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# About a low complexity class of Cellular Automata<sup>\*</sup>

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## Résumé

Extending to all probability measures the notion of  $\mu$ -equicontinuous cellular automata introduced for Bernoulli measures by Gilman, we show that the entropy is null if  $\mu$  is an invariant measure and that the sequence of image measures of a shift ergodic measure by iterations of such automata converges in Cesàro mean to an invariant measure  $\mu_c$ . Moreover this cellular automaton is still  $\mu_c$ -equicontinuous and the set of periodic points is dense in the topological support of the measure  $\mu_c$ . The last property is also true when  $\mu$  is invariant and shift ergodic.

## 1 Introduction, definitions

Let  $A$  be a finite set. We denote by  $A^{\mathbb{Z}}$ , the set of bi-infinite sequences  $x = (x_i)_{i \in \mathbb{Z}}$  where  $x_i \in A$ . We endow  $A^{\mathbb{Z}}$  with the product topology of the discrete topologies on  $A$ . A point  $x \in A^{\mathbb{Z}}$  is called a configuration. The shift  $\sigma: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  is defined by  $\sigma(x) = (x_{i+1})_{i \in \mathbb{Z}}$ . A cellular automaton (CA) is a continuous self-map  $F$  on  $A^{\mathbb{Z}}$  commuting with the shift. The Curtis-Hedlund-Lyndon theorem states that for every cellular automaton  $F$  there exist an integer  $r$  and a block map  $f$  from  $A^{2r+1}$  to  $A$  such that  $F(x)_i = f(x_{i-r}, \dots, x_i, \dots, x_{i+r})$ . The integer  $r$  is called the radius of the cellular automaton. For integers  $i, j$  with  $i \leq j$  we denote by  $x(i, j)$  the word  $x_i \dots x_j$  and by  $x(i, \infty)$  the infinite sequence  $(v_n)_{n \in \mathbb{N}}$  such that for all  $n \in \mathbb{N}$  one has  $v_n = x_{i+n}$ . For any integer  $n \geq 0$  and point  $x \in A^{\mathbb{Z}}$ , we denote by  $B_n(x)$  the set of points  $y$  such that for all  $i \in \mathbb{N}$ , one has  $F^i(x)(-n, n) = F^i(y)(-n, n)$  and by  $C_n(x)$  the set of points  $y$  such that  $y_j = x_j$  with  $-n \leq j \leq n$ . A point  $x \in A^{\mathbb{Z}}$  is called an equicontinuous point if for all positive integer  $n$  there exists another positive integer  $m$  such that  $B_n(x) \supset C_m(x)$ . A point  $x$  is  $\mu$ -equicontinuous if for all  $m \in \mathbb{N}$  one has

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$\lim_{n \rightarrow \infty} \frac{\mu(C_n(x) \cap B_m(x))}{\mu(C_n(x))} = 1$ . In this paper, we call  $\mu$ -equicontinuous CA any cellular automaton with a set of full measure of  $\mu$ -equicontinuous points. Clearly an equicontinuous point which belongs to  $S(\mu) = \{x \in A^{\mathbb{Z}} | \mu(C_n(x)) > 0 | \forall n \in \mathbb{N}\}$ , (the topological support of  $\mu$ ) is also a  $\mu$ -equicontinuous point. When  $\mu$  is a shift ergodic measure, the existence of  $\mu$ -equicontinuous points implies that the cellular automaton is  $\mu$ -equicontinuous (see [2]).

These definitions were motivated by the work of Wolfram (see [6]) that have proposed a first empirical classification based on computer simulations. In [2] Gilman introduces a formal and measurable classification by dividing the set of CA in three parts (CA with equicontinuous points, CA without equicontinuous points but with  $\mu$ -equicontinuous points,  $\mu$ -expansive CA). The Gilman's classes are defined thanks to a Bernoulli measure not necessarily invariant and corresponds to the Wolfram's simulations based on random entry. Here we study some properties of the  $\mu$ -equicontinuous class that allows to construct easily invariant measures (see Theorem 2.1) and we try to describe what kind of dynamic characterizes  $\mu$ -equicontinuous CA when  $\mu$  is an invariant measure. Finally, remark that the comparison between equicontinuity (see some properties of this class in [1] and [4]) and  $\mu$ -equicontinuity take more sense when we study the restriction of the automaton to  $S(\mu)$  (see Section 4 for comments and examples).

## 2 Statement of the results

### 2.1 Gilman's Results

**Proposition 2.1** [3] *If  $\exists x$  and  $m \neq 0$  such that  $B_n(x) \cap \sigma^{-m}B_n(x) \neq \emptyset$  with  $n \geq r$  (the radius of the automaton  $F$ ) then the common sequence  $(F^i(y)(-n, n))_{i \in \mathbb{N}}$  of all points  $y \in B_n(x)$  is ultimately periodic.*

In [3] Gilman states the following result for any Bernoulli measure  $\mu$ . The proof uses only the shift ergodicity of these measures and can be extended to any shift ergodic measure.

**Proposition 2.2** [3] *Let  $\mu$  be a shift ergodic measure. If a cellular automaton  $F$  has a  $\mu$ -equicontinuous point, then for all  $\epsilon > 0$  there exists a  $F$ -invariant closed set  $Y$  such that  $\mu(Y) > 1 - \epsilon$  and the restriction of  $F$  to  $Y$  is equicontinuous.*

### 2.2 New Results

**Proposition 2.3** *The measure entropy  $h_\mu(F)$  of a  $\mu$ -equicontinuous and  $\mu$ -invariant cellular automaton  $F$  (with  $\mu$  not necessarily shift invariant) is equal to zero.*

**Proposition 2.4** *If a cellular automaton  $F$  has some  $\mu$ -equicontinuous points where  $\mu$  is a  $F$ -invariant and shift ergodic measure then the set of  $F$ -periodic points is dense in the topological support of  $\mu$ .*

**Theorem 2.1** *Let  $\mu$  be a shift-ergodic measure. If a cellular automaton  $F$  has some  $\mu$ -equicontinuous points then the sequence  $(\mu_n)_{n \in \mathbb{N}} = (\frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i})_{n \in \mathbb{N}}$  converges vaguely to an invariant measure  $\mu_c$ .*

**Theorem 2.2** *If  $\mu$  is a shift ergodic measure and  $F$  a  $\mu$ -equicontinuous cellular automaton then  $F$  is also a  $\mu_c$ -equicontinuous cellular automaton.*

**Corollary 2.1** *If  $\mu_c = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}$  where  $\mu$  is a shift ergodic measure and  $F$  is a cellular automaton with  $\mu$ -equicontinuous points then the set of  $F$ -periodic points is dense in  $S(\mu_c)$ .*

### 3 Proofs (Sketches)

#### 3.1 Proof of Proposition 2.3

Denote by  $(\alpha_p)_{p \in \mathbb{N}}$  the partition of  $A^{\mathbb{Z}}$  by the  $2p+1$  central coordinates and remark that  $h_\mu(F) = \lim_{p \rightarrow \infty} h_\mu(F, \alpha_p)$  where  $h_\mu(F, \alpha_p)$  denote the measurable entropy with respect to the partition  $\alpha_p$ . Using the Shannon-McMillan-Breiman Theorem, we can show that  $\forall p \in \mathbb{N}$ , there exists  $m \in \mathbb{N}$  such that  $h_\mu(F, \alpha_p) \leq \int \lim_{n \rightarrow \infty} \frac{-\log \mu(B_m(x))}{n} d\mu(x) = 0$ .

#### 3.2 Proof of Theorem 2.1

It is sufficient to show that for all  $x \in S(\mu)$  and  $m \in \mathbb{N}$  the sequence  $(\mu_n(C_m(x)))_{n \in \mathbb{N}}$  converges. From Proposition 2.2 there exists a set  $Y_\epsilon$  of measure greater than  $1 - \epsilon$  such that for all points  $y \in Y_\epsilon$  and positive integer  $k$  the sequences  $(F^n(y)(-k, k))_{n \in \mathbb{N}}$  are eventually periodic with preperiod  $pp_\epsilon(k)$  and period  $p_\epsilon(k)$ . We get that  $\mu_n(C_m(x) \cap Y_\epsilon) = \frac{1}{n} \sum_{i=0}^{pp_\epsilon(k)-1} \mu(F^{-i}(C_m(x)) \cap Y_\epsilon) + \frac{1}{n} \sum_{i=pp_\epsilon(k)}^{n-1} \mu(F^{-i}(C_m(x)) \cap Y_\epsilon)$  for all  $x \in A^{\mathbb{Z}}$  and integer  $k \geq m$ . Remark that the first term tends to 0 and the periodicity of the second one implies that  $\lim_{n \rightarrow \infty} \mu_n(C_m(x) \cap Y_\epsilon) = \frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu(F^{-(i+pp_\epsilon(k))}(C_m(x) \cap Y_\epsilon))$ . Moreover we have  $\lim_{\epsilon \rightarrow 0} \mu_n(C_m(x) \cap Y_\epsilon) = \mu_n(C_m(x))$ . Since for all  $x$  and  $m \in \mathbb{N}$  one has  $|\mu_n(C_m(x) \cap Y_\epsilon) - \mu_n(C_m(x))| \leq \frac{n\epsilon}{n} = \epsilon$  the convergence is uniform with respect to  $\epsilon$ . It follows that we can reverse the limits and obtain that

$$\begin{aligned} \mu_c &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}(C_m(x)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \lim_{\epsilon \rightarrow 0} \mu \circ F^{-i}(C_m(x) \cap Y_\epsilon) \\ &= \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}(C_m(x) \cap Y_\epsilon) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu(F^{-(i+pp_\epsilon(k))}(C_m(x) \cap Y_\epsilon)) = \mu_c(C_m(x)). \end{aligned}$$

The invariance of converging subsequences of  $(\mu_n)_{n \in \mathbb{N}}$  is a classical result.  $\square$

### 3.3 Proof of Proposition 2.4

Since  $\mu$  is a shift ergodic measure and there exist a  $\mu$ -equicontinuous points  $x$ , for all  $m \in \mathbb{N}$  and  $z \in S(\mu)$  there exist  $(i, j) \in \mathbb{N}^2$  such that  $\mu(C_p(z) \cap \sigma^{-(i+p)}B_r(x) \cap \sigma^{j+p}B_r(x) =: S) > 0$  ( $r$  is the radius of the CA). From the Poincaré recurrence theorem, for all  $z \in S(\mu)$ , there exists  $m \in \mathbb{N}$  and  $y \in S$  such that  $F^m(y)(-r-p-i, j+p-r-1) = y(-r-p-i, j+p-r-1)$ . From the Proof of Proposition 2.1 (see [3]), the shift periodic point  $\bar{w} = \dots w w w \dots$  such that  $\bar{w}(-r-p-i, j+p-r-1) = w = y(-r-p-i, j+p-r-1)$  belongs to  $S$  and since the  $F$  orbit of each  $y' \in S \cap \{y'' \in A^{\mathbb{Z}} \mid y''_l = y_l \mid (-r-p-i \leq l \leq j+p-r-1)\}$  share the same central coordinates, it follows that  $F^m(\bar{w})(-r-p-i, j+p-r-1) = w = \bar{w}(-r-p-i, j+p-r-1)$  which implies that  $F^m(\bar{w}) = \bar{w}$  and permit to conclude.

### 3.4 Proof of Theorem 2.2 and Corollary 2.1

Let  $x$  be a  $\mu$ -equicontinuous point. For all  $m \in \mathbb{N}$ , define  $Y_m := \cup_{i,j \in \mathbb{N}^2} (\sigma^{-i-m}B_r(x) \cap \sigma^{j+m}B_r(x))$  ( $r$  is the radius of  $F$ ) and  $\Omega_m = \lim_{n \rightarrow \infty} \cap_{j=0}^n \cup_{i=j}^{\infty} F^i(Y_m)$  (the omega-limit set of  $Y_m$  under  $F$ ). Since  $\mu$  is a shift ergodic measure and  $\mu(B_r(x)) > 0$ , for all  $m \in \mathbb{N}$ , we get that  $\mu(Y_m) = 1$  and consequently  $\mu_c(\Omega_m) = 1$ . Let  $\Lambda(F)$  be the omega-limit set of  $A^{\mathbb{Z}}$ . Using the eventual periodicity of  $(F^n(x)(-r, r))_{n \in \mathbb{N}}$  (see Proposition 2.1), it can be proved that the omega-limit set of  $B_r(x)$  is a finite union of sets  $B_r(z_l) \cap \Lambda(F)$  ( $0 \leq l \leq p-1$ ). This implies that  $\Omega_m = \cup_{z \in [z_0 \dots z_{p-1}]} \cup_{i,j \in \mathbb{N}^2} (\sigma^{-i-m}B_r(z) \cap \sigma^{j+m}B_r(z)) \cap \Lambda(F)$  and it follows that for all  $z \in S(\mu_c)$  and  $k \in \mathbb{N}$ , the inequality  $\mu_c(C_k(z) \cap \Omega_k) > 0$  implies that there always exist a point  $z'$  and integers  $i, j \geq m$  such that  $\mu_c(C_p(z) \cap \sigma^{-(i+p)}B_r(z') \cap \sigma^{j+p}B_r(z')) > 0$ . Using final arguments of the proof of Proposition 2.4, the last inequality is sufficient to show Corollary 2.1. For any measurable set  $E$ , define  $E^{\mu_c} = \{y \in E \mid \lim_{n \rightarrow \infty} \frac{\mu_c(C_n(y) \cap E)}{\mu_c(C_n(y))} = 1\}$ . For all  $m \in \mathbb{N}$ , define  $\Omega'_m := \cup_{z \in [z_0 \dots z_{p-1}]} \cup_{i,j \in \mathbb{N}^2} (\sigma^{-i-m}B_r(z) \cap \sigma^{j+m}B_r(z))^{\mu_c} \cap \Lambda(F)$  and denote by  $\Omega$  the set  $\cap_{m \in \mathbb{N}} \Omega'_m$ . Since for all measurable set  $E$ , one has  $\mu_c(E^{\mu_c}) = \mu_c(E)$ , for all  $m \in \mathbb{N}$ , we get that  $\mu_c(\Omega'_m) = 1$  and consequently  $\mu_c(\Omega) = 1$ . Since for all  $y \in \Omega$  and  $k \in \mathbb{N}$  there exist integers  $i, j \geq k$  and a point  $z'$  such that  $y \in \sigma^{-i}(B_r(z') \cap \sigma^j B_r(z'))^{\mu_c}$ , we obtain that  $y \in B_m^{\mu_c}(y)$  which finish the proof.

## 4 Example of $\mu$ -equicontinuous CA without equicontinuous points

In [2] Gilman gives an example of a  $\mu$ -equicontinuous CA  $F_s$  that has no equicontinuous points. The automaton  $F_s$  act on  $\{0, 1, 2\}^{\mathbb{Z}}$  and is defined thank to the following block map of radius 1 :

$$\left| \begin{array}{c|c|c|c|c|c|c|c|c|c} *00 & *01 & *02 & *10 & *11 & *12 & *20 & *21 & *22 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 2 \end{array} \right|$$

The letter  $*$  stands for any letter in  $\{0, 1, 2\}$ . Considering 0 as a background element, the 2's move straight down, 1's move to the left and 1 and 2 collide annihilate each other. In this case the measure  $\mu$  is a Bernoulli measure on  $\{0, 1, 2\}^{\mathbb{Z}}$  and the existence of  $\mu$ -equicontinuous points depends on the parameters  $p(0), p(1), p(2)$  of this measure. In [2] it is shown that if  $p(2) > p(1)$  then the probability that a 2 is never annihilated is positive and this implies that there exist  $\mu$ -equicontinuous points. Since the existence or non existence of a sufficient number of 1 in the right side can always modify the central coordinates one has  $C_m(x) \not\subset B_n(x)$  for all  $n, m \in \mathbb{N}$  which implies that there is no equicontinuous points.

Remark that using Theorem 2.1 and 2.2 the automaton  $F_s$  is  $\mu_c$ -equicontinuous if  $p(2) > p(1)$  but the restriction of  $F_s$  to  $S(\mu_c)$  always has equicontinuous points ( $S(\mu_c) = \{0, 2\}^{\mathbb{N}}$  and  $F : S(\mu_c) \rightarrow S(\mu_c)$  is the identity). In [5], we describe a more complex CA  $\mathcal{F}$  such that  $\mathcal{F} : S(\mu_c) \rightarrow S(\mu_c)$  is  $\mu_c$ -equicontinuous, without equicontinuous points and the invariant measure  $\mu_c$  is construct thanks to Theorem 2.1.

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