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# Multiplicative noise cleaning via a variational method involving curvelet coefficients

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**Abstract.** Classical ways to denoise images contaminated with multiplicative noise (e.g. speckle noise) are filtering, statistical (Bayesian) methods, variational methods and methods that convert the multiplicative noise into additive noise (using a logarithmic function) in order to apply a shrinkage estimation for the log-image data and transform back the result using an exponential function.

We propose a new method that involves several stages: we apply a reasonable under-optimal hard-thresholding on the curvelet transform of the log-image; the latter is restored using a specialized hybrid variational method combining an  $\ell^1$  data-fitting to the thresholded coefficients and a Total Variation regularization (TV) in the image domain; the restored image is an exponential of the obtained minimizer, weighted so that the mean of the original image is preserved. The minimization stage is realized using a properly adapted fast Douglas-Rachford splitting. The existence of a minimizer of our specialized criterion and the convergence of the minimization scheme are proved. The obtained numerical results outperform the main alternative methods.

## 1 Introduction

In many active imaging systems (e.g. synthetic aperture radar, laser or ultrasound imaging), the data for the unknown image  $S_0 : \Omega \rightarrow \mathbf{R}_+$ ,  $\Omega \subset \mathbf{R}^2$ , are severely corrupted with multiplicative noise. Then several independent measurements for the same image are needed:

$$S_k = S_0 \eta_k + n_k, \quad \forall k \in \{1, \dots, K\}, \quad (1)$$

where  $\eta_k : \Omega \rightarrow \mathbf{R}_+$ , and  $n_k$  represent the multiplicative and a typically zero-mean additive noise,  $\forall k$ . Commonly (see e.g. [27])  $\eta_k$  is modeled as a one-sided exponential probability density function (pdf) (*cf.* Fig. 1(a)):  $\text{pdf}(\eta_k) = \mu e^{-\mu\eta_k} \mathbb{1}_{\mathbf{R}_+}(\eta_k)$  for  $\mu = 1$ . In practice, one takes an average of all measurements, see e.g. Fig. 2(b). Since  $\frac{1}{K} \sum_{k=1}^K n_k \approx 0$ , the data read (*cf.* e.g. [1, 27, 30]):

$$S = \frac{1}{K} \sum_{k=1}^K S_k = S_0 \frac{1}{K} \sum_{k=1}^K \eta_k = S_0 \eta. \quad (2)$$

Usually all  $\eta_k$  are independent. Denoting by  $\Gamma$  the usual Gamma-function, the mean of the noise  $\eta$  in (2) has a Gamma distribution (*cf.* Fig. 1(b)):

$$\eta = \frac{1}{K} \sum_{k=1}^K \eta_k : \quad \text{pdf}(\eta) = \frac{K^K \eta^{K-1} \exp(-K\eta)}{\Gamma(K)}. \quad (3)$$

Various adaptive filters have been proposed, see e.g. [17,31]: they work well when the noise is moderate or weak, i.e. for  $K$  large. Bayesian, variational or diffusion-based methods have been proposed as well; see e.g. [2, 18, 24, 28]. Numerous methods convert the multiplicative noise into additive noise by

$$v = \log S = \log S_0 + \log \eta = u_0 + n, \quad (4)$$

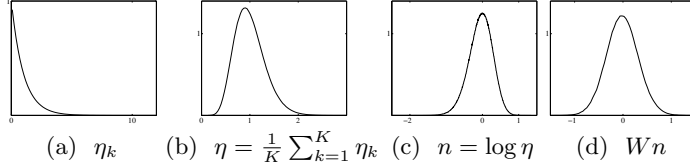
see e.g. [1, 16, 23, 30]. Then the pdf of  $n$  reads (*cf.* Fig. 1(c)):

$$n = \log \eta : \quad \text{pdf}(n) = K^K (\Gamma(K))^{-1} \exp(-K(n - e^n)). \quad (5)$$

One can prove that  $\mathbb{E}[n] = \psi_0(K) - \log K$  and  $\text{Var}[n] = \psi_1(K)$ , where  $\psi_k(z) = \left(\frac{d}{dz}\right)^{k+1} \log \Gamma(z)$  is the polygamma function. A common strategy is to decompose the log-data  $v$  into a multiscale frame for  $L^2(\mathbf{R}^2)$  (an over-complete basis), say  $W \equiv \{w_i : i \in I\}$  where  $I$  is a set of indexes:

$$y = Wv = Wu_0 + Wn. \quad (6)$$

By the Central Limit Theorem, the noise  $Wn$  in  $y$  is nearly Gaussian —*cf.* Fig.



**Fig. 1.** Noise distributions.

1(d). Then coefficients  $y$  are denoised using shrinkage estimators  $\mathcal{T} : \mathbf{R} \rightarrow \mathbf{R}$ :

$$y_{\mathcal{T}}[i] = \mathcal{T}((Wv)[i]), \quad \forall i \in I. \quad (7)$$

Shrinkage functions designed for multiplicative noise were proposed e.g. in [1,30]. Let  $\tilde{W} \equiv \{\tilde{w}_i : i \in I\}$  be a left inverse of  $W$ . Then a denoised log-image  $v_{\mathcal{T}}$  reads

$$v_{\mathcal{T}} = \sum_{i \in I} \mathcal{T}((Wv)[i]) \tilde{w}_i = \sum_{i \in I} \mathcal{T}(y[i]) \tilde{w}_i. \quad (8)$$

Then the sought-after image is of the form  $S_{\mathcal{T}} = \exp v_{\mathcal{T}}$ .

*Our approach.* We apply (4) and consider a tight-frame transform of the log-data. The restored log-image (section 2) minimizes a criterion composed of an  $\ell^1$ -fitting to the (suboptimally) hard-thresholded frame coefficients and a Total Variation (TV) regularization in the image domain. The minimization (section 3) uses a specialized Douglas-Rachford splitting. The full algorithm, involving a bias correction, is given in section 4. Experiments are presented in section 5.

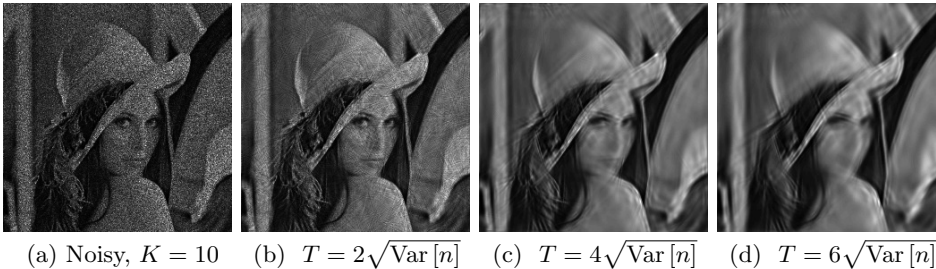
**Some notations.**  $(\cdot)^T$  means transposed,  $(\cdot)^*$  means convex conjugate and  $(\cdot)^{\star}$  means adjoint.

## 2 Restoration of the log-image

Here we consider how to restore a good log-image given data  $v : \Omega \rightarrow \mathbf{R}$  obtained using (4). We focus on methods which, for a given preprocessed data set, lead to convex optimization problems. We comment only variational methods and shrinkage estimators since they underly our specialized hybrid objective function.

### 2.1 Drawbacks of shrinkage restoration and variational methods

*Shrinkage restoration.* The main problems with these methods, sketched in (7)-(8), is that shrinking large coefficients entails an erosion of the spiky features, while shrinking small coefficients yields Gibbs-like oscillations in the vicinity of edges and a loss of details in the textured area. On the other hand, if shrinkage is insufficient, some coefficients bearing mainly noise can remain almost unchanged—we call such coefficients *outliers*—and (8) shows that they yield artifacts with the shape of the functions  $\tilde{w}_i$ , see Fig. 2. Even though various improvements were brought, these artifacts remain visible—see the results on Fig. 3(d) and Fig. 4(c) in Section 5 using the very recent Stein-block thresholding [8].



**Fig. 2.** (a) Noisy Lena obtained according to (1)-(2) for  $K = 10$ . (b)-(d) Restorations exp  $v_{\mathcal{T}_H}$  where data  $v$  are denoised by hard-thresholding of its curvelet coefficients, see (12)-(13), for different choices of  $T$ .

*Variational methods.* In these methods, the restored function minimizes a criterion  $\mathcal{F}_v$  of the form

$$\mathcal{F}_v(u) = \rho \int_{\Omega} \psi(u(t), v(t)) dt + \int_{\Omega} \varphi(|\nabla u(t)|) dt, \quad (9)$$

where  $\psi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  measures closeness to data and  $\varphi(|\nabla u(\cdot)|)$  introduces priors via a trade-off parameter  $\rho > 0$ . A classical choice is  $\psi = (u(\cdot) - v(\cdot))^2$ . It is usually required that the potential function  $\varphi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  promotes images involving edges. Analysing the minimizers of  $\mathcal{F}_v$  as solutions of PDE's on  $\Omega$ , Rudin, Osher and Fatemi [25] exhibited that  $\varphi(|\nabla u(t)|) = |\nabla u(t)|$ , leads to such images, where for any  $z(t) = (z_1(t), z_2(t)) \in \mathbf{R}^2, t \in \Omega$ , one sets  $|z(t)| \stackrel{\text{def}}{=} \sqrt{z_1(t)^2 + z_2(t)^2}$ . The resulting regularization term is known as Total Variation (TV) and will be denoted by  $\|\cdot\|_{\text{TV}}$ . However, whatever smooth data-fitting is chosen, this regularization yields images containing numerous constant regions (called stair-casing effect), hence textures and fine details are removed, see [22]. The method in [2] is of this kind and operates in the image domain; the fitting term is derived

from (3) and the denoised image  $\hat{S}$ , defined by

$$\hat{S} = \arg \min_{\Sigma} \mathcal{F}_S \quad \text{for} \quad \mathcal{F}_S(\Sigma) = \rho(K) \int (\log \Sigma(t) + S(t)/\Sigma(t)) dt + \|\Sigma\|_{\text{TV}}, \quad (10)$$

exhibits constant regions (see section 5). In [26], the regularization  $\|\Sigma\|_{\text{TV}}$  is changed into  $\|\log \Sigma\|_{\text{TV}}$  so as to reformulate the model as a convex problem, and not to over smooth the image parts with higher gray values. To recover the denoised image, we applied  $\hat{S} \propto \exp(\hat{u})$  for

$$\hat{u} = \arg \min_u \quad \text{where} \quad F_v(u) = \rho\|u - v\|^2 + \|u\|_{\text{TV}}. \quad (11)$$

Following [25], various edge-preserving convex functions  $\varphi$  have been proposed; see [3] for a recent overview. Even though  $\varphi'(0) = 0$  alleviates stair-casing, a systematic drawback of the resulting restored images is that the amplitude of edges is underestimated; thus neat edges or spiky areas are subjected to erosion.

## 2.2 Hybrid methods

Hybrid methods, see e.g. [5, 9, 14, 19], combine the information contained in the large coefficients  $y[i]$  obtained according to (6) with priors directly on the image  $u$ . They amount to define the restored function  $\hat{u}$  by

$$\text{minimize } \Phi(u) \quad \text{subject to } \hat{u} \in \{u : |(W(u - v)) [i]| \leq \mu_i, \forall i \in I\}.$$

Using an edge-preserving regularization, such as  $\Phi = \text{TV}$  is a pertinent choice. The selection of parameters  $\{\mu_i\}_{i \in I}$  is more tricky. This choice must take into account the magnitude of the relevant data coefficient  $y[i]$ . However, choosing  $\mu_i$  based solely on  $y[i]$ , as done in these papers, is too rigid since there are either correct data coefficients that incur smoothing ( $\mu_i > 0$ ), or noisy coefficients that are left unchanged ( $\mu_i = 0$ ). A good compromise that we adopt is to determine  $(\mu_i)_{i \in I}$  based both on the data and on the prior term.

## 2.3 A specialized hybrid criterion

Given the log-data  $v$  obtained by (4), we apply a frame transform as in (6) to get  $y = Wv = Wu_0 + Wn$ . The noise contained in the  $i$ -th datum reads  $\langle n, w_i \rangle$ . The low frequency approximation coefficients carry important information on the image. Therefore, a good choice is to keep them intact at this stage. Let  $I_* \subset I$  denote the subset of all such elements of the frame. Then we apply a hard-thresholding operator  $\mathcal{T}_H$  [12] to all coefficients  $I \setminus I_*$ :

$$y_{\mathcal{T}_H}[i] \stackrel{\text{def}}{=} \mathcal{T}_H(y[i]), \quad \forall i \in I \setminus I_*, \quad \text{where} \quad \mathcal{T}_H(t) = \begin{cases} 0 & \text{if } |t| \leq T, \\ t & \text{otherwise,} \end{cases} \quad (12)$$

where  $T$  is an *underoptimal* threshold in order to preserve the information relevant to edges and to some fine details in textured areas, contained in the small coefficients. Let us consider

$$v_{\mathcal{T}_H} = \sum_{i \in I_1} Wv[i] \tilde{w}_i, \quad \text{where} \quad I_1 = \{i \in I : |y[i]| > T\} \cup I_*. \quad (13)$$

The image  $v_{\mathcal{T}_H}$  contains a lot of artifacts with the shape of the  $\tilde{w}_i$  for those  $y[i]$  that are noisy but above the threshold  $T$ , as well as information on the fine

details in the original log-image  $u_0$ . In all cases, whatever the choice of  $T$ , an image of the form  $v_{\mathcal{T}_H}$  is unsatisfactory—see Fig. 2.

The denoised coefficients, denoted by  $\hat{x}$ , are obtained based on the under-thresholded data  $y_{\mathcal{T}_H}$ . We focus on hybrid methods of the form:  $\hat{x} = \arg \min_x F(x)$  for  $F(x) = \Psi(x, y_{\mathcal{T}_H}) + \Phi(\widetilde{W}x)$ , where  $\Psi$  is a data-fitting term in the frame domain and  $\Phi$  is an edge-preserving regularization term in the log-image domain. Let us denote

$$I_0 = I \setminus I_1 = \{i \in I \setminus I_* : |y[i]| \leq T\}. \quad (14)$$

Coefficients  $y[i]$  for  $i \in I_0$  can be of the two types. 1. Coefficients  $y[i]$  bearing mainly noise—then the best choice is  $\hat{x}[i] = 0$ ; 2. Coefficients  $y[i]$  relevant to edges and other details in  $u_0$ . Since  $y[i]$  is difficult to distinguish from the noise, the relevant  $\hat{x}[i]$  should be restored using the edge-preserving prior  $\Phi$ . Note that a careful restoration must find a nonzero  $\hat{x}[i]$  in order to avoid Gibbs-like oscillations in  $\hat{u}$ .

Coefficients  $y[i]$  for  $i \in I_1$  are of the following two types. 1. Large coefficients which carry the main features of the sought-after function. They verify  $y[i] \approx \langle w_i, u_0 \rangle$  and can be kept intact. 2. Coefficients highly contaminated by noise, i.e.  $|y[i]| \gg |\langle w_i, u_0 \rangle|$ . We call them *outliers* because if we had  $\hat{x}[i] = y[i]$ , then  $\hat{u}$  would contain an artifact with the shape of  $\tilde{w}_i$  since by (13) we get  $v_{\mathcal{T}_H} = \sum_{j \setminus i} \hat{x}[j] \tilde{w}_j + y[i] \tilde{w}_i$ . Instead,  $\hat{x}[i]$  must be restored according to the prior  $\Phi$ .

This analysis clearly defines the goals that the minimizer  $\hat{x}$  of  $F$  is expected to achieve: restored coefficients  $\hat{x}[i]$  have to fit  $y_{\mathcal{T}_H}[i]$  exactly if they are coherent with the prior  $\Phi$ , otherwise they have to be restored according to  $\Phi$ . Since [21] it is known that such requirements can be satisfied by criteria  $F$  where  $\Psi$  is non-smooth at the origin (e.g.  $\ell^1$ ), see also [13]. For these reasons, we focus on

$$F(x) = \Psi(x) + \Phi(x), \quad (15)$$

where, for  $\Lambda = \text{diag}(\lambda_i)_{i \in I}$ ,

$$\Psi(x) = \sum_{i \in I_1 \cup I_*} \lambda_i |(x - y)[i]| + \sum_{i \in I_0} \lambda_i |x[i]| = \|\Lambda(x - y_{\mathcal{T}_H})\|_1, \quad (16)$$

$$\Phi(x) = \int_{\Omega} |\nabla \widetilde{W}x| ds = \|\widetilde{W}x\|_{\text{TV}}. \quad (17)$$

In the pre-processing step (12) we do not recommend the use of a shrinkage function other than  $\mathcal{T}_H$  since it will alter *all* the data coefficients without restoring them faithfully. Via  $\mathcal{T}_H$ , we base our restoration on data  $y_{\mathcal{T}_H}$  where all non-thresholded coefficients keep the original information on the sought-after image. The theorem stated next addresses the existence and the uniqueness of a minimizer for  $F$ . Given  $y$ , let  $G_y$  be the (convex) set of all minimizers of  $F$ :

$$G_y \stackrel{\text{def}}{=} \{\hat{x} \in \ell^2(I) : F(\hat{x}) = \min_{x \in \ell^2(I)} F(x)\}. \quad (18)$$

**Theorem 1.** [13] *For  $y \in \ell^2(I)$  and  $T > 0$  given, consider  $F$  as defined in (15), where  $\Omega \in \mathbb{R}^2$  is open, bounded and its boundary  $\partial\Omega$  is Lipschitz. Suppose that  $\{w_i\}_{i \in I}$  is a frame of  $L^2(\Omega)$  and the operator  $\widetilde{W}$  is the pseudo-inverse of  $W$ . Assume also that  $\lambda_{\min} = \min_{i \in I} \lambda_i > 0$ . Then  $G_y$  is nonempty, and for all*

*$\hat{x}_1, \hat{x}_2 \in G_y$ ,  $\nabla \widetilde{W} \hat{x}_1 \propto \nabla \widetilde{W} \hat{x}_2$ , a.e. on  $\Omega$ .*

In words,  $\hat{S}_1 = \widetilde{W}\hat{x}_1$  and  $\hat{S}_2 = \widetilde{W}\hat{x}_2$  have the same level lines, i.e. they differ by a local change of contrast; the latter is usually invisible to the naked eye. The choice of  $\lambda_i$  is investigated in [13]. Following this analysis, we use only two values for  $\lambda_i$ , depending only on the set  $I_\epsilon$  the index  $i$  belongs to. We focus on curvelets transforms of the log-data because (a) such a transform captures efficiently the main features of the data and (b) it is a tight-frame which is helpful for the subsequent numerical stage.

### 3 Minimization for the log-image

Let  $\Gamma_0(\mathcal{H})$  denote the class of proper lower-semicontinuous convex functions on a Hilbert space  $\mathcal{H}$ . Now we focus on the minimization problem

$$\text{find } \hat{x} \text{ such that } F(\hat{x}) = \min_x F \text{ for } F = \Psi + \Phi, \quad (19)$$

where  $\Psi$  and  $\Phi$  are defined in (16)-(17). Clearly,  $\Psi, \Phi \in \Gamma_0(\ell_2(I))$ , hence  $F \in \Gamma_0(\ell_2(I))$ . The set  $G_y$  in (18) is non-empty by Theorem 1 and can be rewritten as  $G_y = \{x \in \ell^2(I) \mid x \in (\partial F)^{-1}(0)\}$ , where  $\partial F$  stands for subdifferential. Minimizing  $F$  amounts to finding a solution to the fixed point equation

$$x = (\text{Id} + \gamma\partial F)^{-1}(x), \quad (20)$$

where  $(\text{Id} + \gamma\partial F)^{-1}$  is the *resolvent operator* associated to  $\partial F$ ,  $\gamma > 0$  is the proximal stepsize and  $\text{Id}$  is the identity map on  $\ell^2(I)$ . Since  $(\text{Id} + \gamma(\partial\Psi + \partial\Phi))^{-1}$  cannot be calculated in closed-form, we focus on splitting methods that use separately the resolvent operators  $(\text{Id} + \gamma\partial\Psi)^{-1}$  and  $(\text{Id} + \gamma\partial\Phi)^{-1}$ .

#### 3.1 Specialized Douglas-Rachford (D-R) splitting algorithm

The D-R family is the most general class of monotone operator splitting methods. Given a sequence  $\mu_t \in (0, 2)$ , D-R methods can be expressed via the recursion

$$x^{(t+1)} = \left[ \left(1 - \frac{\mu_t}{2}\right) \text{Id} + \frac{\mu_t}{2} (2(\text{Id} + \gamma\partial\Psi)^{-1} - \text{Id}) \circ (2(\text{Id} + \gamma\partial\Phi)^{-1} - \text{Id}) \right] x^{(t)}. \quad (21)$$

Since problem (19) has solutions, we have the following convergence result:

**Theorem 2.** *Let  $\gamma > 0$  and  $\mu_t \in (0, 2)$  be such that  $\sum_{t \in \mathbb{N}} \mu_t(2 - \mu_t) = +\infty$ . Take  $x^{(0)} \in \ell^2(I)$  and consider the sequence of iterates defined by (21). Then,  $(x^{(t)})_{t \in \mathbb{N}}$  converges weakly to some point  $\hat{x} \in \ell^2(I)$  and  $(\text{Id} + \gamma\partial\Phi)^{-1}(\hat{x}) \in G_y$ .*

The statement follows from [10, Corollary 5.2]. The sequence  $\mu_t = 1, \forall t \in \mathbb{N}$  fits.

#### 3.2 Proximal calculus

Proximity operators, invented in [20], generalize convex projection.

**Definition 1 (Moreau [20]).** *Let  $\varphi \in \Gamma_0(\mathcal{H})$ . Then  $\forall x \in \mathcal{H}$  the function  $z \mapsto \varphi(z) + \|x - z\|^2/2$ , for  $z \in \mathcal{H}$ , achieves its infimum at a unique point denoted by  $\text{prox}_\varphi x$ . The relevant operator  $\text{prox}_\varphi: \mathcal{H} \rightarrow \mathcal{H}$  is the proximity operator of  $\varphi$ .*

By the minimality condition for  $\text{prox}_\varphi$ , it is easy to see that  $\forall x, p \in \mathcal{H}$  we have  $p = \text{prox}_\varphi x \iff x - p \in \partial\varphi(p) \iff (\text{Id} + \partial\varphi)^{-1} = \text{prox}_\varphi$ . By introducing the reflection operator  $\text{rprox}_\varphi \stackrel{\text{def}}{=} 2\text{prox}_\varphi - \text{Id}$ , the D-R iteration (21) reads

$$x^{(t+1)} = \left[ \left(1 - \frac{\mu_t}{2}\right) \text{Id} + \frac{\mu_t}{2} \text{rprox}_{\gamma\Psi} \circ \text{rprox}_{\gamma\Phi} \right] x^{(t)}. \quad (22)$$

### Proximity operator of $\Psi$

**Lemma 1.** *Let  $x \in \ell^2(I)$ . Then  $\text{prox}_{\gamma\Psi}(x) = \left( y_{\mathcal{T}_H}[i] + \mathcal{T}_S^{\gamma\lambda_i} (x[i] - y_{\mathcal{T}_H}[i]) \right)_{i \in I}$ , where  $\mathcal{T}_S^{\gamma\lambda_i}(z[i]) = \max\{0, z[i] - \gamma\lambda_i \text{sign}(z[i])\}$ .*

The proof is quite standard and can be found in our Report [15]. Note that

$$\text{rprox}_{\gamma\Psi}(x) = 2 \left( y_{\mathcal{T}_H}[i] + \mathcal{T}_S^{\gamma\lambda_i} (x[i] - y_{\mathcal{T}_H}[i]) \right)_{i \in I} - x. \quad (23)$$

**Proximity operator of  $\Phi$ .** Clearly,  $\Phi(x) = \|\cdot\|_{\text{TV}} \circ \widetilde{W}(x)$ . Computing  $\text{prox}_{\gamma\Phi}$  for an arbitrary  $\widetilde{W}$  may be intractable. We assume that

- (w1)  $\widetilde{W} : \ell^2(I) \rightarrow L^2(\Omega)$  is surjective;
- (w2)  $\widetilde{W}W = \text{Id}$  and  $\widetilde{W} = c^{-1}W^*$  for  $0 < c < \infty$ ; note that  $W^*W = c \text{Id}$ ;
- (w3)  $\widetilde{W}$  is bounded.

Let  $\mathcal{X} = L^2(\Omega) \times L^2(\Omega)$ ,  $\langle \cdot, \cdot \rangle_{\mathcal{X}}$  be the inner product in  $\mathcal{X}$  and  $\|\cdot\|_p$ ,  $p \in [1, \infty]$  the  $L^p$ -norm on  $\mathcal{X}$ . Define  $\overline{B}_\infty^\gamma(\mathcal{X})$  as the  $\gamma$ -radius closed  $L^\infty$ -ball in  $\mathcal{X}$ ,  $\overline{B}_\infty^\gamma \stackrel{\text{def}}{=} \{z \in \mathcal{X} : \|z\|_\infty \leq \gamma\} = \{z = (z_1, z_2) \in \mathcal{X} : |z(t)| \leq \gamma, \forall t \in \Omega\}$ , and  $P_{\overline{B}_\infty^\gamma(\mathcal{X})} : \mathcal{X} \rightarrow \overline{B}_\infty^\gamma(\mathcal{X})$  the associated projector.

**Lemma 2.** *Let  $x \in \ell^2(I)$  and  $\overline{B}_\infty^\gamma(\mathcal{X})$  is as defined above. Then:*

$$\text{prox}_{\gamma\Phi}(x) = \left( \text{Id} - W \circ \left( \text{Id} - \text{prox}_{c^{-1}\gamma\|\cdot\|_{\text{TV}}} \right) \circ \widetilde{W} \right) (x); \quad (24)$$

$$\text{prox}_{c^{-1}\gamma\|\cdot\|_{\text{TV}}}(u) = u - P_C(u), \quad (25)$$

$$\text{where } C = \left\{ \text{div}(z) \in L^2(\Omega) \mid z \in \mathcal{C}_c^\infty(\Omega \times \Omega), z \in \overline{B}_\infty^{\gamma/c}(\mathcal{X}) \right\}. \quad (26)$$

**Sketch of the proof.** By (w1),  $\text{range}(\widetilde{W}) = L^2(\Omega)$ . Using that  $\text{domain}(\|\cdot\|_{\text{TV}}) = L^2(\Omega)$ , we find  $\text{cone}(\text{dom}\|\cdot\|_{\text{TV}} - \text{range}\widetilde{W}) = \{0\}$ . Statement (i) follows from applying [11, Proposition 11] whose requirements are satisfied.

If  $\varphi \in \Gamma_0(L^2(\Omega))$  and  $\varphi^*$  is its convex conjugate, the Moreau decomposition [20, Proposition 4.a] asserts

$$\text{prox}_\varphi + \text{prox}_{\varphi^*} = \text{Id}. \quad (27)$$

Since the conjugate function of a norm is the indicator function of the ball of its dual norm,  $(c^{-1}\gamma\|\cdot\|_{\text{TV}})^*(z) = 0$  if  $z \in C$ ,  $+\infty$  if  $z \notin C$ . where  $C$  is given



in (26). Using Definition 1,  $\text{prox}_{(c^{-1}\gamma\|\cdot\|_{\text{TV}})^*} = P_C$ . Identifying  $c^{-1}\gamma\|\cdot\|_{\text{TV}}$  with  $\varphi$  and  $(c^{-1}\gamma\|\cdot\|_{\text{TV}})^*$  with  $\varphi^*$ , equation (27) leads to (ii)<sup>4</sup>.  $\diamond$   
From (24)-(25) we easily find that

$$\text{rprox}_{\gamma\varphi}(x) = \left(\text{Id} - 2W \circ P_C \circ \widetilde{W}\right)(x) . \quad (28)$$

**Calculation of the projection  $P_C$  in (25) on a discrete grid.** In this case,  $W$  is an  $M \times N$  tight frame with  $M = \#I \gg N = \#\Omega$  and assumption (w2) reads  $\widetilde{W}W = \text{Id}$  and  $\widetilde{W} = c^{-1}W^T$ ,  $c \in (0, \infty)$  hence  $W^TW = c \text{Id}$ . The discrete counterpart of  $\mathcal{X}$  is  $\mathcal{X} = \ell^2(\Omega) \times \ell^2(\Omega)$ . We denote the discrete gradient by  $\ddot{\nabla}$  (cf. [6] or [29]) and the discrete divergence  $\text{Div} : \mathcal{X} \rightarrow \ell^2(\Omega)$  is defined as  $\text{Div} = -\ddot{\nabla}^*$ . Moreover,  $C$  in (26) admits a simpler expression:

$$C = \left\{ \text{Div}(z) \in \ell^2(\Omega) \mid z \in \overline{B}_\infty^{\gamma/c}(\mathcal{X}) \right\} , \quad (29)$$

where  $\overline{B}_\infty^{\gamma/c}(\mathcal{X})$  is defined using the new discrete notations. The projection  $P_C$  in (25) does not admit an explicit form so we provide an iterative scheme for its calculation in the next lemma.

**Lemma 3.** *We adapt all assumptions of Lemma 2 to the new discrete setting, as explained above. Consider the forward-backward iteration*

$$z^{(t+1)} = P_{\overline{B}_\infty^1(\mathcal{X})} \left( z^{(t)} + \beta_t \ddot{\nabla} \left( \text{Div}(z^{(t)}) - cu/\gamma \right) \right) \quad (30)$$

$$\text{for } 0 < \inf_t \beta_t \leq \sup_t \beta_t < 1/4 \quad (31)$$

$$\text{where } \forall (i, j) \in \Omega , \quad P_{\overline{B}_\infty^1(\mathcal{X})}(z)[i, j] = \begin{cases} z[i, j] & \text{if } |z[i, j]| \leq 1; \\ z[i, j]/|z[i, j]| & \text{otherwise .} \end{cases} \quad (32)$$

Then

- (i)  $(z^{(t)})_{t \in \mathbb{N}}$  converges to a point  $\hat{z} \in \overline{B}_\infty^1(\mathcal{X})$ ;
- (ii)  $\left( c^{-1}\gamma \text{Div}(z^{(t)}) \right)_{t \in \mathbb{N}}$  converges to  $c^{-1}\gamma \text{Div}(\hat{z}) = (\text{Id} - \text{prox}_{c^{-1}\gamma\|\cdot\|_{\text{TV}}})(u)$ .

The proof of this lemma can be found in our Report [15].

The iteration proposed in (30) to compute the proximity operator of the TV-norm is different from the projection algorithm of [6]. A similar iteration was proposed in [7] and in some other articles. The proof we gave is however simpler as it uses known properties of proximity operators. Note that computing  $\text{prox}_{\|\cdot\|_{\text{TV}}}$  amounts to solving a discrete ROF-denoising. Our iteration to solve this problem is one possibility among others, see e.g. a recent report [4].

A crucial property of the D-R scheme (22) is its robustness to numerical errors that may occur when computing the proximity operators  $\text{prox}_\psi$  and  $\text{prox}_\varphi$ , see [10]. More precisely, let  $a_t \in \ell^2(I)$  be an error term that models the inexact computation of  $\text{prox}_{\gamma\varphi}$  in (24), as the latter is obtained through (30). If the sequence of error terms  $(a_t)_{t \in \mathbb{N}}$  and stepsizes  $(\mu_t)_{t \in \mathbb{N}}$  in Theorem 2 obey  $\sum_{t \in \mathbb{N}} \mu_t \|a_t\| < +\infty$ , then the D-R algorithm (22) converges [10, Corollary 6.2]. In our experiments, using 200 inner iterations in (30) is sufficient to satisfy this requirement.

<sup>4</sup> Note that our argument (27) to compute  $\text{prox}_{c^{-1}\gamma\|\cdot\|_{\text{TV}}}(u)$  is not used in [6], which instead uses conjugates and bi-conjugates of the objective function.

### 3.3 Bias correction to recover the sought-after image

Recall from (4) that  $u_0 = \log S_0$  and set  $\hat{u} = \widetilde{W}\hat{x}^{(N_{\text{DR}})}$  as the estimator of  $u_0$ , where  $N_{\text{DR}}$  is the number of D-R iterations in (22). Unfortunately, the estimator  $\hat{u}$  is prone to bias, i.e.  $\mathbb{E}[\hat{u}] = u_0 - b_{\hat{u}}$ . A problem that classically arises in statistical estimation is how to correct such a bias. More importantly is how this bias affects the estimate after applying the inverse transformation, here the exponential. Our goal is then to ensure that for the estimate  $\hat{S}$  of the image, we have  $\mathbb{E}[\hat{S}] = S_0$ . Expanding  $\hat{S}$  in the neighborhood of  $\mathbb{E}[\hat{u}]$ , we have

$$\mathbb{E}[e^{\hat{u}}] = \exp(\mathbb{E}[\hat{u}](1 + \text{Var}[\hat{u}]/2 + R_2)) = S_0 \exp(-b_{\hat{u}}(1 + \text{Var}[\hat{u}]/2 + R_2)), \quad (33)$$

where  $R_2$  is expectation of the Lagrange remainder in the Taylor series. One can observe that the posterior distribution of  $\hat{u}$  is nearly symmetric, hence  $R_2 \approx 0$ . Then  $b_{\hat{u}} \approx \log(1 + \text{Var}[\hat{u}]/2)$  ensures unbiasedness. Consequently, finite sample (nearly) unbiased estimates of  $u_0$  and  $S_0$  are respectively  $\hat{u} + \log(1 + \text{Var}[\hat{u}]/2)$ , and  $\exp(\hat{u})(1 + \text{Var}[\hat{u}]/2)$ .  $\text{Var}[\hat{u}]$  can be reasonably estimated by  $\psi_1(K)$ , the variance of the noise  $n$  in (4) being given in (1). Thus, given the restored log-image  $\hat{u}$ , our denoised image read:

$$\hat{S} = \exp(\hat{u})(1 + \psi_1(K)/2). \quad (34)$$

## 4 Full algorithm to suppress multiplicative noise

Piecing together Lemmas 1 and 2, and Theorem 2, we write down the full multiplicative noise removal algorithm:

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**Task:** Denoise an image  $S$  corrupted with multiplicative noise according to (2).

**Parameters:** The observed noisy image  $S$ , number of iterations  $N_{\text{DR}}$  (Douglas-Rachford outer iterations) and  $N_{\text{FB}}$  (Forward-Backward inner iterations), step-sizes  $\mu_t \in (0, 2)$ ,  $0 < \beta_t < 1/4$  and  $\gamma > 0$ , tight-frame transform  $W$  and initial threshold  $T$  (e.g.  $T = 2\sqrt{\psi_1(K)}$ ), regularization parameters  $\lambda_{0,1}$  associated to the sets  $I_{0,1}$ .

**Specific operators:**

(a)  $\mathcal{T}_S^{\gamma\lambda_i}(z) = \left( \max\{0, z[i] - \gamma\lambda_i \text{sign}(z[i])\} \right)_{i \in I}, \quad \forall z \in \mathbf{R}^{\#I}.$

(b)  $\forall (i, j) \in \Omega, \quad P_{\overline{B}_\infty^1(\mathcal{X})}(z)[i, j] = \begin{cases} z[i, j] & \text{if } |z[i, j]| \leq 1 \\ z[i, j]/|z[i, j]| & \text{else.} \end{cases}$

(c)  $\ddot{\nabla}$  and  $\text{Div}$ —the discrete versions of the continuous operators  $\nabla$  and  $\text{div}$ .

(d)  $\psi_1(\cdot)$  defined according to (1) (built-in Matlab function).

**Initialization:** Compute  $v = \log S$  and transform coefficients  $y = Wv$ . Hard-threshold  $y$  at  $T$  to get  $y_{\mathcal{T}_H}$ . Choose  $x^{(0)}$ .

**Main iteration:** **For**  $t = 1$  **to**  $N_{\text{DR}}$ ,

(1) Inverse curvelet transform of  $x^{(t)}$  according to  $u^{(t)} = \widetilde{W}x^{(t)}$ .

(2) Initialize  $z^{(0)}$ ; **For**  $s = 0$  **to**  $N_{\text{FB}} - 1$

$$z^{(s+1)} = P_{\overline{B}_\infty^1(\mathcal{X})} \left( z^{(s)} + \beta_t \ddot{\nabla} \left( \text{Div}(z^{(s)}) - \frac{c}{\gamma} u^{(t)} \right) \right).$$

(3) Set  $z^{(t)} = z^{(N_{\text{FB}})}$  and compute  $w^{(t)} = c^{-1}\gamma \text{Div}(z^{(t)})$ .

- (4) Forward curvelet transform:  $\alpha^{(t)} = Ww^{(t)}$ . (5) Compute  $r^{(t)} = \text{rprox}_{\gamma\phi}(x^{(t)}) = x^{(t)} - 2\alpha^{(t)}$ .
- (6) Find  $q^{(t)} = (\text{rprox}_{\gamma\psi} \circ \text{rprox}_{\gamma\phi}) x^{(t)} = 2 \left( y_{\mathcal{T}_H}[i] + \mathcal{T}_S^{\gamma\lambda_i}(r^{(t)}[i] - y_{\mathcal{T}_H}[i]) \right)_{i \in I} - r^{(t)}$ .
- (7) Update  $x^{(t+1)}$  :  $x^{(t+1)} = (1 - \mu_t/2) x^{(t)} + (\mu_t/2) q^{(t)}$ .
- Output:** Denoised image  $\hat{S} = \exp\left(\widetilde{W}x^{(N_{\text{DR}})}\right) (1 + \psi_1(K)/2)$ .
- 

## 5 Experiments

In all experiments, our algorithm was run using second-generation curvelet tight frame along with the following set of parameters:  $\forall t, \mu_t \equiv 1, \beta_t = 0.24, \gamma = 10$  and  $N_{\text{DR}} = 50$ . The initial threshold  $T$  was set to  $2\sqrt{\psi_1(K)}$ . For comparison purposes, some very recent multiplicative noise removal algorithms from the literature are considered: the AA algorithm [2] minimizing the criterion in (10), and the Stein-block denoising method [8] in the curvelet domain, applied on the log transformed image. The latter is a sophisticated shrinkage-based denoiser that thresholds the coefficients by blocks rather than individually, and has been shown to be nearly minimax over a large class of images in presence of various additive bounded noises. We also tried the L2-TV method where the restored log-image  $\hat{u}$  minimizes (11) and the denoised image  $\hat{S}$  involves the bias correction (34). Thanks to the bias correction, it can be seen as an improved version of the first method proposed in the recent Report [26, § 4.1]. For fair comparison, the hyperparameters for all competitors were tweaked to reach their best level of performance on each noisy realization.

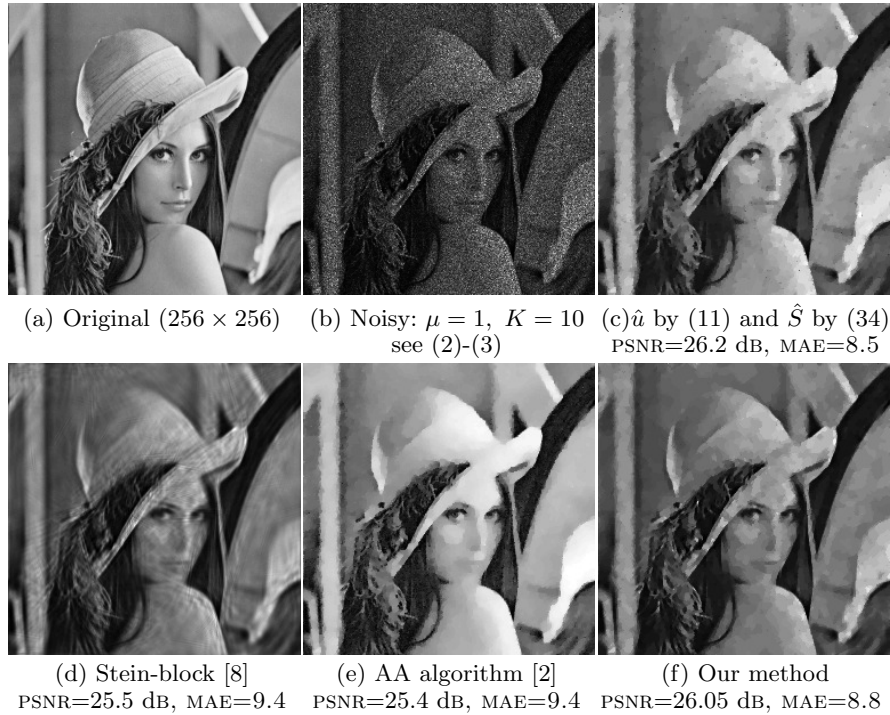
The denoising algorithms were tested on two images: Lena and Boat, all of size  $256 \times 256$  and gray-scale in the range  $[1, 256]$ . For each image, a noisy observation is generated by multiplying the original image by a realization of noise according to (2)-(3) for  $K = 10$ . The running time of our denoising method is 1 minute 3 seconds for 50 iterations on Intel 2.5 GHz Core Duo. The denoising performance of any algorithm is measured in terms of peak signal-to-noise ratio (PSNR) and mean absolute-deviation MAE, namely

$$\text{PSNR} = 20 \log_{10} \sqrt{N} \|S_0\|_{\infty} / \left\| \hat{S} - S_0 \right\|_2 \text{ dB} \quad \text{and} \quad \text{MAE} = \left\| \hat{S} - S_0 \right\|_1 / N.$$

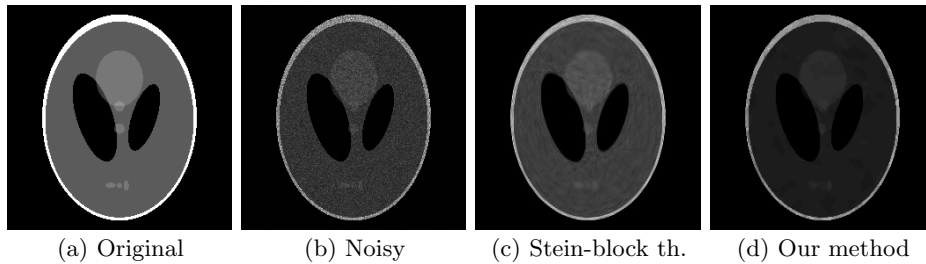
The results are depicted in Figs. 3 and 4. Note that the AA algorithm tends to over-regularize the solution. Our denoiser clearly outperforms its competitors.

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**Fig. 3.** Restoration of (b) using modern methods. Note that (c) is a slightly improved version of [26] and that the restoration in (d) is done in the curvelet domain.



**Fig. 4.** (a) Shepp-Logan phantom ( $256 \times 256$ ). (b) Noisy,  $K = 10$ . (c) Denoised with Stein-block thresholding in the curvelet domain [8] PSNR=24.73dB, MAE=4. (d) Denoised with our algorithm PSNR=31.25dB, MAE=1.87.

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