



Sufficient conditions for the existence and asymptotic behavior of solution to a quasilinear elliptic problem

Dragos-Patru Covei

► To cite this version:

Dragos-Patru Covei. Sufficient conditions for the existence and asymptotic behavior of solution to a quasilinear elliptic problem. *International Journal of Computer Mathematics*, 2011, 88 (12), pp.2637-2647. <10.1080/00207160.2011.554539>. <hal-00712745>

HAL Id: hal-00712745

<https://hal.science/hal-00712745v1>

Submitted on 28 Jun 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization



Sufficient conditions for the existence and asymptotic behavior of solution to a quasilinear elliptic problem

Journal:	<i>International Journal of Computer Mathematics</i>
Manuscript ID:	GCOM-2010-0113-B.R3
Manuscript Type:	Original Article
Date Submitted by the Author:	15-Dec-2010
Complete List of Authors:	Covei, Dragos-Patru; Constantin Brancusi University of Tg-Jiu, Romania, Department of Informatics and Applied Sciences
Keywords:	35J25, 35J60, 35J62, 35J75, 35J66

SCHOLARONE™
Manuscripts

RESEARCH ARTICLE

Sufficient conditions for the existence and asymptotic behavior of solution to a quasilinear elliptic problem

Dragos-Patru Covei^{a,b*}

^a*Department of Informatics and Applied Sciences, Constantin Brancusi University of Tg-Jiu, Bld. Republicii, No 1, 210152, Romania*

^b*Department of Mathematics, West University of Timisoara, C. Coposu Blvd. Number 4, 300223, Timisoara, Romania*

(v3.6 released February 2010)

In this article we combine the existing regularity theory, perturbation method and lower and upper solutions method to study the existence and asymptotic behavior of positive solution to a boundary value problem for the p -Laplacian operator. More exactly, we study the existence and asymptotic behavior of the positive solution to a quasilinear elliptic problem of the form $-\Delta_p u = \lambda a(x)g(u)$ in $D'(\Omega)$, $u > 0$ in Ω , $\lim_{x \rightarrow \partial\Omega} u(x) = 0$. Under some conditions on a and g , we show that there is a non-negative number Λ_0 such that for all $\lambda \in (0, \Lambda_0]$, the problem has a solution u_λ in the sense of distribution, which is bounded from above by some positive numbers $\mu(\lambda)$. Such estimates and the asymptotic behavior are important in computer programs when we know an algorithm for determining the solution.

Keywords: Quasilinear problem; Existence; Asymptotic behavior.

AMS Subject Classification: 35J25, 35J60, 35J62, 35J75, 35J66.

1. Introduction

In this article we study the existence and asymptotic behavior of the solutions for the following quasilinear elliptic problem

$$-\Delta_p u = \lambda a(x)g(u) \text{ in } D'(\Omega), u > 0 \text{ in } \Omega, \lim_{x \rightarrow \partial\Omega} u(x) = 0, \quad (1)$$

where $1 < p < \infty$, $\lambda > 0$ is a parameter, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator of u , $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$.

A solution of the problem (1) will be a function $u \in C^1(\Omega)$ which satisfies

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \int_{\Omega} \lambda a(x)g(u)\varphi dx, \varphi \in C_0^\infty(\Omega), \quad (2)$$

$$u(x) > 0, \text{ for all } x \in \Omega \quad (3)$$

$$\lim_{x \rightarrow \partial\Omega} u(x) = 0. \quad (4)$$

*Corresponding author. Email: coveid@yahoo.com

Such problems has been extensively studied for both bounded or unbounded domains: Ye-Zhou [18], Goncalves-Santos [9], the author [2], Hai-Wang [8], Chai-Zhao [1], Goncalves-Silva [10] and references therein. Our study is motivated by the recent works of [1, 2, 9, 10] where the existence, non-existence and asymptotic behavior of solutions for the problem (1) are solved in $\Omega \subseteq \mathbb{R}^N$.

For instance, when $\Omega \subset \mathbb{R}^N$ in [2] the problem (1) has been considered under the assumptions that $\lambda = 1$, the anisotropic potential function $a : \overline{\Omega} \rightarrow \mathbb{R}$ satisfies the conditions

- (a1) $a \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$;
- (a2) $a(x) > 0$ for all $x \in \overline{\Omega}$,

and that the non-linearity $g : (0, \infty) \rightarrow (0, \infty)$ is a C^1 function such that

- (g1) $t \mapsto \frac{g(t)}{t^{p-1}}$ is decreasing on $(0, \infty)$;
- (g2) $\lim_{t \searrow 0} g(t) = \tau_0$ and $\lim_{t \nearrow \infty} \frac{g(t)}{t^{p-1}} = \tau_\infty$

where $0 \leq \tau_\infty < \infty$ and $0 < \tau_0 \leq \infty$.

The above result has been generalized by Chai-Zhao [1] and Miao-Yang [19] for more general class of functions. After these results Goncalves-Silva [10] deals with more general conditions for the function g but in the case $\Omega \subset \mathbb{R}^N$ they didn't obtain the property (4) for the solution u . In other words the function u is not an entire solution.

Inspired by the above mentioned works we study, in a natural way, the existence and asymptotic behavior of solution to the problem (1). By construction of a suitable lower and upper solution, we intent in this paper to discover more ideas and techniques that in [1], [2] in order to open the access for more general class of function as in [10] in any situation $\Omega \subseteq \mathbb{R}^N$.

Our main result is the following:

THEOREM 1.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Under hypotheses (a1), (a2), (g1) and (g2) there is a non-negative number Λ_0 such that for all $\lambda \in (0, \Lambda_0]$ there is a positive number $\mu := \mu(\lambda)$ and some functions $u := u_\lambda \in C^1(\Omega) \cap L^\infty(\Omega)$ with $u \leq \mu$ satisfying (1) in the sense of distributions. Moreover, there exists constant $l_1 > 0$ such that the solution u satisfies*

$$u(x)^p [g(u(x)/4)2^{p-1} + 1]^{-1/(p-1)} \leq \lambda^{-\frac{1}{p-1}} l_1 \mu^{p-1} \cdot 2^{[(p-1)^2+p]/(p-1)} d(x), \quad x \in \overline{\Omega}, \quad (5)$$

where $d(x)$ denote the distance from $x \in \Omega$ to the boundary $\partial\Omega$.

Our main theorem apply to the class of functions

$$g(u) = u^{-\gamma} + u^\lambda + \tau_\infty u^{p-1}, \quad \gamma > 0 \text{ and } 0 < \lambda < p-1$$

where

$$\lim_{t \searrow 0} g(t) = \infty \text{ and } \lim_{t \nearrow \infty} \frac{g(t)}{t^{p-1}} = \tau_\infty.$$

The result of existence and asymptotic behavior is a generalization of the work [2] in the whole space done for the p-Laplacian equation and of the work [9] done in bounded sets for the Laplace problem. To prove the main result we use the approach of lower and upper solutions employed in [10] and the results obtained

in [3] with some adaptation in the estimates. The key tool is an inequality for the p -Laplacian developed by Diaz-Saà in the 1987's (see reference [6]).

The rest of this paper is organized as follows: in Section 2, we collect several technical lemmas. Section 3 contains the proof of our main result.

2. Preliminary results

We present some results about the eigenvalues problem for the p -Laplacian.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary.

DEFINITION 2.1 *We will say that $\xi \in \mathbb{R}$ is an eigenvalue, if there exists a function $\omega \in W^{1,p}(\Omega)$, $\omega \neq 0$, such that $\omega = 0$ on $\partial\Omega$ and*

$$\int_{\Omega} |\nabla \omega(x)|^{p-2} \nabla \omega(x) \nabla \eta(x) dx = \int_{\Omega} \xi |\omega(x)|^{p-2} \omega(x) \eta(x) dx, \quad (6)$$

for any $\eta \in C_0^\infty(\Omega)$.

DEFINITION 2.2 *The function ω which appears in (6) is called the eigenfunction corresponding to the eigenvalue ξ .*

A first observation about problem (6) is:

Remark 1 If (ω, ξ) is a solution of the problem (6) then for any $\alpha \in \mathbb{R}$ the couple $(\alpha\omega, \xi)$ is the same a solution.

For $1 < p < \infty$ the existence of ξ and ω has been first considered by Garcia Azorero and Peral Alonzo [15]. They showed that there exists an increasing subset $\{\xi_k\}_{k \geq 1}$ of eigenvalue which are strictly positive and such that $\xi_k \rightarrow \infty$ for $k \rightarrow \infty$.

It is well-known that the first eigenvalue which appears in (6) is

$$\xi_1 := \inf \left\{ \frac{\int_{\Omega} |\nabla \omega|^p dx}{\int_{\Omega} |\omega|^p dx} \mid \omega \in W_0^{1,p}(\Omega), \omega \neq 0 \right\} > 0,$$

and that the eigenfunction corresponding to ξ_1 is the minimum of the Euler functional

$$J(\omega) := \frac{1}{p} \int_{\Omega} |\nabla \omega|^p - \frac{\xi_1}{p} \int_{\Omega} |\omega|^p.$$

Lindqvist [12, 13] have obtained the following characterization for the eigenvalue ξ_1 and for the corresponding eigenfunctions:

LEMMA 2.3 *In any bounded domain Ω of \mathbb{R}^N ($N \geq 2$) we have:*

- i) ξ_1 is simple in the sense that if ω_1, ω_2 are the eigenfunctions corresponding to the eigenvalue ξ_1 then there exists a parameter c such that $\omega_1 = c\omega_2$;
- ii) the eigenfunctions corresponding to ξ_1 has a constant sign in the sense that $\omega > 0$ or $\omega < 0$;
- iii) ξ_1 is isolated in the sense that there exists $\kappa > 0$ such that in the interval $(\xi_1, \xi_1 + \kappa)$ there are no other eigenfunctions of (6);
- iv) the non negative eigenfunction corresponding to ξ_1 is unique (after multiplying by a constant).

In the article we note by $\omega_1(\Omega)$ the eigenfunction corresponding to the eigenvalue $\xi_1(\Omega)$ for the problem (6). The $C^{1,\alpha}$ -regularity of the eigenfunctions corresponding to the eigenvalue ξ_1 for the problem (6) is given in the following:

LEMMA 2.4 If $\xi := \xi_1$ in (6) then there exists $\alpha \in (0, 1)$ such that $\omega_1 \in C^{1,\alpha}(\bar{\Omega})$.

The following result can be found in more general form in Diaz-Saà [6] and is the main tool in the proof of the main result.

LEMMA 2.5 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that $G : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is such that:

H1) for a.e. $x \in \Omega$, the function $u \mapsto G(x, u)$ is continuous on $[0, \infty)$ and the function $u \mapsto G(x, u)/u^{p-1}$ is decreasing on $(0, \infty)$;

H2) for all $u \geq 0$, the function $x \mapsto G(x, u)$ belong to $L^\infty(\Omega)$;

H3) $\exists C > 0$ such that $G(x, u) \leq C(u^{p-1} + 1)$ a.e. $x \in \Omega$, $\forall u \geq 0$.

Under these hypotheses, the problem

$$-\Delta_p u = G(x, u), \quad u \geq 0, \quad u \not\equiv 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (7)$$

has almost one solution. If, in addition,

$$a_0(x) = \lim_{s \searrow 0} \frac{G(x, s)}{s^{p-1}} \text{ and } a_\infty(x) = \lim_{s \nearrow \infty} \frac{G(x, s)}{s^{p-1}},$$

is such that

$$-\infty < a_0(x) \leq +\infty \text{ and } -\infty \leq a_\infty(x) < +\infty \quad (8)$$

then the problem (7) has a unique solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover $u \in C^{1,\alpha}(\bar{\Omega})$.

We restate the well-known result of Diaz-Saà which can be found in [6].

LEMMA 2.6 Let Ω be an open set in \mathbb{R}^N . For $i = 1, 2$ let $w_i \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $w_i > 0$ a.e. in Ω , $\Delta_p w_i^{1/p} \in L^\infty(\Omega)$ and $w_1 = w_2$ on $\partial\Omega$. If $w_i/w_j \in L^\infty(\Omega)$ ($i \neq j$, $i, j = 1, 2$) then

$$\int_{\Omega} \left(\frac{-\Delta_p w_1^{1/p}}{w_1^{(p-1)/p}} + \frac{\Delta_p w_2^{1/p}}{w_2^{(p-1)/p}} \right) (w_1 - w_2) dx \geq 0.$$

The next result can be found in more general form in the article of DiBenedetto [5] (see also Tolksdorf [16]).

LEMMA 2.7 Let $N \geq 2$, $p \in (1, \infty)$, Ω be an open set in \mathbb{R}^N and Ω' a subset of Ω such that $\overline{\Omega'} \subset \Omega$. Assume that $h : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function (i.e. measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}$) and that there exists $\gamma > 0$ such that $|h(x, u)| \leq \gamma$. If $u \in W_{loc}^{1,p}(\Omega) \cap L_{loc}^\infty(\Omega)$ satisfies

$$-\Delta_p u = h(x, u) \text{ in } D'(\Omega),$$

then $|\nabla u| \in L_{loc}^\infty(\Omega)$ and for any compact subset $K \subset \Omega'$, there exists $\alpha \in (0, 1)$ and the parameters $C_0, C_1 > 0$ depending only upon $N, p, M = \text{ess sup}_{\Omega'} |u|$, $\gamma(M)$ and $\text{dist}(K, \partial\Omega')$, such that

$$\|\nabla u\|_{L^\infty(K)} \leq C_0 \text{ and } |u_{x_i}(x) - u_{x_i}(y)| \leq C_1 |x - y|^\alpha, \quad x, y \in K, i = \overline{1, N}. \quad (9)$$

The following result can be found in [7].

LEMMA 2.8 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. If (a1) and (a2) hold, then

$$-\Delta_p w_1 = a(x) \text{ in } D'(\Omega), \quad w_1 > 0 \text{ in } \Omega, \quad w_1 = 0 \text{ on } \partial\Omega, \quad (10)$$

has a unique solution $w_1 \in C^{1,\alpha}(\overline{\Omega})$. Moreover, there exist constants $l_1 \geq k_1 > 0$, such that $k_1 d(x) \leq w_1(x) \leq l_1 d(x)$ on $\overline{\Omega}$ where $d(x)$ denote the distance from $x \in \Omega$ to the boundary $\partial\Omega$.

The following lemma can be found in [11].

LEMMA 2.9 Let Ω be a domain in \mathbb{R}^N (but otherwise arbitrary, in particular, not necessarily smooth or bounded). Then for every $k \in \mathbb{N}$ there exist bounded domains Ω_k such that

i) the boundary of Ω_k is of class C^∞ ; ii) $\overline{\Omega}_k \subset \Omega_{k+1} \subset \Omega$; iii) $\bigcup_{k \in \mathbb{N}} \Omega_k = \Omega$.

The first result of this section is the following.

LEMMA 2.10 Make the same assumption on a and Ω as in Lemma 2.8. If in addition (g1) and (g2) hold, then there exists number $\Lambda_0 \in (0, \infty)$ such that for each $\lambda \in (0, \Lambda_0]$ there are a positive number $\mu := \mu(\lambda)$ and a function $v^l := v_\lambda^l \in C^1(\overline{\Omega})$ with $\max_{x \in \overline{\Omega}} v^l(x) = \mu$ satisfying

$$-\Delta_p v^l \geq \lambda a(x) g(v^l) \text{ in } D'(\Omega), \quad v^l > l \text{ in } \Omega, \quad v^l = l \text{ on } \partial\Omega, \quad (11)$$

where $l \geq 0$ is a parameter.

Proof. Let $w_1 \in C^{1,\alpha}(\overline{\Omega})$ be the unique solution of the problem (10) and w a function defined by

$$w(x) := \lambda^{\frac{1}{p-1}} w_1(x) \text{ for all } x \in \overline{\Omega}. \quad (12)$$

Set

$$M := \max_{x \in \overline{\Omega}} w_1(x), \text{ and } \Lambda_0 = \left(\frac{1}{M(p-1)} \right)^{p-1} \frac{1}{\tau_\infty}.$$

Let $\lambda \in (0, \Lambda_0)$. Consider first the function

$$H(y+l) = (g(y+l) + 1)^{1/(p-1)} \text{ for } y > 0 \text{ and } l \geq 0.$$

Note that

$$\begin{aligned} H1) H(y+l) &\geq [g(y+l)]^{1/(p-1)}; & H2) \lim_{y \rightarrow 0+} H(y+l)/y &= \infty; \\ H3) \lim_{y \rightarrow \infty} H(y+l)/y &= (\tau_\infty)^{1/(p-1)}; & H4) y \mapsto H(y+l)/y^{p-1} & \text{ is decreasing on } (0, \infty). \end{aligned}$$

Consider the continuous function

$$\Pi(y) := \frac{1}{My^{p-1}} \int_0^{y-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma, \quad y > l,$$

and observe that

$$\lim_{y \rightarrow \infty} \frac{1}{My^{p-1}} \int_0^{y-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma = \lim_{y \rightarrow \infty} \frac{(y-l)^{p-1}}{M(p-1)y^{p-2}H(y)} = \frac{1}{(p-1)M} \left(\frac{1}{\tau_\infty} \right)^{\frac{1}{p-1}} \quad (13)$$

and

$$\lim_{y \rightarrow l} \frac{1}{My^{p-1}} \int_0^{y-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma = 0. \quad (14)$$

From (13) and (14) one easily deduces that, for every object $\lambda \in (0, \Lambda_0]$ there is some $\mu \in (l, \infty)$ with the property $\Pi(\mu) = \lambda^{1/(p-1)}$, which is equivalent to

$$\frac{1}{\mu^{p-1}} \int_0^{\mu-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma = M\lambda^{\frac{1}{p-1}}. \quad (15)$$

Introduce the function $I: \mathbb{R}^N \times [l, \mu] \rightarrow \mathbb{R}$ by

$$I(x, s) = w(x) - \frac{1}{\mu^{p-1}} \int_0^{s-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma.$$

Then

$$\frac{\partial I(x, s)}{\partial s} = -\frac{1}{\mu^{p-1}} \frac{(s-l)^{p-1}}{H(s)} < 0, \quad s > l. \quad (16)$$

Now, as an application of (16) there is a function $v^l: \bar{\Omega} \rightarrow [l, \mu]$ of class $C^{1,\alpha}(\bar{\Omega})$ such that $I(x, v^l(x)) = 0$ for all $x \in \bar{\Omega}$ and so

$$w(x) = \frac{1}{\mu^{p-1}} \int_0^{v^l(x)-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma, \quad (17)$$

is well defined.

Moreover for

$$\Gamma(t) = \int_0^t \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma$$

we also have

$$v^l(x) = \Gamma^{-1}(\mu^{p-1}w(x)) + l, \quad (18)$$

where Γ^{-1} is the inverse function of Γ and from Lemma 2.8 that

$$\Gamma^{-1}\left(k_1 \lambda^{\frac{1}{p-1}} \mu^{p-1} d(x)\right) \leq v^l(x) - l \leq \Gamma^{-1}\left(l_1 \lambda^{\frac{1}{p-1}} \mu^{p-1} d(x)\right) \quad \text{on } \bar{\Omega}. \quad (19)$$

Letting $x \rightarrow \partial\Omega$ in (19) follows $v^l(x) = l$ on $\partial\Omega$. Of course, we still need to prove that

$$\max_{x \in \bar{\Omega}} v^l(x) = \mu \quad \text{for all } x \in \bar{\Omega}. \quad (20)$$

By (12) we get in the point x_0 , where the maximum is achieved that

$$w(x_0) = \lambda^{\frac{1}{p-1}} M = \frac{1}{\mu^{p-1}} \int_0^{\mu-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma \geq w(x) = \frac{1}{\mu^{p-1}} \int_0^{v^l(x)-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma,$$

and

$$I(x_0, \mu) = w(x_0) - \frac{1}{\mu^{p-1}} \int_0^{\mu-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma = 0.$$

Then $v^l(x) \leq \mu$ for all $x \in \bar{\Omega}$ and so $v^l(x_0) = \mu$.

It remains to show that $v^l(x)$ constructed in (17) is an upper bound for the problem (11).

Differentiating in (17) with respect to x we obtain

$$\Delta_p w = \left[\frac{(v^l - l)^{p-1}}{\mu^{p-1} H(v^l - l)} \right]^{p-1} \Delta_p v^l + \frac{(p-1)|\nabla v^l|^p}{\mu^{2(p-1)}} \left[\frac{(v^l - l)^{p-1}}{H(v^l - l)} \right]^{p-2} \frac{\partial}{\partial v^l} \left[\frac{(v^l - l)^{p-1}}{H(v^l - l)} \right].$$

and using H4) we have

$$-\Delta_p v^l \geq - \left[\frac{\mu^{p-1} H(v^l - l)}{(v^l - l)^{p-1}} \right]^{p-1} \Delta_p w = \lambda a(x) \left[\frac{\mu^{p-1} H(v^l - l)}{(v^l - l)^{p-1}} \right]^{p-1} \geq \lambda a(x) g(v^l),$$

Then the same arguments used in [10] follows that v verifies

$$-\Delta_p v^l \geq -\lambda a(x) g(v^l) \text{ in } \Omega. \quad (21)$$

in the distributions sense and the proof is now complete. This key lemma was suggested in an early version of [10].

3. Proof of the main result

There are several new ideas in the works [1], [2], [10] and [19] which will be exploited in the next proof.

For each $\varepsilon > 0$ consider the function $G_\varepsilon : \Omega \times \mathbb{R}_+ \rightarrow (0, \infty)$ defined by

$$G_\varepsilon(x, s) := \lambda a(x) g(s + \varepsilon) \text{ if } s \geq 0$$

and the associated problem

$$-\Delta_p v = G_\varepsilon(x, v) \text{ in } \Omega, v(x) = 0 \text{ on } \partial\Omega. \quad (22)$$

In order to obtain a solution of the problem (22), it is enough to check the hypotheses of the Diaz-Saa theorem are fulfilled.

$H1$: Since $g \in C^1((0, \infty), (0, \infty))$ and $\varepsilon > 0$, it follows that the mapping $u \mapsto G_\varepsilon(x, u)$ is continuous in $[0, \infty)$ and from

$$\frac{G_\varepsilon(x, u)}{u^{p-1}} = \lambda a(x) \frac{g(u + \varepsilon)}{(u + \varepsilon)^{p-1}} \left(1 + \frac{\varepsilon}{u}\right)^{p-1},$$

using positivity of a and (g1) we deduce that the function $u \rightarrow \frac{G_\varepsilon(x,u)}{u^{p-1}}$ is decreasing on $(0, \infty)$.

$H2$: For all $u \geq 0$, since $a \in C^{0,\alpha}(\overline{\Omega})$, we obtain $x \mapsto a(x)g(u)$ belongs to $L^\infty(\Omega)$.

$H3$: By $\lim_{u \rightarrow \infty} \frac{g(u+\varepsilon)}{u^{p-1}+1} = \lim_{u \rightarrow \infty} \frac{g(u+\varepsilon)}{(u+\varepsilon)^{p-1}} \cdot \frac{(u+\varepsilon)^{p-1}}{u^{p-1}+1} = \tau_\infty$, $a \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^1((0, \infty), (0, \infty))$, there exists $C > 0$ such that $G_\varepsilon(x, u) \leq C(u^{p-1} + 1)$ for all $u \geq 0$.

Observe that

$$a_0(x) = \lim_{u \searrow 0} \frac{G_\varepsilon(x, u)}{u^{p-1}} = +\infty \text{ and } a_\infty(x) = \lim_{u \rightarrow +\infty} \frac{G_\varepsilon(x, u)}{u^{p-1}} = \lambda a(x) \tau_\infty.$$

Thus by results of Diaz-Saà, problem (22) has a unique positive solution $u_\varepsilon \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ such that

$$\begin{aligned} \int_{\Omega} |\nabla u_\varepsilon(x)|^{p-2} \nabla u_\varepsilon \nabla \varphi dx &= \int_{\Omega} G_\varepsilon(x, u) \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega), \\ u_\varepsilon &> 0 \text{ in } \Omega, \\ u_\varepsilon &= 0 \text{ on } \partial\Omega. \end{aligned} \quad (23)$$

We will start by producing a lower solution of the equation (23). To this end, pick $\lambda \in (0, \Lambda_0]$. For simplicity, we denote by $\xi_1(\lambda) := \xi_1(\lambda, \Omega)$ and $\omega_{1,\lambda} := \omega_1(\lambda, \Omega)$.

Notice that under this notation we have

$$-\Delta_p \omega_{1,\lambda} = \xi_1(\lambda) \lambda \omega_{1,\lambda} |\omega_{1,\lambda}|^{p-2} \text{ in } \Omega, \quad \omega_{1,\lambda} > 0 \text{ in } \Omega, \quad \omega_{1,\lambda} = 0 \text{ on } \partial\Omega.$$

Since $\lim_{s \rightarrow 0+} g(s)/s^{p-1} = +\infty$ and $a \in C^\alpha(\overline{\Omega})$ with $a > 0$ on $\overline{\Omega}$ there exists $\delta := \delta(\lambda) > 0$ such that

$$\frac{g(s)}{s^{p-1}} \geq \frac{\xi_1(\lambda)}{\min_{x \in \overline{\Omega}} a(x)} \quad \forall s \in (0, \delta). \quad (24)$$

Let $\underline{u}_1(x) = c\omega_{1,\lambda}(x)$ where c is a constant chosen such that

$$0 < c < \frac{\delta}{\max_{x \in \overline{\Omega}} \omega_{1,\lambda}(x)}.$$

Then $\underline{u} = \underline{u}_1 - \varepsilon$ satisfies

$$\begin{aligned} -\Delta_p \underline{u} &= -\Delta_p \underline{u}_1 = \xi_1(\lambda) \lambda (c\omega_{1,\lambda})^{p-1} \\ &\leq \lambda a(x) g(\underline{u}_1) = \lambda a(x) g(\underline{u} + \varepsilon) = G_\varepsilon(x, \underline{u}) \text{ in } D'(\Omega), \end{aligned}$$

i.e. $\underline{u}(x)$ is a lower solution of (23).

We see by Lemma 2.10 that there exists at least one solution $\overline{u}_\varepsilon(x)$ corresponding to $l = \varepsilon$ which satisfies (11) in the sense of distributions that means

$$\begin{aligned} \int_{\Omega} |\nabla \overline{u}_\varepsilon(x)|^{p-2} \nabla \overline{u}_\varepsilon(x) \nabla \varphi dx &\geq \int_{\Omega} \lambda a(x) g(\overline{u}_\varepsilon(x)) \varphi dx, \quad \varphi \in C_0^\infty(\Omega) \\ \overline{u}_\varepsilon(x) &> \varepsilon \text{ in } \Omega, \\ \overline{u}_\varepsilon(x) &= \varepsilon \text{ on } \partial\Omega. \end{aligned} \quad (25)$$

Taking $\varepsilon > 0$ small enough, recalling that $\overline{u_\varepsilon}(x) = \varepsilon > 0$ on $\partial\Omega$ and setting $\overline{u}(x) = \overline{u_\varepsilon}(x) - \varepsilon$ we have

$$-\Delta_p \overline{u}(x) = -\Delta_p \overline{u_\varepsilon}(x) \geq \lambda a(x) g(\overline{u_\varepsilon}(x)) = G_\varepsilon(x, \overline{u}(x)) \text{ in } D'(\Omega),$$

showing $\overline{u}(x)$ is an upper solution of (22).

We note here that $\overline{u_\varepsilon}(x)$ is a suitable upper solution chosen to meet the lower and upper solution method for singular problems developed recently by Goncalves and Santos [10]. The upper solution suggest the study of (1) for more general class of functions as in [3], [10].

Set $B_{\underline{u}_1, \overline{u_\varepsilon}} = \{x \in \Omega \mid \underline{u}_1(x) > \overline{u_\varepsilon}(x)\}$. We show that $B_{\underline{u}_1, \overline{u_\varepsilon}} = \emptyset$. To this end, notice that $B_{\underline{u}_1, \overline{u_\varepsilon}} \subset \Omega$ is open and, in fact, $B_{\underline{u}_1, \overline{u_\varepsilon}} \subset \subset \Omega$. Assume, to the contrary, that $B_{\underline{u}_1, \overline{u_\varepsilon}} \neq \emptyset$. Let $w_1 := (\underline{u}_1)^p$ and $w_2 := (\overline{u_\varepsilon})^p$.

Now, applying Diaz-Saa's inequality we have

$$\begin{aligned} 0 &\leq \int_{B_{\underline{u}_1, \overline{u_\varepsilon}}} \left[\frac{-\Delta_p w_1^{1/p}}{w_1^{(p-1)/p}} + \frac{\Delta_p w_2^{1/p}}{w_2^{(p-1)/p}} \right] (w_1 - w_2) dx \\ &= \int_{B_{\underline{u}_1, \overline{u_\varepsilon}}} \left[\frac{-\Delta_p \underline{u}_1(x)}{(\underline{u}_1(x))^{p-1}} + \frac{\Delta_p \overline{u_\varepsilon}(x)}{(\overline{u_\varepsilon}(x))^{p-1}} \right] (w_1 - w_2) dx \\ &\leq \int_{B_{\underline{u}_1, \overline{u_\varepsilon}}} \lambda \left[\frac{\xi_1(\lambda) \underline{u}_1^{p-1}}{\underline{u}_1^{p-1}} - a(x) \frac{g(\overline{u_\varepsilon}(x))}{\overline{u_\varepsilon}^{p-1}} \right] (w_1 - w_2) dx \\ &\leq \int_{B_{\underline{u}_1, \overline{u_\varepsilon}}} \lambda \left[\xi_1(\lambda) - a(x) \frac{g(\underline{u}_1(x))}{\underline{u}_1^{p-1}} \right] (w_1 - w_2) dx < 0, \end{aligned}$$

which is impossible. This shows that $B_{\underline{u}_1, \overline{u_\varepsilon}} = \emptyset$, proving the claim. Hence $\underline{u}(x) \leq \overline{u}(x)$ in $\overline{\Omega}$.

Moreover, we can easy prove that

$$\underline{u}(x) < u_\varepsilon(x) \leq \overline{u}(x) \text{ in } \overline{\Omega}. \quad (26)$$

Now, due to Lemma 2.9 we can approximate Ω with smooth bounded domains Ω_k such that

$$\overline{\Omega}_k \subset \Omega_{k+1} \subset \Omega \text{ for all } k \in \mathbb{N} \text{ and } \Omega = \cup_{k=1}^{\infty} \Omega_k.$$

Taking an integer $k \geq 1$ such that $\text{supp}(\varphi) \subset \Omega_k$ we have by (23)

$$\begin{aligned} \int_{\Omega_k} |\nabla u_\varepsilon(x)|^{p-2} \nabla u_\varepsilon(x) \nabla \varphi(x) dx &= \int_{\Omega_k} \lambda a(x) g(u_\varepsilon(x) + \varepsilon) \varphi dx, \\ u_\varepsilon &> 0 \text{ in } \Omega_k, \end{aligned} \quad (27)$$

By Lemma 2.7 we see that there are $\alpha \in (0, 1)$ and the constants $C_0, C_1 > 0$ such that

$$|\nabla u_\varepsilon(x)|_{L^\infty} \leq C_0 \text{ and } |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| \leq C |x - y|^\alpha, \quad x, y \in \overline{\Omega}_k \quad k = 1, 2, \dots$$

Set $\varepsilon := 1/n$ and $u_\varepsilon := u^n$. It follows that $\{u^n\}$ is bounded in $C^{1,\alpha}(\overline{\Omega}_1)$.

Now, since the embedding

$$C^{1,\alpha}(\overline{\Omega}_1) \hookrightarrow C^1(\overline{\Omega}_1)$$

is compact there exists a sequence denoted by $\{u_1^{n_{1,j}}\}_{j=1,2,\dots}$ where $n_{1,j} \rightarrow \infty$ as $j \rightarrow \infty$ such that

$$u_1^{n_{1,j}} \xrightarrow{n_{1,j} \rightarrow \infty} u_1 \text{ in } C^1(\overline{\Omega}_1) \text{ and } u_1 > 0 \text{ in } \overline{\Omega}_1.$$

Hence

$$|\nabla u_1^{n_{1,j}}|^{p-2} \nabla u_1^{n_{1,j}} \xrightarrow{n_{1,j} \rightarrow \infty} |\nabla u_1|^{p-2} \nabla u_1 \text{ in } C(\overline{\Omega}_1).$$

We have by the standard integral convergence theorems, that

$$\int_{\Omega_1} |\nabla u_1^{n_{1,j}}|^{p-2} \nabla u_1^{n_{1,j}} \nabla \varphi(x) dx \xrightarrow{n_{1,j} \rightarrow \infty} \int_{\Omega_1} |\nabla u_1|^{p-2} \nabla u_1 \nabla \varphi(x) dx,$$

and

$$\int_{\Omega_1} \lambda a(x) g(u_1^{n_{1,j}} + 1/n_{1,j}) \varphi dx \xrightarrow{n_{1,j} \rightarrow \infty} \int_{\Omega_1} \lambda a(x) g(u_1) \varphi dx.$$

Then u_1 satisfies (2)-(3) with $\Omega = \Omega_1$.

Repeat the above proof up to the existence of the sequence $\{u_1^{n_{1,j}}\}_{j=1,2,\dots}$ to get a subsequence $\{u_2^{n_{2,j}}\}_{j=1,2,\dots}$ such that

$$u_2^{n_{2,j}} \xrightarrow{n_{2,j} \rightarrow \infty} u_2 \text{ in } C^1(\overline{\Omega}_2) \text{ and } u_2 > 0 \text{ in } \overline{\Omega}_2.$$

Hence

$$|\nabla u_2^{n_{2,j}}|^{p-2} \nabla u_2^{n_{2,j}} \xrightarrow{n_{2,j} \rightarrow \infty} |\nabla u_2|^{p-2} \nabla u_2 \text{ in } C(\overline{\Omega}_2).$$

The likewise u_2 is a solution of (2)-(3) with $\Omega = \Omega_2$ and $u_2|_{\Omega_1} = u_1$.

Choose k large. In the same way as above, we obtain a sequence $\{u_k^{n_{k,j}}\}_{j=1,2,\dots}$ which is a subsequence of $\{u_k^{n_{(k-1),j}}\}$ such that

$$u_k^{n_{k,j}} \xrightarrow{n_{k,j} \rightarrow \infty} u_k \text{ in } C^1(\overline{\Omega}_k) \text{ and } u_k > 0 \text{ in } \overline{\Omega}_k$$

and

$$|\nabla u_k^{n_{k,j}}|^{p-2} \nabla u_k^{n_{k,j}} \xrightarrow{n_{k,j} \rightarrow \infty} |\nabla u_k|^{p-2} \nabla u_k \text{ in } C(\overline{\Omega}_k).$$

Then $u_k|_{\Omega_{k-1}} = u_{k-1}$ and u_k is a solution of (2)-(3) with $\Omega = \Omega_k$. Let $u : \Omega \rightarrow (0, \infty)$ defined by $u(x) := \lim_{k \rightarrow \infty} u_k(x)$. We have $u \in C^1(\Omega)$, $u(x) > 0$ in Ω and $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ in $\overline{\Omega}$ (since $u_k > 0$ in Ω and $\underline{u} \leq u_k \leq \overline{u}$ for every k). Hence $u|_{\partial\Omega} = 0$ because we have $\overline{u}|_{\partial\Omega} = 0$ and so u is the solution of the problem (1).

The asymptotic behavior of solution is given in what follows. Let $\overline{u}_0(x)$ the solution of (11) corresponding to $\varepsilon = 0$. From the above arguments we have

$$w(x) = \frac{1}{\mu^{p-1}} \int_0^{\overline{u}_0(x)} t^{p-1}/H(t) dt \geq \frac{1}{\mu^{p-1}} \int_{\overline{u}_0/2}^{\overline{u}_0} t^{p-1}/H(t) dt := \frac{1}{\mu^{p-1}} \int_{\overline{u}_0/2}^{\overline{u}_0} h_1(t) dt,$$

where $h_1(t) := t^{p-1}/H(t), t > 0$.

Using H1), H4) and the definition of $H(t)$ we find that

$$\begin{aligned} w(x) &\geq \frac{1}{\mu^{p-1}} h_1(\overline{u_0}(x)/2)(\overline{u_0}(x)/2) \\ &\geq \frac{1}{\mu^{p-1}} \left(\frac{1}{2}\right)^{p/(p-1)} \left(\frac{\overline{u_0}(x)}{2}\right)^p \frac{1}{[g(\overline{u_0}(x)/4)(1 + 1/g(\overline{u_0}(x)/2))]^{1/(p-1)}}. \end{aligned}$$

Using g1)

$$w(x) \geq \frac{1}{\mu^{p-1}} \left(\frac{1}{2}\right)^{p/(p-1)} \left(\frac{\overline{u_0}(x)}{4}\right)^p \frac{2^{p+1}}{[g(\overline{u_0}(x)/4)2^{p-1} + 1]^{1/(p-1)}}. \quad (28)$$

From (28), g1) and the fact that $u \leq \overline{u_0}$ we get

$$u(x)^p [g(u(x)/4)2^{p-1} + 1]^{-1/(p-1)} \leq \mu^{p-1} \cdot 2^{[(p-1)^2+p]/(p-1)} w(x) \text{ for all } x \in \overline{\Omega}. \quad (29)$$

Then is sufficient to apply Lemma 2.8 to obtain (5). This ends the proof of Theorem 1.1.

Remark 1 The case $\Omega = \mathbb{R}^N$ can be now treated as in [3], [10].

We can observe from [4] the following:

Remark 2 Let $\tau_0 \in (0, \infty]$ and $\tau_\infty \in [0, \infty)$. If $g : (0, \infty) \rightarrow (0, \infty)$ is a continuous function then the following are equivalently: i) $\lim_{s \searrow 0} g(s) = \tau_0$ and $\lim_{s \nearrow \infty} \frac{g(s)}{s^{p-1}} = \tau_\infty$;

ii) exists $\eta > 0$ such that $\lim_{s \searrow 0} \frac{g(s)}{(s+\eta)^{p-1}} = \frac{\tau_0}{\eta^{p-1}}$ and $\lim_{s \nearrow \infty} \frac{g(s)}{(s+\eta)^{p-1}} = \tau_\infty$.

Acknowledgement. The author would like to thank the referee for valuable suggestions. The work is supported by CNCSIS-UEFISCSU, project number PN II - IDEI 1080/2008 from West University of Timisoara.

References

- [1] X. Chai, W. Niu and P. Zhao, *The existence and non-existence of positive solutions to a singular quasilinear elliptic problem in R^N* , Nonlinear Analysis 71 (2009) 3257-3266.
- [2] D.P. Covei, *Existence and asymptotic behavior of positive solution to a quasilinear elliptic problem in R^N* , Nonlinear Analysis: Theory, Methods & Applications, Volume 69, Issue 8, 15 October 2008, Pages 2615-2622.
- [3] D. P. Covei, *Existence and uniqueness of solutions for the Lane, Emden and Fowler type problem*, Nonlinear Analysis: Theory, Methods & Applications, Volume 72, Issue 5, 1 March 2010, Pages 2684-2693.
- [4] D. P. Covei, *A Lane-Emden-Fowler Type Problem With Singular Nonlinearity*, Journal of Mathematics of Kyoto University, Volume 49, No. 2, 2009.
- [5] E. DiBenedetto, *$C^{1,\alpha}$ -local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis, Volume 7, Issue 8, Pages 827-850, 1983.
- [6] J. I. Diaz and J. E. Saà, *Existence et unicite de solutions positives pour certaines equations elliptiques quasilineaires*, CRAS 305 Serie I, Pages 521-524, 1987.
- [7] Z. Guo, *Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large*, Proceedings of the Royal Society of Edinburgh, 124A, 189-198, 1994.
- [8] D.D. Hai and H. Wang, *Nontrivial solutions for p-Laplacian systems*, Journal of Mathematical Analysis and applications, 330 (2007) 186-194.
- [9] J. V. Goncalves and C. A. P. Santos, *Singular elliptic problems: Existence, non-existence and boundary behavior*, Nonlinear Analysis: Theory, Methods & Applications, Volume 66, Issue 9, Pages 2078-2090, 1 May 2007.
- [10] J. V. Goncalves and F. K. Silva, *Solutions of quasilinear elliptic equations in R^N decaying at infinity to a non-negative number*, Complex Variables and Elliptic Equations, Volume 55, Issue 5 & 6 May 2010, pages 549 - 571.
- [11] S. Krömer and M. Lilli, *Branches of positive solutions of quasilinear elliptic equations on non-smooth domains*, Nonlinear Analysis, Volume 64, Issue 10, Pages 2183-2202, 15 May 2006.

- [12] P. Lindqvist, *On the equation $\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \lambda u |u|^{p-2} = 0$* , Proceedings of the American Mathematical Society, 109, Pages 157-164, 1990.
- [13] P. Lindqvist, *A nonlinear eigenvalue problem*, Lectures at the "Minicorsi di Analisi Matematica" at Padova in June 2000.
- [14] A. Le, *Eigenvalue problems for the p -Laplacian*, Nonlinear Analysis, 64 (2006) 1057-1099.
- [15] J. G. Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p -laplacian nonlinear eigenvalues*, Communications in Partial Differential Equations, Volume 12, No. 12, Pages 1389-1430, 1987.
- [16] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, Journal of Differential Equations, Volume 51, Issue 1, Pages 126-150, January 1984.
- [17] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Applied Mathematics and Optimization, 12 (1984), 191-202.
- [18] D. Ye and F. Zhou, *Invariant criteria for existence of bounded positive solutions*, Discrete and Continuous Dynamical Systems-Series A, Volume: 12, Number 3, March 2005 , Pages 413-424.
- [19] M. Qing and Z. Yang, *Quasilinear elliptic equation involving singular non-linearities*, International Journal of Computer Mathematics, Volume 87, Issue 6 May 2010, pages 1338 - 1348.

RESEARCH ARTICLE

Sufficient conditions for the existence and asymptotic behavior of solution to a quasilinear elliptic problem

Dragos-Patru Covei^{a,b*}

^a*Department of Informatics and Applied Sciences, Constantin Brancusi University of Tg-Jiu, Bld. Republicii, No 1, 210152, Romania*

^b*Department of Mathematics, West University of Timisoara, C. Coposu Blvd. Number 4, 300223, Timisoara, Romania*

(v3.6 released February 2010)

In this article we combine the existing regularity theory, perturbation method and lower and upper solutions method to study the existence and asymptotic behavior of positive solution to a boundary value problem for the p -Laplacian operator. More exactly, we study the existence and asymptotic behavior of the positive solution to a quasilinear elliptic problem of the form $-\Delta_p u = \lambda a(x)g(u)$ in $D'(\Omega)$, $u > 0$ in Ω , $\lim_{x \rightarrow \partial\Omega} u(x) = 0$. Under some conditions on a and g , we show that there is a non-negative number Λ_0 such that for all $\lambda \in (0, \Lambda_0]$, the problem has a solution u_λ in the sense of distribution, which is bounded from above by some positive numbers $\mu(\lambda)$. Such estimates and the asymptotic behavior are important in computer programs when we know an algorithm for determining the solution.

Keywords: Quasilinear problem; Existence; Asymptotic behavior.

AMS Subject Classification: 35J25, 35J60, 35J62, 35J75, 35J66.

1. Introduction

In this article we study the existence and asymptotic behavior of the solutions for the following quasilinear elliptic problem

$$-\Delta_p u = \lambda a(x)g(u) \text{ in } D'(\Omega), u > 0 \text{ in } \Omega, \lim_{x \rightarrow \partial\Omega} u(x) = 0, \quad (1)$$

where $1 < p < \infty$, $\lambda > 0$ is a parameter, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator of u , $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$.

A solution of the problem (1) will be a function $u \in C^1(\Omega)$ which satisfies

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \int_{\Omega} \lambda a(x)g(u)\varphi dx, \varphi \in C_0^\infty(\Omega), \quad (2)$$

$$u(x) > 0, \text{ for all } x \in \Omega \quad (3)$$

$$\lim_{x \rightarrow \partial\Omega} u(x) = 0. \quad (4)$$

*Corresponding author. Email: coveid@yahoo.com

Such problems has been extensively studied for both bounded or unbounded domains: Ye-Zhou [18], Goncalves-Santos [9], the author [2], Hai-Wang [8], Chai-Zhao [1], Goncalves-Silva [10] and references therein. Our study is motivated by the recent works of [1, 2, 9, 10] where the existence, non-existence and asymptotic behavior of solutions for the problem (1) are solved in $\Omega \subseteq \mathbb{R}^N$.

For instance, when $\Omega \subset \mathbb{R}^N$ in [2] the problem (1) has been considered under the assumptions that $\lambda = 1$, the anisotropic potential function $a : \overline{\Omega} \rightarrow \mathbb{R}$ satisfies the conditions

- (a1) $a \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$;
- (a2) $a(x) > 0$ for all $x \in \overline{\Omega}$,

and that the non-linearity $g : (0, \infty) \rightarrow (0, \infty)$ is a C^1 function such that

- (g1) $t \mapsto \frac{g(t)}{t^{p-1}}$ is decreasing on $(0, \infty)$;
- (g2) $\lim_{t \searrow 0} g(t) = \tau_0$ and $\lim_{t \nearrow \infty} \frac{g(t)}{t^{p-1}} = \tau_\infty$

where $0 \leq \tau_\infty < \infty$ and $0 < \tau_0 \leq \infty$.

The above result has been generalized by Chai-Zhao [1] and Miao-Yang [19] for more general class of functions. After these results Goncalves-Silva [10] deals with more general conditions for the function g but in the case $\Omega \subset \mathbb{R}^N$ they didn't obtain the property (4) for the solution u . In other words the function u is not an entire solution.

Inspired by the above mentioned works we study, in a natural way, the existence and asymptotic behavior of solution to the problem (1). By construction of a suitable lower and upper solution, we intent in this paper to discover more ideas and techniques that in [1], [2] in order to open the access for more general class of function as in [10] in any situation $\Omega \subseteq \mathbb{R}^N$.

Our main result is the following:

THEOREM 1.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Under hypotheses (a1), (a2), (g1) and (g2) there is a non-negative number Λ_0 such that for all $\lambda \in (0, \Lambda_0]$ there is a positive number $\mu := \mu(\lambda)$ and some functions $u := u_\lambda \in C^1(\Omega) \cap L^\infty(\Omega)$ with $u \leq \mu$ satisfying (1) in the sense of distributions. Moreover, there exists constant $l_1 > 0$ such that the solution u satisfies*

$$u(x)^p [g(u(x)/4)2^{p-1} + 1]^{-1/(p-1)} \leq \lambda^{-\frac{1}{p-1}} l_1 \mu^{p-1} \cdot 2^{[(p-1)^2+p]/(p-1)} d(x), \quad x \in \overline{\Omega}, \quad (5)$$

where $d(x)$ denote the distance from $x \in \Omega$ to the boundary $\partial\Omega$.

Our main theorem apply to the class of functions

$$g(u) = u^{-\gamma} + u^\lambda + \tau_\infty u^{p-1}, \gamma > 0 \text{ and } 0 < \lambda < p - 1$$

where

$$\lim_{t \searrow 0} g(t) = \infty \text{ and } \lim_{t \nearrow \infty} \frac{g(t)}{t^{p-1}} = \tau_\infty.$$

The result of existence and asymptotic behavior is a generalization of the work [2] in the whole space done for the p-Laplacian equation and of the work [9] done in bounded sets for the Laplace problem. To prove the main result we use the approach of lower and upper solutions employed in [10] and the results obtained

in [3] with some adaptation in the estimates. The key tool is an inequality for the p-Laplacian developed by Diaz-Saà in the 1987's (see reference [6]).

The rest of this paper is organized as follows: in Section 2, we collect several technical lemmas. Section 3 contains the proof of our main result.

2. Preliminary results

We present some results about the eigenvalues problem for the p-Laplacian.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary.

DEFINITION 2.1 *We will say that $\xi \in \mathbb{R}$ is an eigenvalue, if there exists a function $\omega \in W^{1,p}(\Omega)$, $\omega \neq 0$, such that $\omega = 0$ on $\partial\Omega$ and*

$$\int_{\Omega} |\nabla \omega(x)|^{p-2} \nabla \omega(x) \nabla \eta(x) dx = \int_{\Omega} \xi |\omega(x)|^{p-2} \omega(x) \eta(x) dx, \quad (6)$$

for any $\eta \in C_0^\infty(\Omega)$.

DEFINITION 2.2 *The function ω which appears in (6) is called the eigenfunction corresponding to the eigenvalue ξ .*

A first observation about problem (6) is:

Remark 1 If (ω, ξ) is a solution of the problem (6) then for any $\alpha \in \mathbb{R}$ the couple $(\alpha\omega, \xi)$ is the same a solution.

For $1 < p < \infty$ the existence of ξ and ω has been first considered by Garcia Azorero and Peral Alonzo [15]. They showed that there exists an increasing subset $\{\xi_k\}_{k \geq 1}$ of eigenvalue which are strictly positive and such that $\xi_k \rightarrow \infty$ for $k \rightarrow \infty$.

It is well-known that the first eigenvalue which appears in (6) is

$$\xi_1 := \inf \left\{ \frac{\int_{\Omega} |\nabla \omega|^p dx}{\int_{\Omega} |\omega|^p dx} \mid \omega \in W_0^{1,p}(\Omega), \omega \neq 0 \right\} > 0,$$

and that the eigenfunction corresponding to ξ_1 is the minimum of the Euler functional

$$J(\omega) := \frac{1}{p} \int_{\Omega} |\nabla \omega|^p - \frac{\xi_1}{p} \int_{\Omega} |\omega|^p.$$

Lindqvist [12, 13] have obtained the following characterization for the eigenvalue ξ_1 and for the corresponding eigenfunctions:

LEMMA 2.3 *In any bounded domain Ω of \mathbb{R}^N ($N \geq 2$) we have:*

- i) ξ_1 is simple in the sense that if ω_1, ω_2 are the eigenfunctions corresponding to the eigenvalue ξ_1 then there exists a parameter c such that $\omega_1 = c\omega_2$;
- ii) the eigenfunctions corresponding to ξ_1 has a constant sign in the sense that $\omega > 0$ or $\omega < 0$;
- iii) ξ_1 is isolated in the sense that there exists $\kappa > 0$ such that in the interval $(\xi_1, \xi_1 + \kappa)$ there are no other eigenfunctions of (6);
- iv) the non negative eigenfunction corresponding to ξ_1 is unique (after multiplying by a constant).

In the article we note by $\omega_1(\Omega)$ the eigenfunction corresponding to the eigenvalue $\xi_1(\Omega)$ for the problem (6). The $C^{1,\alpha}$ -regularity of the eigenfunctions corresponding to the eigenvalue ξ_1 for the problem (6) is given in the following:

LEMMA 2.4 If $\xi := \xi_1$ in (6) then there exists $\alpha \in (0, 1)$ such that $\omega_1 \in C^{1,\alpha}(\overline{\Omega})$.

The following result can be found in more general form in Diaz-Saà [6] and is the main tool in the proof of the main result.

LEMMA 2.5 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that $G : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is such that:

- H1) for a.e. $x \in \Omega$, the function $u \mapsto G(x, u)$ is continuous on $[0, \infty)$ and the function $u \mapsto G(x, u)/u^{p-1}$ is decreasing on $(0, \infty)$;
 - H2) for all $u \geq 0$, the function $x \mapsto G(x, u)$ belong to $L^\infty(\Omega)$;
 - H3) $\exists C > 0$ such that $G(x, u) \leq C(u^{p-1} + 1)$ a.e. $x \in \Omega$, $\forall u \geq 0$.
- Under these hypotheses, the problem

$$-\Delta_p u = G(x, u), \quad u \geq 0, \quad u \not\equiv 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (7)$$

has almost one solution. If, in addition,

$$a_0(x) = \lim_{s \searrow 0} \frac{G(x, s)}{s^{p-1}} \text{ and } a_\infty(x) = \lim_{s \nearrow \infty} \frac{G(x, s)}{s^{p-1}},$$

is such that

$$-\infty < a_0(x) \leq +\infty \text{ and } -\infty \leq a_\infty(x) < +\infty \quad (8)$$

then the problem (7) has a unique solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover $u \in C^{1,\alpha}(\overline{\Omega})$.

We restate the well-known result of Diaz-Saà which can be found in [6].

LEMMA 2.6 Let Ω be an open set in \mathbb{R}^N . For $i = 1, 2$ let $w_i \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $w_i > 0$ a.e. in Ω , $\Delta_p w_i^{1/p} \in L^\infty(\Omega)$ and $w_1 = w_2$ on $\partial\Omega$. If $w_i/w_j \in L^\infty(\Omega)$ ($i \neq j$, $i, j = 1, 2$) then

$$\int_{\Omega} \left(\frac{-\Delta_p w_1^{1/p}}{w_1^{(p-1)/p}} + \frac{\Delta_p w_2^{1/p}}{w_2^{(p-1)/p}} \right) (w_1 - w_2) dx \geq 0.$$

The next result can be found in more general form in the article of DiBenedetto [5] (see also Tolksdorf [16]).

LEMMA 2.7 Let $N \geq 2$, $p \in (1, \infty)$, Ω be an open set in \mathbb{R}^N and Ω' a subset of Ω such that $\overline{\Omega'} \subset \Omega$. Assume that $h : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function (i.e. measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}$) and that there exists $\gamma > 0$ such that $|h(x, u)| \leq \gamma$. If $u \in W_{loc}^{1,p}(\Omega) \cap L_{loc}^\infty(\Omega)$ satisfies

$$-\Delta_p u = h(x, u) \text{ in } D'(\Omega),$$

then $|\nabla u| \in L_{loc}^\infty(\Omega)$ and for any compact subset $K \subset \Omega'$, there exists $\alpha \in (0, 1)$ and the parameters $C_0, C_1 > 0$ depending only upon $N, p, M = \text{ess sup}_{\Omega'} |u|$, $\gamma(M)$ and $\text{dist}(K, \partial\Omega')$, such that

$$\|\nabla u\|_{L^\infty(K)} \leq C_0 \text{ and } |u_{x_i}(x) - u_{x_i}(y)| \leq C_1 |x - y|^\alpha, \quad x, y \in K, i = \overline{1, N}. \quad (9)$$

The following result can be found in [7].

LEMMA 2.8 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. If (a1) and (a2) hold, then*

$$-\Delta_p w_1 = a(x) \text{ in } D'(\Omega), \quad w_1 > 0 \text{ in } \Omega, \quad w_1 = 0 \text{ on } \partial\Omega, \tag{10}$$

has a unique solution $w_1 \in C^{1,\alpha}(\overline{\Omega})$. Moreover, there exist constants $l_1 \geq k_1 > 0$, such that $k_1 d(x) \leq w_1(x) \leq l_1 d(x)$ on $\overline{\Omega}$ where $d(x)$ denote the distance from $x \in \Omega$ to the boundary $\partial\Omega$.

The following lemma can be found in [11].

LEMMA 2.9 *Let Ω be a domain in \mathbb{R}^N (but otherwise arbitrary, in particular, not necessarily smooth or bounded). Then for every $k \in \mathbb{N}$ there exist bounded domains Ω_k such that*

$$i) \text{ the boundary of } \Omega_k \text{ is of class } C^\infty; \quad ii) \overline{\Omega}_k \subset \Omega_{k+1} \subset \Omega; \quad iii) \bigcup_{k \in \mathbb{N}} \Omega_k = \Omega.$$

The first result of this section is the following.

LEMMA 2.10 *Make the same assumption on a and Ω as in Lemma 2.8. If in addition (g1) and (g2) hold, then there exists number $\Lambda_0 \in (0, \infty)$ such that for each $\lambda \in (0, \Lambda_0]$ there are a positive number $\mu := \mu(\lambda)$ and a function $v^l := v_\lambda^l \in C^1(\overline{\Omega})$ with $\max_{x \in \overline{\Omega}} v^l(x) = \mu$ satisfying*

$$-\Delta_p v^l \geq \lambda a(x) g(v^l) \text{ in } D'(\Omega), \quad v^l > l \text{ in } \Omega, \quad v^l = l \text{ on } \partial\Omega, \tag{11}$$

where $l \geq 0$ is a parameter.

Proof.. Let $w_1 \in C^{1,\alpha}(\overline{\Omega})$ be the unique solution of the problem (10) and w a function defined by

$$w(x) := \lambda^{\frac{1}{p-1}} w_1(x) \text{ for all } x \in \overline{\Omega}. \tag{12}$$

Set

$$M := \max_{x \in \overline{\Omega}} w_1(x), \text{ and } \Lambda_0 = \left(\frac{1}{M(p-1)} \right)^{p-1} \frac{1}{\tau_\infty}.$$

Let $\lambda \in (0, \Lambda_0)$. Consider first the function

$$H(y+l) = (g(y+l)+1)^{1/(p-1)} \text{ for } y > 0 \text{ and } l \geq 0.$$

Note that

$$\begin{array}{ll} H1) H(y+l) \geq [g(y+l)]^{1/(p-1)}; & H2) \lim_{y \rightarrow 0+} H(y+l)/y = \infty; \\ H3) \lim_{y \rightarrow \infty} H(y+l)/y = (\tau_\infty)^{1/(p-1)}; & H4) y \mapsto H(y+l)/y^{p-1} \\ & \text{is decreasing on } (0, \infty). \end{array}$$

Consider the continuous function

$$\Pi(y) := \frac{1}{My^{p-1}} \int_0^{y-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma, \quad y > l,$$

and observe that

$$\lim_{y \rightarrow \infty} \frac{1}{M y^{p-1}} \int_0^{y-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma = \lim_{y \rightarrow \infty} \frac{(y-l)^{p-1}}{M(p-1)y^{p-2}H(y)} = \frac{1}{(p-1)M} \left(\frac{1}{\tau_\infty} \right)^{\frac{1}{p-1}} \quad (13)$$

and

$$\lim_{y \rightarrow l} \frac{1}{M y^{p-1}} \int_0^{y-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma = 0. \quad (14)$$

From (13) and (14) one easily deduces that, for every object $\lambda \in (0, \Lambda_0]$ there is some $\mu \in (l, \infty)$ with the property $\Pi(\mu) = \lambda^{1/(p-1)}$, which is equivalent to

$$\frac{1}{\mu^{p-1}} \int_0^{\mu-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma = M \lambda^{\frac{1}{p-1}}. \quad (15)$$

Introduce the function $I: \mathbb{R}^N \times [l, \mu] \rightarrow \mathbb{R}$ by

$$I(x, s) = w(x) - \frac{1}{\mu^{p-1}} \int_0^{s-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma.$$

Then

$$\frac{\partial I(x, s)}{\partial s} = -\frac{1}{\mu^{p-1}} \frac{(s-l)^{p-1}}{H(s)} < 0, \quad s > l. \quad (16)$$

Now, as an application of (16) there is a function $v^l: \overline{\Omega} \rightarrow [l, \mu]$ of class $C^{1,\alpha}(\overline{\Omega})$ such that $I(x, v^l(x)) = 0$ for all $x \in \overline{\Omega}$ and so

$$w(x) = \frac{1}{\mu^{p-1}} \int_0^{v^l(x)-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma, \quad (17)$$

is well defined.

Moreover for

$$\Gamma(t) = \int_0^t \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma$$

we also have

$$v^l(x) = \Gamma^{-1}(\mu^{p-1}w(x)) + l, \quad (18)$$

where Γ^{-1} is the inverse function of Γ and from Lemma 2.8 that

$$\Gamma^{-1}\left(k_1 \lambda^{-\frac{1}{p-1}} \mu^{p-1} d(x)\right) \leq v^l(x) - l \leq \Gamma^{-1}\left(l_1 \lambda^{-\frac{1}{p-1}} \mu^{p-1} d(x)\right) \text{ on } \overline{\Omega}. \quad (19)$$

Letting $x \rightarrow \partial\Omega$ in (19) follows $v^l(x) = l$ on $\partial\Omega$. Of course, we still need to prove that

$$\max_{x \in \overline{\Omega}} v^l(x) = \mu \text{ for all } x \in \overline{\Omega}. \quad (20)$$

By (12) we get in the point x_0 , where the maximum is achieved that

$$w(x_0) = \lambda^{\frac{1}{p-1}} M = \frac{1}{\mu^{p-1}} \int_0^{\mu-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma \geq w(x) = \frac{1}{\mu^{p-1}} \int_0^{v^l(x)-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma,$$

and

$$I(x_0, \mu) = w(x_0) - \frac{1}{\mu^{p-1}} \int_0^{\mu-l} \frac{\sigma^{p-1}}{H(\sigma+l)} d\sigma = 0.$$

Then $v^l(x) \leq \mu$ for all $x \in \overline{\Omega}$ and so $v^l(x_0) = \mu$.

It remains to show that $v^l(x)$ constructed in (17) is an upper bound for the problem (11).

Differentiating in (17) with respect to x we obtain

$$\Delta_p w = \left[\frac{(v^l - l)^{p-1}}{\mu^{p-1} H(v^l - l)} \right]^{p-1} \Delta_p v^l + \frac{(p-1)|\nabla v^l|^p}{\mu^{2(p-1)}} \left[\frac{(v^l - l)^{p-1}}{H(v^l - l)} \right]^{p-2} \frac{\partial}{\partial v^l} \left[\frac{(v^l - l)^{p-1}}{H(v^l - l)} \right].$$

and using H4) we have

$$-\Delta_p v^l \geq - \left[\frac{\mu^{p-1} H(v^l - l)}{(v^l - l)^{p-1}} \right]^{p-1} \Delta_p w = \lambda a(x) \left[\frac{\mu^{p-1} H(v^l - l)}{(v^l - l)^{p-1}} \right]^{p-1} \geq \lambda a(x) g(v^l),$$

Then the same arguments used in [10] follows that v verifies

$$-\Delta_p v^l \geq -\lambda a(x) g(v^l) \text{ in } \Omega. \quad (21)$$

in the distributions sense and the proof is now complete. This key lemma was suggested in an early version of [10].

3. Proof of the main result

There are several new ideas in the works [1], [2], [10] and [19] which will be exploited in the next proof.

For each $\varepsilon > 0$ consider the function $G_\varepsilon : \Omega \times \mathbb{R}_+ \rightarrow (0, \infty)$ defined by

$$G_\varepsilon(x, s) := \lambda a(x) g(s + \varepsilon) \text{ if } s \geq 0$$

and the associated problem

$$-\Delta_p v = G_\varepsilon(x, v) \text{ in } \Omega, v(x) = 0 \text{ on } \partial\Omega. \quad (22)$$

In order to obtain a solution of the problem (22), it is enough to check the hypotheses of the Diaz-Saà theorem are fulfilled.

H1 : Since $g \in C^1((0, \infty), (0, \infty))$ and $\varepsilon > 0$, it follows that the mapping $u \mapsto G_\varepsilon(x, u)$ is continuous in $[0, \infty)$ and from

$$\frac{G_\varepsilon(x, u)}{u^{p-1}} = \lambda a(x) \frac{g(u + \varepsilon)}{(u + \varepsilon)^{p-1}} \left(1 + \frac{\varepsilon}{u}\right)^{p-1},$$

using positivity of a and (g1) we deduce that the function $u \rightarrow \frac{G_\varepsilon(x,u)}{u^{p-1}}$ is decreasing on $(0, \infty)$.

H2 : For all $u \geq 0$, since $a \in C^{0,\alpha}(\overline{\Omega})$, we obtain $x \mapsto a(x)g(u)$ belongs to $L^\infty(\Omega)$.

H3 : By $\lim_{u \rightarrow \infty} \frac{g(u+\varepsilon)}{u^{p-1}+1} = \lim_{u \rightarrow \infty} \frac{g(u+\varepsilon)}{(u+\varepsilon)^{p-1}} \cdot \frac{(u+\varepsilon)^{p-1}}{u^{p-1}+1} = \tau_\infty$, $a \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^1((0, \infty), (0, \infty))$, there exists $C > 0$ such that $G_\varepsilon(x, u) \leq C(u^{p-1} + 1)$ for all $u \geq 0$.

Observe that

$$a_0(x) = \lim_{u \searrow 0} \frac{G_\varepsilon(x, u)}{u^{p-1}} = +\infty \text{ and } a_\infty(x) = \lim_{u \rightarrow +\infty} \frac{G_\varepsilon(x, u)}{u^{p-1}} = \lambda a(x) \tau_\infty.$$

Thus by results of Diaz-Saà, problem (22) has a unique positive solution $u_\varepsilon \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ such that

$$\begin{aligned} \int_{\Omega} |\nabla u_\varepsilon(x)|^{p-2} \nabla u_\varepsilon \nabla \varphi dx &= \int_{\Omega} G_\varepsilon(x, u) \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega), \\ u_\varepsilon &> 0 \text{ in } \Omega, \\ u_\varepsilon &= 0 \text{ on } \partial\Omega. \end{aligned} \quad (23)$$

We will start by producing a lower solution of the equation (23). To this end, pick $\lambda \in (0, \Lambda_0]$. For simplicity, we denote by $\xi_1(\lambda) := \xi_1(\lambda, \Omega)$ and $\omega_{1,\lambda} := \omega_1(\lambda, \Omega)$.

Notice that under this notation we have

$$-\Delta_p \omega_{1,\lambda} = \xi_1(\lambda) \lambda \omega_{1,\lambda} |\omega_{1,\lambda}|^{p-2} \text{ in } \Omega, \omega_{1,\lambda} > 0 \text{ in } \Omega, \omega_{1,\lambda} = 0 \text{ on } \partial\Omega.$$

Since $\lim_{s \rightarrow 0+} g(s)/s^{p-1} = +\infty$ and $a \in C^\alpha(\overline{\Omega})$ with $a > 0$ on $\overline{\Omega}$ there exists $\delta := \delta(\lambda) > 0$ such that

$$\frac{g(s)}{s^{p-1}} \geq \frac{\xi_1(\lambda)}{\min_{x \in \overline{\Omega}} a(x)} \quad \forall s \in (0, \delta). \quad (24)$$

Let $\underline{u}_1(x) = c\omega_{1,\lambda}(x)$ where c is a constant chosen such that

$$0 < c < \frac{\delta}{\max_{x \in \overline{\Omega}} \omega_{1,\lambda}(x)}.$$

Then $\underline{u} = \underline{u}_1 - \varepsilon$ satisfies

$$\begin{aligned} -\Delta_p \underline{u} &= -\Delta_p \underline{u}_1 = \xi_1(\lambda) \lambda (c\omega_{1,\lambda})^{p-1} \\ &\leq \lambda a(x) g(\underline{u}_1) = \lambda a(x) g(\underline{u} + \varepsilon) = G_\varepsilon(x, \underline{u}) \text{ in } D'(\Omega), \end{aligned}$$

i.e. $\underline{u}(x)$ is a lower solution of (23).

We see by Lemma 2.10 that there exists at least one solution $\overline{u}_\varepsilon(x)$ corresponding to $l = \varepsilon$ which satisfies (11) in the sense of distributions that means

$$\begin{aligned} \int_{\Omega} |\nabla \overline{u}_\varepsilon(x)|^{p-2} \nabla \overline{u}_\varepsilon(x) \nabla \varphi dx &\geq \int_{\Omega} \lambda a(x) g(\overline{u}_\varepsilon(x)) \varphi dx, \quad \varphi \in C_0^\infty(\Omega) \\ \overline{u}_\varepsilon(x) &> \varepsilon \text{ in } \Omega, \\ \overline{u}_\varepsilon(x) &= \varepsilon \text{ on } \partial\Omega. \end{aligned} \quad (25)$$

Taking $\varepsilon > 0$ small enough, recalling that $\overline{u_\varepsilon}(x) = \varepsilon > 0$ on $\partial\Omega$ and setting $\overline{u}(x) = \overline{u_\varepsilon}(x) - \varepsilon$ we have

$$-\Delta_p \overline{u}(x) = -\Delta_p \overline{u_\varepsilon}(x) \geq \lambda a(x)g(\overline{u_\varepsilon}(x)) = G_\varepsilon(x, \overline{u}(x)) \text{ in } D'(\Omega),$$

showing $\overline{u}(x)$ is an upper solution of (22).

We note here that $\overline{u_\varepsilon}(x)$ is a suitable upper solution chosen to meet the lower and upper solution method for singular problems developed recently by Goncalves and Santos [10]. The upper solution suggest the study of (1) for more general class of functions as in [3], [10].

Set $B_{\underline{u}_1, \overline{u_\varepsilon}} = \{x \in \Omega \mid \underline{u}_1(x) > \overline{u_\varepsilon}(x)\}$. We show that $B_{\underline{u}_1, \overline{u_\varepsilon}} = \emptyset$. To this end, notice that $B_{\underline{u}_1, \overline{u_\varepsilon}} \subset \Omega$ is open and, in fact, $B_{\underline{u}_1, \overline{u_\varepsilon}} \subset\subset \Omega$. Assume, to the contrary, that $B_{\underline{u}_1, \overline{u_\varepsilon}} \neq \emptyset$. Let $w_1 := (\underline{u}_1)^p$ and $w_2 := (\overline{u_\varepsilon})^p$.

Now, applying Diaz-Saà's inequality we have

$$\begin{aligned} 0 &\leq \int_{B_{\underline{u}_1, \overline{u_\varepsilon}}} \left[\frac{-\Delta_p w_1^{1/p}}{w_1^{(p-1)/p}} + \frac{\Delta_p w_2^{1/p}}{w_2^{(p-1)/p}} \right] (w_1 - w_2) dx \\ &= \int_{B_{\underline{u}_1, \overline{u_\varepsilon}}} \left[\frac{-\Delta_p \underline{u}_1(x)}{(\underline{u}_1(x))^{p-1}} + \frac{\Delta_p \overline{u_\varepsilon}(x)}{(\overline{u_\varepsilon}(x))^{p-1}} \right] (w_1 - w_2) dx \\ &\leq \int_{B_{\underline{u}_1, \overline{u_\varepsilon}}} \lambda \left[\frac{\xi_1(\lambda) \underline{u}_1^{p-1}}{\underline{u}_1^{p-1}} - a(x) \frac{g(\overline{u_\varepsilon}(x))}{\overline{u_\varepsilon}^{p-1}} \right] (w_1 - w_2) dx \\ &\leq \int_{B_{\underline{u}_1, \overline{u_\varepsilon}}} \lambda \left[\xi_1(\lambda) - a(x) \frac{g(\underline{u}_1(x))}{\underline{u}_1^{p-1}} \right] (w_1 - w_2) dx < 0, \end{aligned}$$

which is impossible. This shows that $B_{\underline{u}_1, \overline{u_\varepsilon}} = \emptyset$, proving the claim. Hence $\underline{u}(x) \leq$

$\overline{u}(x)$ in $\overline{\Omega}$.

Moreover, we can easy prove that

$$\underline{u}(x) < u_\varepsilon(x) \leq \overline{u}(x) \text{ in } \overline{\Omega}. \tag{26}$$

Now, due to *Lemma 2.9* we can approximate Ω with smooth bounded domains Ω_k such that

$$\overline{\Omega}_k \subset \Omega_{k+1} \subset \Omega \text{ for all } k \in \mathbb{N} \text{ and } \Omega = \cup_{k=1}^\infty \Omega_k.$$

Taking an integer $k \geq 1$ such that $\text{supp}(\varphi) \subset \Omega_k$ we have by (23)

$$\begin{aligned} \int_{\Omega_k} |\nabla u_\varepsilon(x)|^{p-2} \nabla u_\varepsilon(x) \nabla \varphi(x) dx &= \int_{\Omega_k} \lambda a(x) g(u_\varepsilon(x) + \varepsilon) \varphi dx, \\ u_\varepsilon &> 0 \text{ in } \Omega_k, \end{aligned} \tag{27}$$

By *Lemma 2.7* we see that there are $\alpha \in (0, 1)$ and the constants $C_0, C_1 > 0$ such that

$$|\nabla u_\varepsilon(x)|_{L^\infty} \leq C_0 \text{ and } |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| \leq C |x - y|^\alpha, \quad x, y \in \overline{\Omega}_k \quad k = 1, 2, \dots$$

Set $\varepsilon := 1/n$ and $u_\varepsilon := u^n$. It follows that $\{u^n\}$ is bounded in $C^{1,\alpha}(\overline{\Omega}_1)$.

Now, since the embedding

$$C^{1,\alpha}(\overline{\Omega}_1) \hookrightarrow C^1(\overline{\Omega}_1)$$

is compact there exists a sequence denoted by $\{u_1^{n_{1,j}}\}_{j=1,2,\dots}$ where $n_{1,j} \rightarrow \infty$ as $j \rightarrow \infty$ such that

$$u_1^{n_{1,j}} \xrightarrow{n_{1,j} \rightarrow \infty} u_1 \text{ in } C^1(\overline{\Omega}_1) \text{ and } u_1 > 0 \text{ in } \overline{\Omega}_1.$$

Hence

$$|\nabla u_1^{n_{1,j}}|^{p-2} \nabla u_1^{n_{1,j}} \xrightarrow{n_{1,j} \rightarrow \infty} |\nabla u_1|^{p-2} \nabla u_1 \text{ in } C(\overline{\Omega}_1).$$

We have by the standard integral convergence theorems, that

$$\int_{\Omega_1} |\nabla u_1^{n_{1,j}}|^{p-2} \nabla u_1^{n_{1,j}} \nabla \varphi(x) dx \xrightarrow{n_{1,j} \rightarrow \infty} \int_{\Omega_1} |\nabla u_1|^{p-2} \nabla u_1 \nabla \varphi(x) dx,$$

and

$$\int_{\Omega_1} \lambda a(x) g(u_1^{n_{1,j}} + 1/n_{1,j}) \varphi dx \xrightarrow{n_{1,j} \rightarrow \infty} \int_{\Omega_1} \lambda a(x) g(u_1) \varphi dx.$$

Then u_1 satisfies (2)-(3) with $\Omega = \Omega_1$.

Repeat the above proof up to the existence of the sequence $\{u_1^{n_{1,j}}\}_{j=1,2,\dots}$ to get a subsequence $\{u_2^{n_{2,j}}\}_{j=1,2,\dots}$ such that

$$u_2^{n_{2,j}} \xrightarrow{n_{2,j} \rightarrow \infty} u_2 \text{ in } C^1(\overline{\Omega}_2) \text{ and } u_2 > 0 \text{ in } \overline{\Omega}_2.$$

Hence

$$|\nabla u_2^{n_{2,j}}|^{p-2} \nabla u_2^{n_{2,j}} \xrightarrow{n_{2,j} \rightarrow \infty} |\nabla u_2|^{p-2} \nabla u_2 \text{ in } C(\overline{\Omega}_2).$$

The likewise u_2 is a solution of (2)-(3) with $\Omega = \Omega_2$ and $u_2|_{\Omega_1} = u_1$.

Choose k large. In the same way as above, we obtain a sequence $\{u_k^{n_{k,j}}\}_{j=1,2,\dots}$ which is a subsequence of $\{u_k^{n_{(k-1),j}}\}$ such that

$$u_k^{n_{k,j}} \xrightarrow{n_{k,j} \rightarrow \infty} u_k \text{ in } C^1(\overline{\Omega}_k) \text{ and } u_k > 0 \text{ in } \overline{\Omega}_k$$

and

$$|\nabla u_k^{n_{k,j}}|^{p-2} \nabla u_k^{n_{k,j}} \xrightarrow{n_{k,j} \rightarrow \infty} |\nabla u_k|^{p-2} \nabla u_k \text{ in } C(\overline{\Omega}_k).$$

Then $u_k|_{\Omega_{k-1}} = u_{k-1}$ and u_k is a solution of (2)-(3) with $\Omega = \Omega_k$. Let $u : \Omega \rightarrow (0, \infty)$ defined by $u(x) := \lim_{k \rightarrow \infty} u_k(x)$. We have $u \in C^1(\Omega)$, $u(x) > 0$ in Ω and $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ in $\overline{\Omega}$ (since $u_k > 0$ in Ω and $\underline{u} \leq u_k \leq \overline{u}$ for every k). Hence $u|_{\partial\Omega} = 0$ because we have $\overline{u}|_{\partial\Omega} = 0$ and so u is the solution of the problem (1).

The asymptotic behavior of solution is given in what follows. Let $\overline{u}_0(x)$ the solution of (11) corresponding to $\varepsilon = 0$. From the above arguments we have

$$w(x) = \frac{1}{\mu^{p-1}} \int_0^{\overline{u}_0(x)} t^{p-1}/H(t) dt \geq \frac{1}{\mu^{p-1}} \int_{\overline{u}_0/2}^{\overline{u}_0} t^{p-1}/H(t) dt := \frac{1}{\mu^{p-1}} \int_{\overline{u}_0/2}^{\overline{u}_0} h_1(t) dt,$$

where $h_1(t) := t^{p-1}/H(t), t > 0$.

Using H1), H4) and the definition of $H(t)$ we find that

$$\begin{aligned} w(x) &\geq \frac{1}{\mu^{p-1}} h_1(\overline{u_0}(x)/2)(\overline{u_0}(x)/2) \\ &\geq \frac{1}{\mu^{p-1}} \left(\frac{1}{2}\right)^{p/(p-1)} \left(\frac{\overline{u_0}(x)}{2}\right)^p \frac{1}{[g(\overline{u_0}(x)/4)(1 + 1/g(\overline{u_0}(x)/2))]^{1/(p-1)}}. \end{aligned}$$

Using g1)

$$w(x) \geq \frac{1}{\mu^{p-1}} \left(\frac{1}{2}\right)^{p/(p-1)} \left(\frac{\overline{u_0}(x)}{4}\right)^p \frac{2^{p+1}}{[g(\overline{u_0}(x)/4)2^{p-1} + 1]^{1/(p-1)}}. \quad (28)$$

From (28), g1) and the fact that $u \leq \overline{u_0}$ we get

$$u(x)^p [g(u(x)/4)2^{p-1} + 1]^{-1/(p-1)} \leq \mu^{p-1} \cdot 2^{[(p-1)^2+p]/(p-1)} w(x) \text{ for all } x \in \overline{\Omega}. \quad (29)$$

Then is sufficient to apply Lemma 2.8 to obtain (5). This ends the proof of Theorem 1.1.

Remark 1 The case $\Omega = \mathbb{R}^N$ can be now treated as in [3], [10].

We can observe from [4] the following:

Remark 2 Let $\tau_0 \in (0, \infty]$ and $\tau_\infty \in [0, \infty)$. If $g : (0, \infty) \rightarrow (0, \infty)$ is a continuous function then the following are equivalently:

- i) $\lim_{s \searrow 0} g(s) = \tau_0$ and $\lim_{s \nearrow \infty} \frac{g(s)}{s^{p-1}} = \tau_\infty$;
- ii) exists $\eta > 0$ such that $\lim_{s \searrow 0} \frac{g(s)}{(s+\eta)^{p-1}} = \frac{\tau_0}{\eta^{p-1}}$ and $\lim_{s \nearrow \infty} \frac{g(s)}{(s+\eta)^{p-1}} = \tau_\infty$.

Acknowledgement. *The author would like to thank the referee for valuable suggestions. The work is supported by CNCSIS-UEFISCSU, project number PN II - IDEI 1080/2008 from West University of Timisoara.*

References

- [1] X. Chai, W. Niu and P. Zhao, *The existence and non-existence of positive solutions to a singular quasilinear elliptic problem in \mathbb{R}^N* , Nonlinear Analysis 71 (2009) 3257-3266.
- [2] D.P. Covei, *Existence and asymptotic behavior of positive solution to a quasilinear elliptic problem in \mathbb{R}^N* , Nonlinear Analysis: Theory, Methods & Applications, Volume 69, Issue 8, 15 October 2008, Pages 2615-2622.
- [3] D. P. Covei, *Existence and uniqueness of solutions for the Lane, Emden and Fowler type problem*, Nonlinear Analysis: Theory, Methods & Applications, Volume 72, Issue 5, 1 March 2010, Pages 2684-2693.
- [4] D. P. Covei, *A Lane-Emden-Fowler Type Problem With Singular Nonlinearity*, Journal of Mathematics of Kyoto University, Volume 49, No. 2, 2009.
- [5] E. DiBenedetto, *$C^{1,\alpha}$ -local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis, Volume 7, Issue 8, Pages 827-850, 1983.
- [6] J. I. Diaz and J. E. Saà, *Existence et unicite de solutions positives pour certaines equations elliptiques quasilineaires*, CRAS 305 Serie I, Pages 521-524, 1987.
- [7] Z. Guo, *Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large*, Proceedings of the Royal Society of Edinburgh, 124A, 189-198, 1994.
- [8] D.D. Hai and H. Wang, *Nontrivial solutions for p-Laplacian systems*, Journal of Mathematical Analysis and applications, 330 (2007) 186-194.
- [9] J. V. Goncalves and C. A. P. Santos, *Singular elliptic problems: Existence, non-existence and boundary behavior*, Nonlinear Analysis: Theory, Methods & Applications, Volume 66, Issue 9, Pages 2078-2090, 1 May 2007.
- [10] J. V. Goncalves and F. K. Silva, *Solutions of quasilinear elliptic equations in \mathbb{R}^N decaying at infinity to a non-negative number*, Complex Variables and Elliptic Equations, Volume 55, Issue 5 & 6 May 2010, pages 549 - 571.

[11] S. Krömer and M. Lilli, *Branches of positive solutions of quasilinear elliptic equations on non-smooth domains*, Nonlinear Analysis, Volume 64, Issue 10, Pages 2183-2202, 15 May 2006.

[12] P. Lindqvist, *On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda u|u|^{p-2}=0$* , Proceedings of the American Mathematical Society, 109, Pages 157-164, 1990.

[13] P. Lindqvist, *A nonlinear eigenvalue problem*, Lectures at the "Minicorsi di Analisi Matematica" at Padova in June 2000.

[14] A. Le, *Eigenvalue problems for the p -Laplacian*, Nonlinear Analysis, 64 (2006) 1057-1099.

[15] J. G. Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p -laplacian nonlinear eigenvalues*, Communications in Partial Differential Equations, Volume 12, No. 12, Pages 1389-1430, 1987.

[16] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, Journal of Differential Equations, Volume 51, Issue 1, Pages 126-150, January 1984.

[17] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Applied Mathematics and Optimization, 12 (1984), 191-202.

[18] D. Ye and F. Zhou, *Invariant criteria for existence of bounded positive solutions*, Discrete and Continuous Dynamical Systems-Series A, Volume: 12, Number 3, March 2005 , Pages 413-424.

[19] M. Qing and Z. Yang, *Quasilinear elliptic equation involving singular non-linearities*, International Journal of Computer Mathematics, Volume 87, Issue 6 May 2010, pages 1338 - 1348.