$$\varphi^* \partial = \sum_{\{i_1, i_2\} \subset \{1, \dots, l\}} 2\lambda_{i_2} \partial_{\{i_1, i_2\}} + \sum_{\{j_1, j_2\} \subset \{1, \dots, h\}} 2\lambda_{j_2}^\vee \partial_{\{j_1, j_2\}}.$$

Clearly, we have an equality of sets

$$\varphi^{-1}(\Delta) = \bigcup_{\{i_1, i_2\} \subset \{1, \dots, l\}} \Delta_{\{i_1, i_2\}} \cup \bigcup_{\{j_1, j_2\} \subset \{1, \dots, h\}} \Delta_{\{j_1, j_2\}},$$

and we claim that the multiplicity of $\Delta_{\{i_1, i_2\}}$ is $2\lambda_{i_2}$. To see why, let (\mathbf{x}, \mathbf{y}) be a generic point in $\Delta_{\{i_1, i_2\}}$: we have $x_{i_1} = x_{i_2}$ but no other equality among the x_i 's and the y_j 's. Thus the scheme represented by $\varphi(\mathbf{x}, \mathbf{y})$ is a union of λ_{i_2} double points and $n - 2\lambda_{i_2}$ other distinct points. Near the point $\varphi(\mathbf{x}, \mathbf{y})$, $S^{[n]}$ is isomorphic to $(S^{[2]})^{\lambda_{i_2}} \times S^{n-2\lambda_{i_2}}$. Thus the multiplicity of our component may be deduced from the case of $S^{[2]}$: in this case the multiplicity was 2 in view of the computation we made at the beginning of the proof. Thus the multiplicity is $2\lambda_{i_2}$ as claimed.

Since $z^* \varphi^* \partial = p^* \partial$, it remains only to show that $2\sum_{\{i_1, i_2\} \subset \{1, \dots, l\}} \lambda_{i_2} = \sum_{j=1}^{h(\lambda)} \lambda_j^\vee (\lambda_j^\vee - 1)$. The first sum is equal to $\sum_{1 \leq i_1 < i_2 \leq l, 1 \leq j \leq \lambda_{i_2}} 2$. In this sum, when $j = j_0$ is fixed, i_2 is such that $\lambda_{i_2} \geq j_0$, which forces $i_2 \leq \lambda_{j_0}^\vee$. Thus $\sum_{1 \leq i_1 < i_2 \leq l, j \leq \lambda_{i_2}, j=j_0} 2 = \lambda_{j_0}^\vee (\lambda_{j_0}^\vee - 1)$. Our proof is now complete. \square

Corollary 30 *The equivariant class of Δ_2 in $A_T^*(S^{[n]})$ is $\partial = -2c_1(\mathcal{O}_X^{[n]})$.*

Proof. By Proposition 29, the two classes have the same restriction on the T -fixed points of $S^{[n]}$ and the restriction morphism is injective. \square

If $f : A \rightarrow A$ is any operator, we now give a formula for the commutator $[\partial, f]$. To express this formula, let us introduce the following notation:

Notation 31 *If $f : A \rightarrow A$ is an endomorphism, define $\Delta_{f,\lambda,\mu} \in K$ for λ, μ partitions by the formula*

$$f(\text{fix}(\lambda)) = \sum_{\mu} \Delta_{f,\lambda,\mu} \text{fix}(\mu).$$

For $c = (a, b) \in \mathbb{N}^2$, let $w(c) = aU + bV$ be the weight of the corresponding monomial. Corollary 30 immediately implies:

Corollary 32 *Let $f : A \rightarrow A$ be any operator and let $\lambda \subset \mu$ be two partitions. We have*

$$\Delta_{[\partial,f],\lambda,\mu} = -2\Delta_{f,\lambda,\mu} \sum_{c \in \mu \setminus \lambda} w(c).$$

4.3 Computation of the operator q_i for all i

In the previous sections, we computed q_1, q_{-1} and ∂ on the basis $\text{fix}(\lambda)$. We introduce an auxiliary operator ρ and give formulas for higher q_i 's in terms of q_1, q_{-1} and ρ . This yields an inductive procedure to compute q_i on the basis $\text{fix}(\lambda)$.

Definition 33 *Let $R^n \subset S^{[n,n+1]}$ be the closure of the set of pairs of schemes (z_n, z_{n+1}) with z_n reduced, $z_n \subset z_{n+1}$ and $z_n = (z_{n+1})_{\text{red}}$.*

Let $\rho : \bigoplus_n A_T^*(S^{[n]}) \rightarrow \bigoplus_n A_T^{*+1}(S^{[n+1]})$ be the morphism associated with the correspondence $\Pi_n [R^n]$. It has conformal and cohomological degree 1.

The following theorem gives a complete computation of the operators q_i .

Theorem 34 *We have*

$$\begin{aligned} (i-1)q_i &= \rho q_{i-1} - q_{i-1} \rho & \text{for } i > 1 \\ (i+1)q_i &= \rho^\vee q_{i+1} - q_{i+1} \rho^\vee & \text{for } i < -1 \\ 2\rho &= \partial q_1 - q_1 \partial \\ 2\rho^\vee &= q_{-1} \partial - \partial q_{-1} \end{aligned}$$

Proof. The non equivariant version of the first statement is proved in [Le99, Theorem 3.5]. Our formula can be proved geometrically as follows. Let π_1, π_2, π_3 be the projections of $S^{[n]} \times S^{[n+i-1]} \times S^{[n+i]}$ on each factor and, for $a, b \in \{1, 2, 3\}$, let π_{ab} be the projection on two factors. To compute the composition ρq_{i-1} , we have to understand the intersection $\pi_{12}^{-1}(Q_{i-1}^n) \cap \pi_{23}^{-1}(R^{n+i-1})$. There are two irreducible components in this intersection. One, say E_1 , is the closure of the set of triples of the form $(z_n, z_n \amalg w_{i-1}, z_{n+1} \amalg w_{i-1})$, where z_n is a reduced subscheme of length n ,

w_{i-1} is a punctual subscheme of length $i - 1$ with support not belonging to z_n , and z_{n+1} is a subscheme of length $n + 1$ containing z_n and having the same support as z_n .

Another component denoted E_2 is the closure of the set of triples of the form $(z_n, z_n \amalg w_{i-1}, z_n \amalg w_i)$, where z_n is again a reduced subscheme of length n and w_{i-1} resp. w_i are punctual subschemes of length $i - 1$ resp. i with common support not belonging to z_n . The component E_2 has multiplicity $i - 1$ and $\pi_{13}(E_2) = Q_i^n$. We claim that these are all the components of the intersection $\pi_{12}^{-1}(Q_i^n) \cap \pi_{23}^{-1}(R^{n+i-1})$. This can be seen using arguments similar to the detailed proof of Proposition 38; details will be skipped here.

Consider now the composition $q_{i-1}\rho$ and the product $S^{[n]} \times S^{[n+1]} \times S^{[i]}$. The intersection $\pi_{12}^{-1}(R^n) \cap \pi_{23}^{-1}(Q_i^{n+1})$ has only one component E'_1 which is the closure of the set of triples $(z_n, z_{n+1}, z_{n+1} \amalg w_{i-1})$, with the same notations as for the component E_1 . In the commutator $\rho q_{i-1} - q_{i-1}\rho$ the components E_1 and E'_1 cancel each other, and we get the formula.

The third statement is proved by a similar argument. The correspondences in $S^{[n]} \times S^{[n+1]}$ corresponding to both compositions ∂q_1 and $q_1\partial$ contain the closure of the set of pairs $(z_n, z_n \amalg w_1)$ where z_n is a non-reduced subscheme of length n , and these cancel each other. The composition ∂q_1 moreover contains the closure of the set of pairs (z_n, z_{n+1}) with z_n reduced and $\text{supp}(z_{n+1}) = \text{supp}(z_n)$, namely, the correspondence R^n , with mutliplicity 2.

The second and the fourth equalities are obtained from the first and the third equalities using duality and the fact that ∂ is self-dual. \square

Applying this theorem and Corollary 32, we deduce the following formula for the operator ρ :

Corollary 35

$$\rho(\text{fix}(\lambda)) = - \sum_{\mu \in \lambda[1]} \frac{\text{Coker}(\lambda, \mu)}{\text{Ker}(\lambda, \mu)} w(\mu \setminus \lambda) \text{fix}(\mu) \quad (3)$$

Example 36 *Applying the result of this subsection recursively, we obtain the following base changes between the basis $\text{nak}(\lambda)$ and $\text{fix}(\lambda)$ in conformal degree 2 and 3:*

$$\begin{aligned} \text{nak}(1, 1) &= 2\text{fix}(1, 1) + 2\text{fix}(2) \\ \text{nak}(2) &= -2U\text{fix}(1, 1) - 2V\text{fix}(2) \end{aligned}$$

$$\begin{aligned} \text{nak}(3) &= 6V^2\text{fix}(3) + 3UV\text{fix}(2, 1) + 6U^2\text{fix}(1, 1, 1) \\ \text{nak}(2, 1) &= -6V\text{fix}(3) - 2(U + V)\text{fix}(2, 1) - 6U\text{fix}(1, 1, 1) \\ \text{nak}(1, 1, 1) &= 6\text{fix}(3) + 6\text{fix}(2, 1) + 6\text{fix}(1, 1, 1) \end{aligned}$$

The denominators of the fractions of the intermediate computations simplify and the final base change is polynomial. This is because $\text{nak}(\lambda)$ lies in the subring $A_T^(S^{[n]})$ of $A_K^*(S^{[n]})$.*

4.4 Commutation relations

In this subsection, we compute the commutators between the different q_i 's.

We note that it is not possible to keep the proof by Nakajima. Indeed, the equivariant pushforward of a class under a non proper contracting morphism is not zero and the vanishing arguments of Nakajima are not valid in our context. This non vanishing feature is crucial for us because this is precisely the contribution of such contracting morphisms that will give the non commutativity $[q_{-1}, q_1] = \frac{1}{UV} Id$.

4.4.1 Commutation with q_1

Our first goal is to study the commutator $[q_1, q_i]$. This will follow from a geometric argument studying directly the correspondences.

Recall (Definition 24) that we denoted by $Q_i^n \subset S^{[n]} \times S^{[n+i]}$ Nakajima's correspondence. Consider the product $S^{[n]} \times S^{[n+1]} \times S^{[n+i+1]}$ and for $a, b \in \{n, n+1, n+i+1\}$ the projection $\pi_{a,b} : S^{[n]} \times S^{[n+1]} \times S^{[n+i+1]} \rightarrow S^{[a]} \times S^{[b]}$. Let us denote by $\mathcal{I}_{1i} \subset S^{[n]} \times S^{[n+1]} \times S^{[n+i+1]}$ the intersection $\pi_{n,n+1}^{-1}(Q_1^n) \cap \pi_{n+1,n+i+1}^{-1}(Q_i^{n+1})$.

Let us introduce some piece of notation:

Notation 37 *Let $w, z \subset S$ be two subschemes. Assume that $w \subset z$ or $w \supset z$. If $w \subset z$ assume moreover that the support of $\mathcal{O}_z/\mathcal{O}_w$ is a point: in this case we denote by $\text{supp}(w \neq z)$ this point. If $w \supset z$ assume that the support of $\mathcal{O}_w/\mathcal{O}_z$ is a point: we denote by $\text{supp}(w \neq z)$ this point.*

Moreover, given a subscheme z and a point x , we denote by w_x the largest punctual subscheme of w whose support is x .

We denote by $l(w)$ the length of w .

Let $E_1 = \{(w, z, t) \in \mathcal{I}_{1i}, z \cap t \text{ reduced}, \text{supp}(w \neq z) \neq \text{supp}(z \neq t)\}$ and denote by E_2 the set $\{(w, z, t) \in \mathcal{I}_{1i}, z \cap t \text{ reduced}, \text{supp}(w \neq z) = \text{supp}(z \neq t)\}$.

Proposition 38 *The intersection $\mathcal{I}_{1i} = \pi_{n,n+1}^{-1}(Q_1^n) \cap \pi_{n+1,n+i+1}^{-1}(Q_i^{n+1})$ is proper. If $i > 0$, then $\mathcal{I}_{1i} = \overline{E_1}$ and \mathcal{I}_{1i} is reduced irreducible of dimension $2n + i + 3$. If $i < 0$, then $\mathcal{I}_{1i} = \overline{E_1} \cup \overline{E_2}$ a union of two reduced subschemes of dimension $2n + i + 3$.*

Proof. By Proposition 17, $\pi_{n,n+1}^{-1}(Q_1^n)$ is smooth and thus locally a complete intersection. Therefore each irreducible component of \mathcal{I}_{1i} has codimension at most $4n+i+1$ in $S^{[n]} \times S^{[n+1]} \times S^{[n+i+1]}$, and so has dimension at least $2n + i + 3$.

If $i < 0$, let $e = 2$ and if $i > 0$, let $e = 1$. To prove that \mathcal{I}_{1i} has exactly e reduced components and the other claims of the proposition, it suffices to describe a set of subschemes $E(p, q) \subset \mathcal{I}_{1i}$ and $E(p) \subset \mathcal{I}_{1i}$ with the following conditions:

- $\mathcal{I}_{1i} = \coprod_{p,q} E(p, q) \amalg \coprod_p E(p)$ realizes \mathcal{I}_{1i} as a disjoint union.
- Exactly e elements among the subschemes $E(p, q)$ and $E(p)$ have the expected dimension $2n + i + 3$.
- These e strata are reduced.
- The other strata have dimension less than $2n + i + 3$.

The components in the intersection will then be the closures of the maximal strata. For $p \geq 0$, $q \geq 0$, $i \neq 0$, $q + i \geq 0$, let $E(p, q)$ be the set

$$\{(w, z, t) \in \mathcal{I}_{1i}, \text{supp}(w \neq z) \neq \text{supp}(z \neq t), l(w_{\text{supp}(w \neq z)}) = p, l(w_{\text{supp}(z \neq t)}) = q\}.$$

For $p \geq 0$, $i \neq 0$, $p + 1 + i \geq 0$, let

$$E(p) := \{(w, z, t) \in \mathcal{I}_{1i}, \text{supp}(w \neq z) = \text{supp}(z \neq t) = x, l(w_x) = p\}.$$

Let (w, z, t) in $E(p, q)$. Let $x = \text{supp}(w \neq z)$ and $y = \text{supp}(z \neq t)$. Let $w_1 \subset w$ the largest subscheme whose support does not contain x nor y . Since $w = w_1 \cup w_x \cup w_y$, $z = w_1 \cup z_x \cup w_y$, $t = w_1 \cup z_x \cup t_y$, the triple (w, z, t) is characterized by the data $w_1, (w_x, z_x), w_y, t_y$. Since $l(w_1) = n - p - q$, w_1 moves in dimension $2n - 2p - 2q$. The pair (w_x, z_x) with $w_x \subset z_x$, $l(w_x) = l(z_x) - 1 = p$ moves in dimension $p + 2$ by Proposition 20. The scheme w_y with $l(w_y) = q$ moves in dimension q if $q = 0$ and $q + 1$ if $q > 0$. Given w_y , the scheme t_y with $l(t_y) = q + i$, $t_y \supset w_y$ (case $i > 0$), $t_y \subset w_y$ (case $i < 0$) moves in dimension $q + i$ if $q + i = 0$, $q + i + 1$ if $q + i > 0$ and $q = 0$, at most $q + i - 1$ if $q + i > 0$ and $q > 0$.

Summing up, in any case, the dimension of $E(p, q)$ is at most $2n + i + 3$, and the equality $\dim E(p, q) = 2n + i + 3$ is realized only when $i > 0$, $p = 0$, $q = 0$, and when $i < 0$, $p = 0$, $q = -i$.

Let (w, z, t) in $E(p)$. Let $x = \text{supp}(w \neq z)$. Let $w_1 \subset w$ the largest subscheme whose support does not contain x . Since $w = w_1 \cup w_x$, $z = w_1 \cup z_x$, $t = w_1 \cup t_x$, the triple (w, z, t) is characterized by the data $w_1, (w_x, z_x), t_x$. Since $l(w_1) = n - p$, w_1 moves in dimension $2n - 2p$. The pair w_x, z_x with $w_x \subset z_x$ and $l(w_x) = l(z_x) - 1 = p$ moves in dimension $p + 2$. The scheme t_x with $l(t_x) = p + 1 + i$, $t_x \supset z_x$ (case $i > 0$), $t_x \subset z_x$ (case $i < 0$) moves in dimension $p + 1 + i$ if $p + 1 + i = 0$, at most $p + i$ if $p + 1 + i > 0$.

Summing up, in any case, the dimension of $E(p)$ is at most $2n + i + 3$, and the equality $\dim S(p, q) = 2n + i + 3$ is realized only when $i < 0$, $p + 1 + i = 0$.

By construction, a point (w, z, t) in a stratum of maximal dimension is such that $z \cap t$ is reduced. The result follows. \square

We now consider the product $S^{[n]} \times S^{[n+i]} \times S^{[n+i+1]}$ and the three projections $\pi_{n, n+i}, \pi_{n, n+i+1}, \pi_{n+i, n+i+1}$ defined as above. We denote by \mathcal{I}_{i1} the intersection $\pi_{n, n+i}^{-1}(Q_i^n) \cap \pi_{n+i, n+i+1}^{-1}(Q_1^{n+i})$.

Let $E' = \{(w, z, t) \in \mathcal{I}_{i1}, z \cap t \text{ reduced}, \text{supp}(w \neq z) \neq \text{supp}(z \neq t)\}$.

Proposition 39 *The intersection $\pi_{n, n+i}^{-1}(Q_i^n) \cap \pi_{n+i, n+i+1}^{-1}(Q_1^{n+i})$ is proper. More precisely $\mathcal{I}_{i1} = \overline{E'}$ and \mathcal{I}_{i1} is reduced irreducible of dimension $2n + i + 3$.*

Proof. The proof is similar to the proof of Proposition 38. We introduce a stratification of \mathcal{I}_{i1} in the form $\mathcal{I}_{i1} = \coprod_{p, q} E(p, q) \amalg \coprod_q E(q)$. For $p \geq 0$, $q \geq 0$, $i \neq 0$, $q + i \geq 0$, let $E(p, q)$ be the set

$$\{(w, z, t) \in \mathcal{I}_{i1}, \text{supp}(w \neq z) \neq \text{supp}(z \neq t), l(w_{\text{supp}(w \neq z)}) = q, l(w_{\text{supp}(z \neq t)}) = p\}.$$

The only stratum of expected dimension $2n + i + 3$ is $E(0, 0)$ when $i > 0$ and $E(-i, 0)$ when $i < 0$. The study of the strata $E(p, q)$ is rigorously similar to the mentioned Proposition 38 and we skip it. For $q \geq 0$, $i \neq 0$, $q + i \geq 0$, let

$$E(q) := \{(w, z, t) \in \mathcal{I}_{i1}, \text{supp}(w \neq z) = \text{supp}(z \neq t) = x, l(w_x) = q\}.$$

Let (w, z, t) in $E(q)$. Let $x = \text{supp}(w \neq z)$. Let $w_1 \subset w$ the largest subscheme whose support does not contain x . Since $w = w_1 \cup w_x$, $z = w_1 \cup z_x$, $t = w_1 \cup t_x$, the triple (w, z, t) is characterized by the data $w_1, (z_x, t_x), w_x$. Since $l(w_1) = n - q$, w_1 moves in dimension $2n - 2q$. The pair (z_x, t_x) with $z_x \subset t_x$, $l(z_x) = l(t_x) - 1 = q + i$ moves in dimension $q + i + 2$ by Proposition 20. Given z_x , the scheme w_x with $l(w_x) = q$, $w_x \subset z_x$ (case $i > 0$), $w_x \supset z_x$ (case $i < 0$) moves in dimension q if $q = 0$, at most $q - 1$ if $q > 0$.

Summing up, in any case, the dimension of $E(q)$ is at most $2n + i + 2$. There is no stratum $E(q)$ of the expected dimension $2n + i + 3$. \square

In the proof of the next proposition we will use the following easy lemma:

Lemma 40 *Let $\pi : S \rightarrow pt$ be the projection of S to a point. Then $\pi_* 1 = 1/UV$.*

Proof. This is a direct application of Theorem 5. \square

Proposition 41 *We have $[q_{-1}, q_1] = \frac{1}{UV} Id$. Moreover, for $i \neq -1$, we have $[q_i, q_1] = 0$.*

Proof. The composition $q_i q_1$ resp. $q_1 q_i$ corresponds to $(\pi_{n,n+i+1})_*(\pi_{n,n+1}^*[Q_1^n] \cup \pi_{n+1,n+i+1}^*[Q_i^{n+1}])$ resp. $(\pi_{n,n+i+1})_*(\pi_{n,n+i}^*[Q_i^n] \cup \pi_{n+i,n+i+1}^*[Q_1^{n+1}])$.

By Propositions 38 and 39, the intersection $\pi_{n,n+1}^{-1}(Q_1^n) \cap \pi_{n+1,n+i+1}^{-1}(Q_i^{n+1})$ resp. $\pi_{n,n+i}^{-1}(Q_i^n) \cap \pi_{n+i,n+i+1}^{-1}(Q_1^{n+1})$ is proper and therefore the cup product $\pi_{n,n+1}^*[Q_1^n] \cup \pi_{n+1,n+i+1}^*[Q_i^{n+1}]$ resp. $\pi_{n,n+i}^*[Q_i^n] \cup \pi_{n+i,n+i+1}^*[Q_1^{n+1}]$ is equal to the class of \mathcal{I}_{1i} resp. \mathcal{I}_{i1} .

Moreover, when $i > 0$, these two propositions show that \mathcal{I}_{1i} and \mathcal{I}_{i1} are birational to $S^{[n]} \times S \times S_{punc}^{[i]}$, where the indice *punc* refers to the punctual Hilbert scheme. So there is a commutative T -equivariant diagram:

$$\begin{array}{ccc} \mathcal{I}_{1i} & \overset{\text{-----}}{\dashrightarrow} & \mathcal{I}_{i1} \\ & \searrow \pi_{n,n+i+1} & \swarrow \pi_{n,n+i+1} \\ & S^{[n]} \times S^{[n+i+1]} & \end{array}$$

From this it follows that, when $i > 0$, the correspondence

$$(\pi_{n,n+i+1})_*(\pi_{n,n+1}^*[Q_1^n] \cup \pi_{n+1,n+i+1}^*[Q_i^{n+1}]) - (\pi_{n,n+i+1})_*(\pi_{n,n+i}^*[Q_i^n] \cup \pi_{n+i,n+i+1}^*[Q_1^{n+1}])$$

defining the commutator morphism $[q_1, q_i]$ is zero.

When $i < 0$, there is an extra component in \mathcal{I}_{1i} , namely $\overline{E(-i-1)}$ with the notations of Proposition 38. Let $c = \overline{E(-i-1)}$ denote the class of this component. The commutator $[q_i, q_1]$ is then defined by the correspondence $(\pi_{n,n+i+1})_*(c)$.

If $i < -1$, the morphism $\pi_{n,n+i+1} : \overline{E(-i-1)} \rightarrow S^{[n]} \times S^{[n+i+1]}$ is a proper morphism with fibers of positive dimension. It follows that the correspondence $(\pi_{n,n+i+1})_*(c)$ defining the morphism is equal to zero.

If $i = -1$, the morphism $\pi_{n,n+i+1} : \overline{E(-i-1)} \rightarrow S^{[n]} \times S^{[n+i+1]}$ is not proper any more. It is birational to the morphism $\varphi : \Delta \times S \rightarrow S^{[n]} \times S^{[n]}$ where $\Delta \subset S^{[n]} \times S^{[n]}$ is the diagonal and $\varphi(w, w, x) = (w, w)$. It follows that the correspondence $(\pi_{n,n+i+1})_*(c)$ defining the morphism is equal to $\frac{1}{UV}[\Delta]$ and the proposition follows. \square

Proposition 42 *Let i, j be positive integers. Then $q_i q_j = q_j q_i$.*

Proof. By Theorem 34, we have $(i-1)q_i = \rho q_{i-1} - q_{i-1} \rho$ and $j q_{j+1} = \rho q_j - q_j \rho$. From this it follows that

$$(i-1)[q_i, q_j] = [\rho, [q_{i-1}, q_j]] - j[q_{i-1}, q_{j+1}] .$$

By Proposition 41 and induction on i , we may assume that $[q_{i-1}, q_j] = 0$ and $[q_{i-1}, q_{j+1}] = 0$. Thus the proposition is proved. \square

4.4.2 Commuting ρ and ρ^\vee

We now compute the commutator $[\rho, \rho^\vee]$. This is the technical key point of the computation of the commutation relations involving higher q_i 's.

Proposition 43 *We have:*

$$[\rho, \rho^\vee] = \bigoplus_{n \geq 0} 2n \operatorname{Id}_{A_K^*(S^{[n]})}.$$

The heart of the proof is to get rid of an excess intersection component. To this aim, we use some standard intersection theory formulas to break up the initial intersection product into several pieces. After this rewriting, some of the intersections that show up are transverse and easy to compute. The other pieces (responsible for the excess intersections) are intersections with Cartier divisors. They can be handled with Chern class formalism.

Proof. Let us first compute the correspondence $\rho \rho^\vee$ in the equivariant Chow ring of $S^{[n]} \times S^{[n]}$.

On the product $S^{[n]} \times S^{[n-1]} \times S^{[n]}$ we denote by π_i and π_{ij} ($i, j \in \{1, 2, 3\}$) the natural projections. Let $C = \pi_{12}^{-1}((R^\vee)^n)$ and $D = \pi_{23}^{-1}(R^{n-1})$. Let E_1 resp. E_2 in $S^{[n]} \times S^{[n-1]} \times S^{[n]}$ be the closure of the set of triples (z_n, z_{n-1}, z'_n) with z_{n-1} reduced, z_n and z'_n non reduced, $z_{n-1} \subset z_n$, $z_{n-1} \subset z'_n$ and $\operatorname{supp}(z_n \neq z_{n-1}) = \operatorname{supp}(z'_n \neq z_{n-1})$ resp. $\operatorname{supp}(z_n \neq z_{n-1}) \neq \operatorname{supp}(z'_n \neq z_{n-1})$. Let $F_i := \pi_{13}(E_i)$: a generic element in F_1 resp. F_2 is a couple (z_n, z'_n) where z_n and z'_n have the same support, both have exactly one double point and the double points have the same resp. different support. The generic elements in E_1, E_2 are depicted in the following picture:



Proposition 44 *The intersection $C \cap D$ is generically transverse and equal to the union $E_1 \cup E_2$.*

Proof. The codimension of C and D in the product $S^{[n]} \times S^{[n-1]} \times S^{[n]}$ is $2n-1$. It follows that the components of $C \cap D$ have dimension at least $2n$.

Moreover $C \cap D \subset L$, where L parametrizes the triples (z_n, z_{n-1}, z'_n) with $z_n \supset z_{n-1}, z'_n \supset z_{n-1}, \operatorname{supp}(z_n) = \operatorname{supp}(z_{n-1}), \operatorname{supp}(z'_n) = \operatorname{supp}(z_{n-1})$. The locus $L_k \subset L$ parametrizing the triples (z_n, z_{n-1}, z'_n) with z_{n-1} supported by k points is such that $\overline{L_{n-1}} = E_1 \cup E_2$ has pure dimension $2n$. For $k < n-1$, $\dim L_k < 2n$. Thus the generic point of any component of $C \cap D$

is in L_{n-1} . The reverse inclusion $L_{n-1} \subset C \cap D$ is obvious, so that $C \cap D = \overline{L_{n-1}} = E_1 \cup E_2$. The intersection is proper since both E_1 and E_2 have dimension $2n$. The intersection is transverse along L_{n-1} , thus generically transverse. \square

Since the restrictions of π_{13} to E_1 and E_2 are birational on their image, it follows from the proposition that

$$\rho\rho^\vee = [F_1] + [F_2]. \quad (4)$$

Now we compute $\rho^\vee\rho$. We use similar notations for π_i, π_{ij} on $S^{[n]} \times S^{[n+1]} \times S^{[n]}$, and moreover we denote by η_1, η_2 the two projections from $S^{[n]} \times S^{[n]}$ to $S^{[n]}$. First of all we consider the variety

$$Q' = \pi_{12}^{-1}(Q_1^n) \cap \pi_{23}^{-1}(Q_{-1}^{n+1}).$$

We want to prove that Q' admits two irreducible components.

Lemma 45 *Let $L_k \subset S^{[k-1]} \times S^{[k]} \times S^{[k-1]}$ be the locus parametrizing the triples (z_{k-1}, z_k, z'_{k-1}) with $z_{k-1} \subset z_k, z'_{k-1} \subset z_k$ and z_k supported at the origin. Then for $k \geq 2$, $\dim L_k \leq 2k - 3$.*

Proof. The pair (z_{k-1}, z_k) moves in dimension at most $k - 1$. When z_{k-1} and z_k are fixed, z'_{k-1} moves in dimension at most $k - 2$. \square

An element in Q' is a triple (z_n, z_{n+1}, z'_n) with $z_n \subset z_{n+1}$ and $z_{n+1} \supset z'_n$. Let $Q'_2 \subset Q'$ the closed locus defined by the condition $z_n = z'_n$ and \underline{Q}'_1 the open locus defined by the condition $z_n \neq z'_n$. Let Q'_1 be the closure of \underline{Q}'_1 .

Proposition 46 *The varieties Q'_1 and Q'_2 are irreducible of dimension $\dim Q'_1 = \dim Q'_2 = 2n + 2$. The irreducible components of Q' are Q'_1 and Q'_2 . Moreover, the intersection $Q' = \pi_{12}^{-1}(Q_1^n) \cap \pi_{23}^{-1}(Q_{-1}^{n+1})$ is generically transverse.*

Proof. The claims concerning Q'_2 are true since Q'_2 is isomorphic to $S^{[n, n+1]}$ by projection on the first two factors.

As for \underline{Q}'_1 , let us denote by $p = z_{n+1} \setminus z_n$ and $p' = z_{n+1} \setminus z'_n$ the natural residual points defined by a triple $(z_n, z_{n+1}, z'_n) \in \underline{Q}'_1$.

The irreducibility of $S_0^{[k, k+1]}$ implies that the locus $L_0 \subset \underline{Q}'_1$ with $p \neq p'$ is irreducible of dimension $2n + 2$.

For $m \geq 1$, we define the locus L_m in \underline{Q}'_1 by the conditions $p = p'$ and $\text{length}(z_{n+1})_p = m$. It follows from Lemma 45 that it has dimension at most $(2m - 1) + 2(n + 1 - m) = 2n + 1$ when $m \geq 2$. When $m = 1$, $L_1 = \emptyset$.

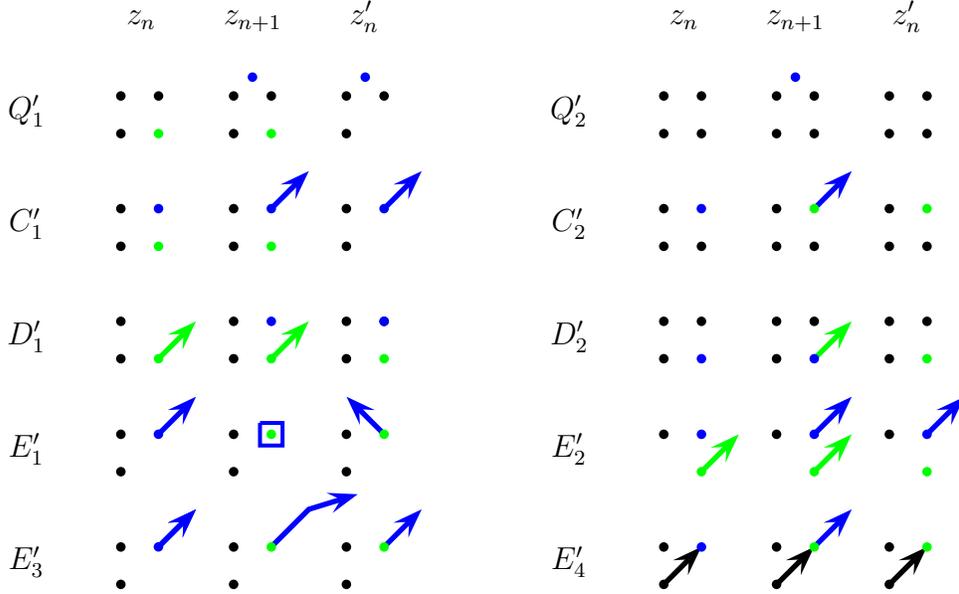
Since the codimension of the intersection is bounded by the sum of the codimensions, the components of Q' have dimension at least $2n + 2$.

By construction $\underline{Q}'_1 = \cup_{m \geq 2} L_m \cup L_0$ and $Q' = \underline{Q}'_1 \cup Q'_2$. The dimensions computed above show that the generic points of Q' coincide with the generic points of L_0 and Q'_2 . Moreover, for $m \geq 2$, $L_m \subset \overline{L_0}$ otherwise there would be in Q' a component of dimension less than $2n + 2$.

The transversality of the intersection $\pi_{12}^{-1}(Q_1^n) \cap \pi_{23}^{-1}(Q_{-1}^{n+1})$ is easily verified at the generic points of Q'_1 and Q'_2 . \square

We denote by C'_1 resp. C'_2 the closures of the sets of triples (z_n, z_{n+1}, z'_n) where $\text{supp}(z_{n+1} \neq z_n) \in z_n$ and $\text{supp}(z_{n+1} \neq z_n) \neq \text{supp}(z_{n+1} \neq z'_n)$ resp. $\text{supp}(z_{n+1} \neq z_n) = \text{supp}(z_{n+1} \neq z'_n)$. We

denote by D'_1 resp. D'_2 the closures of the sets of triples (z_n, z_{n+1}, z'_n) where $\text{supp}(z_{n+1} \neq z'_n) \in z'_n$ and $\text{supp}(z_{n+1} \neq z_n) \neq \text{supp}(z_{n+1} \neq z'_n)$ resp. $\text{supp}(z_{n+1} \neq z_n) = \text{supp}(z_{n+1} \neq z'_n)$. The varieties $Q'_1, Q'_2, C'_1, C'_2, D'_1, D'_2$ (as well as the following varieties E'_1, E'_2, E'_3, E'_4) are depicted in the following array :



Intersecting with Chern classes of line bundles, in particular with Cartier divisors, commutes with the intersection product in the Chow ring ([Fu98], Example 8.1.6). It follows that if X and Y are smooth in Z smooth, if $\Delta_X \subset X$ and $\Delta_Y \subset Y$ are Cartier divisors with restrictions R_X and R_Y on the (not necessarily smooth) generically transverse intersection $X \cap Y$, then the intersection $[\Delta_X] \cdot_Z [\Delta_Y]$ computed in the Chow ring $A^*(Z)$ is equal to $i_*(R_X \cdot_{X \cap Y} R_Y)$ where $i : X \cap Y \rightarrow Z$ is the natural injection. Since we are working with divisors, one can replace smoothness of X and Y with smooth in codimension one. Moreover, there are equivariant analogs of these statements.

According to Proposition 46, one can apply the above with $Z = S^{[n]} \times S^{[n+1]} \times S^{[n]}$, $X = \pi_{12}^{-1}(Q_1^n)$, $Y = \pi_{23}^{-1}(Q_{-1}^{n+1})$, $X \cap Y = Q' = Q'_1 \cup Q'_2$, $\Delta_X = \pi_{12}^{-1}(R^n)$, $\Delta_Y = \pi_{23}^{-1}((R^\vee)^{n+1})$. For the restrictions of the divisors, we use the notation $R_{X_i} = \Delta_X \cap Q_i$ and $R_{Y_i} = \Delta_Y \cap Q_i$. We obtain:

$$\rho^\vee \rho = (\pi_{13})_* i_* ((R_{X1} + R_{X2}) \cdot (R_{Y1} + R_{Y2}))$$

where the intersection product takes place in Q' .

Proposition 47 $R_{X1} = [C'_1]$, $R_{X2} = 2[C'_2]$, $R_{Y1} = [D'_1]$, $R_{Y2} = 2[D'_2]$.

Proof. This is clear set theoretically. The multiplicities are computed in local coordinates at a generic point. \square

Since R^n is a divisor on the smooth variety $S^{[n, n+1]}$, Δ_X is a Cartier divisor, and so are its restrictions R_{X_i} . Thus $[C'_1]$ and $2[C'_2]$ are Cartier divisors, and similarly for D'_1 and D'_2 . Thus our task now is to compute the product of the divisors $([C'_1] + 2[C'_2]) \cdot ([D'_1] + 2[D'_2])$ in $A_T^* Q'$. Note however that $C'_2 = D'_2$, so that the corresponding intersection is certainly not proper. In

fact we compute $2[C'_2] \cdot ([D'_1] + 2[D'_2])$ by another method. We know that $\rho^\vee = \frac{1}{2}(q_{-1}\delta - \delta q_{-1})$. So in $A_T^*(Q')$ we have $2([D'_1] + 2[D'_2]) = \pi_2^*\partial_{n+1} - \pi_3^*\partial_n$. The pushforward $\pi_{13,*}([C'_2] \cup \pi_3^*\partial_n)$ can be computed thanks to the projection formula: this is $\pi_{13,*}[C'_2] \cup \eta_2^*\partial_n$. But since π_{13} is proper and contractant when restricted to C'_2 , $\pi_{13,*}[C'_2] = 0$.

To compute $\pi_{13,*}([C'_2] \cup \pi_2^*\partial_{n+1})$ we observe that the general fibers of π_{13} over $\pi_{13}(C'_2)$ are isomorphic n copies of \mathbb{P}^1 and $\pi_2^*\partial_{n+1}$ restricts to a line bundle isomorphic to $\mathcal{O}(-2)$ on each \mathbb{P}^1 (in fact the class of the diagonal is $-2c_1^T(\mathcal{O}^{[2]})$ if $\mathcal{O}^{[2]}$ denotes the tautological bundle). Thus we get $\pi_{13,*}([C'_2] \cup \pi_2^*\partial_{n+1}) = -2nId_{S^{[n]}}$.

To compute the other products we consider geometric intersections. Let E'_1, E'_2, E'_3, E'_4 be the closures of some sets of triples (z_n, z_{n+1}, z'_n) . To define these triples we use the following conventions: p_i, p'_i will be punctual subschemes of length i and w_j will be reduced subschemes of length j . Moreover, unless otherwise stated, these subschemes will be generic (among punctual subschemes) and their supports disjoint.

Let p_2, p'_2 share the same support, and let p_3 be the 2-fat point having the same support as p_2 and p'_2 . A generic triple (z_n, z_{n+1}, z'_n) in E'_1 is given as follows: $z_n = w_{n-2} \amalg p_2$, $z'_n = w_{n-2} \amalg p'_2$ and $z_{n+1} = w_{n-2} \amalg p_3$.

Let $p_1 \subset p_2, p'_1 \subset p'_2, w_{n-3}$ be generic. A generic triple (z_n, z_{n+1}, z'_n) in E'_2 is given as follows: $z_n = w_{n-3} \amalg p_1 \amalg p'_2, z_{n+1} = w_{n-3} \amalg p_2 \amalg p'_2, z'_n = w_{n-3} \amalg p_2 \amalg p'_1$.

Let $p_2 \subset p_3, w_{n-2}$ be generic. A generic triple (z_n, z_{n+1}, z'_n) in E'_3 is given as follows: $z_n = z'_n = w_{n-2} \amalg p_2$ and $z_{n+1} = w_{n-2} \amalg p_3$.

Let $p_1 \subset p_2$, let p'_2 and w_{n-3} be generic. A generic triple (z_n, z_{n+1}, z'_n) in E'_4 is given as follows: $z_n = z'_n = w_{n-3} \amalg p_1 \amalg p'_2, z_{n+1} = w_{n-3} \amalg p_2 \amalg p'_2$.

Lemma 48 *We have the set theoretic intersection $C'_1 \cap (D'_1 \cup D'_2) = E'_1 \cup E'_2 \cup E'_3 \cup E'_4$.*

Proof. Let \mathcal{I} denote an irreducible component in the intersection of the lemma. We know that \mathcal{I} has dimension at least $2n$. Let $\xi = (z_n, z_{n+1}, z'_n)$ be a generic point in one of these components. Let $x = \text{supp}(z_{n+1} \neq z_n)$ and $y = \text{supp}(z_{n+1} \neq z'_n)$. Let p be the length of z_n at x and q the length of z'_n at y . Since $\mathcal{I} \subset C'_1$, $p \geq 1$. Since $\mathcal{I} \subset D'_1 \cup D'_2$, $q \geq 1$. Finally, since $\mathcal{I} \subset C'_1$, z'_n is non reduced.

Assume first that $x \neq y$. Then the triple ξ is defined by the inclusions $(z_n)_{|x} \subset (z_{n+1})_{|x}$ and $(z'_n)_{|y} \subset (z_{n+1})_{|y}$ and the intersection $z_n \cap (S \setminus \{x, y\})$ which has length $n - p - q - 1$. So the dimension of the set of such triples is $(p+2) + (q+2) + 2(n-p-q-1) = 2n+2-p-q$, so that $p+q=2$. Therefore $p=1=q$, and so $\mathcal{I} = E'_2$.

From now on, we assume that $x = y$, so $q = p$. If $p = 1$, since z'_n is non reduced, we have $\mathcal{I} = E'_4$.

Let us see that $p \leq 2$. We denote by f the dimension of the set of schemes of length p included in $(z_{n+1})_{|x}$. Since the support of such a subscheme of $(z_{n+1})_{|x}$ is x we have $f \leq p-1$. Let $r : \mathcal{I} \dashrightarrow S_0^{[p,p+1]}$ which maps a triple (z_n, z_{n+1}, z'_n) to the pair $((z_n)_{|x}, (z_{n+1})_{|x})$. We denote by d the dimension of $r(\mathcal{I})$. By Proposition 20 $\dim S_0^{[p,p+1]} = p+2$ so we have $d \leq p+2$. Moreover $\dim \mathcal{I} = 2n - 2p + d + f \geq 2n$. Summing up, we have

$$d + f \geq 2p, \quad d \leq p + 2, \quad f \leq p - 1. \quad (5)$$

If $f = p-1$, then any scheme of length p supported at x is included in $(z_{n+1})_{|x}$, and this implies that $p = 2$ and $(z_{n+1})_{|x}$ is a 2-fat point. In this case $\mathcal{I} = E'_1$.

Let us assume that $f \leq p - 2$. Equation (5) implies that $f = p - 2$ and $d = p + 2$, so $r(\mathcal{I}) = S_0^{[n, n+1]}$. If we assume that $p > 2$, we get $f > 0$ and therefore $(z_{n+1})|_x$ cannot be curvilinear, contradicting $r(\mathcal{I}) = S_0^{[n, n+1]}$. Thus we have $p = 2$ and $\mathcal{I} = E'_3$. \square

Lemma 49 *There exist integers a, b, c, d such that $[C'_1] \cup [D'_1] = [E'_1] + [E'_2] + a[E'_3] + b[E'_4]$ and $[C'_1] \cup [D'_2] = c[E'_3] + d[E'_4]$.*

Proof. Note that a generic point in E'_1 and E'_2 is a smooth point in Q' (this will for example be a consequence of our following parametrization of Q' near such a point). Let us first compute the intersection number of C'_1 and D'_1 along E'_1 . A generic point in C'_1 resp. D'_1, E'_1 can be obtained by disjoint union of $n - 2$ distinct points and a generic point in the same variety in the case $n = 2$, thus it is enough to consider the case where $n = 2$. We consider the particular point $\xi = (z_2, z_3, z'_2)$ where z_2 resp. z_3, z'_2 is the subscheme of the plane defined by the equations (X, Y^2) resp. $(X^2, XY, Y^2), (X^2, Y)$. Note that the projection $S^{[2]} \times S^{[3]} \times S^{[2]} \rightarrow S^{[2]} \times S^{[2]}$ restricts to an isomorphism on its image in a neighborhood of ξ in Q' , since for $\varepsilon = (y_2, y_3, y'_2)$ in such a neighborhood, y_3 is the scheme-theoretic union of y_2 and y'_2 . Thus Q' is locally isomorphic to the set of pairs (y_2, y'_2) of subschemes of length 2 which meet. Note that for both y_2 and y'_2 there is a unique line containing it. Moreover since all our intersection computations are invariant under translations, we may assume that the intersection point of these two lines is the origin.

We parameterize pairs of subschemes (y_2, y'_2) near (z_2, z'_2) such that these two lines meet at the origin by stating that y_2 resp. y'_2 corresponds to the ideal $(X + aY, Y^2 + bY + c)$ resp. $(X^2 + dX + e, Y + fX)$. Then Q' is defined by the fact that the origin belongs to y_2 and y'_2 , namely by the equations $c = e = 0$ (thus Q' is locally an affine space).

Inside this variety, C'_1 resp. D'_1 is defined by the fact that y'_2 resp. y_2 is non reduced. Thus it is defined by the equation $d = 0$ resp. $b = 0$. We thus see that the intersection of C'_1 and D'_1 is transverse along a generic point in E'_1 .

Around a generic point in E'_2 things are easier because the projection $Q' \rightarrow S^{[n+1]}, \xi \mapsto z_{n+1}$ is locally an isomorphism. Thus Q' is locally isomorphic to the product $S^{[n-3]} \times S^{[2]} \times S^{[2]}$, and (z_{n-3}, z_2, z'_2) in this product belongs to C'_1 resp. D'_1 if and only if z_2 resp. z'_2 is punctual. So the intersection $C'_1 \cap D'_1$ is transverse at such a point.

Now the lemma follows from Lemma 48. \square

We have $\pi_{13,*}[E'_3] = \pi_{13,*}[E'_4] = 0$ since the restriction of π_{13} to E'_3 and E'_4 is proper and contractant. We have $\pi_{13,*}[E'_1] = [F_1]$ and $\pi_{13,*}[E'_2] = [F_2]$. Therefore this gives $\rho^\vee \rho = [F_1] + [F_2] - 2n \text{Id}$. Since by (4), $\rho \rho^\vee = [F_1] + [F_2]$, the proposition is proved. \square

4.4.3 The commutator $[q_i, q_j]$

We can now compute the commutator $[q_i, q_j]$ for all i, j .

Lemma 50 *We have $[q_{-1}, \rho] = 0$.*

Proof. First let us compute the correspondence ρq_{-1} . Consider the product $S^{[n]} \times S^{[n-1]} \times S^{[n]}$ and the natural projections on this product. Let $C := \pi_{12}^{-1}(Q_{-1}^n) \cap \pi_{23}^{-1}(R^{n-1})$. It is the closure

of the set of triples (z_n, z_{n-1}, z'_n) with z_n reduced, $z_{n-1} \subset z_n \cap z'_n$, and z'_n having a point of length 2. Let $F \subset S^{[n]} \times S^{[n]}$ be the closure of the set of pairs (z_n, z'_n) with z_n reduced, $(z'_n)_{red} \subset z_n$ and z'_n having a point of length 2 and simple points otherwise. Since the restriction of π_{13} to C is birational, the morphism $\rho_{q_{-1}}$ is given by the correspondence F .

Now we compute the correspondence $q_{-1}\rho$. The corresponding intersection has been studied in the proof of Proposition 43. With these notations we have $\pi_{23}^*[Q_{-1}^{n+1}] \cup \pi_{12}^*[R^n] = [C'_1] + 2[C'_2]$. Moreover the restriction of π_{13} to C'_1 is birational with image F and the restriction of π_{13} to C'_2 is proper contractant. Thus the morphism $q_{-1}\rho$ is also given by the correspondence F , and the lemma is proved. \square

Recall the convention that $q_0 = 0$.

Proposition 51 *Let i be arbitrary. We have $[\rho, q_i] = |i| q_{i+1}$.*

Proof. If $i \geq 0$ this is Theorem 34. If $i = -1$ this is Lemma 50. Let us assume that $i = -j$ with $j \geq 2$. The Jacobi identity reads:

$$[[q_{-j+1}, \rho^\vee], \rho] + [[\rho, q_{-j+1}], \rho^\vee] + [[\rho^\vee, \rho], q_{-j+1}] = 0.$$

Theorem 34 yields $[q_{-j+1}, \rho^\vee] = (j-1)q_{-j}$. We may assume by induction that $[\rho, q_{-j+1}] = (j-1)q_{-j+2}$. Finally by Proposition 43 we have $[[\rho^\vee, \rho], q_{-j+1}] = 2(j-1)q_{-j+1}$. Thus we get:

$$(j-1)[q_{-j}, \rho] + (j-1)(j-2)q_{-j+1} + 2(j-1)q_{-j+1} = 0,$$

hence the proposition is proved. \square

Theorem 52 *Let i and j be any integers. We have*

$$[q_i, q_{-j}] = \begin{cases} 0 & \text{if } i \neq j \\ \frac{i(-1)^{i+1}}{UV} Id & \text{if } i = j \end{cases}$$

Proof. Since q_{-i} is the adjoint of q_i and $[q_i, q_{-j}] = 0$ if $i \geq 0$ and $j \leq 0$ by Proposition 42, we may assume that $i, j \geq 1$. Moreover the proposition will be true if $i = 1$ or $j = 1$ by Proposition 41. Thus we assume $i, j \geq 2$. Once again we apply Jacobi identity:

$$[[\rho, q_{i-1}], q_{-j}] + [[q_{-j}, \rho], q_{i-1}] + [[q_{i-1}, q_{-j}], \rho] = 0. \quad (6)$$

By induction we may assume that the commutator $[q_{i-1}, q_{-j}]$ is given by the proposition. Therefore it is either 0 or a scalar; in both cases it will commute with ρ , so the last term vanishes. By Proposition 51, $[q_{-j}, \rho] = -jq_{-j+1}$ and $[\rho, q_{i-1}] = (i-1)q_i$.

Therefore equation (6) reads $(i-1)[q_i, q_{-j}] = j[q_{-j+1}, q_{i-1}]$. If $i \neq j$, the second term vanishes by induction and so $[q_i, q_{-j}] = 0$. If $i = j$ we get $[q_i, q_{-i}] = i \frac{(-1)^{i+1}}{UV} Id$ as we wanted to prove. \square

5 Class of the small diagonal

Let Δ_i be the locus in $S^{[n]}$ where at least i points share the same support. In particular Δ_2 is the big diagonal, and Δ_n is the small diagonal. In Corollary 30, we proved the equivariant formula for the big diagonal $[\Delta_2] = -2c_1(\mathcal{O}^{[n]})$, which is analogous to Lehn's formula valid in the classical setting. In this section, we prove an equivariant formula for the small diagonal.

Theorem 53 *The T -equivariant class of Δ_n is: $[\Delta_n] = (-1)^{n-1} n c_{n-1}(\mathcal{O}^{[n]})$.*

The projection from the equivariant Chow ring to the classical Chow ring gives obviously the analogous formula in the classical setting.

Remark 54 *Given u an equivariant line bundle over S , let $c(u) \in \bigoplus A_T^*(S^{[n]})$ denote $([u^{[n]}])_n$, where $[\cdot]$ denotes total equivariant Chern polynomial and $u^{[n]}$ is the bundle over $S^{[n]}$ tautologically defined by u . More generally we have the following formula:*

$$c(u) = \exp \left(\sum_{m \geq 1} \frac{(-1)^{m-1}}{m} q_m \right) \cdot \phi .$$

In this formula ϕ denotes the fundamental class in $S^{[0]}$.

Proof. This formula is Lehn's Theorem 4.6 [Le99] and we explain why his proof is valid in our equivariant context. Lehn introduces the operator $\mathfrak{C}(u) := c(u) \cdot q_1 \cdot c(u)^{-1}$ and shows [Le99, Theorem 4.2] that

$$\mathfrak{C}(u) = q_1(c(u)) + \rho . \tag{7}$$

The proof of this theorem relies on the exact sequence [Le99, (11)] which is equivariant and his Lemma 3.9, for which we proved an equivariant version (Corollary 32). Thus the relation (7) holds in the equivariant context. Lehn's proof of [Le99, Corollary 4.3] is purely algebraic and therefore we also have $c(u) = \exp(\mathfrak{C}(u)) \cdot \phi$. Finally, the proof of [Le99, Theorem 4.6] uses this relation together with the commutation relations of q_i and ρ , which we also proved in Theorem 34. \square

We now give another proof of Theorem 53, as a straightforward consequence of an explicit expression of q_n (Theorem 55) which we believe is interesting in itself. The class $[\Delta_n]$ is equal to $q_n \cdot \phi$.

Recall Notation 31. If $\lambda \subset \mathbb{N}^2$ is a set of cardinal n and $M : \{1, \dots, n\} \rightarrow \lambda$ is a bijection, let $M^- : \{1, \dots, n-1\} \rightarrow \lambda \setminus M(n)$ be the restriction of M and $M^+ : \{1, \dots, n-1\} \rightarrow \lambda \setminus M(1)$ the bijection defined by $M^+(i) = M(i+1)$. Let $w : \lambda \rightarrow \mathbb{Q}[U, V]$ be the map sending (a, b) to the linear form $aU + bV$ corresponding to the weight of the monomial $X^a Y^b$ for the T -action. Let

$$P_M = \frac{(-1)^{n-1}}{(n-1)!} \sum_{i=1}^{i=n} (-1)^{i-1} \binom{n-1}{i-1} w(M(1)) \cdots w(M(i-1)) w(M(i+1)) \cdots w(M(n))$$

if $n > 1$ and $P_M = 1$ if $n = 1$.

Theorem 55 *We have the relation*

$$\Delta_{q_n, \lambda, \mu} = \sum_M P_M \prod_{i=0}^{n-1} \Delta_{q_1, \lambda_i, \lambda_{i+1}} \quad ,$$

where M runs through the standard skew Young diagrams of shape $\mu \setminus \lambda$, and λ_i is the partition defined by $\lambda_i = \lambda \cup \{M(1), \dots, M(i)\}$.

Note that $\sum_M \prod_{i=0}^{n-1} \Delta_{q_1, \lambda_i, \lambda_{i+1}} = \Delta_{q_1^n, \lambda, \mu}$. If λ is empty and M is a tableau of shape μ , all the terms but the first in the sum defining P_M are zero, and $P_M = \frac{(-1)^{n-1}}{(n-1)!} c_{n-1}(\mathcal{O}^{[n]})_{\text{fix}(\mu)}$. Since $q_1^n(\phi) := q_1 \circ \dots \circ q_1(\phi) = n! \in A_T^*(S^{[n]})$, Theorem 53 is a consequence of Theorem 55.

If $M : \{1, \dots, n\} \rightarrow \lambda$ is a standard skew young diagram of shape λ , define Q_M by $Q_M = 1$ if $n = 1$ and recursively by the formula $Q_M = \frac{1}{n-1}(-w(M(n))Q_{M^-} + w(M(1))Q_{M^+})$.

Lemma 56 *For every standard skew Young diagram $M : \{1, \dots, n\} \rightarrow \lambda$, $P_M = Q_M$.*

Proof. This is obvious if $n = 1$ or $n = 2$. To simplify the notation, we denote $w(M(k))$ by m_k . For n general, we have

$$\begin{aligned} (n-1)Q_M &= -m_n Q_{M^-} + m_1 Q_{M^+} \\ &= -m_n P_{M^-} + m_1 P_{M^+} \\ &= -\frac{(-1)^{n-2}}{(n-2)!} \left(m_n \sum_{i=1}^{i=n-1} (-1)^{i-1} \binom{n-2}{i-1} m_1 \dots \hat{m}_i \dots m_{n-1} \right) \\ &\quad + m_1 \sum_{i=2}^{i=n} (-1)^i \binom{n-2}{i-2} m_2 \dots \hat{m}_i \dots m_n \\ &= \frac{(-1)^{n-1}}{(n-2)!} \left(\sum_{i=2}^{i=n-1} (-1)^{i-1} m_1 \dots \hat{m}_i \dots m_n \left(\binom{n-2}{i-1} + \binom{n-2}{i-2} \right) \right) \\ &\quad + m_2 \dots m_n + (-1)^{n-1} m_1 \dots m_{n-1} \\ &= \frac{(-1)^{n-1}}{(n-2)!} \sum_{i=1}^{i=n} (-1)^{i-1} \binom{n-1}{i-1} m_1 \dots \hat{m}_i \dots m_n \\ &= (n-1)P_M \end{aligned}$$

□

Lemma 57 *Let λ and μ be two Young diagrams of cardinal n and $n+1$ with $\lambda \subset \mu$. Then $\Delta_{\rho, \lambda, \mu} = -w(\mu \setminus \lambda) \Delta_{q_1, \lambda, \mu}$.*

Proof. This is a direct consequence of the formula $2\rho = \partial q_1 - q_1 \partial$ and the formula for ∂ given in Proposition 29. □

We now prove the formula for $\Delta_{q_n, \lambda, \mu}$ from Theorem 55. The formula is clearly true for $n = 1$. Suppose that the formula for q_{n-1} is true. Since $(n-1)q_n = \rho q_{n-1} - q_{n-1}\rho$, we get:

$$\begin{aligned}
(n-1)\Delta_{q_n, \lambda, \mu} &= \sum_{p_n \in \text{Corners}(\mu)} \Delta_{q_{n-1}, \lambda, \mu \setminus p_n} \Delta_{\rho, \mu \setminus p_n, \mu} \\
&\quad - \sum_{p_1 \in \text{OutsideCorners}(\lambda) \cap \mu} \Delta_{\rho, \lambda, \lambda \cup p_1} \Delta_{q_{n-1}, \lambda \cup \{p_1\}, \mu} \\
\stackrel{\text{Lemma 57}}{=} &\sum_{p_n \in \text{Corners}(\mu)} -\Delta_{q_{n-1}, \lambda, \mu \setminus p_n} \Delta_{q_1, \mu \setminus p_n, \mu} w(p_n) \\
&\quad + \sum_{p_1 \in \text{OutsideCorners}(\lambda) \cap \mu} \Delta_{q_1, \lambda, \lambda \cup p_1} w(p_1) \Delta_{q_{n-1}, \lambda \cup \{p_1\}, \mu} \\
\stackrel{\text{induction}}{=} &\sum_{M \text{ standard}} \left(\prod_{j=0}^{n-1} \Delta_{q_1, \lambda_j, \lambda_{j+1}} \right) (-w(M(n))P_{M^-} + w(M(1))P_{M^+}) \\
\stackrel{P_M = Q_M}{=} &(n-1) \sum_{M \text{ standard}} \left(\prod_{j=0}^{n-1} \Delta_{q_1, \lambda_j, \lambda_{j+1}} \right) P_M
\end{aligned}$$

6 Base change formulas

The goal of this section is to compute the base change formula from es to nak and its inverse (recall Section 3 for the bases es and nak of A). In particular, we prove that in the classical setting, these two bases are equal up to a constant (Theorem 64).

6.1 Equivariant operators $q_{i, X}$

The basis $\text{nak}(\lambda)$ is defined using creation operators. The basis $\text{es}(\lambda)$ is defined via a Bialynicki-Birula stratification. However, one can introduce operators $q_{i, X}$ such that $\text{es}(\lambda)$ is defined using creation operators too. The goal of this section is to introduce the operators $q_{i, X}$ and to compute a base change inductive formula between $q_{i, X}$ and q_i (Theorem 60).

The operator $q_{i, X}$ means “adding i points on a vertical line”. More formally, $q_{i, X} : A_K^*(S^{[n]}) \rightarrow A_K^*(S^{[n+i]})$ is defined by the Fourier transform along the correspondence $Q_{i, X} \subset S^{[n]} \times S^{[n+i]}$, where $Q_{i, X}$ is the closure of the set of pairs $(z_n, z_n \amalg x_i)$ where $z_n \in S^{[n]}$, $x_i \in S^{[i]}$, x_i is included in the vertical line Δ_{x_0} with equation $X = x_0$ for some $x_0 \in k$, and z_n and x_i have disjoint support. We denote by $\pi_n : Q_{i, X} \rightarrow S^{[n]}$ resp. $\pi_{n+i} : Q_{i, X} \rightarrow S^{[n+i]}$ the natural projections.

First of all these operators allow the computation of the Ellingsrud-Stromme cells:

Proposition 58 *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition. Then we have*

$$q_{\lambda_1, X} \circ \dots \circ q_{\lambda_l, X}(\phi) = \prod_i (\lambda_i^\vee - \lambda_{i+1}^\vee)! \cdot \text{es}_\lambda .$$

Proof. To prove this result by induction on l , it is enough to show that $q_{i, X}(\text{es}_\lambda) = k \text{es}_\mu$, where μ is the partition obtained inserting one part equal to i in λ and k is the number of parts equal to i in μ .

To this end we apply the definition of $q_{i,X}$. For $n = |\lambda|$, we have $q_{i,X}(\text{es}_\lambda) = \pi_{n+i,*}\pi_n^*(ES_\lambda)$. Recall that the Bialynicki-Birula cell decomposition by ES_λ is associated to the injection $k^* \rightarrow T, t \mapsto (t^{-1}, t^{-d})$. Let (z_n, z_{n+i}) be a point belonging to $\pi_n^{-1}(ES_\lambda)$, and assume that z_n resp. z_{n+i} belongs to the open cell corresponding to the partition λ' resp. μ' . We claim that $l(\mu') \leq l(\lambda) + 1$. In fact, since the whole construction is k^* -invariant, we also have $(x_{\lambda'}, x_{\mu'}) \in \pi_n^{-1}(ES_\lambda)$. On the other hand, for a generic $(z_n, z_{n+i}) \in \pi_n^{-1}(ES_\lambda)$, there exists $x \in k$ such that $(X - x) \cdot I(z_n) \subset I(z_{n+i})$ and thus we get $X \cdot I(x_{\lambda'}) \subset I(x_{\mu'})$. Therefore $l(\mu') \leq l(\lambda') + 1$. Since $x_{\lambda'} \in ES_\lambda$, $l(\lambda') \leq l(\lambda)$, thus $l(\mu') \leq l(\lambda) + 1$.

Now, given such μ' , we have $\dim(ES_{\mu'}) \leq n + l(\lambda) + i + 1$. In fact, the dimension of ES_λ is equal to $n + l(\lambda)$. Let C be a component of $\pi_n^{-1}(ES_\lambda)$. The dimension of C is at least $n + l(\lambda) + i + 1$. Thus, if the restriction $C \rightarrow ES_{\mu'}$ is not dominant, it is contractant and it follows that $\pi_{n+i,*}[C] = 0$. If it is dominant, then arguing on the generic points the only possibility is that $\mu' = \mu$ and that C is the component which is the closure of the set of points (z_n, z_{n+i}) with z_n generic in ES_λ and z_{n+i} obtained adding i points on a vertical line to z_n . Let C be this component.

The morphism $\pi_n : C \rightarrow ES_\lambda$ is submersive at a generic point in C , so C is a reduced component of $\pi_n^{-1}(ES_\lambda)$, thus $\pi_{n+i,*}\pi_n^*[ES_\lambda] = \pi_{n+i,*}[C]$.

Moreover, given a generic element $z_{n+i} \in ES_\mu$, there are k vertical lines containing exactly i points. Thus there are k couples (z_n, z_{n+i}) in the fiber $q^{-1}(z_{n+i})$: z_n is obtained from z_{n+i} removing one of these lines. Thus the restriction of π_{n+i} to C has degree k with image ES_μ , which proves the claim. \square

Let $\Delta := \Delta_0$ denote the vertical line with equation $X = 0$. Let $S_\Delta^{[n]}$ denote the subvariety of $S^{[n]}$ parameterizing subschemes with support included in Δ . If λ is a partition of weight n and length l , let $S_{\Delta,\lambda}^{[n]}$ denote the closure in $S_\Delta^{[n]}$ of the variety of schemes $z = z_1 \amalg \cdots \amalg z_l$, where z_i has length λ_i and is supported on one point in Δ .

Proposition 59 *The varieties $S_{\Delta,\lambda}^{[n]}$ are the irreducible components of $S_\Delta^{[n]}$, which is therefore equidimensional of dimension n .*

Proof. Let λ be a partition of weight n and length l , and let i such that $1 \leq i \leq n$. The variety parameterizing schemes of length λ_i supported on one fixed point is irreducible by [Bri77, ES87]. Thus so is the variety parameterizing schemes of length λ_i supported on one point in Δ . Thus each $S_{\Delta,\lambda}^{[n]}$ is irreducible of dimension n . Since we have $S_\Delta^{[n]} = \bigcup_\lambda S_{\Delta,\lambda}^{[n]}$, the proposition is proved. \square

Theorem 60 *We have the following formula:*

$$i q_{i,X} = (-1)^{i+1} q_i + U \cdot \sum_{j=1}^{i-1} (-1)^j q_j \circ q_{i-j,X}$$

With the help of this theorem one can compute all the operators $q_{i,X}$ by induction on i .

To prove the theorem we define auxiliary operators. For $i \geq 0$ and $j \geq 2$ let $q_{i,j,X}$ be the operator corresponding to “adding i points on a same vertical line plus one punctual scheme of length j whose support is on this line”. Formally, $q_{i,j,X}$ is defined by an incidence $Q_{i,j,X}$ in

$S^{[n]} \times S^{[n+i+j]}$ where a generic point in $Q_{i,j,X}$ is of the form $(z_n, z_n \amalg \{x_1, \dots, x_i\} \amalg t_j)$ where the x_k 's are distinct points on a vertical line Δ not meeting z_n and t_j is a length j punctual scheme supported on $\Delta \setminus \{x_1, \dots, x_i\}$. Let us moreover use the convention that $q_{i,1,X} = (i+1)q_{i+1,X}$, $q_{-1,j,X} = 0$, and $q_{0,X} = -1/U$.

The theorem is a consequence of the following proposition because this proposition implies that the right hand side is equal to $q_{i-1,1,X} = i q_{i,X}$.

Proposition 61 *For $i \geq 0$ and $j \geq 1$, we have the following relation in $A_T^*(S^{[n]} \times S^{[n+i]})$:*

$$-U q_j \circ q_{i,X} = q_{i,j,X} + q_{i-1,j+1,X}$$

Proof. If $i = 0$ then the proposition is just formal thanks to the conventions we made above. Let us assume that $i > 0$. Recall Definition 24. There is a morphism $s : Q_j \rightarrow S$ mapping the pair (z_n, z_{n+j}) to the support of $\mathcal{O}_{z_{n+j}}/\mathcal{O}_{z_n}$. Let Δ be the line in S with equation $X = 0$. Let $Q_j(-U) \subset Q_j$ denote the divisor containing the set of pairs (z_n, z_{n+j}) in Q_j with $s(z_n, z_{n+j}) \in \Delta$. We have in $A_T^*(S^{[n]} \times S^{[n+j]})$ the relation $[Q_j(-U)] = s^*[\Delta] = -U \cdot [Q_j]$. Similarly there is a morphism $x : Q_{i,X} \rightarrow \mathbb{A}^1$, mapping a pair (z_n, z_{n+i}) to the common X -coordinate of the points in $z_{n+i} \setminus z_n$. Thus we use similar notations and define $Q_{i,X}(-U) := x^{-1}(0) \subset Q_{i,X}$. In the Chow ring, $[Q_{i,X}(-U)] = -U[Q_{i,X}]$.

Consider the product $S^{[n]} \times S^{[n+i]} \times S^{[n+i+j]}$ and the projections π_a, π_{ab} . Let I be the intersection $\pi_{12}^{-1}(Q_{i,X}(-U)) \cap \pi_{23}^{-1}(Q_j(-U))$. Every proper component of I has dimension $2n + i + j$.

Let C be a component of I which contributes to the composition $q_j \circ q_{i,X}$, ie. a component with $\pi_{13,*}[C] \neq 0$. Our first task is to prove that for a generic element $(z_n, z_{n+i}, z_{n+i+j})$ of C , the support of z_n is disjoint from Δ .

If $z \subset S$ is a subscheme of dimension 0, we denote by z_Δ the union of the components of z supported on Δ . Let I_k be the locally closed set of pairs (z_n, z_{n+i+j}) in $S^{[n]} \times S^{[n+i+j]}$ such that the length of $(z_n)_\Delta$ is k , and $z_{n+i+j} \supset z_n \cap (S \setminus \Delta)$, the support of $\mathcal{O}_{z_{n+i+j}}/\mathcal{O}_{z_n} \cap (S \setminus \Delta)$ is included in Δ . Then I_k is birational to $S_\Delta^{[k]} \times S^{[n-k]} \times S_\Delta^{[k+i+j]}$, and thus has dimension $2n + i + j$.

We denote by k the integer such that for a generic triple $(z_n, z_{n+i}, z_{n+i+j})$ in C , the length of $(z_n)_\Delta$ is k . Since, $\pi_{13}(C) \subset \bar{I}_k$, $\dim \pi_{13}(C) \leq 2n + i + j$. Moreover, if $k > 0$, since z_{n+i+j} has to contain z_n , $\pi_{13}(C)$ cannot contain \bar{I}_k , and thus $\dim \pi_{13}(C) < 2n + i + j$. Since π_{13} is proper we deduce that $\pi_{13,*}[C] = 0$ in this case.

Let us now assume that $\dim \pi_{13}(C) = 2n + i + j$. We thus have $k = 0$ and $\dim C = 2n + i + j$. For a generic element $(z_n, z_{n+i}, z_{n+i+j})$ in C , $(z_{n+i+j})_\Delta$ has length $i + j$, thus we have a well-defined rational map $C \dashrightarrow S_\Delta^{[i+j]}$, with $2n$ -dimensional fibers. Let D be the closure of the image of this rational map. Since $\dim D = i + j$, D is a component of $S_\Delta^{[i+j]}$; let us denote λ the partition such that $D = S_{\Delta,\lambda}^{[i+j]}$. By definition of I , λ must be dominated by the partition $(j, 1^i)$. It is clear that D can contain $S_{\Delta,\mu}^{[i+j]}$ only if $\mu = (j, 1^i)$ or $\mu = (j+1, 1^{i-1})$. Therefore I has exactly two components which are not contracted by π_{13} .

To describe these components let us consider some subschemes $z_n, x_{i-1}, x_i, p_j, p_{j+1}$ satisfying the following conditions. The lengths of these subschemes are given by their indices. The support of z_n does not meet Δ , whereas the other subschemes have support included in Δ . The subschemes p_j, p_{j+1} are punctual whereas x_{i-1} and x_i are reduced. Finally $p_j \subset p_{j+1}$, $x_{i-1} \subset x_i$,

and the support of p_{j+1} is not included in x_i . With these conditions let I_1 resp. I_2 be the closure of the set of triples $(z_n, z_{n+i}, z_{n+i+j})$ where $z_{n+i} = z_n \amalg x_i$ and $z_{n+i+j} = z_n \amalg x_i \amalg p_j$ resp. $z_{n+i+j} = z_n \amalg x_{i-1} \amalg p_{j+1}$. The restriction of π_{13} to I_2 is birational with image $Q_{i-1, j+1, X}(-U)$. We have $\pi_{13}(I_1) = Q_{i, j, X}(-U)$. If $j > 1$ then the restriction of π_{13} to I_1 is birational whereas if $j = 1$ it has degree $i + 1$. In view of our convention for $q_{i, 1, X}$, this proves the proposition. \square

6.2 Base change formulas

Definition 62 *If $\lambda \in \mathcal{P}_n$ is a partition, we define the operators $q_\lambda = \prod_{i \in \lambda} q_i$, $q_{\lambda, X} = \prod_{i \in \lambda} q_{i, X}$, and the constant $z_\lambda = \prod \lambda_i \prod (\lambda_i^\vee - \lambda_{i+1}^\vee)!$. Let $j \in \lambda$. With the notation with multiplicity $\lambda = (1^{\alpha_1}, \dots, r^{\alpha_r})$, we denote by $\lambda \setminus j$ the partition $(1^{\alpha_1}, \dots, j^{\alpha_j-1}, \dots, r^{\alpha_r})$, with multiplicity one less for j . We let $t_\lambda = \sum_{j \in \lambda} \frac{j}{(\alpha_j-1)!} \frac{(l(\lambda)-1)!}{\prod_{i \neq j} (\alpha_i)!}$ and $u_\lambda = \prod_i (\lambda_i^\vee - \lambda_{i+1}^\vee)!$.*

By definition of q_λ and $q_{\lambda, X}$, the base change formulas from q_λ to $q_{\lambda, X}$ are determined by the decomposition of q_n in terms of the operators $q_{\lambda, X}$ and similarly for the inverse base change. In particular, the following theorem gives a full base change at the level of operators. Since $\text{es}(\lambda) = \frac{1}{u_\lambda} q_{\lambda, X}(\phi)$ and since $\text{nak}(\lambda) = q_\lambda(\phi)$ the theorem applied to the vacuum also yields the corresponding base changes between $\text{es}(\lambda)$ and $\text{nak}(\lambda)$.

Theorem 63

$$q_{i, X} = (-1)^{i+1} \sum_{|\lambda|=i} z_\lambda^{-1} U^{l(\lambda)-1} q_\lambda$$

$$q_i = (-1)^{i+1} \sum_{|\lambda|=i} t_\lambda U^{l(\lambda)-1} q_{\lambda, X}$$

Proof. By induction, the case $i = 1$ being obvious.

$$\begin{aligned} i \, q_{i, X} & \stackrel{\text{Theorem 60}}{=} (-1)^{i+1} q_i + U \sum_{j=1}^{i-1} (-1)^j q_j \circ q_{i-j, X} \\ & \stackrel{\text{induction hypothesis}}{=} (-1)^{i+1} q_i + \sum_{j=1}^{i-1} (-1)^j U (-1)^{i-j+1} \sum_{|\lambda|=i-j} z_\lambda^{-1} U^{l(\lambda)-1} q_j \circ q_\lambda \\ & = (-1)^{i+1} q_i + (-1)^{i+1} \sum_{|\mu|=i, l(\mu)>1} c_\mu q_\mu \end{aligned}$$

with

$$c_\mu = \sum_{j \in \mu} z_{\mu \setminus j}^{-1} U^{l(\mu)-1}$$

Since $\sum_{j \in \mu} \frac{z_\mu}{z_{\mu \setminus j}} = |\mu|$, we obtain $c_\mu = |\mu| (-1)^{|\mu|+1} z_\mu^{-1} U^{l(\mu)-1}$, as required for the induction.

The proof of the second formula is similar : the difficulty is to guess the formula for q_i , then the induction is straightforward. Indeed, we start with the formula of theorem 60 $(-1)^{i+1} q_i = -i q_{i, X} + U \sum_{j=1}^{i-1} (-1)^j q_j \circ q_{i-j, X}$, and we replace q_j on the right hand side by the induction

formula. With the value of t_λ in the definition above and the the formula for q_i , the induction follows. □

We now project the previous theorem from the equivariant Chow ring to the classical Chow ring. All the constructions made so far in the equivariant setting can be realized in the classical setting. We denote by q_n^{cla} and $q_{n,X}^{cla}$ the corresponding operators on the classical Chow ring. Similarly, we denote by $\text{nak}^{cla}(\lambda)$ and $\text{es}^{cla}(\lambda)$ the bases of the classical Chow ring induced by these operators.

Theorem 64

$$\begin{aligned} q_n^{cla} &= (-1)^{n+1} n q_{n,X}^{cla} \\ \text{nak}^{cla}(\lambda) &= (-1)^{|\lambda|+l(\lambda)} \left(\prod_{i \in \lambda} i\right) \text{es}^{cla}(\lambda) \end{aligned}$$

Proof. In the classical setting, $U = 0$ and the first formula for the operators is the projection of the corresponding formula in the equivariant setting. Applying the operators to the vacuum yields the second formula. □

References

[BB76] Bialynicki-Birula, A., *Some properties of the decompositions of algebraic varieties determined by actions of a torus*. Bull. Acad. Pol. Sci. Math. astron. Phys. **24**, (No. 9) 667-674 (1976)

[Bri77] Briancon, J., *Description de HilbⁿCx, y*. Invent. Math. **41** (1977), no. 1, 45-89

[Bri97] Brion, M. *Equivariant Chow groups for torus actions*. Transform. Groups **2** (1997), no. 3, 225-267

[Che98] Cheah, J. *Cellular decompositions for nested Hilbert schemes of points*. Pacific J. Math. **183** (1998), no. 1, 39-90

[EG98] Edidin, D., Graham, W., *Localization in equivariant intersection theory and the Bott residue formula*, Amer. J. Math. **120** (1998), no. 3, 619-636

[ES87] Ellingsrud, G; Stromme, S *On the homology of the Hilbert scheme of points in the plane*. Invent. Math. **87** (1987), no. 2, 343-352

[Ev04] Evain, L., *Irreducible components of the equivariant punctual Hilbert schemes*. Adv. Math. **185** (2004), no. 2, 328–346

[Ev07] Evain, L., *The Chow ring of punctual Hilbert schemes on toric surfaces*. Transform. Groups **12** (2007), no. 2, 227–249.

[Fu98] Fulton, W., *Intersection Theory*. Springer Verlag, (1998)

[GL80] Gaffney, T., Lazarsfeld, R. *On the ramification of branched coverings of Pⁿ*. Invent. Math. **59** (1980), no. 1, 53-58.

- [Laz80] Lazarsfeld, R. *Branched coverings of projective space* PhD Thesis, Brown University, 1980
- [Gro61] Grothendieck, A. *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*. Séminaire Bourbaki, Vol. 6, Exp. No. 221, 249-276, Soc. Math. France, Paris, 1995.
- [Le99] Lehn, M. *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*. Invent. Math. **136** (1999), no. 1, 157-207.
- [Na99] Nakajima, H. *Lectures on Hilbert schemes of points on surfaces*. American Mathematical Society, 1999.
- [SV09] Schiffmann, O., Vasserot, E., *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of \mathbb{A}^2* , arXiv:0905.2555
- [Va01] Vasserot, E. *Sur l'anneau de cohomologie du schéma de Hilbert de \mathbf{C}^2* . C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 1, 7-12.