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| Jean-Yves Charbonnel. On the Commuting variety of a reductive Lie algebra.. 2012. hal-00711467v2

**HAL Id: hal-00711467**

**<https://hal.science/hal-00711467v2>**

Preprint submitted on 31 Oct 2012 (v2), last revised 29 Dec 2014 (v3)

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# ON THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA

JEAN-YVES CHARBONNEL

ABSTRACT. The commuting variety of a reductive Lie algebra  $\mathfrak{g}$  is the underlying variety of a well defined subscheme of  $\mathfrak{g} \times \mathfrak{g}$ . In this note, it is proved that this scheme is normal. In particular, its ideal of definition is a prime ideal.

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## 1. INTRODUCTION

In this note, the base field  $\mathbb{k}$  is algebraically closed of characteristic 0,  $\mathfrak{g}$  is a reductive Lie algebra of finite dimension,  $\ell$  is its rank, and  $G$  is its adjoint group.

1.1. **Notations.** • For  $V$  a module over a  $\mathbb{k}$ -algebra, its symmetric and exterior algebras are denoted by  $S(V)$  and  $\wedge(V)$  respectively. If  $E$  is a subset of  $V$ , the submodule of  $V$  generated by  $E$  is denoted by  $\text{span}(E)$ . When  $V$  is a vector space over  $\mathbb{k}$ , the grassmanian of all  $d$ -dimensional subspaces of  $V$  is denoted by  $\text{Gr}_d(V)$ .

• All topological terms refer to the Zariski topology. If  $Y$  is a subset of a topological space  $X$ , let denote by  $\overline{Y}$  the closure of  $Y$  in  $X$ . For  $Y$  an open subset of the algebraic variety  $X$ ,  $Y$  is called a *big open subset* if the codimension of  $X \setminus Y$  in  $X$  is bigger than 2. For  $Y$  a closed subset of an algebraic variety  $X$ , its dimension is the biggest dimension of its irreducible components and its codimension in  $X$  is the smallest codimension in  $X$  of its irreducible components. For  $X$  an algebraic variety,  $\mathbb{k}[X]$  is the algebra of regular functions on  $X$ .

• All the complexes considered in this note are graded complexes over  $\mathbb{Z}$  of vector spaces and their differentials are homogeneous of degree  $-1$  and they are denoted by  $d$ . As usual, the gradation of the complex is denoted by  $C_\bullet$ .

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Date: October 31, 2012.

1991 Mathematics Subject Classification. 14A10, 14L17, 22E20, 22E46 .

Key words and phrases. polynomial algebra, complex, commuting variety, Cohen-Macaulay, homology, projective dimension, depth.

- The dimension of the Borel subalgebras of  $\mathfrak{g}$  is denoted by  $\mathfrak{b}_{\mathfrak{g}}$ . Let set  $n := \mathfrak{b}_{\mathfrak{g}} - \ell$  so that  $\dim \mathfrak{g} = 2\mathfrak{b}_{\mathfrak{g}} - \ell_{\mathfrak{g}} = 2n + \ell$ .
- The dual  $\mathfrak{g}^*$  of  $\mathfrak{g}$  identifies with  $\mathfrak{g}$  by a given non degenerate, invariant, symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g} \times \mathfrak{g}$  extending the Killing form of  $[\mathfrak{g}, \mathfrak{g}]$ .
- For  $x \in \mathfrak{g}$ , let denote by  $\mathfrak{g}^x$  the centralizer of  $x$  in  $\mathfrak{g}$ . The set of regular elements of  $\mathfrak{g}$  is

$$\mathfrak{g}_{\text{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}$$

The subset  $\mathfrak{g}_{\text{reg}}$  of  $\mathfrak{g}$  is a  $G$ -invariant open subset of  $\mathfrak{g}$ . According to [V72],  $\mathfrak{g} \setminus \mathfrak{g}_{\text{reg}}$  is equidimensional of codimension 3.

- Let denote by  $S(\mathfrak{g})^{\mathfrak{g}}$  the algebra of  $\mathfrak{g}$ -invariant elements of  $S(\mathfrak{g})$ . Let  $p_1, \dots, p_{\ell}$  be homogeneous generators of  $S(\mathfrak{g})^{\mathfrak{g}}$  of degree  $d_1, \dots, d_{\ell}$  respectively. Let choose the polynomials  $p_1, \dots, p_{\ell}$  so that  $d_1 \leq \dots \leq d_{\ell}$ . For  $i = 1, \dots, d_{\ell}$  and  $(x, y) \in \mathfrak{g} \times \mathfrak{g}$ , let consider a shift of  $p_i$  in direction  $y$ :  $p_i(x + ty)$  with  $t \in \mathbb{k}$ . Expanding  $p_i(x + ty)$  as a polynomial in  $t$ , one obtains

$$(1) \quad p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y) t^m; \quad \forall (t, x, y) \in \mathbb{k} \times \mathfrak{g} \times \mathfrak{g}$$

where  $y \mapsto (m!)p_i^{(m)}(x, y)$  is the derivate at  $x$  of  $p_i$  at the order  $m$  in the direction  $y$ . The elements  $p_i^{(m)}$  defined by (1) are invariant elements of  $S(\mathfrak{g}) \otimes_{\mathbb{k}} S(\mathfrak{g})$  under the diagonal action of  $G$  in  $\mathfrak{g} \times \mathfrak{g}$ . Let remark that  $p_i^{(0)}(x, y) = p_i(x)$  while  $p_i^{(d_i)}(x, y) = p_i(y)$  for all  $(x, y) \in \mathfrak{g} \times \mathfrak{g}$ .

*Remark 1.1.* The family  $\mathcal{P}_x := \{p_i^{(m)}(x, \cdot); 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$  for  $x \in \mathfrak{g}$ , is a Poisson-commutative family of  $S(\mathfrak{g})$  by Mishchenko-Fomenko [MF78]. One says that the family  $\mathcal{P}_x$  is constructed by the *argument shift method*.

- Let  $i \in \{1, \dots, \ell\}$  be. For  $x$  in  $\mathfrak{g}$ , let denote by  $\varepsilon_i(x)$  the element of  $\mathfrak{g}$  given by

$$\langle \varepsilon_i(x), y \rangle = \frac{d}{dt} p_i(x + ty) \Big|_{t=0}$$

for all  $y$  in  $\mathfrak{g}$ . Thereby,  $\varepsilon_i$  is an invariant element of  $S(\mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  under the canonical action of  $G$ . According to [Ko63, Theorem 9], for  $x$  in  $\mathfrak{g}$ ,  $x$  is in  $\mathfrak{g}_{\text{reg}}$  if and only if  $\varepsilon_1(x), \dots, \varepsilon_{\ell}(x)$  are linearly independent. In this case,  $\varepsilon_1(x), \dots, \varepsilon_{\ell}(x)$  is a basis of  $\mathfrak{g}^x$ .

- Let denote by  $\varepsilon_i^{(m)}$ , for  $0 \leq m \leq d_i - 1$ , the elements of  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  defined by the equality:

$$(2) \quad \varepsilon_i(x + ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x, y) t^m, \quad \forall (t, x, y) \in \mathbb{k} \times \mathfrak{g} \times \mathfrak{g}$$

and let set:

$$V_{x,y} := \text{span}(\{\varepsilon_i^{(0)}(x, y), \dots, \varepsilon_i^{(d_i-1)}(x, y), i = 1, \dots, \ell\})$$

for  $(x, y)$  in  $\mathfrak{g} \times \mathfrak{g}$ .

**1.2. Main result and main idea.** Since  $\mathfrak{g}$  identifies with its dual,  $S(\mathfrak{g})$  is the algebra of polynomial functions on  $\mathfrak{g}$ . The commuting variety  $\mathcal{C}(\mathfrak{g})$  of  $\mathfrak{g}$  is the subvariety of elements  $(x, y)$  of  $\mathfrak{g} \times \mathfrak{g}$  such that  $[x, y] = 0$ . Let  $I_{\mathfrak{g}}$  be the ideal of  $S(\mathfrak{g} \times \mathfrak{g})$  generated by the functions  $(x, y) \mapsto \langle v, [x, y] \rangle$  with  $v$  in  $\mathfrak{g}$ . Then  $\mathcal{C}(\mathfrak{g})$  is the underlying subvariety of the subscheme of  $\mathfrak{g} \times \mathfrak{g}$  defined by  $I_{\mathfrak{g}}$ . It is a well known and long standing open question whether or not this scheme is reduced. The main result of this note is the following theorem:

**Theorem 1.2.** *The subscheme of  $\mathfrak{g} \times \mathfrak{g}$  defined by  $I_{\mathfrak{g}}$  is normal. Furthermore,  $I_{\mathfrak{g}}$  is a prime ideal of  $S(\mathfrak{g} \times \mathfrak{g})$ .*

According to a result of R. W. Richardson [Ri79],  $\mathcal{C}(\mathfrak{g})$  is irreducible. So the last assertion of the theorem is a consequence of the first one. In [Di79], J. Dixmier gave a partial answer to this question. In fact, he proved that  $I_{\mathfrak{g}}$  and its radical have the same part in degree 1. The main tool of our proof uses the main argument of the Dixmier's proof: for a finitely generated module  $M$  over  $S(\mathfrak{g} \times \mathfrak{g})$ ,  $M = 0$  if the codimension of its support is at least  $l + 2$  with  $l$  the projective dimension of  $M$  (see Appendix A).

Let introduce the characteristic submodule of  $\mathfrak{g}$ , denoted by  $B_{\mathfrak{g}}$ . By definition,  $B_{\mathfrak{g}}$  is a submodule of  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  and an element  $\varphi$  of  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  is in  $B_{\mathfrak{g}}$  if and only if for all  $(x, y)$  in  $\mathfrak{g} \times \mathfrak{g}$ ,  $\varphi(x, y)$  is in the sum of subspaces  $\mathfrak{g}^{ax+by}$  with  $(a, b)$  in  $\mathbb{k}^2 \setminus \{0\}$ . This is a free module of rank  $b_{\mathfrak{g}}$ . Moreover, for all  $\varphi$  in  $B_{\mathfrak{g}}$  and for all  $(x, y)$  in  $\mathfrak{g} \times \mathfrak{g}$ ,  $\langle \varphi(x, y), [x, y] \rangle = 0$ . The first step of the proof of Theorem 1.2 is the following proposition:

**Proposition 1.3.** *For  $i$  positive integer, the module  $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{b_{\mathfrak{g}}} (B_{\mathfrak{g}})$  has projective dimension at most  $i$ .*

The second step of the proof of Theorem 1.2 is the following theorem:

**Theorem 1.4.** *The ideal  $I_{\mathfrak{g}}$  is radical and its projective dimension is  $2n - 1$ .*

Then Theorem 1.2 follows easily from this theorem and [Po08, Theorem 1].

## 2. CHARACTERISTIC MODULE

For  $(x, y)$  in  $\mathfrak{g} \times \mathfrak{g}$ , let set:

$$V'_{x,y} = \sum_{(a,b) \in \mathbb{k}^2 \setminus \{0\}} \mathfrak{g}^{ax+by}$$

By definition, the characteristic module  $B_{\mathfrak{g}}$  of  $\mathfrak{g}$  is the submodule of elements  $\varphi$  of  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  such that  $\varphi(x, y)$  is in  $V'_{x,y}$  for all  $(x, y)$  in  $\mathfrak{g} \times \mathfrak{g}$ . In this section, some properties of  $B_{\mathfrak{g}}$  are given.

**2.1.** Let denote by  $\Omega_{\mathfrak{g}}$  the subset of elements  $(x, y)$  of  $\mathfrak{g} \times \mathfrak{g}$  such that  $P_{x,y}$  has dimension 2 and such that  $P_{x,y} \setminus \{0\}$  is contained in  $\mathfrak{g}_{\text{reg}}$ . According to [CMo08, Corollary 10],  $\Omega_{\mathfrak{g}}$  is a big open subset of  $\mathfrak{g} \times \mathfrak{g}$ .

**Proposition 2.1.** *Let  $(x, y)$  be in  $\mathfrak{g} \times \mathfrak{g}$  such that  $P_{x,y} \cap \mathfrak{g}_{\text{reg}}$  is not empty.*

(i) *Let  $O$  be an open subset of  $\mathbb{k}^2$  such that  $ax + by$  is in  $\mathfrak{g}_{\text{reg}}$  for all  $(a, b)$  in  $O$ . Then  $V_{x,y}$  is the sum of the  $\mathfrak{g}^{ax+by}$ 's,  $(a, b) \in O$ .*

(ii) *The spaces  $[x, V_{x,y}]$  and  $[y, V_{x,y}]$  are equal.*

(iii) *The space  $[x, V_{x,y}]$  is orthogonal to  $V_{x,y}$ . Furthermore,  $(x, y)$  is in  $\Omega_{\mathfrak{g}}$  if and only if  $[x, V_{x,y}]$  is the orthogonal complement of  $V_{x,y}$  in  $\mathfrak{g}$ .*

(iv) *The space  $V_{x,y}$  is contained in  $V'_{x,y}$ . Moreover,  $V_{x,y} = V'_{x,y}$  if  $(x, y)$  is in  $\Omega_{\mathfrak{g}}$ .*

- (v) The element  $(x, y)$  of  $\mathfrak{g} \times \mathfrak{g}$  is in  $\Omega_{\mathfrak{g}}$  if and only if  $V_{x,y}$  has dimension  $b_{\mathfrak{g}}$ .  
 (vi) For  $g$  in  $G$ , for  $i = 1, \dots, \ell$  and for  $m = 0, \dots, d_i - 1$ ,  $\varepsilon_i^{(m)}$  is a  $G$ -equivariant map.

*Proof.* (i) For pairwise different elements  $t_1, \dots, t_{d_i-1}$  of  $\mathbb{k} \setminus \{0\}$ , the  $\varepsilon_i^{(m)}(x, y)$ 's,  $m = 0, \dots, d_i - 1$  are linear combinations of the  $\varepsilon_i(x + t_j y)$ 's,  $j = 1, \dots, d_i - 1$  for  $i = 1, \dots, \ell$ . Furthermore, for all  $z$  in  $\mathfrak{g}_{\text{reg}}$ ,  $\varepsilon_1(z), \dots, \varepsilon_{\ell}(z)$  is a basis of  $\mathfrak{g}^z$  by [Ko63, Theorem 9], whence the assertion since the maps  $\varepsilon_1, \dots, \varepsilon_{\ell}$  are homogeneous.

(ii) Let  $O$  be an open subset of  $(\mathbb{k} \setminus \{0\})^2$  such that  $ax + by$  is in  $\mathfrak{g}_{\text{reg}}$  for all  $(a, b)$  in  $O$ . For all  $(a, b)$  in  $O$ ,  $[x, \mathfrak{g}^{ax+by}] = [y, \mathfrak{g}^{ax+by}]$  since  $[ax + by, \mathfrak{g}^{ax+by}] = 0$  and since  $ab \neq 0$ , whence the assertion by (i).

(iii) results from [Bol91, Theorem 2.1].

(iv) According to [Ko63, Theorem 9], for all  $z$  in  $\mathfrak{g}$  and for  $i = 1, \dots, \ell$ ,  $\varepsilon_i(z)$  is in  $\mathfrak{g}^z$ . Hence for all  $t$  in  $\mathbb{k}$ ,  $\varepsilon_i(x + ty)$  is in  $V'_{x,y}$ . So  $\varepsilon_i^{(m)}(x, y)$  is in  $V'_{x,y}$  for all  $m$ , whence  $V_{x,y} \subset V'_{x,y}$ .

Let suppose that  $(x, y)$  is in  $\Omega_{\mathfrak{g}}$ . According to [Ko63, Theorem 9], for all  $(a, b)$  in  $\mathbb{k}^2 \setminus \{0\}$ ,  $\varepsilon_1(ax + by), \dots, \varepsilon_{\ell}(ax + by)$  is a basis of  $\mathfrak{g}^{ax+by}$ . Hence  $\mathfrak{g}^{ax+by}$  is contained in  $V_{x,y}$ , whence the assertion.

(v) Since  $V_{x,y}$  only depends on  $P_{x,y}$ , one can suppose  $x$  regular. Then  $\mathfrak{g}^x$  is contained in  $V_{x,y}$  by [Ko63, Theorem 9] and  $[x, V_{x,y}]$  has dimension  $\dim V_{x,y} - \ell$ . As a result, by (iii),

$$2\dim V_{x,y} - \ell \leq \dim \mathfrak{g}$$

and the equality holds if and only if  $[x, V_{x,y}]$  is the orthogonal complement of  $V_{x,y}$  in  $\mathfrak{g}$ , whence the assertion by (iii) again.

(vi) Let  $i$  be in  $\{1, \dots, \ell\}$ . Since  $p_i$  is  $G$ -invariant,  $\varepsilon_i$  is a  $G$ -equivariant map. As a result, its 2-polarizations  $\varepsilon_i^{(0)}, \dots, \varepsilon_i^{(d_i-1)}$  are  $G$ -equivariant for the diagonal action of  $G$  in  $\mathfrak{g} \times \mathfrak{g}$ .  $\square$

**Theorem 2.2.** (i) The module  $B_{\mathfrak{g}}$  is a free module of rank  $b_{\mathfrak{g}}$  whose a basis is the sequence  $\varepsilon_i^{(0)}, \dots, \varepsilon_i^{(d_i-1)}$ ,  $i = 1, \dots, \ell$ .

(ii) For  $\varphi$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ ,  $\varphi$  is in  $B_{\mathfrak{g}}$  if and only if  $p\varphi \in B_{\mathfrak{g}}$  for some  $p$  in  $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$ .

(iii) For all  $\varphi$  in  $B_{\mathfrak{g}}$  and for all  $(x, y)$  in  $\mathfrak{g} \times \mathfrak{g}$ ,  $\varphi(x, y)$  is orthogonal to  $[x, y]$ .

*Proof.* (i) and (ii) According to Proposition 2.1,(iv),  $\varepsilon_i^{(m)}$  is in  $B_{\mathfrak{g}}$  for all  $(i, m)$ . Moreover, according to Proposition 2.1,(v), these elements are linearly independent over  $S(\mathfrak{g} \times \mathfrak{g})$  since the sum of the degrees  $d_1, \dots, d_{\ell}$  equal  $b_{\mathfrak{g}}$  by [Bou02, Ch. V, §5, Proposition 3]. Let  $\varphi$  be an element of  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  such that  $p\varphi$  is in  $B_{\mathfrak{g}}$  for some  $p$  in  $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$ . Since  $\Omega_{\mathfrak{g}}$  is a big open subset of  $\mathfrak{g} \times \mathfrak{g}$ , for all  $(x, y)$  in a dense open subset of  $\Omega_{\mathfrak{g}}$ ,  $\varphi(x, y)$  is in  $V_{x,y}$  by Proposition 2.1,(iv). According to Proposition 2.1,(v), the map

$$\Omega_{\mathfrak{g}} \longrightarrow \text{Gr}_{b_{\mathfrak{g}}}(\mathfrak{g}) \quad (x, y) \longmapsto V_{x,y}$$

is regular. So,  $\varphi(x, y)$  is in  $V_{x,y}$  for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$  and for some regular functions  $a_{i,m}$ ,  $i = 1, \dots, \ell$ ,  $m = 0, \dots, d_i - 1$  on  $\Omega_{\mathfrak{g}}$ ,

$$\varphi(x, y) = \sum_{i=1}^{\ell} \sum_{m=0}^{d_i-1} a_{i,m}(x, y) \varepsilon_i^{(m)}(x, y)$$

for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$ . Since  $\Omega_{\mathfrak{g}}$  is a big open subset of  $\mathfrak{g} \times \mathfrak{g}$  and since  $\mathfrak{g} \times \mathfrak{g}$  is normal, the  $a_{i,m}$ 's have a regular extension to  $\mathfrak{g} \times \mathfrak{g}$ . Hence  $\varphi$  is a linear combination of the  $\varepsilon_i^{(m)}$ 's with coefficients in  $S(\mathfrak{g} \times \mathfrak{g})$ . As

a result, the sequence  $\varepsilon_i^{(m)}$ ,  $i = 1, \dots, \ell$ ,  $m = 0, \dots, d_i - 1$  is a basis of the module  $B_{\mathfrak{g}}$  and  $B_{\mathfrak{g}}$  is the subset of elements  $\varphi$  of  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  such that  $p\varphi \in B_{\mathfrak{g}}$  for some  $p$  in  $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$ .

(iii) Let  $\varphi$  be in  $B_{\mathfrak{g}}$ . According to (i) and Proposition 2.1,(iii) and (iv), for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$ ,  $[x, \varphi(x, y)]$  is orthogonal to  $V_{x,y}$ . Then, since  $y$  is in  $V_{x,y}$ ,  $[x, \varphi(x, y)]$  is orthogonal to  $y$  and  $\langle \varphi(x, y), [x, y] \rangle = 0$ , whence the assertion.  $\square$

**2.2.** Let also denote by  $\langle \cdot, \cdot \rangle$  the natural extension of  $\langle \cdot, \cdot \rangle$  to the module  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ .

**Proposition 2.3.** *Let  $C_{\mathfrak{g}}$  be the orthogonal complement of  $B_{\mathfrak{g}}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ .*

(i) *For  $\varphi$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ ,  $\varphi$  is in  $C_{\mathfrak{g}}$  if and only if  $\varphi(x, y)$  is in  $[x, V_{x,y}]$  for all  $(x, y)$  in a nonempty open subset of  $\mathfrak{g} \times \mathfrak{g}$ .*

(ii) *The module  $C_{\mathfrak{g}}$  is free of rank  $b_{\mathfrak{g}} - \ell$ . Furthermore, the sequence of maps*

$$(x, y) \mapsto [x, \varepsilon_i^{(1)}(x, y)], \dots, (x, y) \mapsto [x, \varepsilon_i^{(d_i-1)}(x, y)], \quad i = 1, \dots, \ell$$

*is a basis of  $C_{\mathfrak{g}}$ .*

(iii) *The orthogonal complement of  $C_{\mathfrak{g}}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  equals  $B_{\mathfrak{g}}$ .*

*Proof.* (i) Let  $\varphi$  be in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ . If  $\varphi$  is in  $C_{\mathfrak{g}}$ , then  $\varphi(x, y)$  is orthogonal to  $V_{x,y}$  for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$ . Then, according to Proposition 2.1,(iii),  $\varphi(x, y)$  is in  $[x, V_{x,y}]$  for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$ . Conversely, let suppose that  $\varphi(x, y)$  is in  $[x, V_{x,y}]$  for all  $(x, y)$  in a nonempty open subset  $V$  of  $\mathfrak{g} \times \mathfrak{g}$ . By Proposition 2.1,(iii) again, for all  $(x, y)$  in  $V \cap \Omega_{\mathfrak{g}}$ ,  $\varphi(x, y)$  is orthogonal to the  $\varepsilon_i^{(m)}(x, y)$ 's,  $i = 1, \dots, \ell$ ,  $m = 0, \dots, d_i - 1$ , whence the assertion by Theorem 2.1.

(ii) Let  $C$  be the submodule of  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  generated by the maps

$$(x, y) \mapsto [x, \varepsilon_i^{(1)}(x, y)], \dots, (x, y) \mapsto [x, \varepsilon_i^{(d_i-1)}(x, y)], \quad i = 1, \dots, \ell$$

According to (i),  $C$  is a submodule of  $C_{\mathfrak{g}}$ . This module is free of rank  $b_{\mathfrak{g}} - \ell$  since  $[x, V_{x,y}]$  has dimension  $b_{\mathfrak{g}} - \ell$  for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$  by Proposition 2.1, (v). According to (i), for  $\varphi$  in  $C_{\mathfrak{g}}$ , for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$ ,

$$\varphi(x, y) = \sum_{i=1}^{\ell} \sum_{m=1}^{d_i-1} a_{i,m}(x, y) [x, \varepsilon_i^{(m)}(x, y)]$$

with the  $a_{i,m}$ 's regular on  $\Omega_{\mathfrak{g}}$  and uniquely defined by this equality. Since  $\Omega_{\mathfrak{g}}$  is a big open subset of  $\mathfrak{g} \times \mathfrak{g}$  and since  $\mathfrak{g} \times \mathfrak{g}$  is normal, the  $a_{i,m}$ 's have a regular extension to  $\mathfrak{g} \times \mathfrak{g}$ . As a result,  $\varphi$  is in  $C$ , whence the assertion.

(iii) Let  $\varphi$  be in the orthogonal complement of  $C_{\mathfrak{g}}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ . According to (ii), for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$ ,  $\varphi(x, y)$  is orthogonal to  $[x, V_{x,y}]$ . Hence by Proposition 2.1,(iii),  $\varphi(x, y)$  is in  $V_{x,y}$  for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$ . So, by Theorem 2.1,  $\varphi$  is in  $B_{\mathfrak{g}}$ , whence the assertion.  $\square$

Let denote by  $\mathcal{B}$  and  $\mathcal{C}$  the localizations of  $B_{\mathfrak{g}}$  and  $C_{\mathfrak{g}}$  on  $\mathfrak{g} \times \mathfrak{g}$  respectively. For  $(x, y)$  in  $\mathfrak{g} \times \mathfrak{g}$ , let  $C_{x,y}$  be the image of  $C_{\mathfrak{g}}$  by the evaluation map at  $(x, y)$ .

**Lemma 2.4.** *There exists an affine open cover  $\mathcal{O}$  of  $\Omega_{\mathfrak{g}}$  verifying the following condition: for all  $O$  in  $\mathcal{O}$ , there exist some subspaces  $E$  and  $F$  of  $\mathfrak{g}$ , depending on  $O$ , such that*

$$\mathfrak{g} = E \oplus V_{x,y} = F \oplus C_{x,y}$$

*for all  $(x, y)$  in  $O$ . Moreover, for all  $(x, y)$  in  $O$ , the orthogonal complement of  $V_{x,y}$  in  $\mathfrak{g}$  equals  $C_{x,y}$ .*

*Proof.* According to Proposition 2.1,(iii) and (v), for all  $(x, y)$  in  $\Omega_{\mathfrak{g}}$ ,  $V_{x,y}$  and  $C_{x,y}$  have dimension  $b_{\mathfrak{g}}$  and  $b_{\mathfrak{g}} - \ell$  respectively so that the maps

$$\Omega_{\mathfrak{g}} \longrightarrow \text{Gr}_{b_{\mathfrak{g}}}(\mathfrak{g}) \quad (x, y) \longmapsto V_{x,y} \quad \Omega_{\mathfrak{g}} \longrightarrow \text{Gr}_{b_{\mathfrak{g}}-\ell}(\mathfrak{g}) \quad (x, y) \longmapsto C_{x,y}$$

are regular, whence the assertion.  $\square$

### 3. TORSION AND PROJECTIVE DIMENSION

Let  $E$  and  $E^{\#}$  be the quotients of  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  by  $B_{\mathfrak{g}}$  and  $C_{\mathfrak{g}}$  respectively. For  $i$  positive integer, let denote by  $E_i$  the quotient of  $\wedge^i(E)$  by its torsion module.

**3.1.** Let  $B_{\mathfrak{g}}^*$  and  $C_{\mathfrak{g}}^*$  be the duals of  $B_{\mathfrak{g}}$  and  $C_{\mathfrak{g}}$ .

**Lemma 3.1.** (i) *The  $S(\mathfrak{g} \times \mathfrak{g})$ -modules  $E$  and  $E^{\#}$  have projective dimension at most 1.*

(ii) *The  $S(\mathfrak{g} \times \mathfrak{g})$ -modules  $E$  and  $E^{\#}$  are torsion free.*

(iii) *The modules  $C_{\mathfrak{g}}$  and  $B_{\mathfrak{g}}$  are the duals of  $E$  and  $E^{\#}$  respectively.*

(iv) *The canonical morphisms from  $E$  to  $C_{\mathfrak{g}}^*$  is an embedding.*

*Proof.* (i) By definition, the short sequences of  $S(\mathfrak{g} \times \mathfrak{g})$ -modules,

$$0 \longrightarrow B_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g} \longrightarrow E \longrightarrow 0$$

$$0 \longrightarrow C_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g} \longrightarrow E^{\#} \longrightarrow 0$$

are exact. Hence  $E$  and  $E^{\#}$  have projective dimension at most 1 since  $B_{\mathfrak{g}}$  and  $C_{\mathfrak{g}}$  are free modules by Theorem 2.2 and Proposition 2.3,(ii).

(ii) The module  $E$  is torsion free by Theorem 2.2,(ii). By definition, for  $\varphi$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ ,  $\varphi$  is in  $C_{\mathfrak{g}}$  if  $p\varphi$  is in  $C_{\mathfrak{g}}$  for some  $p$  in  $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$ , whence  $E^{\#}$  is torsion free.

(iii) According to the exact sequences of (i), the dual of  $E$  is the orthogonal complement of  $B_{\mathfrak{g}}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  and the dual of  $E^{\#}$  is the orthogonal complement of  $C_{\mathfrak{g}}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ , whence the assertion since  $C_{\mathfrak{g}}$  is the orthogonal complement of  $B_{\mathfrak{g}}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  by definition and since  $B_{\mathfrak{g}}$  is the orthogonal complement of  $C_{\mathfrak{g}}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  by Proposition 2.3,(iii).

(iv) Let  $\bar{\omega}$  be in the kernel of the canonical morphism from  $E$  to  $C_{\mathfrak{g}}^*$ . Let  $\omega$  be a representative of  $\bar{\omega}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ . According to Proposition 2.3,(iii),  $B_{\mathfrak{g}}$  is the orthogonal complement of  $C_{\mathfrak{g}}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$  so that  $\omega$  is in  $B_{\mathfrak{g}}$ , whence the assertion.  $\square$

Let  $\iota$  be the morphism

$$S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g} \longrightarrow C_{\mathfrak{g}}^* \quad v \longmapsto (\mu \mapsto \langle v, \mu \rangle)$$

**Lemma 3.2.** *The submodule  $\iota(S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g})$  of  $C_{\mathfrak{g}}^*$  has projective dimension at most 1.*

*Proof.* According to Proposition 2.3,(iii),  $B_{\mathfrak{g}}$  is the kernel of  $\iota$ , whence the assertion since  $B_{\mathfrak{g}}$  is a free module by Theorem 2.2.  $\square$

Let set

$$\varepsilon = \wedge_{i=1}^{\ell} \varepsilon_i^{(0)} \wedge \cdots \wedge \varepsilon_i^{(d_i-1)}$$

and for  $i$  positive integer, let denote by  $\theta_i$  the morphism

$$S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge^i(\mathfrak{g}) \longrightarrow \wedge^i(\mathfrak{g}) \wedge \wedge^{\mathfrak{b}_\mathfrak{g}}(\mathbf{B}_\mathfrak{g}) \quad \varphi \longmapsto \varphi \wedge \varepsilon$$

**Proposition 3.3.** *Let  $i$  be a positive integer.*

- (i) *The morphism  $\theta_i$  defines through the quotient an isomorphism from  $E_i$  onto  $\wedge^i(\mathfrak{g}) \wedge \wedge^{\mathfrak{b}_\mathfrak{g}}(\mathbf{B}_\mathfrak{g})$ .*
- (ii) *The short sequence of  $S(\mathfrak{g} \times \mathfrak{g})$ -modules*

$$0 \longrightarrow \mathbf{B}_\mathfrak{g} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow \mathfrak{g} \otimes_{\mathbb{k}} E_i \longrightarrow E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow 0$$

is exact.

*Proof.* (i) For  $j$  positive integer, let denote by  $\pi_j$  the canonical map from  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge^j(\mathfrak{g})$  to  $\wedge^j(E)$ . Let  $\omega$  be in the kernel of  $\pi_i$ . Let  $O$  be an element of the affine open cover of  $\Omega_\mathfrak{g}$  of Lemma 2.4 and let  $W$  be a subspace of  $\mathfrak{g}$  such that

$$\mathfrak{g} = W \oplus V_{x,y}$$

for all  $(x, y)$  in  $O$  so that  $\pi_1$  induces an isomorphism

$$\mathbb{k}[O] \otimes_{\mathbb{k}} W \longrightarrow \mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E$$

Moreover,  $\mathbf{B}_\mathfrak{g}$  is the kernel of  $\pi_1$ . Then, from the equality

$$\mathbb{k}[O] \otimes_{\mathbb{k}} \wedge^i(\mathfrak{g}) = \bigoplus_{j=0}^i \wedge^j(W) \wedge \mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^{i-j}(\mathbf{B}_\mathfrak{g})$$

it results that the restriction of  $\omega$  to  $O$  is in  $\mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^{i-1}(\mathfrak{g}) \wedge \mathbf{B}_\mathfrak{g}$ . Hence the restriction of  $\omega \wedge \varepsilon$  to  $O$  equals 0 and  $\omega$  is in the kernel of  $\theta_i$  since  $\wedge^i(\mathfrak{g}) \wedge \wedge^{\mathfrak{b}_\mathfrak{g}}(\mathbf{B}_\mathfrak{g})$  has no torsion as a submodule of a free module. As a result,  $\theta_i$  defines through the quotient a morphism from  $\wedge^i(E)$  to  $\wedge^i(\mathfrak{g}) \wedge \wedge^{\mathfrak{b}_\mathfrak{g}}(\mathbf{B}_\mathfrak{g})$ . Let denote it by  $\vartheta'_i$ . Since  $\wedge^i(\mathfrak{g}) \wedge \wedge^{\mathfrak{b}_\mathfrak{g}}(\mathbf{B}_\mathfrak{g})$  is torsion free, the torsion submodule of  $\wedge^i(E)$  is contained in the kernel of  $\vartheta'_i$ . Hence  $\vartheta'_i$  defines through the quotient a morphism from  $E_i$  to  $\wedge^i(\mathfrak{g}) \wedge \wedge^{\mathfrak{b}_\mathfrak{g}}(\mathbf{B}_\mathfrak{g})$ . Denoting it by  $\vartheta_i$ ,  $\vartheta'_i$  and  $\vartheta_i$  are surjective since  $\theta_i$  is too.

Let  $\bar{\omega}$  be in the kernel of  $\vartheta'_i$  and let  $\omega$  be a representative of  $\bar{\omega}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge^i(\mathfrak{g})$ . Then  $\omega \wedge \varepsilon = 0$  so that the restriction of  $\omega$  to the above open subset  $O$  is in  $\mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^{i-1}(\mathfrak{g}) \wedge \mathbf{B}_\mathfrak{g}$ . As a result, the restriction of  $\bar{\omega}$  to  $O$  equals 0. So,  $\bar{\omega}$  is in the torsion submodule of  $\wedge^i(E)$ , whence the assertion.

- (ii) By definition, the sequence

$$0 \longrightarrow \mathbf{B}_\mathfrak{g} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g} \longrightarrow E \longrightarrow 0$$

is exact. Then the sequence

$$\mathrm{Tor}_1^{S(\mathfrak{g} \times \mathfrak{g})}(E, E_i) \longrightarrow \mathbf{B}_\mathfrak{g} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow \mathfrak{g} \otimes_{\mathbb{k}} E_i \longrightarrow E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow 0$$

is exact. By definition,  $E_i$  is torsion free. As a result,  $\mathbf{B}_\mathfrak{g} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$  is torsion free since  $\mathbf{B}_\mathfrak{g}$  is a free module. Then, since  $\mathrm{Tor}_1^{S(\mathfrak{g} \times \mathfrak{g})}(E, E_i)$  is a torsion module, its image in  $\mathbf{B}_\mathfrak{g} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$  equals 0, whence the assertion.  $\square$



**3.2.** For  $i$  positive integer,  $\langle \cdot, \cdot \rangle$  has a canonical extension to  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge^i(\mathfrak{g})$  denoted again by  $\langle \cdot, \cdot \rangle$ .

**Lemma 3.4.** *Let  $i$  be a positive integer. Let  $T_i$  be the torsion module of  $E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$  and let  $T'_i$  be its inverse image by the canonical morphism  $\mathfrak{g} \otimes_{\mathbb{k}} E_i \rightarrow E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$ .*

(i) *The canonical morphism from  $\wedge^i(E)$  to  $\wedge^i(C_{\mathfrak{g}}^*)$  defines through the quotient an embedding of  $E_i$  into  $\wedge^i(C_{\mathfrak{g}}^*)$ .*

(ii) *The module of  $T'_i$  is the intersection of  $\mathfrak{g} \otimes_{\mathbb{k}} E_i$  and  $B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*)$ .*

(iii) *The module  $T'_i$  is isomorphic to  $\text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^{\#}, E_i)$ .*

*Proof.* (i) According to Lemma 3.1,(iii), there is a canonical morphism from  $\wedge^i(E)$  to  $\wedge^i(C_{\mathfrak{g}}^*)$ . Let  $\bar{\omega}$  be in its kernel and let  $\omega$  be a representative of  $\bar{\omega}$  in  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge^i(\mathfrak{g})$ . Then  $\omega$  is orthogonal to  $\wedge^i(C_{\mathfrak{g}})$  with respect to  $\langle \cdot, \cdot \rangle$ . So for  $O$  as in Lemma 2.4, the restriction of  $\omega$  to  $O$  is in  $\mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^{i-1}(\mathfrak{g}) \wedge B_{\mathfrak{g}}$ . Hence the restriction of  $\bar{\omega}$  to  $O$  equals 0. In other words,  $\bar{\omega}$  is in the torsion module of  $\wedge^i(E)$ , whence the assertion since  $\wedge^i(C_{\mathfrak{g}}^*)$  is a free module.

(ii) Since  $\wedge^i(C_{\mathfrak{g}}^*)$  is a free module, by Proposition 3.3,(ii), there is a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i & \longrightarrow & \mathfrak{g} \otimes_{\mathbb{k}} E_i & \longrightarrow & E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*) & \longrightarrow & \mathfrak{g} \otimes_{\mathbb{k}} \wedge^i(C_{\mathfrak{g}}^*) & \longrightarrow & E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*) & \longrightarrow & 0 \end{array}$$

Moreover, the tow first vertical arrows are embeddings. Hence  $T'_i$  is the intersection of  $\mathfrak{g} \otimes_{\mathbb{k}} E_i$  and  $B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*)$ .

(iii) According to the identification of  $\mathfrak{g}$  with its dual,  $\mathfrak{g} \otimes_{\mathbb{k}} E_i = \text{Hom}_{\mathbb{k}}(\mathfrak{g}, E_i)$ . Moreover, according to the short exact sequence of  $S(\mathfrak{g} \times \mathfrak{g})$ -modules

$$0 \longrightarrow C_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g} \longrightarrow E^{\#} \longrightarrow 0$$

the sequence of  $S(\mathfrak{g} \times \mathfrak{g})$ -modules

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^{\#}, E_i) \longrightarrow \text{Hom}_{\mathbb{k}}(\mathfrak{g}, E_i) \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, E_i) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^{\#}, E_i)$$

is exact. For  $\varphi$  in  $\text{Hom}_{\mathbb{k}}(\mathfrak{g}, E_i)$ ,  $\varphi$  is in the kernel of the third arrow if and only if  $C_{\mathfrak{g}}$  is contained in the kernel of  $\varphi$  or equivalently  $\varphi$  is in  $B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*)$ , whence the assertion by (ii).  $\square$

The following corollary results from Lemma 3.4.

**Corollary 3.5.** *Let  $i$  be a positive integer and let  $\bar{E}_i$  be the quotient of  $E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$  by its torsion module. Then the short sequence of  $S(\mathfrak{g} \times \mathfrak{g})$ -modules*

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^{\#}, E_i) \longrightarrow \mathfrak{g} \otimes_{\mathbb{k}} E_i \longrightarrow \bar{E}_i \longrightarrow 0$$

is exact.

**3.3.** For  $i$  positive integer, let  $d_i$  and  $d'_i$  be the projective dimensions of  $E_i$  and  $\text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i)$ .

**Lemma 3.6.** Let  $i$  be a positive integer and let  $d''_i$  be the projective dimension of  $\text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i)$ .

(i) The integer  $d'_i$  is at most  $\sup\{d''_i - 2, d_i\}$ .

(ii) For  $P$  a projective module over  $S(\mathfrak{g} \times \mathfrak{g})$ ,  $\text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P)$  is canonically isomorphic to  $A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P$  with  $A_{\mathfrak{g}}$  the quotient of  $C_{\mathfrak{g}}^*$  by  $\iota(S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g})$ .

(iii) The integer  $d''_i$  is at most  $d_i + 2$ .

*Proof.* (i) From the short exact sequence

$$0 \longrightarrow C_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g} \longrightarrow E^\# \longrightarrow 0$$

one deduces the exact sequence

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i) \longrightarrow \text{Hom}_{\mathbb{k}}(\mathfrak{g}, E_i) \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, E_i) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) \longrightarrow 0$$

whence the two short exact sequences

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i) \longrightarrow \text{Hom}_{\mathbb{k}}(\mathfrak{g}, E_i) \longrightarrow Z \longrightarrow 0$$

$$0 \longrightarrow Z \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, E_i) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) \longrightarrow 0$$

with  $Z$  the image of the arrow

$$\text{Hom}_{\mathbb{k}}(\mathfrak{g}, E_i) \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, E_i)$$

Denoting by  $d$  the projective dimension of  $Z$ , one deduces the inequalities

$$d'_i \leq \sup\{d - 1, d_i\} \quad d \leq \sup\{d''_i - 1, d_i\}$$

since  $C_{\mathfrak{g}}$  is a free module, whence the assertion.

(ii) From the short exact sequence

$$0 \longrightarrow C_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g} \longrightarrow E^\# \longrightarrow 0$$

one deduces the exact sequence

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, P) \longrightarrow \text{Hom}_{\mathbb{k}}(\mathfrak{g}, P) \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, P) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P) \longrightarrow 0$$

Since  $C_{\mathfrak{g}}$  is a free module,  $\text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, P)$  is canonically isomorphic to  $C_{\mathfrak{g}}^* \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P$  and there is a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbb{k}}(\mathfrak{g}, P) & \longrightarrow & \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, P) \\ \downarrow & & \downarrow \\ \mathfrak{g} \otimes_{\mathbb{k}} P & \xrightarrow{\iota \otimes \text{id}_P} & C_{\mathfrak{g}}^* \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P \end{array}$$

whenever the vertical arrows are isomorphisms, whence the assertion since  $P$  is projective.

(iii) Let

$$0 \longrightarrow P_{d_i} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow E_i \longrightarrow 0$$

be a projective resolution of  $E_i$ . For  $j$  nonnegative integer, let  $Z_j$  be the space of cycles of this complex. Since  $E^\#$  has projective dimension at most 1, for all module  $M$  over  $S(\mathfrak{g} \times \mathfrak{g})$ ,  $\text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^j(E^\#, M) = 0$  for all integer at least 2, whence the exact sequences

$$\begin{aligned} \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, Z_0) &\longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P_0) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) \longrightarrow 0 \\ \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, Z_j) &\longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P_j) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, Z_{j-1}) \longrightarrow 0 \end{aligned}$$

for all positive integer  $j$  and whence the long exact sequence

$$0 \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P_{d_i}) \longrightarrow \cdots \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P_0) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) \longrightarrow 0$$

Then by (ii), there is an exact sequence

$$0 \longrightarrow A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P_{d_i} \longrightarrow \cdots \longrightarrow A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P_0 \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) \longrightarrow 0$$

According to Lemma 3.2,  $A_{\mathfrak{g}}$  has projective dimension at most 2. Hence  $d_i''$  is at most  $d_i + 2$ .  $\square$

The following corollary results from Lemma 3.6,(i) and (iii) and Corollary 3.5.

**Corollary 3.7.** *Let  $i$  be a positive integer. Then  $\overline{E}_i$  has projective dimension at most  $d_i + 1$ .*

**3.4.** For  $i$  a positive integer and for  $M$  a  $S(\mathfrak{g} \times \mathfrak{g})$ -module, let consider on  $M^{\otimes i}$  the canonical action of the symmetric group  $\mathfrak{S}_i$ . For  $\sigma$  in  $\mathfrak{S}_i$ , let denote by  $\epsilon(\sigma)$  its signature. Let  $M_{\text{sign}}^{\otimes i}$  be the submodule of elements  $a$  of  $M^{\otimes i}$  such that  $\sigma.a = \epsilon(\sigma)a$  for all  $\sigma$  in  $\mathfrak{S}_i$  and let  $\delta_i$  be the endomorphism of  $M^{\otimes i}$ ,

$$a \longmapsto \delta_i(a) = \frac{1}{i!} \sum_{\sigma \in \mathfrak{S}_i} \epsilon(\sigma) \sigma.a$$

Then  $\delta_i$  is a projection of  $M^{\otimes i}$  onto  $M_{\text{sign}}^{\otimes i}$ .

For  $L$  submodule of  $C_{\mathfrak{g}}^*$ , let denote by  $L_i$  the image of  $L^{\otimes i}$  by the canonical map from  $L^{\otimes i}$  to  $(C_{\mathfrak{g}}^*)^{\otimes i}$  and let set  $L_{i,\text{sign}} := L_i \cap (C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i}$ . Let  $\overline{\wedge}^i(L)$  be the quotient of  $\wedge^i(L)$  by its torsion module. For  $i \geq 2$ , let identify  $\mathfrak{S}_{i-1}$  with the stabilizer of  $i$  in  $\mathfrak{S}_i$  and let denote by  $L_{i-1,\text{sign},1}$  the submodule of elements  $a$  of  $L_i$  such that  $\sigma.a = \epsilon(\sigma)a$  for all  $\sigma$  in  $\mathfrak{S}_{i-1}$ .

**Lemma 3.8.** *Let  $i$  be a positive integer and let  $L$  be a submodule of  $C_{\mathfrak{g}}^*$ .*

(i) *The module  $L_i$  is the quotient of  $L^{\otimes i}$  by its torsion module.*

(ii) *The module  $L_{i,\text{sign}}$  is isomorphic to  $\overline{\wedge}^i(L)$ .*

(iii) *For  $i \geq 2$ , the module  $L_{i,\text{sign}}$  is a direct factor of  $L_{i-1,\text{sign},1}$ .*

(iv) *For  $i \geq 2$ , the module  $L_{i-1,\text{sign},1}$  is isomorphic to the quotient of  $\overline{\wedge}^{i-1}(L) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$  by its torsion module.*

*Proof.* (i) Let  $L_1$  and  $L_2$  be submodules of a free module  $F$  over  $S(\mathfrak{g} \times \mathfrak{g})$ . From the short exact sequence

$$0 \longrightarrow L_2 \longrightarrow F \longrightarrow F/L_2 \longrightarrow 0$$

one deduces the exact sequence

$$\text{Tor}_{S(\mathfrak{g} \times \mathfrak{g})}^1(L_1, L_2) \longrightarrow L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L_2 \longrightarrow L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} F \longrightarrow L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} (F/L_2) \longrightarrow 0$$

Since  $F$  is free,  $L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} F$  is torsion free. Hence the kernel of the second arrow is the torsion module of  $L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L_2$  since  $\text{Tor}_{S(\mathfrak{g} \times \mathfrak{g})}^1(L_1, L_2)$  is a torsion module, whence the assertion by induction on  $i$ .

(ii) There is a commutative diagram

$$\begin{array}{ccc} L^{\otimes i} & \longrightarrow & (C_{\mathfrak{g}}^*)^{\otimes i} \\ \delta_i \downarrow & & \downarrow \delta_i \\ L_{\text{sign}}^{\otimes i} & \longrightarrow & (C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i} \end{array}$$

so that  $L_{i,\text{sign}}$  is the image of  $L_{\text{sign}}^{\otimes i}$  by the canonical morphism  $L^{\otimes i} \longrightarrow (C_{\mathfrak{g}}^*)^{\otimes i}$ , whence a commutative diagram

$$\begin{array}{ccc} L_{\text{sign}}^{\otimes i} & \longrightarrow & \wedge^i(L) \\ \downarrow & & \downarrow \\ (C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i} & \longrightarrow & \wedge^i(C_{\mathfrak{g}}^*) \end{array}$$

According to (i), the kernel of the left down arrow is the torsion module of  $L_{\text{sign}}^{\otimes i}$  so that the kernel of the right down arrow is the torsion module of  $\wedge^i(L)$  since the horizontal arrows are isomorphisms. Moreover, the image of  $L_{i,\text{sign}}$  in  $\wedge^i(C_{\mathfrak{g}}^*)$  is the image of  $\wedge^i(L)$ . Hence  $\wedge^i(L)$  is isomorphic to  $L_{i,\text{sign}}$ .

(iii) Let denote by  $Q_i$  the kernel of the endomorphism  $\delta_i$  of  $(C_{\mathfrak{g}}^*)^{\otimes i}$ . Since  $\delta_i$  is a projection onto  $(C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i}$  such that  $\delta_i(L_i)$  is contained in  $L_{i,\text{sign}}$ ,

$$(C_{\mathfrak{g}}^*)^{\otimes i} = (C_{\mathfrak{g}}^*)^{\otimes i} \oplus Q_i \quad L_i = L_{i,\text{sign}} \oplus Q_i \cap L_i$$

whence

$$L_{i-1,\text{sign},1} = L_{i,\text{sign}} \oplus Q_i \cap L_{i-1,\text{sign},1}$$

since  $L_{i,\text{sign}}$  is a submodule of  $L_{i-1,\text{sign},1}$ .

(iv) Let  $L'_i$  be the image of  $L_{i-1,\text{sign},1}$  by the canonical morphism  $(C_{\mathfrak{g}}^*)^{\otimes i} \rightarrow \wedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\mathfrak{g}}^*$ . Then  $L'_i$  is contained in  $\wedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$  since  $\wedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$  is a submodule of  $\wedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\mathfrak{g}}^*$ . Moreover, the morphism  $L_{i-1,\text{sign},1} \rightarrow L'_i$  is an isomorphism since the morphism

$$(C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes(i-1)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\mathfrak{g}}^* \longrightarrow \wedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\mathfrak{g}}^*$$

is too. From (ii), it results the commutative diagram

$$\begin{array}{ccc} L_{\text{sign}}^{\otimes(i-1)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L & \longrightarrow & \overline{\wedge^{i-1}(L)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L \\ \downarrow & & \downarrow \\ L_{i-1,\text{sign},1} & \longrightarrow & L'_i \end{array}$$

with the right down arrow surjective. According to (i), the kernel of the left down arrow is the torsion module of  $L_{\text{sign}}^{\otimes(i-1)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$ . Hence the kernel of the right down arrow is the torsion module of  $\wedge^{i-1}(L) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$ , whence the assertion.  $\square$

**Proposition 3.9.** *Let  $i$  be a positive integer. Then  $E_i$  and  $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$  have projective dimension at most  $i$ .*

*Proof.* According to Proposition 3.3,(i), the modules  $E_i$  and  $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$  are isomorphic. Let prove by induction on  $i$  that  $E_i$  has projective dimension at most  $i$ . By Lemma 3.1,(i), it is true for  $i = 1$ . Let suppose that it is true for  $i - 1$ . According to Corollary 3.7,  $\overline{E}_i$  has projective dimension at most  $i$ . Then by Lemma 3.8, for  $L = E$ ,  $E_i$  has projective dimension at most  $i$  since  $E$  is a submodule of  $C_{\mathfrak{g}}^*$  by Lemma 3.1,(iv) and since  $E_i = \overline{\wedge^i(E)}$  by Lemma 3.4,(i).  $\square$

#### 4. MAIN RESULTS

Let  $I_{\mathfrak{g}}$  be the ideal of  $S(\mathfrak{g} \times \mathfrak{g})$  generated by the functions  $(x, y) \mapsto \langle v, [x, y] \rangle$  with  $v$  in  $\mathfrak{g}$ . The nullvariety of  $I_{\mathfrak{g}}$  in  $\mathfrak{g} \times \mathfrak{g}$  is  $\mathcal{C}(\mathfrak{g})$ .

**Lemma 4.1.** *Let  $J_{\mathfrak{g}}$  be the radical of  $I_{\mathfrak{g}}$ . Then the support of  $J_{\mathfrak{g}}/I_{\mathfrak{g}}$  in  $\mathfrak{g} \times \mathfrak{g}$  is strictly contained in  $\mathcal{C}(\mathfrak{g})$ .*

*Proof.* Since  $\mathcal{C}(\mathfrak{g})$  is the nullvariety of  $I_{\mathfrak{g}}$  in  $\mathfrak{g} \times \mathfrak{g}$ . The support of  $J_{\mathfrak{g}}/I_{\mathfrak{g}}$  is contained in  $\mathcal{C}(\mathfrak{g})$ . Let  $x_0$  be a regular element of  $\mathfrak{g}$  and let  $V$  be a complement of  $\mathfrak{g}^{x_0}$  in  $\mathfrak{g}$ . Since the map

$$\mathfrak{g}_{\text{reg}} \longrightarrow \text{Gr}_{\ell}(\mathfrak{g}) \quad x \longmapsto \mathfrak{g}^x$$

is regular, for some open subset  $O'$  of  $\mathfrak{g}_{\text{reg}}$ , containing  $x_0$ ,

$$\mathfrak{g} = V \oplus \mathfrak{g}^x$$

for all  $x$  in  $O'$ . Let  $F$  be a complement of  $[x_0, V]$  in  $\mathfrak{g}$ . Then the map

$$O' \longrightarrow \text{Gr}_{2n}(\mathfrak{g}) \quad x \mapsto [x, V]$$

is regular and for some affine open subset  $O$  of  $O'$ , containing  $x_0$ ,

$$\mathfrak{g} = F \oplus [x, V]$$

for all  $x$  in  $O$ . Let set:

$$I_O := \mathbb{k}[O \times \mathfrak{g}] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} I_{\mathfrak{g}} \quad J_O := \mathbb{k}[O \times \mathfrak{g}] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} J_{\mathfrak{g}}$$

and let prove  $I_O = J_O$ . It will result that  $\{x_0\} \times \mathfrak{g}^{x_0}$  is not contained in the support of  $J_{\mathfrak{g}}/I_{\mathfrak{g}}$ .

According to the identification of  $\mathfrak{g}$  with its dual,

$$\mathbb{k}[O \times \mathfrak{g}] = \mathbb{k}[O] \otimes_{\mathbb{k}} S(\mathfrak{g})$$

Let denote by  $v_1, \dots, v_{2n}$  and  $f_1, \dots, f_{\ell}$  some basis of  $V$  and  $F$  respectively. For  $(i, j)$  in  $\mathbb{N}^{2n} \times \mathbb{N}^{\ell}$ , let  $v_{i,j}$  be the element of  $\mathbb{k}[O] \otimes_{\mathbb{k}} S(\mathfrak{g})$ ,

$$x \longmapsto v_{i,j}(x) := [x, v_1]^{i_1} \cdots [x, v_{2n}]^{i_{2n}} f_1^{j_1} \cdots f_{\ell}^{j_{\ell}}$$

so that  $v_{i,j}, (i, j) \in \mathbb{N}^{2n} \times \mathbb{N}^{\ell}$  is a basis of the  $\mathbb{k}[O]$ -module  $\mathbb{k}[O] \otimes_{\mathbb{k}} S(\mathfrak{g})$ . In particular,  $v_{i,j}$  is in  $I_O$  if

$$|i| := i_1 + \cdots + i_{2n} > 0$$

Let  $\varphi$  be in  $J_O$ . For all  $x$  in  $O$ ,  $\varphi(x)$  is in the ideal of definition of  $\mathfrak{g}^x$  in  $\mathfrak{g}$ . Since  $[x, \mathfrak{g}]$  is the orthogonal complement of  $\mathfrak{g}^x$  in  $\mathfrak{g}$ ,  $\varphi(x)$  is in the ideal of  $S(\mathfrak{g})$  generated by  $[x, V]$  for all  $x$  in  $O$ . Hence  $\varphi$  is a linear

combination with coefficients in  $\mathbb{k}[O]$  of the  $v_{i,j}$ 's with  $|i| > 0$ . As a result,  $J_O$  is contained in  $I_O$ , whence the lemma.  $\square$

Let  $d$  be the  $S(\mathfrak{g} \times \mathfrak{g})$ -derivation of the algebra  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge(\mathfrak{g})$  such that  $dv$  is the function  $(x, y) \mapsto \langle v, [x, y] \rangle$  on  $\mathfrak{g} \times \mathfrak{g}$  for all  $v$  in  $\mathfrak{g}$ . According to Theorem 2.2,(iii),  $\wedge(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$  is a subcomplex of the complex so defined. The gradation on  $\wedge(\mathfrak{g})$  induces a gradation on  $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge(\mathfrak{g})$  so that  $\wedge(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$  is a graded complex denoted by  $C_{\bullet}(\mathfrak{g})$ .

**Lemma 4.2.** *The support of the homology of  $C_{\bullet}(\mathfrak{g})$  is contained in  $\mathcal{C}(\mathfrak{g})$ .*

*Proof.* Let  $(x_0, y_0)$  be in  $\mathfrak{g} \times \mathfrak{g} \setminus \mathcal{C}(\mathfrak{g})$  and let  $v$  be in  $\mathfrak{g}$  such that  $\langle v, [x_0, y_0] \rangle \neq 0$ . For some affine open subset  $O$  of  $\mathfrak{g} \times \mathfrak{g}$ , containing  $(x_0, y_0)$ ,  $\langle v, [x, y] \rangle \neq 0$  for all  $(x, y)$  in  $O$ . Then  $dv$  is an invertible element of  $\mathbb{k}[O]$ . For  $c$  a cycle of  $\mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\bullet}(\mathfrak{g})$ ,

$$d(v \wedge c) = (dv)c$$

so that  $c$  is a boundary of  $\mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\bullet}(\mathfrak{g})$ .  $\square$

**Theorem 4.3.** (i) *The complex  $C_{\bullet}(\mathfrak{g})$  has no homology in degree bigger than  $b_{\mathfrak{g}}$ .*

(ii) *The ideal  $I_{\mathfrak{g}}$  has projective dimension  $2n - 1$ .*

(iii) *The algebra  $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$  is Cohen-Macaulay.*

(iv) *The ideal  $I_{\mathfrak{g}}$  is prime.*

(v) *The projective dimension of the module  $\wedge^n(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$  equals  $n$ .*

*Proof.* (i) Let  $Z$  be the space of cycles of degree  $b_{\mathfrak{g}} + 1$  of  $C_{\bullet}(\mathfrak{g})$ . Let set:

$$C_i := \begin{cases} C_{b_{\mathfrak{g}}+1+i}(\mathfrak{g}) & \text{if } i = 1, \dots, n-1 \\ Z & \text{if } i = 0 \end{cases}$$

$$C_{\bullet} := C_0 \oplus \dots \oplus C_{n-1}$$

Then  $C_{\bullet}$  is a subcomplex of  $C_{\bullet}(\mathfrak{g})$  whose gradation is deduced from the gradation of  $C_{\bullet}(\mathfrak{g})$  by translation by  $b_{\mathfrak{g}} + 1$ . According to Lemma 4.2, the support of its homology is contained in  $\mathcal{C}_{\mathfrak{g}}$ . In particular, its codimension in  $\mathfrak{g} \times \mathfrak{g}$  is

$$4n + 2\ell - (2n + 2\ell) = 2n = n + n - 1 + 1$$

According to Proposition 3.9, for  $i = 1, \dots, n-1$ ,  $C_i$  has projective dimension at most  $i$ . Hence, by Corollary A.3,  $C_{\bullet}$  is acyclic and  $Z$  has projective dimension at most  $2n - 2$ , whence the assertion.

(ii) and (iii) Since  $\mathbf{B}_{\mathfrak{g}}$  is a free module of rank  $b_{\mathfrak{g}}$ ,  $\wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$  is a free module of rank 1. By definition, the short sequence

$$0 \longrightarrow Z \longrightarrow \mathfrak{g} \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}}) \longrightarrow I_{\mathfrak{g}} \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}}) \longrightarrow 0$$

is exact, whence the short exact sequence

$$0 \longrightarrow Z \longrightarrow \mathfrak{g} \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}}) \longrightarrow I_{\mathfrak{g}} \longrightarrow 0$$

Moreover, by Proposition 3.9,  $\mathfrak{g} \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$  has projective dimension at most 1. Then, by (i),  $I_{\mathfrak{g}}$  has projective dimension at most  $2n - 1$ . As a result the  $S(\mathfrak{g} \times \mathfrak{g})$ -module  $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$  has projective dimension

at most  $2n$ . Then by Auslander-Buchsbaum's theorem [Bou98, §3, n°3, Théorème 1], the depth of the graded  $S(\mathfrak{g} \times \mathfrak{g})$ -module  $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$  is at least

$$4b_{\mathfrak{g}} - 2\ell - 2n = 2b_{\mathfrak{g}}$$

so that, according to [Bou98, §1, n°3, Proposition 4], the depth of the graded algebra  $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$  is at least  $2b_{\mathfrak{g}}$ . In other words,  $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$  is Cohen-Macaulay since it has dimension  $2b_{\mathfrak{g}}$ . Moreover, since the graded algebra  $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$  has depth  $2b_{\mathfrak{g}}$ , the graded  $S(\mathfrak{g} \times \mathfrak{g})$ -module  $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$  has depth  $2b_{\mathfrak{g}}$  and projective dimension  $2n$ . Hence  $I_{\mathfrak{g}}$  has projective dimension  $2n - 1$ .

(iii) According to Lemma 4.1, the support in  $\mathfrak{g} \times \mathfrak{g}$  of the module  $J_{\mathfrak{g}}/I_{\mathfrak{g}}$  is strictly contained in  $\mathcal{C}(\mathfrak{g})$ . So its codimension in  $\mathfrak{g} \times \mathfrak{g}$  is at least  $2n + 1$  since  $\mathcal{C}(\mathfrak{g})$  is irreducible. Then by (ii) and Proposition A.2.(ii),  $I_{\mathfrak{g}} = J_{\mathfrak{g}}$ , whence the assertion since  $\mathcal{C}(\mathfrak{g})$  is irreducible by [Ri79].

(iv) By (i),  $I_{\mathfrak{g}}$  has projective dimension  $2n - 1$ . Hence, according to Proposition 3.9 and according to (ii) and Corollary A.3,  $\wedge^n(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$  has projective dimension  $n$ .  $\square$

**Corollary 4.4.** *The subscheme of  $\mathfrak{g} \times \mathfrak{g}$  defined by  $I_{\mathfrak{g}}$  is normal.*

*Proof.* According to Theorem 4.3.(iii), the subscheme of  $\mathfrak{g} \times \mathfrak{g}$  defined by  $I_{\mathfrak{g}}$  is Cohen-Macaulay. According to [Po08, Theorem 1], it is smooth in codimension 1. So by Serre's normality criterion [Bou98, §1, n°10, Théorème 4], it is normal.  $\square$

#### APPENDIX A. PROJECTIVE DIMENSION AND COHOMOLOGY

Let recall in this section classical results. Let  $X$  be a Cohen-Macaulay irreducible affine algebraic variety and let  $S$  be a closed subset of codimension  $p$  of  $X$ . Let  $P_{\bullet}$  be a complex of finitely generated projective  $\mathbb{k}[X]$ -modules whose length  $l$  is finite and let  $\varepsilon$  be an augmentation morphism of  $P_{\bullet}$  whose image is  $R$ , whence an augmented complex of  $\mathbb{k}[X]$ -modules,

$$0 \longrightarrow P_l \longrightarrow P_{l-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} R \longrightarrow 0$$

Let denote by  $\mathcal{P}_{\bullet}$ ,  $\mathcal{R}$ ,  $\mathcal{K}_0$  the localizations on  $X$  of  $P_{\bullet}$ ,  $R$ , the kernel of  $\varepsilon$  respectively and let denote by  $\mathcal{K}_i$  the kernel of the morphism  $\mathcal{P}_i \longrightarrow \mathcal{P}_{i-1}$  for  $i$  positive integer.

**Lemma A.1.** *Let suppose that  $S$  contains the support of the homology of the augmented complex  $P_{\bullet}$ .*

(i) *For all positive integer  $i < p - 1$  and for all projective  $\mathcal{O}_X$ -module  $\mathcal{P}$ ,  $H^i(X \setminus S, \mathcal{P})$  equals zero.*

(ii) *For all nonnegative integer  $j \leq l$  and for all positive integer  $i < p - j$ , the cohomology group  $H^i(X \setminus S, \mathcal{K}_{l-j})$  equals zero.*

*Proof.* (i) Let  $i < p - 1$  be a positive integer. Since the functor  $H^i(X \setminus S, \bullet)$  commutes with the direct sum, it suffices to prove  $H^i(X \setminus S, \mathcal{O}_X) = 0$ . Since  $S$  is a closed subset of  $X$ , one has the relative cohomology long exact sequence

$$\cdots \longrightarrow H_S^i(X, \mathcal{O}_X) \longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow H^i(X \setminus S, \mathcal{O}_X) \longrightarrow H_S^{i+1}(X, \mathcal{O}_X) \longrightarrow \cdots$$

Since  $X$  is affine,  $H^i(X, \mathcal{O}_X)$  equals zero and  $H^i(X \setminus S, \mathcal{O}_X)$  is isomorphic to  $H_S^{i+1}(X, \mathcal{O}_X)$ . Since  $X$  is Cohen-Macaulay, the codimension  $p$  of  $S$  in  $X$  equals the depth of its ideal of definition in  $\mathbb{k}[X]$  [MA86, Ch. 6, Theorem 17.4]. Hence, according to [Gro67, Theorem 3.8],  $H_S^{i+1}(X, \mathcal{O}_X)$  and  $H^i(X \setminus S, \mathcal{O}_X)$  equal zero since  $i + 1 < p$ .

(ii) Let  $j$  be a nonnegative integer. Since  $S$  contains the support of the homology of the complex  $P_\bullet$ , for all nonnegative integer  $j$ , one has the short exact sequence of  $\mathcal{O}_{X \setminus S}$ -modules

$$0 \longrightarrow \mathcal{K}_{j+1}|_{X \setminus S} \longrightarrow \mathcal{P}_{j+1}|_{X \setminus S} \longrightarrow \mathcal{K}_j|_{X \setminus S} \longrightarrow 0$$

whence the long exact sequence of cohomology

$$\cdots \longrightarrow \mathrm{H}^i(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^i(X \setminus S, \mathcal{K}_j) \longrightarrow \mathrm{H}^{i+1}(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^{i+1}(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \cdots .$$

Then, by (i), for  $i < p - 2$  positive integer, the cohomology groups  $\mathrm{H}^i(X \setminus S, \mathcal{K}_j)$  and  $\mathrm{H}^{i+1}(X \setminus S, \mathcal{K}_{j+1})$  are isomorphic since  $\mathcal{P}_{j+1}$  is a projective module. Since  $\mathcal{P}_i = 0$  for  $i > l$ ,  $\mathcal{K}_{l-1}$  and  $\mathcal{P}_l$  have isomorphic restrictions to  $X \setminus S$ . In particular, by (i), for  $i < p - 1$  positive integer,  $\mathrm{H}^i(X \setminus S, \mathcal{K}_{l-1})$  equal zero. Then, by induction on  $j$ , for  $i < p - j$  positive integer,  $\mathrm{H}^i(X \setminus S, \mathcal{K}_{l-j})$  equals zero.  $\square$

**Proposition A.2.** *Let  $R'$  be a  $\mathbb{k}[X]$ -module containing  $R$ . Let suppose that the following conditions are verified:*

- (1)  $p$  is at least  $l + 2$ ,
- (2)  $X$  is normal,
- (3)  $S$  contains the support of the homology of the augmented complex  $P_\bullet$ .

(i) *The complex  $P_\bullet$  is a projective resolution of  $R$  of length  $l$ .*

(ii) *Let suppose that  $R'$  is torsion free and let suppose that  $S$  contains the support in  $X$  of  $R'/R$ . Then  $R' = R$ .*

*Proof.* (i) Let  $j$  be a positive integer. One has to prove that  $\mathrm{H}^0(X, \mathcal{K}_j)$  is the image of  $\mathcal{P}_{j+1}$ . By Condition (3), the short sequence of  $\mathcal{O}_{X \setminus S}$ -modules

$$0 \longrightarrow \mathcal{K}_{j+1}|_{X \setminus S} \longrightarrow \mathcal{P}_{j+1}|_{X \setminus S} \longrightarrow \mathcal{K}_j|_{X \setminus S} \longrightarrow 0$$

is exact, whence the cohomology long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_j) \longrightarrow \mathrm{H}^1(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \cdots$$

By Lemma A.1, (ii),  $\mathrm{H}^1(X \setminus S, \mathcal{K}_{j+1})$  equals 0 since  $1 < p - l + j + 1$ , whence the short exact sequence

$$0 \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_j) \longrightarrow 0$$

Since the codimension of  $S$  in  $X$  is at least 2 and since  $X$  is irreducible and normal, the restriction morphism from  $\mathcal{P}_{j+1}$  to  $\mathrm{H}^0(X \setminus S, \mathcal{P}_{j+1})$  is an isomorphism. Let  $\varphi$  be in  $\mathrm{H}^0(X, \mathcal{K}_j)$ . Then there exists an element  $\psi$  of  $\mathcal{P}_{j+1}$  whose image  $\psi'$  in  $\mathrm{H}^0(X, \mathcal{K}_j)$  has the same restriction to  $X \setminus S$  as  $\varphi$ . Since  $\mathcal{P}_j$  is a projective module and since  $X$  is irreducible,  $\mathcal{P}_j$  is torsion free. Then  $\varphi = \psi'$  since  $\varphi - \psi'$  is a torsion element of  $\mathcal{P}_j$ , whence the assertion.

(ii) Let  $\mathcal{R}'$  be the localization of  $R'$  on  $X$ . Arguing as in (i), since  $S$  contains the support of  $R'/R$  and since  $1 < p - l$ , the short sequence

$$0 \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_0) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{P}_0) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{R}') \longrightarrow 0$$

is exact. Moreover, the restriction morphism from  $\mathcal{P}_0$  to  $\mathrm{H}^0(X \setminus S, \mathcal{P}_0)$  is an isomorphism since the codimension of  $S$  in  $X$  is at least 2 and since  $X$  is irreducible and normal. Let  $\varphi$  be in  $\mathcal{R}'$ . Then for some  $\psi$  in  $\mathcal{P}_0$ ,  $\varphi - \varepsilon(\psi)$  is a torsion element of  $\mathcal{R}'$ . So  $\varphi = \varepsilon(\psi)$  since  $\mathcal{R}'$  is torsion free, whence the assertion.  $\square$



**Corollary A.3.** *Let  $C_\bullet$  be a homology complex of finite type of  $\mathbb{k}[X]$ -modules whose length  $l$  is finite and positive. For  $j = 0, \dots, l$ , let denote by  $Z_j$  the space of cycles of degree  $j$  of  $C_\bullet$ . Let suppose that the following conditions are verified:*

- (1)  $S$  contains the support of the homology of the complex  $C_\bullet$ ,
- (2) for all  $i$ ,  $C_i$  is a submodule of a free module,
- (3) for  $i = 1, \dots, l$ ,  $C_i$  has projective dimension at most  $d$ ,
- (4)  $X$  is normal and  $l + d \leq p - 1$ ,
- (5)  $Z_0$  equals  $C_0$ .

Then  $C_\bullet$  is acyclic and for  $j = 0, \dots, l$ ,  $Z_j$  has projective dimension at most  $l + d - j - 1$ .

*Proof.* Let prove by induction on  $l - j$  that the complex

$$0 \longrightarrow C_l \longrightarrow \cdots \longrightarrow C_{j+1} \longrightarrow Z_j \longrightarrow 0$$

is acyclic and that  $Z_j$  has projective dimension at most  $l + d - j - 1$ . For  $j = l$ ,  $Z_j$  equals zero since  $C_l$  is torsion free by Condition (2) and since  $Z_l$  a submodule of  $C_l$ , supported by  $S$  by Condition (1). Let suppose  $j \leq l - 1$  and let suppose the statement true for  $j + 1$ . By Condition (2),  $C_{j+1}$  has a projective resolution  $P_\bullet$  whose length is at most  $d$  and whose terms are finitely generated. By induction hypothesis,  $Z_{j+1}$  has a projective resolution  $Q_\bullet$  whose length is at most  $l + d - j - 2$  and whose terms are finitely generated, whence an augmented complex  $R_\bullet$  of projective modules whose length is  $l + d - j - 1$ ,

$$0 \longrightarrow Q_{l+d-j-2} \oplus P_{l+d-j-1} \longrightarrow \cdots \longrightarrow Q_0 \oplus P_1 \longrightarrow P_0 \longrightarrow Z_j \longrightarrow 0.$$

Denoting by  $d$  the differentials of  $Q_\bullet$  and  $P_\bullet$ , the restriction to  $Q_i \oplus P_{i+1}$  of the differential of  $R_\bullet$  is the map

$$(x, y) \mapsto (dx, dy + (-1)^i \delta(x)),$$

with  $\delta$  the map which results from the injection of  $Z_{j+1}$  into  $C_{j+1}$ . Since  $P_\bullet$  and  $Q_\bullet$  are projective resolutions, the complex  $R_\bullet$  is a complex of projective modules having no homology in positive degree. Hence the support of the homology of the augmented complex  $R_\bullet$  is contained in  $S$  by Condition (1). Then, by Proposition A.2 and Conditions (3) and (4),  $R_\bullet$  is a projective resolution of  $Z_j$  of length is  $l + d - j - 1$  since  $Z_j$  is a submodule of a free module by Condition (2), whence the corollary by Condition (5).  $\square$

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