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Rational curves with one place at infinity *

Abdallah Assi[†]

Abstract: Let \mathbb{K} be an algebraically closed field of characteristic zero. Given a polynomial $f(x, y) \in \mathbb{K}[x, y]$ with one place at infinity, we prove that either f is equivalent to a coordinate, or the family $(f_\lambda)_{\lambda \in \mathbb{K}}$ has at most two rational elements.

1 Introduction and notations

Let \mathbb{K} be an algebraically closed field of characteristic zero, and let $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ be a monic reduced polynomial of $\mathbb{K}[x][y]$. For all $\lambda \in \mathbb{K}$, we set $f_\lambda = f - \lambda$. Hence we get a family of polynomials $(f_\lambda)_{\lambda \in \mathbb{K}}$. We shall suppose that f_λ is a reduced polynomial for all $\lambda \in \mathbb{K}$. Let g be a nonzero polynomial of $\mathbb{K}[x][y]$. We define the intersection multiplicity of f with g , denoted $\text{int}(f, g)$, to be the rank of the \mathbb{K} -vector space $\frac{\mathbb{K}[x][y]}{(f, g)}$. Note that $\text{int}(f, g)$ is also the x -degree of the y -resultant of f and g . Let $p = (a, b) \in V(f) \cap V(g)$, where V denotes the set of zeros in \mathbb{K}^2 . By setting $\bar{x} = x - a, \bar{y} = y - b$, we may assume that $p = (0, 0)$. We define the intersection multiplicity of f with g at p , denoted $\text{int}_p(f, g)$, to be the rank of the \mathbb{K} -vector space $\frac{\mathbb{K}[[x, y]]}{(f, g)}$. Note that $\text{int}(f, g) = \sum_{p \in V(f) \cap V(g)} \text{int}_p(f, g)$. We define the local Milnor number of f at p , denoted $\mu_p(f)$, to be the intersection multiplicity $\text{int}_p(f_x, f_y)$, where f_x (resp. f_y) denotes the x -derivative (resp. the y -derivative) of f . We set $\mu(f) = \text{int}(f_x, f_y)$ and we recall that $\mu = \sum_{\lambda \in \mathbb{K}} \sum_{p \in V(f_\lambda)} \mu_p(f_\lambda)$. The number of places at p of f , denoted r_p , is defined to be the number of irreducible components of f in $\mathbb{K}[[x]][y]$.

Assume, after possibly a change of variables, that $\deg_x a_i(x) < n - i$ for all $i = 1, \dots, n$ (where \deg_x denotes the x -degree). In particular f has one point at infinity defined by $y = 0$. Let $h_f(x, y, u) = u^n f\left(\frac{x}{u}, \frac{y}{u}\right)$. The local equation of at infinity of f is nothing but $F(y, u) = h_f(1, y, u) \in \mathbb{K}[[u]][y]$. We define the Milnor number at infinity of f , denoted μ_∞ , to be the

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rank of the \mathbb{K} -vector space $\frac{\mathbb{K}[[u]][y]}{(F_u, F_y)}$. The number of places at infinity of f , denoted r_∞ , is defined to be the number of irreducible components of $F(y, u)$ in $\mathbb{K}[[u]][y]$.

Let the notations be as above and let $R(x, \lambda) = P_0(\lambda)x^i + \dots + P_i(\lambda)$ be the y -resultant of f_λ, f_y . We say that $(f_\lambda)_\lambda$ is regular if $P_0(\lambda) \in \mathbb{K}^*$. Note that $(f_\lambda)_\lambda$ is regular if and only if $\text{int}(f_\lambda, f_y) = i$ for all $\lambda \in \mathbb{K}$. Suppose that $(f_\lambda)_\lambda$ is not regular, and let $\lambda_1, \dots, \lambda_s$ be the set of roots of $P_0(\lambda)$. We set $I(f) = \{\lambda_1, \dots, \lambda_s\}$, and we call $I(f)$ the set of irregular values of $(f_\lambda)_\lambda$. Set $A_f = \sum_{k=1}^s (i - \text{int}(f - \lambda_k, f_y))$. For all $\lambda \in \mathbb{K} - I(f)$, we have $\text{int}(f_\lambda, f_y) = \mu + n - 1 + A_f$, where $\mu = \text{int}(f_x, f_y)$ (see [5]).

Note that $A_f = \sum_{\lambda \in \mathbb{K}} (i - \text{int}(f_\lambda, f_y))$, in particular $(f_\lambda)_\lambda$ is regular if and only if $A_f = 0$. On the other hand, given $a \in \mathbb{K}$, if $\text{int}(f_a, f_y) = \mu + n - 1$, then either $(f_\lambda)_\lambda$ is regular or $I(f) = \{a\}$.

2 Curves with one place at infinity

In this section we recall some of the basic properties of curves with one place at infinity. Let $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ be an irreducible monic polynomial of $\mathbb{K}[x][y]$, and assume, after possibly a change of variables, that $a_1(x) = 0$ and that $\text{dey}_x a_k(x) < k$ for all $k = 1, \dots, n$, in such a way that f has only one point at infinity. Let $F(X, Y) = f(X^{-1}, y) \in \mathbb{K}[X^{-1}, y] \subseteq \mathbb{K}((X))[y]$, where $\mathbb{K}((X))$ denotes the field of meromorphic series over \mathbb{K} . Assume that f has one place at infinity, i.e. the projective curve defined by the homogeneous equation $h_f(x, y, u) = f(\frac{x}{u}, \frac{y}{u})u^n$ is analytically irreducible at the point at infinity $(1 : 0 : 0)$. In particular F is an irreducible polynomial in $\mathbb{K}((X))[y]$. Let t be an indeterminate. By Newton theorem, there is $y(t) \in \mathbb{K}((t))$ such that $F(t^n, y(t)) = 0$ and

$$F(t^n, y) = \prod_{w^n=1} (y - y(wt)).$$

Let $y(t) = \sum_i a_i t^i$. We set $\text{supp}(y(t)) = \{i, a_i \neq 0\}$. Clearly $\text{supp}(y(t)) = \text{supp}(y(wt))$ for all w such that $w^n = 1$. We denote this set by $\text{supp}(F)$ and we recall that $\text{gcd}(n, \text{supp}(F)) = 1$.

Given f as above, we will associate with F the sequences $(m_k)_{k \geq 0}$, $(d_k)_{k \geq 1}$ and $(r_k)_{k \geq 0}$ defined by:

$$\begin{aligned} m_0 &= d_1 = r_0 = -n, \quad m_1 = r_1 = \inf(\text{supp}(F)) \text{ and for all } k \geq 2, \\ d_k &= \text{gcd}(m_0, \dots, m_{k-1}) = \text{gcd}(d_{k-1}, m_{k-1}) \\ m_k &= \inf \{i \in \text{supp}(F); i \text{ is not divisible by } d_k\} \\ r_k &= r_{k-1} \frac{d_{k-1}}{d_k} + m_k - m_{k-1}. \end{aligned}$$

Since $\text{gcd}(n, \text{supp}(F)) = 1$, there is $h \in \mathbb{N}$ such that $d_{h+1} = 1$. We denote by convention $m_{h+1} = r_{h+1} = +\infty$.

Let g be a nonzero polynomial of $\mathbb{K}[x][y]$. We define the intersection multiplicity of f with g , denoted $\text{int}(f, g)$, to be the x -degree of the y -resultant of f and g , and we recall that $\text{int}(f, g)$ is the rank of the \mathbb{K} -vector space $\frac{\mathbb{K}[x][y]}{(f, g)}$. The set $\{\text{int}(f, g) | 0 \neq g \in \mathbb{K}[x][y]\}$ is a semigroup of \mathbb{N} . We denote this set by $\Gamma_\infty(f)$ and we call it the semigroup associated with f . Let $G(X, y) = g(X^{-1}, y) \in \mathbb{K}[X^{-1}][y]$. We set $\text{Int}(F, G) = -\text{int}(f, g)$ and we call it the intersection multiplicity of F with G . It is also the X -order of the y -resultant of F and G . The set $\{\text{Int}(F, G) | 0 \neq G \in \mathbb{K}[X^{-1}][y]\}$ is a semigroup of $-\mathbb{N}$. We denote this set by $\Gamma(F)$ and we call it the semigroup associated with F . Note that $\text{Int}(F, G) = O_t G(t^n, y(t))$ -where O_t denotes the t -order-, and that this order does not depend on the choice of the root $y(t)$ of $F(t^n, y) = 0$.

Remark 2.1 (see [1]) With the notations above, for all $k = 0, \dots, h$, $r_k < 0$. Furthermore, $\Gamma_\infty(f) = \langle -r_0, -r_1, \dots, -r_h \rangle$ (resp. $\Gamma(F) = \langle r_0, r_1, \dots, r_h \rangle$).

Let d be a positive integer and assume that d divides n . Let g be a monic polynomial of $\mathbb{K}[x][y]$, of degree $\frac{n}{d}$ in y . There exist unique polynomials $\alpha_1(x, y), \dots, \alpha_d(x, y) \in \mathbb{K}[x][y]$ such that:

$$(*) \quad f = g^d + \alpha_1(x, y)g^{d-1} + \dots + \alpha_d(x, y)$$

and for all $1 \leq k \leq d$, if $\alpha_k(x, y) \neq 0$, then $\deg_y(\alpha_k(x, y)) < \frac{n}{d}$, where \deg_y denotes the y -degree.

We call $(*)$ the g -adic expansion of f . There is a unique monic polynomial g of degree $\frac{n}{d}$ in y such that in the g -adic expansion of f , $\alpha_1 = 0$. We call such a polynomial the d -th approximate root of f , and we denote it by $\text{App}_d(f)$. If $F(X, y) = f(X^{-1}, y)$, then the d -th approximate root of F , defined in the same way as for f , is $\text{App}_d(F) = \text{App}_d(f)(X^{-1}, y)$

For all $1 \leq k \leq h$, let $g_k = \text{App}_{d_k}(f)$ and let $G_k(X, y) = g_k(X^{-1}, y) = \text{App}_{d_k}(F)$ (in particular $g_1 = G_1 = g$ and $g_{h+1} = f, G_{h+1} = F$), then we have:

Lemma 2.2 (See [1]) For all $k = 1, \dots, h + 1$, we have:

i) $\text{Int}(F, G_k) = r_k$ and $\text{int}(f, g_k) = -r_k$.

ii) g_k is a polynomial with one place at infinity and $\Gamma_\infty(g_k) = \langle -\frac{r_0}{d_k}, \dots, -\frac{r_{k-1}}{d_k} \rangle$.

Let the notations be as in Section 1. The integer $\mu = \text{int}(f_x, f_y)$ is the conductor of $\Gamma_\infty(f)$ in \mathbb{N} , i. e., for all $N \geq \mu, N \in \Gamma_\infty(f)$. Furthermore, if we set $e_k = \frac{d_k}{d_{k+1}}$ for all $1 \leq k \leq h$, then we have

$$\mu = \sum_{k=1}^h (e_k - 1)r_k - n + 1$$

and it follows from Dedekind formula (see also [8]) that

$$\text{int}(f, f_y) = \mu + n - 1 = \sum_{k=1}^h (e_k - 1)r_k$$

Lemma 2.3 Let $g_k, k = 1, \dots, h+1$, be the set of approximate roots of f . For all $k = 2, \dots, h$, we have

$$\text{int}(f_x, f_y) = d_k \text{int}(g_{k_x}, g_{k_y}) + \sum_{i=k}^h (e_i - 1)r_i - d_k + 1.$$

Proof. Since g_k is a polynomial with one place at infinity, it follows that

$$\text{int}(g_k, g_{k_y}) = \text{int}(g_{k_x}, g_{k_y}) + \frac{n}{d_k} - 1,$$

on the other hand,

$$\text{int}(f, f_y) = \sum_{k=1}^n (e_k - 1)r_k = d_k \text{int}(g_k, g_{k_y}) + \sum_{i=k}^h (e_i - 1)r_i.$$

This proves our assertion ■

Write

$$g_2 = \text{App}_{d_2}(f) = y^{\frac{n}{d_2}} + ax^{\frac{m}{d_2}} + \sum_{\frac{n}{d_2}i + \frac{m}{d_2}j < \frac{n}{d_2} \frac{m}{d_2}} a_{ij} x^i y^j$$

where $a \neq 0$. Since $\gcd(\frac{n}{d_2}, \frac{m}{d_2}) = 1$, then $\text{int}(g_{2_x}, g_{2_y}) = (\frac{n}{d_2} - 1)(\frac{m}{d_2} - 1)$. In particular, $\text{int}(g_{2_x}, g_{2_y}) > 0$ if and only if $\gcd(n, m) = d_2 < \min(n, m)$, and by the Lemma above, if $\text{int}(g_{2_x}, g_{2_y}) > 0$, then so is for μ . In particular we get the following:

Lemma 2.4 Let the notations be as above. If $\mu = 0$, then so is for $\text{int}(g_{k_x}, g_{k_y})$ for all $1 \leq k \leq h$, and either m divides n or n divides m . In particular, if $\gcd(n, m) = d_2 < \min(n, m)$, then $\mu > 0$.

Proof. Obvious ■

Remark 2.5 Let the notations be as above, and let $B = \{\underline{\theta} = (\theta_1, \dots, \theta_h, \theta_{h+1}) | 0 \leq \theta_i < e_i \text{ for all } i = 1, \dots, h\}$, $B' = \{\underline{\theta} \in B | \theta_{h+1} = 0\}$. Let $g \in \mathbb{K}[x][y]$. The polynomial g can be uniquely written as

$$g = \sum_{\underline{\theta} \in B} c_{\underline{\theta}}(x) g_1^{\theta_1} \dots g_h^{\theta_h} f^{\theta_{h+1}}.$$

Suppose that g does not divide f , in particular $B' \neq \emptyset$. By [1], There is a unique $\underline{\theta}_0 \in B'$ such that $\deg_x c_{\underline{\theta}_0}(x)(-r_0) + \sum_{i=1}^h \theta_{0i}(-r_i) = \max\{\deg_x c_{\underline{\theta}}(x)(-r_0) + \sum_{i=1}^h \theta_i(-r_i)\}$. With these notations, $\text{int}(f, g) = \deg_x c_{\underline{\theta}_0}(x)(-r_0) + \sum_{i=1}^h \theta_{0i}(-r_i)$

3 Abhyankar-Moh Lemma

Let $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ be a nonzero reduced polynomial of $\mathbf{K}[x, y]$ and assume, after possibly a change of variables, that $a_1(x) = 0$ and also that $i > \deg_x(a_i(x))$ for all $2 \leq i \leq n$, in such a way that $y = 0$ is the only point at infinity of f . Let the notations be as in Section 1. If δ_p (resp. δ_∞) denotes the order of the conductor of f at p (resp. at the point at infinity), then $2\delta_p = \mu_p + r_p - 1$ (resp. $2\delta_\infty = \mu_\infty + r_\infty - 1$) (see [6]). If $\mu = \text{int}(f_x, f_y)$, then we set $\mu(f) = \sum_{p \in V(f)} \mu_p$ and $\bar{\mu}(f) = \mu - \mu(f)$.

Assume that f is an irreducible polynomial, and let $g(f)$ be the genus of the normalized curve of $V(f)$. By the genus formula we have:

$$2g(f) + \left(\sum_{p \in V(f)} 2\delta_p \right) + 2\delta_\infty = (n-1)(n-2).$$

Now $\text{int}(f, f_y) = \mu + n - 1 + A(f)$, where $A(f)$ is a nonnegative integer and $A(f) = 0$ if and only if $(f_\lambda)_{\lambda \in \mathbb{K}}$ has at most one irregular value at infinity (see Section 1). On the other hand, the local intersection multiplicity of f with f_y at the point at infinity is $\mu_\infty + n - 1$. In particular $\mu + \mu_\infty = (n-1)(n-2)$, and consequently:

$$2g(f) + \left(\sum_{p \in V(f)} 2\delta_p \right) + 2\delta_\infty = \mu(f) + \bar{\mu}(f) + \mu_\infty + A(f)$$

We finally get:

$$(**) \quad 2g(f) + \sum_{p \in V(f)} (r_p - 1) + r_\infty - 1 = \bar{\mu}(f) + A(f)$$

In particular $g(f) = \sum_{p \in V(f)} (r_p - 1) + r_\infty - 1 = 0$ if and only if $A(f) = \bar{\mu}(f) = 0$. Roughly speaking, f is a rational unibranch curve (at infinity as well as at finite distance) if and only if the pencil $(f_\lambda)_{\lambda \in \mathbb{K}}$ has at most one irregular value at infinity, and for all $\lambda \neq 0$, f_λ is a smooth curve. This with Lemma 2.4. implies the following:

Proposition 3.1 (Abhyankar-Moh Lemma) Let $x(t), y(t)$ be two monic polynomials of $\mathbb{K}[t]$ and let $n = \deg_t(x(t))$, $m = \deg_t(y(t))$. Assume that $m < n$. If $\mathbb{K}[x(t), y(t)] = \mathbb{K}[t]$, then m divides n .

Proof. Let $f \in \mathbb{K}[x][y]$ be the generator of the Kernel of the map $\mathbb{K}[x][y] \mapsto \mathbb{K}[t]$ which sends x, y to $x(t), y(t)$ respectively. The polynomial f is a rational polynomial with one place at infinity, and $\mu(f) = 0$, hence $\mu = \bar{\mu}(f)$. It follows from (**) that $\mu = \bar{\mu}(f) = A(f) = 0$, in particular, by Lemma 2.4., $d_2 = \gcd(n, m) = m$ ■

4 Rational one place curves

Let f be a rational polynomial with one place at infinity, and let the notations be as in Section 1. It follows from the equality (**) of Section 3 that $\sum_{p \in V(f)} (r_p - 1) = \bar{\mu}(f)$. We shall prove the following:

Theorem 4.1 Let the notations be as above and let $(f_\lambda)_\lambda$ be a pencil of polynomials defined by f . If f is rational, then exactly one of the following hold:

- i) For all $\lambda \in \mathbb{K}$, f_λ is rational, and $\sigma(f)$ is a coordinate of \mathbb{K}^2 for some automorphism σ of \mathbb{K}^2 .
- ii) The polynomial $f - \lambda$ is rational for at most one $\lambda_1 \neq 0$, i.e. the pencil $(f_\lambda)_\lambda$ has at most two rational elements.
- iii) Assume that the pencil $(f_\lambda)_\lambda$ has two rational elements f and f_{λ_1} . We have $\mu(f) = \mu(f_{\lambda_1}) = \frac{\mu}{2}$, furthermore, given a singular point p of $V(f)$ (resp. $V(f_{\lambda_1})$), f (resp. f_{λ_1}) has two places at p and $\mu_p(f) = 1$ (resp. $\mu_p(f_{\lambda_1}) = 1$). In particular, f (resp. f_{λ_1}) has exactly $\frac{\mu}{2}$ singular points.

We shall first prove the following Lemma:

Lemma 4.2 Let $H = y^N + a_1(x)y^{N-1} + \dots + a_N(x)$ be a non zero reduced polynomial of $\mathbb{K}[[x]][y]$, and let $H = H_1 \dots H_r$ be the decomposition of H into irreducible components of $\mathbb{K}[[x]][y]$. Let $\mu_{(0,0)}$ denotes the Milnor number of H at $(0,0)$ (i.e. $\mu_{(0,0)}$ is the rank of the \mathbb{K} -vector space $\frac{\mathbb{K}[[x]][y]}{(H_x, H_y)}$). We have the following:

- i) $\mu_{(0,0)} \geq r - 1$.
- ii) If $r \geq 3$, then $\mu_{(0,0)} > r - 1$.
- iii) If $r = 2$ and $\mu_{(0,0)} = r - 1 = 1$, then (H_1, H_2) is a local system of coordinates at $(0,0)$.

Proof. We have $\text{int}_{(0,0)}(H, H_y) = \mu_{(0,0)} + N - 1$, but

$$\text{int}_{(0,0)}(H, H_y) = \sum_{i=1}^r \text{int}(H_i, H_{i_y}) + 2 \sum_{i \neq j} \text{int}_{(0,0)}(H_i, H_j)$$

$$= \sum_{i=1}^r \text{int}[(H_{i_x}, H_{i_y}) + \deg_y H_i - 1] + 2 \sum_{i \neq j} \text{int}_{(0,0)}(H_i, H_j)$$

hence

$$\mu_{(0,0)} + N - 1 = \left(\sum_{i=1}^r \text{int}(H_{i_x}, H_{i_y}) \right) + N - r + 2 \sum_{i \neq j} \text{int}_{(0,0)}(H_i, H_j).$$

Finally we have $\mu_{(0,0)} = \left(\sum_{i=1}^r \text{int}(H_{i_x}, H_{i_y}) \right) - r + 1 + 2 \sum_{i \neq j} \text{int}_{(0,0)}(H_i, H_j)$. Now for all $1 \leq i \leq r$, $\text{int}_{(0,0)}(H_{i_x}, H_{i_y}) \geq 0$ and $\sum_{i \neq j} \text{int}_{(0,0)}(H_i, H_j) \geq C_2^r = \frac{r(r-1)}{2}$, hence $\mu_{(0,0)} \geq r(r-1) - (r-1) = (r-1)^2$ and i), ii) follow immediately. Assume that $r = 2$. If $\mu_{(0,0)} = r - 1$, then $\text{int}_{(0,0)}(H_{1_x}, H_{1_y}) = \text{int}_{(0,0)}(H_{2_x}, H_{2_y}) = 0$ and $\text{int}_{(0,0)}(H_1, H_2) = 1$. This implies iii) ■

Proof of Theorem 4.1. If $\mu(f) = 0$, then $\mu = 0$, and by Proposition 3.1., either m divides n or n divides m . Furthermore, $\sigma(f)$ is a coordinate of \mathbb{K}^2 for some automorphism σ of \mathbb{K}^2 . Assume that $\mu(f) > 0$ and let p_1, \dots, p_s be the set of singular points of $V(f)$. Let r_i denotes the number of places of f at p_i for all $1 \leq i \leq s$. By Lemma 4.2., for all $1 \leq i \leq s$, $\mu_{p_i} \geq r_i - 1$, on the other hand, equality (***) of Section 3 implies that $\sum_{i=1}^s (\mu_{p_i} + r_i - 1) = \mu$, in particular $\mu \leq \sum_{i=1}^s 2\mu_{p_i} = 2\mu(f)$, hence $\mu(f) \geq \frac{\mu}{2}$. If f_{λ_1} is rational for some $\lambda_1 \neq 0$, then the same argument as above implies that $\mu(f_{\lambda_1}) \geq \frac{\mu}{2}$. This is possible only for at most one $\lambda_1 \neq 0$, hence ii) follows immediately.

Assume that f_{λ_1} is rational for some $\lambda_1 \neq 0$. Clearly this holds only if $\mu(f) = \mu(f_{\lambda_1}) = \frac{\mu}{2}$. We shall prove iii) for f . Let the notations be above. For all $1 \leq i \leq s$, we have $\mu_{p_i} = r_i - 1$, hence, by lemma 4.2., $r_i = 2$, $\text{int}_{p_i}(H_{1_x}, H_{1_y}) = \text{int}_{p_i}(H_{2_x}, H_{2_y}) = 0$ and $\text{int}_{p_i}(H_1, H_2) = 1$. This implies iii) ■

As a corollary we get the following:

Corollary 4.3 Let f be as above and let $(f_\lambda)_\lambda$ be a pencil of polynomials defined by f . Assume that f is a rational polynomial, and that $\mu(f) > 0$. Let p_1, \dots, p_s be the set of singular points of f .

i) If $r_{p_i} = 1$ (resp. $r_{p_i} \geq 3$) for some $1 \dots i \dots s$, then f is the only rational point of the pencil $(f_\lambda)_\lambda$.

ii) If $r_{p_i} = 2$ for all $1 \leq i \leq s$ but $s \neq \frac{\mu}{2}$, then f is the only rational point of the pencil $(f_\lambda)_\lambda$.

Proof. This is an immediate application of Theorem 4.1. ■

Proposition 4.4 Let $f \neq g$ be two polynomials of $\mathbb{K}[x][y]$ and assume that f, g are parametrized by polynomials of $\mathbb{K}[t]$. Under these hypotheses, exactly one of the following conditions hold:

i) $f = g + \lambda_1$ for some $\lambda_1 \in \mathbb{K}^*$, and f is equivalent to a coordinate, i.e. $\sigma(f)$ is a coordinate of \mathbb{K}^2 for some automorphism σ of \mathbb{K}^2 .

ii) $f = g + \lambda_1$ for some $\lambda_1 \in \mathbb{K}^*$, $\mu(f) = \mu(f_{\lambda_1}) = \frac{\text{int}(f_x, f_y)}{2} > 0$, and f, g satisfy the condition iii) of Theorem 4.1.

iii) $\text{int}(f, g) > 0$, i.e. f, g meet in a least one point of \mathbb{K}^2 .

Proof. The polynomial f (resp. g) has one place at infinity. Assume that $\deg_y g \geq \deg_y f$ and let the notations be as in Section 2. Let g_1, \dots, g_h, f be the set of approximate roots of f and write, with the notations of Remark 2.5., $g = \sum_{\underline{\theta} \in B} c_{\underline{\theta}}(x) g_1^{\theta_1} \dots g_h^{\theta_h} f^{\theta_{h+1}}$. Since g has one place at infinity, then f does not divide g , in particular there is a unique $\underline{\theta}_0$ such that $c_{\underline{\theta}_0} \neq 0, \theta_{0h+1} = 0$, and $\text{int}(f, g) = -r_0 \deg_x c_{\underline{\theta}_0}(x) + \sum_{k=1}^h \theta_{0k}(-r_k) = \max\{-r_0 \deg_x c_{\underline{\theta}}(x) + \sum_{k=1}^h \theta_i(-r_i) \mid \underline{\theta} \in B'\}$. Clearly $\text{int}(f, g) \geq 0$. If $\text{int}(f, g) = 0$, then $c_{\underline{\theta}_0}(x) \in \mathbb{K}^*$ and $\theta_{01} = \dots = \theta_{0h} = 0$, in particular, since f does not divide g , $g = f + c, c \in \mathbb{K}^*$. Now apply Theorem 4.1. ■

Remark 4.5 Let $(x(t), y(t)) = (t^3 - 3t, t^2 - 2)$ and $(X(s), Y(s)) = (s^3 + 3s, s^2 + 2)$, and let $f(x, y) = \text{res}_t(x - x(t), y - y(t))$ (resp. $g(x, y) = \text{res}_s(x - X(s), y - Y(s))$). We have $(x(t) - X(s), y(t) - Y(s)) = \mathbb{K}[t, s]$, hence $\text{int}(f, g) = 0$. In fact,

$$f(x, y) = y^3 - x^2 - 3y + 2 = -x^2 + (y + 2)(y - 1)^2$$

and

$$g(x, y) = y^3 - x^2 - 3y - 2 = -x^2 + (y - 2)(y + 1)^2,$$

hence $f = g + 4$. The genus of a generic element of the family $(f_\lambda)_\lambda$ is 1, and $f, f - 4$ are the two rational elements of this family. Note that $\mu = 2$ and $\mu(f) = \mu(f - 4) = 1$. This example shows that the bound of Theorem 4.1. is sharp.

Remark 4.6 Let $(f_\lambda)_\lambda$ be a pencil of algebraic plane curves and assume that $f - \lambda$ is irreducible for all $\lambda \in \mathbb{K}$. If the generic element of the pencil is rational, then for all $\lambda \in \mathbb{K}$, $f - \lambda$ is rational and irreducible. In this case, by [7], f has one place at infinity and $\sigma(f)$ is a coordinate of \mathbb{K}^2 for some automorphism σ of \mathbb{K} . Assume that the genus of the generic element of the pencil $(f_\lambda)_\lambda$ is greater or equal than one. Similarly to the case of curves with one place at infinity, it is natural to address the following question:

Question: Is there an integer $c \in \mathbb{N}$ such that, given a pencil of irreducible polynomials $(f_\lambda)_\lambda$, if $\mu + A_f > 0$, then the number of rational elements in the pencil is bounded by c ?

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