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The bigraft algebras

Anthony Mansuy

*Laboratoire de Mathématiques, Université de Reims
Moulin de la Housse - BP 1039 - 51687 REIMS Cedex 2, France
e-mail : anthony.mansuy@univ-reims.fr*

Abstract. In this paper, we introduce the notion of bigraft algebra, generalizing the notions of left and right graft algebras. We give a combinatorial description of the free bigraft algebra generated by one generator and we endow this algebra with a Hopf algebra structure, and a pairing. Next, we study the Koszul dual of the bigraft operad and we give a combinatorial description of the free dual bigraft algebra generated by one generator. With the help of a rewriting method, we prove that the bigraft operad is Koszul. Finally, we define the notion of infinitesimal bigraft bialgebra and we prove a rigidity theorem for connected infinitesimal bigraft bialgebras.

Résumé. Dans ce papier, nous introduisons la notion d'algèbre bigreffe, généralisant les notions d'algèbres de greffes à gauche et à droite. Nous donnons une description combinatoire de l'algèbre bigreffe libre engendrée par un générateur et nous munissons cette algèbre d'une structure d'algèbre de Hopf et d'un couplage. Nous étudions ensuite le dual de Koszul de l'opérade bigreffe et nous donnons une description combinatoire de l'algèbre bigreffe dual engendrée par un générateur. A l'aide d'une méthode de réécriture, nous prouvons que l'opérade bigreffe est Koszul. Enfin, nous définissons la notion de bialgèbre bigreffe infinitésimale et nous prouvons un théorème de rigidité pour les bialgèbres bigreffe infinitésimales connexes.

Keywords. Planar rooted trees, Koszul quadratic operads, Infinitesimal Hopf algebra.

AMS Classification. 05C05, 16W30, 18D50.

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Introduction

The Connes-Kreimer Hopf algebra of rooted trees is introduced and studied in [CK1, Mo]. This commutative, non cocommutative Hopf algebra is used to study a problem of Renormalisation in Quantum Fields Theory, as explained in [CK2, CK3]. A non commutative version, the Hopf algebra of planar rooted trees, is introduced in [F2, Hol]. With a total order on the vertices, we obtain the Hopf algebra of ordered trees.

In [Me], F. Menous study some sets of probabilities, called induced averages by J. Ecalle, associated to a random walk on \mathbb{R} . For this, he constructs, with a formalism similar to mould calculus, a set of ordered trees using two grafting operators. In [Ma], the author study more generally these two grafting operators and constructs a Hopf subalgebra \mathcal{B} of the Hopf algebra of ordered trees. He equips \mathcal{B} with two operations \succ and \prec (respectively the left graft and the right graft) and, after having defined the concept of bigraft algebra, he proves that \mathcal{B} is a bigraft algebra.

The aim of this text is an algebraic study of the bigraft algebras introduced in [Ma]. A bigraft algebra (\mathcal{BG} -algebra for short) is an associative algebra $(A, *)$ equipped with a left graft \succ and a right graft \prec such that $(A, *, \succ)$ is a left graft algebra, $(A, *, \prec)$ is a right graft algebra and with the entanglement relation:

$$(x \succ y) \prec z = x \succ (y \prec z). \quad (1)$$

We construct a decorated version of the Hopf algebra of planar rooted trees denoted \mathcal{BT} (we decorate the edges with two possible decorations). We equip \mathcal{BT} with a Hopf algebra structure and a pairing. We prove that the augmentation ideal \mathcal{M} of \mathcal{BT} is the free \mathcal{BG} -algebra on one generator and we deduce the dimension of the bigraft operad \mathcal{BG} and a combinatorial description of the composition.

Afterward, we give a presentation of the Koszul dual $\mathcal{BG}^!$ of the bigraft operad \mathcal{BG} , and a combinatorial description of the free $\mathcal{BG}^!$ -algebra on one generator. We also give a presentation of the homology of a \mathcal{BG} -algebra. With the help of a rewriting method (see [DK, Hof, LV]), we prove that \mathcal{BG} is Koszul and we give the PBW basis of the bigraft operad and its Koszul dual.

We finally study the compatibilities of products $*$, \succ , \prec and the coproduct $\tilde{\Delta}_{Ass}$ of deconcatenation. This leads to the definition of infinitesimal bigraft bialgebras. We prove that the subspace corresponding to primitive elements of \mathcal{M} is a \mathcal{L} -algebra, that is to say a \mathbb{K} -vector space with two operations \succ and \prec satisfying the entanglement relation (1) (see [Le1, Le2] for more details on \mathcal{L} -algebras). We deduce a combinatorial description of the free \mathcal{L} -algebra generated by one generator. Finally, we prove that $(Ass, \mathcal{BG}, \mathcal{L})$ is a good triple of operads (see [Lo] for the notion of triple of operads).

This text is organised as follows: in the first section, we recall several facts on the Hopf algebra of planar trees and on right graft algebras. The Hopf algebra of planar decorated trees \mathcal{BT} and its pairing are introduced in section 2. In section 3, we define the notion of \mathcal{BG} -algebra and we give a combinatorial description of the free \mathcal{BG} -algebra on one generator as the augmentation ideal \mathcal{M} of the Hopf algebra \mathcal{BT} . Section 4 is devoted to the study of the Koszul dual of the bigraft operad. In particular, we prove that the bigraft operad is Koszul with the rewriting method. The last section deals with the notion of the infinitesimal bigraft bialgebras and we prove that $(Ass, \mathcal{BG}, \mathcal{L})$ is a good triple of operads.

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Notations.

1. We shall denote by \mathbb{K} a commutative field, of any characteristic. Every vector space, algebra, coalgebra, etc, will be taken over \mathbb{K} . Given a set X , we denote by $\mathbb{K}[X]$ the vector space spanned by X .
2. Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a graded vector space. We denote by $V^{\otimes} = \bigoplus_{n=0}^{\infty} V_n^*$ the graded dual of V . If H is a graded Hopf algebra, H^{\otimes} is also a graded Hopf algebra.
3. Let V and W be two \mathbb{K} -vector spaces. We note $\tau : V \otimes W \rightarrow W \otimes V$ the unique \mathbb{K} -linear map called *the flip* such that $\tau(v \otimes w) = w \otimes v$ for all $v \otimes w \in V \otimes W$.
4. Let (A, Δ, ε) be a counitary coalgebra. Let $1 \in A$, non zero, such that $\Delta(1) = 1 \otimes 1$. We then define the noncounitary coproduct:

$$\tilde{\Delta} : \begin{cases} Ker(\varepsilon) & \rightarrow Ker(\varepsilon) \otimes Ker(\varepsilon), \\ a & \mapsto \Delta(a) - a \otimes 1 - 1 \otimes a. \end{cases}$$

1 Left graft algebra and right graft algebra

1.1 Presentation

Definition 1 1. A left graft algebra (or \mathcal{LG} -algebra for short) is a \mathbb{K} -vector space A together with two \mathbb{K} -linear maps $*, \succ : A \otimes A \rightarrow A$ respectively called product and left graft, satisfying the following relations : for all $x, y, z \in A$,

$$\begin{aligned} (x * y) * z &= x * (y * z), \\ (x * y) \succ z &= x \succ (y \succ z), \\ (x \succ y) * z &= x \succ (y * z). \end{aligned}$$

2. A right graft algebra (or \mathcal{RG} -algebra for short) is a \mathbb{K} -vector space A together with two \mathbb{K} -linear maps $*, \prec : A \otimes A \rightarrow A$ respectively called product and right graft, satisfying the following relations : for all $x, y, z \in A$,

$$\begin{aligned} (x * y) * z &= x * (y * z), \\ (x \prec y) \prec z &= x \prec (y * z), \\ (x * y) \prec z &= x * (y \prec z). \end{aligned}$$

For example:

$$\Delta \left(\begin{array}{c} \downarrow \\ \vee \end{array} \right) = \begin{array}{c} \downarrow \\ \vee \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \downarrow \\ \vee \end{array} + . \otimes \vee + \downarrow \otimes \downarrow + . \otimes \downarrow + .. \otimes \downarrow + \downarrow \otimes ..$$

1.2.2 Hopf algebras of planar trees and universal property

We now recall the construction of the noncommutative generalization of the Connes-Kreimer Hopf algebra [F2, Hol].

A *planar forest* is a rooted forest F such that the set of the roots of F is totally ordered and, for any vertex $v \in V(F)$, the set $\{w \in V(F) \mid w \rightarrow v\}$ is totally ordered. Planar forests are represented such that the total orders on the set of roots and the sets $\{w \in V(F) \mid w \rightarrow v\}$ for any $v \in V(F)$ is given from left to right. We denote by \mathbb{T}_P the set of the planar trees.

Examples.

1. Planar rooted trees of degree ≤ 5 :

$$., \downarrow, \vee, \downarrow \downarrow, \vee \downarrow, \downarrow \vee, \vee \vee, \downarrow \downarrow \downarrow, \downarrow \vee \downarrow, \downarrow \downarrow \vee, \downarrow \downarrow \downarrow \downarrow, \downarrow \vee \downarrow \downarrow, \downarrow \downarrow \vee \downarrow, \downarrow \downarrow \downarrow \vee, \downarrow \downarrow \downarrow \downarrow \downarrow, \downarrow \vee \downarrow \downarrow \downarrow, \downarrow \downarrow \vee \downarrow \downarrow, \downarrow \downarrow \downarrow \vee \downarrow, \downarrow \downarrow \downarrow \downarrow \vee, \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow.$$

2. Planar rooted forests of degree ≤ 4 :

$$1, ., ., ., \downarrow, ., ., ., \downarrow \downarrow, ., \vee, \downarrow, ., ., ., \downarrow \downarrow \downarrow, ., \vee \downarrow, ., \vee \downarrow \downarrow, \downarrow \downarrow \downarrow \downarrow, \vee \downarrow, \downarrow \vee, \downarrow \downarrow \downarrow \downarrow, \downarrow \downarrow \downarrow \downarrow \downarrow.$$

If $v \models V(F)$, then $Lea_v(F)$ and $Roo_v(F)$ are naturally planar forests. It is proved in [F2] that the space \mathcal{H}_P generated by planar forests is a Hopf algebra. Its product is given by the concatenation of planar forests and its coproduct is defined for any rooted forest F by:

$$\Delta(F) = \sum_{v \models V(F)} Lea_v(F) \otimes Roo_v(F) = F \otimes 1 + 1 \otimes F + \sum_{v \models V(F)} Lea_v(F) \otimes Roo_v(F).$$

For example:

$$\begin{aligned} \Delta \left(\begin{array}{c} \downarrow \\ \vee \end{array} \right) &= \begin{array}{c} \downarrow \\ \vee \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \downarrow \\ \vee \end{array} + . \otimes \vee + \downarrow \otimes \downarrow + . \otimes \downarrow + .. \otimes \downarrow + \downarrow \otimes ., \\ \Delta \left(\begin{array}{c} \downarrow \\ \vee \downarrow \end{array} \right) &= \begin{array}{c} \downarrow \\ \vee \downarrow \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \downarrow \\ \vee \downarrow \end{array} + . \otimes \vee + \downarrow \otimes \downarrow + . \otimes \downarrow + .. \otimes \downarrow + \downarrow \otimes . \end{aligned}$$

We define the operator $B_P : \mathcal{H}_P \rightarrow \mathcal{H}_P$, which associates, to a forest $F \in \mathcal{H}_P$, the tree obtained by grafting the roots of the trees of F on a common root. For example, $B_P(\downarrow) = \begin{array}{c} \downarrow \\ \vee \end{array}$, and $B_P(\downarrow \downarrow) = \begin{array}{c} \downarrow \\ \vee \downarrow \end{array}$. It is shown in [Mo] that (\mathcal{H}_P, B_P) is an initial object in the category of couples (A, L) , where A is an algebra and $L : A \rightarrow A$ any linear operator. Explicitly, one has

Theorem 2 *Let A be any algebra and let $L : A \rightarrow A$ be a linear map. Then there exists a unique algebra morphism $\phi : \mathcal{H}_P \rightarrow A$, such that $\phi \circ B_P = L \circ \phi$.*

Remarks.

1. Note that ϕ is inductively defined in the following way: for all trees $T_1, \dots, T_n \in \mathcal{H}_P$,

$$\begin{cases} \phi(1) &= 1, \\ \phi(T_1 \dots T_n) &= \phi(T_1) \dots \phi(T_n), \\ \phi(B_P(T_1 \dots T_n)) &= L(\phi(T_1) \dots \phi(T_n)). \end{cases}$$

2. $\Delta : \mathcal{H}_P \rightarrow \mathcal{H}_P \otimes \mathcal{H}_P$ is the unique algebra morphism such that

$$\Delta \circ B_P = B_P \otimes 1 + (id \otimes B_P) \circ \Delta.$$

1.2.3 The free right graft algebra

We here recall some results of [F3].

Let $F, G \in \mathcal{H}_P$ two nonempty planar forests. We set $F = F_1 \dots F_n$ and $F_n = B_P(H)$. We define:

$$F \prec G = F_1 \dots F_{n-1} B_P(HG).$$

In other terms, G is grafted on the root of the last tree of F , on the right. In particular, $\bullet \prec G = B_P(G)$.

Examples.

$\mathbf{1} \prec \dots = \text{V}$	$\dots \prec \mathbf{1} = \dots \mathbf{1}$	$\dots \prec \dots = \dots \text{V}$	$\dots \prec \dots = \dots \text{V}$
$\mathbf{1} \prec \mathbf{1} \cdot = \text{V}$	$\mathbf{1} \cdot \prec \mathbf{1} = \mathbf{1} \mathbf{1}$	$\mathbf{1} \cdot \prec \dots = \mathbf{1} \text{V}$	$\dots \prec \mathbf{1} \cdot = \dots \text{V}$
$\mathbf{1} \prec \mathbf{1} \cdot = \text{V}$	$\mathbf{1} \cdot \prec \mathbf{1} = \mathbf{1} \mathbf{1}$	$\mathbf{1} \cdot \prec \dots = \mathbf{1} \text{V}$	$\dots \prec \mathbf{1} \cdot = \dots \text{V}$
$\mathbf{1} \prec \text{V} = \text{V}$	$\text{V} \prec \mathbf{1} = \text{V}$	$\text{V} \prec \dots = \text{V}$	$\dots \prec \text{V} = \dots \text{V}$
$\mathbf{1} \prec \mathbf{1} = \text{V}$	$\mathbf{1} \prec \mathbf{1} = \text{V}$	$\mathbf{1} \prec \dots = \text{V}$	$\dots \prec \mathbf{1} = \dots \mathbf{1}$

We denote by \mathcal{M}_P the augmentation ideal of \mathcal{H}_P , that is to say the vector space generated by the nonempty forests of \mathcal{H}_P . We extend $\prec: \mathcal{M}_P \otimes \mathcal{M}_P \rightarrow \mathcal{M}_P$ by linearity.

Remark. \prec is not associative :

$$\bullet \prec (\bullet \prec \bullet) = \mathbf{1} \neq \text{V} = (\bullet \prec \bullet) \prec \bullet$$

The following result is proved in [F3].

Theorem 3 (\mathcal{M}_P, m, \prec) is the free \mathcal{RG} -algebra generated by \bullet (m is the concatenation product).

1.3 Relations with the coproduct

First, we recall the definition of a dipterous algebra (see [LR]):

Definition 4 A (right) dipterous algebra is a \mathbb{K} -vector space A equipped with two binary operations denoted $*$ and \prec satisfying the following relations : for all $x, y, z \in A$,

$$(x * y) * z = x * (y * z), \quad (2)$$

$$(x \prec y) \prec z = x \prec (y * z). \quad (3)$$

Dipterous algebras do not have unit for the product $*$. If A and B are two dipterous algebras, we say that a \mathbb{K} -linear map $f: A \rightarrow B$ is a dipterous morphism if $f(x * y) = f(x) * f(y)$ and $f(x \prec y) = f(x) \prec f(y)$ for all $x, y \in A$. We denote by Dipt-alg the category of dipterous algebras.

Remark. A \mathcal{RG} -algebra is also a dipterous algebra. We get the following canonical functor: $\mathcal{RG}\text{-alg} \rightarrow \text{Dipt-alg}$.

As \mathcal{RG} -algebras are not unitary objects, we need to extend the usual tensor product in order to obtain a copy of A and B in the tensor product of A and B .

Definition 5 Let A, B be two vector spaces. Then :

$$A\overline{\otimes}B = (A \otimes \mathbb{K}) \oplus (A \otimes B) \oplus (\mathbb{K} \otimes B).$$

Let A be a \mathcal{RG} -algebra. We extend $\prec: A \otimes A \rightarrow A$ to a map $\prec: A\overline{\otimes}A \rightarrow A$ in the following way : for all $a \in A$, $a \prec 1 = a$ and $1 \prec a = 0$. Moreover, we extend the product of A to a map from $(A \oplus \mathbb{K}) \otimes (A \oplus \mathbb{K})$ to $A \oplus \mathbb{K}$ by putting $1.a = a.1 = a$ for all $a \in A$ and $1.1 = 1$. Note that $1 \prec 1$ is not defined.

Proposition 6 Let A and B be two \mathcal{RG} -algebras. Then $A\overline{\otimes}B$ is given a structure of dipterous algebra in the following way : for $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,

$$\begin{aligned} (a \otimes b) * (a' \otimes b') &= (a * a') \otimes (b * b'), \\ (a \otimes b) \prec (a' \otimes b') &= (a * a') \otimes (b \prec b'), \text{ if } b \text{ or } b' \in B, \\ (a \otimes 1) \prec (a' \otimes 1) &= (a \prec a') \otimes 1. \end{aligned}$$

Proof. The associativity of $*$: $(A\overline{\otimes}B) \otimes (A\overline{\otimes}B) \rightarrow A\overline{\otimes}B$ (that is to say the relation (2)) is obvious. We prove (3) : for all $a, a', a'' \in A$ and $b, b', b'' \in B$,

$$\begin{aligned} (a \otimes b) \prec ((a' \otimes b') * (a'' \otimes b'')) &= (a \otimes b) \prec ((a' * a'') \otimes (b' * b'')) \\ &= (a * (a' * a'')) \otimes (b \prec (b' * b'')) \\ &= ((a * a') * a'') \otimes ((b \prec b') \prec b'') \\ &= ((a * a') \otimes (b \prec b')) \prec (a'' \otimes b'') \\ &= ((a \otimes b) \prec (a' \otimes b')) \prec (a'' \otimes b''). \end{aligned}$$

This calculation is still true if b, b' or b'' is equal to 1 or if $b' = b'' = 1$ and $b \in B$. If $b = b'' = 1$ and $b' \in B$,

$$\begin{aligned} (a \otimes 1) \prec ((a' \otimes b') * (a'' \otimes 1)) &= (a \otimes 1) \prec ((a' * a'') \otimes b') \\ &= (a * (a' * a'')) \otimes (1 \prec b') \\ &= 0, \\ ((a \otimes 1) \prec (a' \otimes b')) \prec (a'' \otimes 1) &= ((a * a') \otimes (1 \prec b')) \prec (a'' \otimes 1) \\ &= 0. \end{aligned}$$

If $b = b' = 1$ and $b'' \in B$, then a and a' are not equal to 1 and

$$\begin{aligned} (a \otimes 1) \prec ((a' \otimes 1) * (a'' \otimes b'')) &= (a \otimes 1) \prec ((a' * a'') \otimes b'') \\ &= (a * (a' * a'')) \otimes (1 \prec b'') \\ &= 0, \\ ((a \otimes 1) \prec (a' \otimes 1)) \prec (a'' \otimes b'') &= ((a \prec a') \otimes 1) \prec (a'' \otimes b'') \\ &= ((a * a') \prec a'') \otimes (1 \prec b'') \\ &= 0. \end{aligned}$$

Finally if $b = b' = b'' = 1$, then a, a' and a'' are not equal to 1 and

$$\begin{aligned} (a \otimes 1) \prec ((a' \otimes 1) * (a'' \otimes 1)) &= (a \otimes 1) \prec ((a' * a'') \otimes 1) \\ &= (a \prec (a' * a'')) \otimes 1 \\ &= ((a \prec a') \prec a'') \otimes 1 \\ &= ((a \prec a') \otimes 1) \prec (a'' \otimes 1) \\ &= ((a \otimes 1) \prec (a' \otimes 1)) \prec (a'' \otimes 1). \end{aligned}$$

In all cases, the relation (3) is satisfied and $A\overline{\otimes}B$ is a dipterous algebra. \square

Remarks.

1. $A \otimes \mathbb{K}$ is a dipterous subalgebra of $A \overline{\otimes} B$ which is isomorphic to A , and $\mathbb{K} \otimes B$ is a dipterous subalgebra of $A \overline{\otimes} B$ which is isomorphic to B .
2. Suppose that $A, B \neq \{0\}$. $(A \overline{\otimes} B, *, \prec)$ is a \mathcal{RG} -algebra if and only if $\prec: A \otimes A \rightarrow A$ and $\prec: B \otimes B \rightarrow B$ are zero. Indeed, if $A \overline{\otimes} B$ is a \mathcal{RG} -algebra then for all $a, a' \in A$ and $b, b' \in B$,

$$\begin{aligned}
(a \prec a') \otimes b &= ((a \prec a') \otimes 1) * (1 \otimes b) \\
&= ((a \otimes 1) \prec (a' \otimes 1)) * (1 \otimes b) \\
&= (a \otimes 1) \prec (a' \otimes b) \\
&= (a * a') \otimes (1 \prec b) \\
&= 0,
\end{aligned}$$

therefore, by taking $b \neq 0$, $a \prec a' = 0$ for all $a, a' \in A$. Moreover,

$$\begin{aligned}
a \otimes (b \prec b') &= (a \otimes b) \prec (1 \otimes b') \\
&= ((1 \otimes b) * (a \otimes 1)) \prec (1 \otimes b') \\
&= (1 \otimes b) * (a \otimes (1 \prec b')) \\
&= 0,
\end{aligned}$$

therefore, by taking $a \neq 0$, $b \prec b' = 0$ for all $b, b' \in B$.

Reciprocally, if $\prec: A \otimes A \rightarrow A$ and $\prec: B \otimes B \rightarrow B$ are zero, it is clear that $(A \overline{\otimes} B, *, \prec)$ is a \mathcal{RG} -algebra.

Proposition 7 For all tree $T \in \mathcal{M}_P$ and for every forest $F \in \mathcal{M}_P$, $\Delta(T \prec F) = \Delta(T) \prec \Delta(F)$.

Remark. As $\mathcal{M}_P \overline{\otimes} \mathcal{M}_P$ is not a \mathcal{RG} -algebra, if $T \in \mathcal{M}_P$ is a forest, $\Delta(T \prec F) \neq \Delta(T) \prec \Delta(F)$ in general. For example,

$$\begin{aligned}
\Delta((..) \prec .) &= \Delta(. \mathbf{1}) \\
&= . \mathbf{1} \otimes 1 + 1 \otimes . \mathbf{1} + \mathbf{1} \otimes . + . \otimes \mathbf{1} + .. \otimes . + . \otimes .. \\
\Delta(..) \prec \Delta(.) &= (.. \otimes 1 + 1 \otimes .. + 2. \otimes .) \prec (. \otimes 1 + 1 \otimes .) \\
&= . \mathbf{1} \otimes 1 + 1 \otimes . \mathbf{1} + . \otimes .. + 2.. \otimes . + 2. \otimes \mathbf{1}.
\end{aligned}$$

Proof. Let T and F are two planar trees $\in \mathcal{M}_P$. We denote $\Delta(T) = T \otimes 1 + 1 \otimes T + \sum_T T^{(1)} \otimes T^{(2)}$ and $\Delta(F) = F \otimes 1 + 1 \otimes F + \sum_F F^{(1)} \otimes F^{(2)}$. Then

$$\begin{aligned}
\Delta(T) \prec \Delta(F) &= \left(T \otimes 1 + 1 \otimes T + \sum_T T^{(1)} \otimes T^{(2)} \right) \prec \left(F \otimes 1 + 1 \otimes F + \sum_F F^{(1)} \otimes F^{(2)} \right) \\
&= (T \prec F) \otimes 1 + T \otimes F + \sum_T T^{(1)} F \otimes T^{(2)} + 1 \otimes (T \prec F) \\
&\quad + \sum_T T^{(1)} \otimes (T^{(2)} \prec F) + \sum_F F^{(1)} \otimes (T \prec F^{(2)}) \\
&\quad + \sum_{T, F} T^{(1)} F^{(1)} \otimes (T^{(2)} \prec F^{(2)}) \\
&= \Delta(T \prec F).
\end{aligned}$$

If $F = F_1 \dots F_n \in \mathcal{M}_P$ is a forest and T is again a tree $\in \mathcal{M}_P$,

$$\begin{aligned} \Delta(T) \prec \Delta(F) &= \Delta(T) \prec (\Delta(F_1) \dots \Delta(F_n)) \\ &= (\dots ((\Delta(T) \prec \Delta(F_1)) \prec \Delta(F_2)) \dots \prec \Delta(F_{n-1})) \prec \Delta(F_n) \\ &= \Delta((\dots ((T \prec F_1) \prec F_2) \dots \prec F_{n-1}) \prec F_n) \\ &= \Delta(T \prec F), \end{aligned}$$

as $A \overline{\otimes} B$ is a dipterous algebra for the second equality. \square

1.4 A rigidity theorem for the right graft algebras

We now give a rigidity theorem for the right graft algebras (from [F3]).

Recall that if A and B are two \mathcal{RG} -algebras then $A \overline{\otimes} B$ is a dipterous algebra (proposition 6). We define another coproduct Δ_{Ass} on \mathcal{H}_P (the deconcatenation) in the following way: for every forest $F \in \mathcal{H}_P$,

$$\Delta_{Ass}(F) = \sum_{F_1, F_2 \in \mathcal{H}_P, F_1 F_2 = F} F_1 \otimes F_2.$$

We now have defined two products, namely m and \succ and one coproduct on \mathcal{M}_P , namely $\tilde{\Delta}_{Ass}$, obtained from Δ_{Ass} by subtracting its primitive part. The following properties sum up the different compatibilities.

Proposition 8 *For all $x, y \in \mathcal{M}_P$;*

$$\begin{cases} \tilde{\Delta}_{Ass}(xy) &= (x \otimes 1) \tilde{\Delta}_{Ass}(y) + \tilde{\Delta}_{Ass}(x)(1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{Ass}(x \prec y) &= \tilde{\Delta}_{Ass}(x) \prec (1 \otimes y). \end{cases}$$

In particular, with the first equality, $(\mathcal{M}_P, \Delta_{Ass})$ is an infinitesimal bialgebra (see [LR]).

Proof. We can restrict ourselves to $F, G \in \mathcal{H}_P$ two nonempty forests. We put $F = F_1 \dots F_n$, $G = G_1 \dots G_m$ where the F_i 's and the G_i 's are trees and $F_n = B_P(H)$. Hence :

$$\begin{aligned} \tilde{\Delta}_{Ass}(FG) &= \sum_{H_1, H_2 \in \mathcal{M}_P, H_1 H_2 = FG} H_1 \otimes H_2 \\ &= \sum_{H_1, H_2 \in \mathcal{M}_P, H_1 H_2 = G} F H_1 \otimes H_2 + \sum_{H_1, H_2 \in \mathcal{M}_P, H_1 H_2 = F} H_1 \otimes H_2 G + F \otimes G \\ &= (F \otimes 1) \tilde{\Delta}_{Ass}(G) + \tilde{\Delta}_{Ass}(F)(1 \otimes G) + F \otimes G. \end{aligned}$$

It is the same case as in [LR]. Moreover,

$$\begin{aligned} \tilde{\Delta}_{Ass}(F \prec G) &= \tilde{\Delta}_{Ass}(F_1 \dots F_{n-1} B_P(HG)) \\ &= \sum_{i=1}^{n-2} F_1 \dots F_i \otimes F_{i+1} \dots F_{n-1} B_P(HG) + F_1 \dots F_{n-1} \otimes B_P(HG) \\ &= \sum_{i=1}^{n-2} F_1 \dots F_i \otimes (F_{i+1} \dots F_{n-1} F_n) \prec G + F_1 \dots F_{n-1} \otimes F_n \prec G \\ &= \tilde{\Delta}_{Ass}(F) \prec (1 \otimes G). \end{aligned}$$

\square

This justifies the following definition :

Definition 9 *A \mathcal{RG} -infinitesimal bialgebra is a family $(A, m, \prec, \tilde{\Delta}_{Ass})$ where A is a \mathbb{K} -vector space, $m, \prec: A \otimes A \rightarrow A$ and $\tilde{\Delta}_{Ass}: A \rightarrow A \otimes A$ are \mathbb{K} -linear maps, with the following compatibilities :*

1. (A, m, \prec) is a \mathcal{RG} -algebra.

2. For all $x, y \in A$:

$$\begin{cases} \tilde{\Delta}_{Ass}(xy) &= (x \otimes 1)\tilde{\Delta}_{Ass}(y) + \tilde{\Delta}_{Ass}(x)(1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{Ass}(x \prec y) &= \tilde{\Delta}_{Ass}(x) \prec (1 \otimes y). \end{cases} \quad (4)$$

With theorem 3 and proposition 8, we have immediately :

Proposition 10 $(\mathcal{M}_P, m, \prec, \tilde{\Delta}_{Ass})$ is a \mathcal{RG} -infinitesimal bialgebra.

Remark. If A is a \mathcal{RG} -infinitesimal bialgebra, we denote $Prim(A) = Ker(\tilde{\Delta}_{Ass})$. In the \mathcal{RG} -infinitesimal bialgebra \mathcal{M}_P , $Prim(\mathcal{M}_P) = \mathbb{K}[\mathbb{T}_P \setminus \{1\}]$. We denote by \mathcal{P}_P the primitive part of \mathcal{M}_P .

Let us recall the definition of a magmatic algebra :

Definition 11 A magmatic algebra (or *Mag-algebra for short*) is a \mathbb{K} -vector space A equipped with a binary operation \bullet , without any relation.

We do not suppose that *Mag*-algebras have units. If A and B are two *Mag*-algebras, we say that a \mathbb{K} -linear map $f : A \rightarrow B$ is a *Mag*-morphism if $f(x \bullet y) = f(x) \bullet f(y)$ for all $x, y \in A$. We denote by *Mag*-alg the category of *Mag*-algebras.

Proposition 12 For any \mathcal{RG} -infinitesimal bialgebra, its primitive part is a *Mag*-algebra.

Proof. Let A be a \mathcal{RG} -infinitesimal bialgebra. If $x \in Prim(A)$ and $y \in A$ then $\tilde{\Delta}_{Ass}(x \prec y) = \tilde{\Delta}_{Ass}(x) \prec (1 \otimes y) = 0$ by (4). So $(Prim(A), \prec)$ is a *Mag*-algebra. \square

Proposition 13 (\mathcal{P}_P, \prec) is the free *Mag*-algebra generated by \cdot .

Proof. With proposition 12, we have immediately that (\mathcal{P}_P, \prec) is a *Mag*-algebra. Let us prove that (\mathcal{P}_P, \prec) is the free *Mag*-algebra generated by \cdot .

Let A be a *Mag*-algebra and let $a \in A$. Let us prove that there exists a unique morphism of *Mag*-algebras $\phi : \mathcal{P}_P \rightarrow A$, such that $\phi(\cdot) = a$. We define $\phi(T)$ for any nonempty tree $T \in \mathcal{P}_P$ inductively on the degree of T by:

$$\begin{cases} \phi(\cdot) &= a, \\ \phi(B_P(T_1 \dots T_k)) &= (\dots((a \prec \phi(T_1)) \prec \phi(T_2)) \dots \prec \phi(T_{k-1})) \prec \phi(T_k) \text{ if } k \geq 1. \end{cases}$$

This map is linearly extended into a map $\phi : \mathcal{P}_P \rightarrow A$. Let us show that it is a morphism of *Mag*-algebras. Let F, G be two nonempty trees. Let us prove that $\phi(F \prec G) = \phi(F) \prec \phi(G)$. We put $F = B_P(H_1 \dots H_k)$. Then:

$$\begin{aligned} \phi(F \prec G) &= \phi(B_P(H_1 \dots H_k G)) \\ &= (\dots((a \prec \phi(H_1)) \prec \phi(H_2)) \dots \prec \phi(H_k)) \prec \phi(G) \\ &= \phi(B_P(H_1 \dots H_k)) \prec \phi(G) \\ &= \phi(F) \prec \phi(G) \end{aligned}$$

So ϕ is a morphism of *Mag*-algebras.

Let $\phi' : \mathcal{P}_P \rightarrow A$ be another morphism of $\mathcal{M}ag$ -algebras such that $\phi'(\cdot) = a$. Then $T_1 \dots T_k \in \mathcal{P}_P$,

$$\begin{aligned} \phi'(B_P(T_1 \dots T_k)) &= \phi'((\dots((\cdot \prec T_1) \prec T_2) \dots \prec T_{k-1}) \prec T_k) \\ &= (\dots((a \prec \phi'(T_1)) \prec \phi'(T_2)) \dots \prec \phi'(T_{k-1})) \prec \phi'(T_k). \end{aligned}$$

So $\phi = \phi'$. □

The following result is proved in [F3] :

Theorem 14 *The triple of operads $(\mathcal{A}ss, \mathcal{RG}, \mathcal{M}ag)$ is a good triple of operads.*

2 Hopf algebra of planar decorated trees

2.1 Construction

To start, we shall need decorated versions of the Hopf algebra \mathcal{H}_P described in the section 1.2.2. If $T \in \mathcal{H}_P$ is a tree, we note $E(T)$ the set of edges of T .

Definition 15 *Let \mathcal{D} be a nonempty set.*

1. *A planar rooted tree decorated by \mathcal{D} is a couple (T, d) , where T is a planar tree and $d : E(T) \rightarrow \mathcal{D}$ is any map.*
2. *A planar rooted forest decorated by \mathcal{D} is a noncommutative monomial in planar rooted trees decorated by \mathcal{D} .*

Let T be a planar rooted tree decorated by \mathcal{D} . If $v, w \in V(T)$ and $d \in \mathcal{D}$, we shall denote $v \xrightarrow{d} w$ if there is an edge in T from v to w decorated by d .

Note. This definition is different from [F2], where vertices of the trees are decorated.

Examples.

1. Planar rooted trees decorated by \mathcal{D} with degree smaller than 4:

$$1, \quad \cdot, \quad \mathbf{1}^a, a \in \mathcal{D}, \quad \mathbf{1}^b_a, (a, b) \in \mathcal{D}^2, \quad \mathbf{1}^c_{ab}, \mathbf{1}^c_a, \mathbf{1}^c_b, \mathbf{1}^c_a, \mathbf{1}^c_b, (a, b, c) \in \mathcal{D}^3.$$

2. Planar rooted forests decorated by \mathcal{D} with degree smaller than 3:

$$1, \quad \cdot, \quad \dots, \mathbf{1}^a, a \in \mathcal{D}, \quad \dots, \mathbf{1}^a, \mathbf{1}^b_a, (a, b) \in \mathcal{D}^2.$$

If $\mathbf{v} \models V(F)$, then $Lea_{\mathbf{v}}(F)$ and $Roo_{\mathbf{v}}(F)$ are naturally planar decorated forests. The space $\mathcal{H}_P^{\mathcal{D}}$ generated by planar forests decorated by \mathcal{D} is a Hopf algebra. Its product is given by the concatenation of planar decorated forests and its coproduct is defined for any rooted decorated forest F by:

$$\Delta(F) = \sum_{\mathbf{v} \models V(F)} Lea_{\mathbf{v}}(F) \otimes Roo_{\mathbf{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\mathbf{v} \models V(F)} Lea_{\mathbf{v}}(F) \otimes Roo_{\mathbf{v}}(F).$$

For example: if $(a, b, c) \in \mathcal{D}^3$

$$\begin{aligned} \Delta \left(\mathbf{1}^c_{ab} \right) &= \mathbf{1}^c_{ab} \otimes 1 + 1 \otimes \mathbf{1}^c_{ab} + \cdot \otimes \mathbf{1}^b_a + \mathbf{1}^c \otimes \mathbf{1}^b + \cdot \otimes \mathbf{1}^c_a + \dots \otimes \mathbf{1}^a + \mathbf{1}^c \otimes \dots, \\ \Delta \left(\mathbf{1}^c_b \right) &= \mathbf{1}^c_b \otimes 1 + 1 \otimes \mathbf{1}^c_b + \cdot \otimes \mathbf{1}^b_a + \mathbf{1}^c \otimes \mathbf{1}^a + \cdot \otimes \mathbf{1}^c_b + \dots \otimes \mathbf{1}^b + \mathbf{1}^c \otimes \dots \end{aligned}$$

induction hypothesis and $T \in \mathbb{T}$. □

Let $\mathcal{BT} = \mathbb{K}[\mathbb{T}]$ be the subalgebra of $\mathcal{H}_P^{\mathcal{D}}$ generated by \mathbb{T} . We denote by $f_n^{\mathcal{BT}}$ the number of forests of degree n in \mathcal{BT} , $t_n^{\mathcal{BT}}$ the number of trees of degree n in \mathcal{BT} and $F_{\mathcal{BT}}(x) = \sum_{n \geq 0} f_n^{\mathcal{BT}} x^n$, $T_{\mathcal{BT}}(x) = \sum_{n \geq 1} t_n^{\mathcal{BT}} x^n$ the formal series associated. It is possible to calculate $F_{\mathcal{BT}}(x)$ and $T_{\mathcal{BT}}(x)$:

Proposition 17 *The formal series $F_{\mathcal{BT}}$ and $T_{\mathcal{BT}}$ are given by:*

$$\begin{aligned} F_{\mathcal{BT}}(x) &= \frac{3}{-1 + 4 \cos^2 \left(\frac{1}{3} \arcsin \left(\sqrt{\frac{27x}{4}} \right) \right)}, \\ T_{\mathcal{BT}}(x) &= \frac{4}{3} \sin^2 \left(\frac{1}{3} \arcsin \left(\sqrt{\frac{27x}{4}} \right) \right). \end{aligned}$$

Proof. \mathcal{BT} is freely generated by the trees, therefore

$$F_{\mathcal{BT}}(x) = \frac{1}{1 - T_{\mathcal{BT}}(x)}. \quad (5)$$

We have the following relations:

$$\begin{aligned} t_1^{\mathcal{BT}} &= 1, \\ t_n^{\mathcal{BT}} &= \sum_{k=1}^n \sum_{a_1 + \dots + a_k = n-1} (k+1) t_{a_1}^{\mathcal{BT}} \dots t_{a_k}^{\mathcal{BT}} \text{ if } n \geq 2. \end{aligned}$$

Then

$$T_{\mathcal{BT}}(x) - x = \sum_{k=1}^{\infty} (k+1) x T_{\mathcal{BT}}(x)^k = x F \circ T_{\mathcal{BT}}(x),$$

where $F(h) = \sum_{k=1}^{\infty} (k+1) h^k = \frac{2h - h^2}{(1-h)^2}$. We deduce the following equality:

$$T_{\mathcal{BT}}(x)^3 - 2T_{\mathcal{BT}}(x)^2 + T_{\mathcal{BT}}(x) = x. \quad (6)$$

So $T_{\mathcal{BT}}(x)$ is the inverse for the composition of $x^3 - 2x^2 + x$, that is to say

$$T_{\mathcal{BT}}(x) = \frac{4}{3} \sin^2 \left(\frac{1}{3} \arcsin \left(\sqrt{\frac{27x}{4}} \right) \right).$$

Remark that, with (6), we obtain an inductive definition of the coefficients t_n :

$$\begin{cases} t_1^{\mathcal{BT}} = 1, \\ t_n^{\mathcal{BT}} = 2 \sum_{i,j \geq 1 \text{ and } i+j=n} t_i^{\mathcal{BT}} t_j^{\mathcal{BT}} - \sum_{i,j,k \geq 1 \text{ and } i+j+k=n} t_i^{\mathcal{BT}} t_j^{\mathcal{BT}} t_k^{\mathcal{BT}} \text{ if } n \geq 2. \end{cases}$$

We deduce from (5) the equality

$$F_{\mathcal{BT}}(x) = \frac{3}{-1 + 4 \cos^2 \left(\frac{1}{3} \arcsin \left(\sqrt{\frac{27x}{4}} \right) \right)}.$$

□

This gives:

n	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathcal{BT}}$	1	2	7	30	143	728	3876	21318	120175	690690
$f_n^{\mathcal{BT}}$	1	3	12	55	273	1428	7752	43263	246675	1430715

These are the sequences A006013 and A001764 in [Sl].

Proposition 18 *Let $\dagger : \mathcal{BT} \rightarrow \mathcal{BT}$ be the \mathbb{K} -linear map built by induction as follows : $1^\dagger = 1$ and for all $F, G \in \mathbb{F}$, $(FG)^\dagger = G^\dagger F^\dagger$ and $(B(F \otimes G))^\dagger = B(G^\dagger \otimes F^\dagger)$. Then \dagger is an involution over \mathcal{BT} .*

Proof. Let us prove that $(F^\dagger)^\dagger = F$ for all $F \in \mathbb{F}$ by induction on the degree n of F . If $n = 0$, $F = 1$ and this is obvious. Suppose that $n \geq 1$. We have two cases :

1. If $F = B(G \otimes H)$ is a tree, with $G, H \in \mathbb{F}$ such that $|G|, |H| < n$. Then $(F^\dagger)^\dagger = (B(H^\dagger \otimes G^\dagger))^\dagger = B((G^\dagger)^\dagger \otimes (H^\dagger)^\dagger) = B(G \otimes H) = F$, using the induction hypothesis for G and H .
2. If $F = GH$ is a forest, with $G, H \in \mathbb{F} \setminus \{1\}$ such that $|G|, |H| < n$. Then $(F^\dagger)^\dagger = (H^\dagger G^\dagger)^\dagger = (G^\dagger)^\dagger (H^\dagger)^\dagger = GH = F$, using again the induction hypothesis for G and H .

In all cases, $(F^\dagger)^\dagger = F$. □

Remark. We can rewrite the relations between \dagger and the product m and B as follows:

$$\begin{aligned} m \circ (\dagger \otimes \dagger) &= \dagger \circ m \circ \tau, \\ B \circ (\dagger \otimes \dagger) &= \dagger \circ B \circ \tau. \end{aligned}$$

where τ is the flip defined in the introduction.

Examples.

$\mathbf{1} \dots \xrightarrow{\dagger} \dots \mathbf{1}_r$	$i\mathbf{V}_l \mathbf{1}_r^l \xrightarrow{\dagger} \mathbf{1}_l^r r\mathbf{V}_r$	$i\mathbf{V}_r^l \xrightarrow{\dagger} \mathbf{1}_r^l$
$i\mathbf{V}_r^l \cdot i\mathbf{V}_r \xrightarrow{\dagger} i\mathbf{V}_r \cdot i\mathbf{V}_r^l$	$i\mathbf{V}_l^r \xrightarrow{\dagger} i\mathbf{V}_r^l$	$r\mathbf{V}_r^l \mathbf{1}_r^l \xrightarrow{\dagger} \mathbf{1}_l^r r\mathbf{V}_r$
$i\mathbf{V}_r \mathbf{1}_r \mathbf{1}_l i\mathbf{V}_r \xrightarrow{\dagger} i\mathbf{V}_r \mathbf{1}_r \mathbf{1}_l i\mathbf{V}_r$	$\cdot \mathbf{1}_r \mathbf{1}_l^l \xrightarrow{\dagger} \mathbf{1}_r^l \mathbf{1}_l \cdot$	$r\mathbf{V}_r^l \mathbf{1}_r^l \xrightarrow{\dagger} \mathbf{1}_l^r i\mathbf{V}_r^l$

2.2 Hopf algebra structure on \mathcal{BT}

As in the case (\mathcal{H}_P, B_P) (see section 1.2.2), (\mathcal{BT}, B) is an initial object in the category of couples (A, L) where A is an algebra and $L : A \otimes A \rightarrow A$ any \mathbb{K} -linear map:

Theorem 19 *Let A be any algebra and let $L : A \otimes A \rightarrow A$ be a \mathbb{K} -linear map. Then there exists a unique algebra morphism $\phi : \mathcal{BT} \rightarrow A$ such that $\phi \circ B = L \circ (\phi \otimes \phi)$.*

Proof. *Existence.* We define an element $a_F \in A$ for any forest $F \in \mathcal{BT}$ by induction on the degree of F as follows:

1. $a_1 = 1_A$.
2. If F is a tree, there exists two forests $G, H \in \mathcal{BT}$ with $|G| + |H| = |F| - 1$ such that $F = B(G \otimes H)$; then $a_F = L(a_G \otimes a_H)$.
3. If F is not a tree, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}$ and $|F_i| < |F|$; then $a_F = a_{F_1} \dots a_{F_n}$.

Let $\phi : \mathcal{BT} \rightarrow A$ be the unique linear morphism such that $\phi(F) = a_F$ for any forest F . Given two forests F and G , let us prove that $\phi(FG) = \phi(F)\phi(G)$. If $F = 1$ or $G = 1$, this is trivial. If not, $F = F_1 \dots F_n$ and $G = G_1 \dots G_m \neq 1$ and therefore

$$\phi(FG) = a_{F_1} \dots a_{F_n} a_{G_1} \dots a_{G_m} = \phi(F)\phi(G).$$

Therefore ϕ is an algebra morphism. On the other hand, for all forests F, G ,

$$\phi \circ B(F \otimes G) = a_{B(F \otimes G)} = L(a_F \otimes a_G) = L \circ (\phi \otimes \phi)(F \otimes G),$$

therefore $\phi \circ B = L \circ (\phi \otimes \phi)$.

Uniqueness. Let $\psi : \mathcal{BT} \rightarrow A$ be an algebra morphism such that $\psi \circ B = L \circ (\psi \otimes \psi)$. Let us prove that $\psi(F) = a_F$ for any forest F by induction on the degree of F . If $F = 1$, $\psi(F) = 1_A = a_1$. If $|F| \geq 1$, we have two cases:

1. If F is a tree then $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$ and by induction hypothesis,

$$\psi(F) = \psi \circ B(G \otimes H) = L \circ (\psi \otimes \psi)(G \otimes H) = L(a_G \otimes a_H) = a_F.$$

2. If F is not a tree, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}$ and by induction hypothesis,

$$\psi(F) = \psi(F_1) \dots \psi(F_n) = a_{F_1} \dots a_{F_n} = a_F.$$

So $\phi(F) = \psi(F)$ for any forest F and $\phi = \psi$. □

We now define

$$\varepsilon : \begin{cases} \mathcal{BT} & \rightarrow \mathbb{K} \\ F \in \mathbb{F} & \mapsto \delta_{F,1}. \end{cases}$$

ε is an algebra morphism satisfying $\varepsilon \circ B = 0$.

Theorem 20 *Let $\Delta : \mathcal{BT} \rightarrow \mathcal{BT} \otimes \mathcal{BT}$ be the unique algebra morphism such that $\Delta \circ B = L \circ (\Delta \otimes \Delta)$ with*

$$L : \begin{cases} (\mathcal{BT} \otimes \mathcal{BT}) \otimes (\mathcal{BT} \otimes \mathcal{BT}) & \rightarrow \mathcal{BT} \otimes \mathcal{BT} \\ (x \otimes y) \otimes (z \otimes t) & \mapsto B(x \otimes z) \otimes \varepsilon(yt)1 + xz \otimes B(y \otimes t). \end{cases}$$

Equipped with this coproduct, \mathcal{BT} is a graded connected Hopf algebra.

Proof. We use the Sweedler's notation for \mathcal{BT} : for all $x \in \mathcal{BT}$, $\Delta(x) = \sum_x x^{(1)} \otimes x^{(2)}$. Then, if $G, H \in \mathbb{F}$,

$$\begin{aligned} \Delta(B(G \otimes H)) &= \sum_{G,H} B(G^{(1)} \otimes H^{(1)}) \otimes \varepsilon(G^{(2)} H^{(2)})1 + \sum_{G,H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \\ &= B(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}). \end{aligned}$$

Thanks to theorem 19, Δ is well defined. To prove that \mathcal{BT} is a graded connected Hopf algebra, we must prove that Δ is coassociative, counitary, and homogeneous of degree 0.

We first show that ε is a counit for Δ . Let us prove that $(\varepsilon \otimes Id) \circ \Delta(F) = F$ for any forest $F \in \mathbb{F}$ by induction on the degree of F . If $F = 1$,

$$(\varepsilon \otimes Id) \circ \Delta(1) = \varepsilon(1)1 = 1.$$

If F is a tree, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$. Then,

$$\begin{aligned}
(\varepsilon \otimes Id) \circ \Delta(F) &= (\varepsilon \otimes Id) \left(B(G \otimes H) \otimes 1 + \sum_{G, H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \right) \\
&= \varepsilon \circ B(G \otimes H) \otimes 1 + \sum_{G, H} \varepsilon(G^{(1)} H^{(1)}) B(G^{(2)} \otimes H^{(2)}) \\
&= 0 + \sum_{G, H} B(\varepsilon(G^{(1)}) G^{(2)} \otimes \varepsilon(H^{(1)}) H^{(2)}) \\
&= B(G \otimes H) \\
&= F.
\end{aligned}$$

If not, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}$. As $(\varepsilon \otimes Id) \circ \Delta$ is an algebra morphism and using the induction hypothesis,

$$\begin{aligned}
(\varepsilon \otimes Id) \circ \Delta(F) &= (\varepsilon \otimes Id) \circ \Delta(F_1) \dots (\varepsilon \otimes Id) \circ \Delta(F_n) \\
&= F_1 \dots F_n \\
&= F.
\end{aligned}$$

Let us show that $(Id \otimes \varepsilon) \circ \Delta(F) = F$ for any forest $F \in \mathbb{F}$ by induction. If $F = 1$,

$$(Id \otimes \varepsilon) \circ \Delta(1) = 1\varepsilon(1) = 1.$$

If F is a tree, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$ and then

$$\begin{aligned}
(Id \otimes \varepsilon) \circ \Delta(F) &= (Id \otimes \varepsilon) \left(B(G \otimes H) \otimes 1 + \sum_{G, H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \right) \\
&= B(G \otimes H) + \sum_{G, H} G^{(1)} H^{(1)} \varepsilon \circ B(G^{(2)} \otimes H^{(2)}) \\
&= B(G \otimes H) \\
&= F.
\end{aligned}$$

If not, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}$. As $(Id \otimes \varepsilon) \circ \Delta$ is an algebra morphism and using the induction hypothesis,

$$\begin{aligned}
(Id \otimes \varepsilon) \circ \Delta(F) &= (Id \otimes \varepsilon) \circ \Delta(F_1) \dots (Id \otimes \varepsilon) \circ \Delta(F_n) \\
&= F_1 \dots F_n \\
&= F.
\end{aligned}$$

Therefore ε is a counit for Δ .

Let us prove that Δ is coassociative. More precisely, we show that $(\Delta \otimes Id) \circ \Delta(F) = (Id \otimes \Delta) \circ \Delta(F)$ for any forest $F \in \mathbb{F}$ by induction on the degree of F . If $F = 1$ this is obvious. If F is a tree, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$. Then

$$\begin{aligned}
(\Delta \otimes Id) \circ \Delta(F) &= (\Delta \otimes Id) \circ \Delta \circ B(G \otimes H) \\
&= (\Delta \otimes Id) \left(B(G \otimes H) \otimes 1 + \sum_{G, H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \right) \\
&= B(G \otimes H) \otimes 1 \otimes 1 + \sum_{G, H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \otimes 1 \\
&\quad + \sum_{G, H, G^{(1)}, H^{(1)}} (G^{(1)})^{(1)} (H^{(1)})^{(1)} \otimes (G^{(1)})^{(2)} (H^{(1)})^{(2)} \otimes B(G^{(2)} \otimes H^{(2)}),
\end{aligned}$$

$$\begin{aligned}
(Id \otimes \Delta) \circ \Delta(F) &= (Id \otimes \Delta) \circ \Delta \circ B(G \otimes H) \\
&= (Id \otimes \Delta) \left(B(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \right) \\
&= B(G \otimes H) \otimes 1 \otimes 1 + \sum_{G,H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \otimes 1 \\
&\quad + \sum_{G,H,G^{(2)},H^{(2)}} G^{(1)} H^{(1)} \otimes (G^{(2)})^{(1)} (H^{(2)})^{(1)} \otimes B((G^{(2)})^{(2)} \otimes (H^{(2)})^{(2)}).
\end{aligned}$$

We conclude with the coassociativity of Δ applied to G and H . If F is not a tree, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}$. As $(\Delta \otimes Id) \circ \Delta$ and $(Id \otimes \Delta) \circ \Delta$ are algebra morphisms and using the induction hypothesis,

$$\begin{aligned}
&(\Delta \otimes Id) \circ \Delta(F) \\
&= (\Delta \otimes Id) \circ \Delta(F_1) \dots (\Delta \otimes Id) \circ \Delta(F_n) \\
&= \sum_{F_1, \dots, F_n} \sum_{F_1^{(1)}, \dots, F_n^{(1)}} (F_1^{(1)})^{(1)} \dots (F_n^{(1)})^{(1)} \otimes (F_1^{(1)})^{(2)} \dots (F_n^{(1)})^{(2)} \otimes F_1^{(2)} \dots F_n^{(2)} \\
&= \sum_{F_1, \dots, F_n} \sum_{F_1^{(2)}, \dots, F_n^{(2)}} F_1^{(1)} \dots F_n^{(1)} \otimes (F_1^{(2)})^{(1)} \dots (F_n^{(2)})^{(1)} \otimes (F_1^{(2)})^{(2)} \dots (F_n^{(2)})^{(2)} \\
&= (Id \otimes \Delta) \circ \Delta(F_1) \dots (Id \otimes \Delta) \circ \Delta(F_n) \\
&= (Id \otimes \Delta) \circ \Delta(F).
\end{aligned}$$

Therefore Δ is coassociative.

Let us show that Δ is homogeneous of degree 0. Easy induction, using the fact that L is homogeneous of degree 1. Note that it can also be proved using proposition 22. As it is graded and connected, it had an antipode. This ends the proof. \square

Remark. It was shown that for all $x, y \in \mathcal{BT}$,

$$\Delta \circ B(x \otimes y) = B(x \otimes y) \otimes 1 + (m \otimes B) \circ \Delta(x \otimes y),$$

with $\Delta(x \otimes y) = (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)(x \otimes y)$, where τ is the flip.

Proposition 21 For all forests $F \in \mathcal{BT}$, $\Delta(F^\dagger) = (\dagger \otimes \dagger) \circ \Delta(F)$.

Proof. By induction on the degree n of F . If $n = 0$, $F = 1$ and this is obvious. Suppose that $n \geq 1$. We have two cases :

1. If $F = B(G \otimes H)$ is a tree, with $G, H \in \mathbb{F}$ such that $|G|, |H| < n$. Then

$$\begin{aligned}
\Delta(F^\dagger) &= \Delta(B(H^\dagger \otimes G^\dagger)) \\
&= B(H^\dagger \otimes G^\dagger) \otimes 1 + (m \otimes B) \circ \Delta(H^\dagger \otimes G^\dagger) \\
&= B(G \otimes H)^\dagger \otimes 1^\dagger \\
&\quad + (m \otimes B) \circ (Id \otimes \tau \otimes Id) \circ (((\dagger \otimes \dagger) \circ \Delta) \otimes ((\dagger \otimes \dagger) \circ \Delta))(H \otimes G) \\
&= (\dagger \otimes \dagger)(B(G \otimes H) \otimes 1) \\
&\quad + ((m \circ (\dagger \otimes \dagger)) \otimes (B \circ (\dagger \otimes \dagger))) \circ (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)(H \otimes G) \\
&= (\dagger \otimes \dagger)(B(G \otimes H) \otimes 1) \\
&\quad + (\dagger \otimes \dagger) \circ (m \otimes B) \circ (\tau \otimes \tau) \circ (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)(H \otimes G) \\
&= (\dagger \otimes \dagger)(B(G \otimes H) \otimes 1 + (m \otimes B) \circ (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)(G \otimes H)) \\
&= (\dagger \otimes \dagger) \circ \Delta(F),
\end{aligned}$$

using the induction hypothesis in the third equality.

2. If $F = GH$ is a forest, with $G, H \in \mathbb{F} \setminus \{1\}$ such that $|G|, |H| < n$. Then

$$\begin{aligned}
\Delta(F^\dagger) &= \Delta(H^\dagger G^\dagger) \\
&= (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (\Delta(H^\dagger) \otimes \Delta(G^\dagger)) \\
&= (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (((\dagger \otimes \dagger) \circ \Delta(H)) \otimes ((\dagger \otimes \dagger) \circ \Delta(G))) \\
&= ((m \circ (\dagger \otimes \dagger)) \otimes (m \circ (\dagger \otimes \dagger))) \circ (Id \otimes \tau \otimes Id) \circ (\Delta(H) \otimes \Delta(G)) \\
&= (\dagger \otimes \dagger) \circ (m \otimes m) \circ (\tau \otimes \tau) \circ (Id \otimes \tau \otimes Id) \circ (\Delta(H) \otimes \Delta(G)) \\
&= (\dagger \otimes \dagger) \circ (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (\Delta(G) \otimes \Delta(H)) \\
&= (\dagger \otimes \dagger) \circ \Delta(F),
\end{aligned}$$

using the induction hypothesis in the third equality.

In all cases, $\Delta(F^\dagger) = (\dagger \otimes \dagger) \circ \Delta(F)$. □

We now give a combinatorial description of this coproduct:

Proposition 22 *Let $F \in \mathbb{F}$. Then $\Delta(F) = F \otimes 1 + 1 \otimes F + \sum_{v \models V(F)} Lea_v(F) \otimes Roo_v(F)$.*

Proof. Let Δ' be the unique algebra morphism from \mathcal{BT} to $\mathcal{BT} \otimes \mathcal{BT}$ defined by the formula of proposition 22. It is easy to show that, if $F, G \in \mathbb{F}$:

$$\begin{cases} \Delta'(1) &= 1 \otimes 1, \\ \Delta'(B(F \otimes G)) &= B(F \otimes G) \otimes 1 + (m \otimes B) \circ \Delta'(F \otimes G). \end{cases}$$

By unicity in theorem 20, $\Delta' = \Delta$. □

Theorem 23 *Let A be a Hopf algebra and $L : A \otimes A \rightarrow A$ a linear morphism such that for all $a, b \in A$,*

$$\Delta \circ L(a \otimes b) = L(a \otimes b) \otimes 1 + (m \otimes L) \circ \Delta(a \otimes b).$$

Therefore the unique algebra morphism $\phi : \mathcal{BT} \rightarrow A$ such that $\phi \circ B = L \circ (\phi \otimes \phi)$ (theorem 19) is a Hopf algebra morphism.

Proof. Let us prove that $\Delta \circ \phi(F) = (\phi \otimes \phi) \circ \Delta(F)$ for any forest $F \in \mathcal{BT}$ by induction on the degree of F . If $F = 1$, this is obvious. If F is a tree, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$. Then

$$\begin{aligned}
\Delta \circ \phi(F) &= \Delta \circ \phi \circ B(G \otimes H) \\
&= \Delta \circ L \circ (\phi \otimes \phi)(G \otimes H) \\
&= L(\phi(G) \otimes \phi(H)) \otimes 1 + \sum_{\phi(G), \phi(H)} (\phi(G))^{(1)} (\phi(H))^{(1)} \otimes L((\phi(G))^{(2)} \otimes (\phi(H))^{(2)}) \\
&= \phi \circ B(G \otimes H) \otimes \phi(1) + \sum_{G, H} \phi(G^{(1)} H^{(1)}) \otimes \phi \circ B(G^{(2)} \otimes H^{(2)}) \\
&= (\phi \otimes \phi) \left(B(G \otimes H) \otimes 1 + \sum_{G, H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \right) \\
&= (\phi \otimes \phi) \circ \Delta(F),
\end{aligned}$$

using the induction hypothesis for the fourth equality. If F is not a tree, $F = F_1 \dots F_n$ and then

$$\begin{aligned}
\Delta \circ \phi(F) &= \Delta \circ \phi(F_1) \dots \Delta \circ \phi(F_n) \\
&= (\phi \otimes \phi) \circ \Delta(F_1) \dots (\phi \otimes \phi) \circ \Delta(F_n) \\
&= (\phi \otimes \phi) \circ \Delta(F),
\end{aligned}$$

using the induction hypothesis for the second equality and $\Delta \circ \phi$ and $(\phi \otimes \phi) \circ \Delta$ are algebra morphisms for the first and third equality.

Let us prove that $\varepsilon \circ \phi = \varepsilon$. As $\varepsilon \circ \phi$ and ε are algebra morphisms, it suffices to show that $\varepsilon \circ \phi(F) = \varepsilon(F)$ for any tree $F \in \mathbb{T}$. If $F = 1$, this is obvious. If not, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$. $\varepsilon(F) = 0$ and

$$\begin{aligned}\varepsilon \circ \phi(F) &= \varepsilon \circ \phi \circ B(G \otimes H) \\ &= \varepsilon \circ L \circ (\phi \otimes \phi)(G \otimes H).\end{aligned}$$

Let us prove that $\varepsilon \circ L = 0$: for all $a, b \in A$,

$$\begin{aligned}\Delta \circ L(a \otimes b) &= L(a \otimes b) \otimes 1 + \sum_{a,b} a^{(1)} b^{(1)} \otimes L(a^{(2)} \otimes b^{(2)}) \\ (\varepsilon \otimes Id) \circ \Delta \circ L(a \otimes b) &= \varepsilon \circ L(a \otimes b) 1 + \sum_{a,b} \varepsilon(a^{(1)} b^{(1)}) L(a^{(2)} \otimes b^{(2)}) \\ &= \varepsilon \circ L(a \otimes b) + L(a \otimes b) \\ &= L(a \otimes b)\end{aligned}$$

Therefore $\varepsilon \circ L = 0$ and $\varepsilon \circ \phi = \varepsilon$. □

2.3 A pairing on \mathcal{BT}

Let $\gamma : \mathbb{K}[\mathbb{T}] \rightarrow \mathbb{K}[\mathbb{F}] \otimes \mathbb{K}[\mathbb{F}]$ be the \mathbb{K} -linear map homogeneous of degree -1 such that for all $F, G \in \mathbb{F}$, $\gamma(B(F \otimes G)) = (-1)^{|G|} F \otimes G$. From γ , we define the \mathbb{K} -linear map Γ by:

$$\Gamma : \begin{cases} \mathcal{BT} & \rightarrow \mathcal{BT} \otimes \mathcal{BT}, \\ T_1 \dots T_n & \rightarrow \Delta(T_1) \dots \Delta(T_{n-1}) \gamma(T_n). \end{cases}$$

The following result is immediate :

- Lemma 24** 1. Γ is homogeneous of degree -1 .
2. For all $x, y \in \mathcal{BT}$, $\Gamma(xy) = \Delta(x)\Gamma(y) + \varepsilon(y)\Gamma(x)$.

Let $\Gamma^* : \mathcal{BT}^{\otimes} \otimes \mathcal{BT}^{\otimes} \rightarrow \mathcal{BT}^{\otimes}$ be the transpose of Γ .

- Lemma 25** 1. Γ^* is homogeneous of degree 1 .
2. For all $f, g \in \mathcal{BT}^{\otimes}$,

$$\Delta(\Gamma^*(f \otimes g)) = \Gamma^*(f \otimes g) \otimes 1 + (m \otimes \Gamma^*) \circ \Delta(f \otimes g).$$

Proof.

1. Because Γ is homogeneous of degree -1 .
2. Let $f, g \in \mathcal{BT}^{\otimes}$. For all $x, y \in \mathcal{BT}$:

$$\begin{aligned}\Delta(\Gamma^*(f \otimes g))(x \otimes y) &= (f \otimes g)(\Gamma(xy)) \\ &= (f \otimes g)(\Delta(x)\Gamma(y) + \varepsilon(y)\Gamma(x)) \\ &= \varepsilon(y)(f \otimes g)(\Gamma(x)) + \Delta(f \otimes g)(\Delta(x) \otimes \Gamma(y)) \\ &= (\Gamma^*(f \otimes g) \otimes 1 + (m \otimes \Gamma^*) \circ \Delta(f \otimes g))(x \otimes y).\end{aligned}$$

□

Proposition 26 *The unique algebra morphism $\Phi : \mathcal{BT} \rightarrow \mathcal{BT}^{\otimes}$ such that $\Phi \circ B = \Gamma^* \circ (\Phi \otimes \Phi)$ is a graded Hopf algebra morphism.*

Proof. By the universal property (theorem 23) and the second point of Lemma 25, Φ is a Hopf algebra morphism. Let us prove that it is homogeneous of degree 0. Let F be a forest in \mathcal{BT} with n vertices. $\Phi(F)$ is homogeneous of degree n . If $n = 0$, then $F = 1$ and $\Phi(F) = 1$ is homogeneous of degree 0. Assume the result is true for all forests with $k < n$ vertices. If F is a tree, $F = B(G \otimes H)$. Then $\Phi(F) = \Gamma^* \circ (\Phi(G) \otimes \Phi(H))$. By the induction hypothesis, $\Phi(G) \otimes \Phi(H)$ is homogeneous of degree $n - 1$. As Γ^* is homogeneous of degree 1, $\Phi(F)$ is homogeneous of degree $n - 1 + 1 = n$. If $F = T_1 \dots T_k$, $k \geq 2$. Then $\Phi(F) = \Phi(T_1) \dots \Phi(T_k)$ is homogeneous of degree $|T_1| + \dots + |T_k| = n$ by the induction hypothesis. \square

Notations. For all $x \in \mathcal{BT}$, we note $\Delta(x) = \sum_x x^{(1)} \otimes x^{(2)}$ and $\Gamma(x) = \sum_x x_{(1)} \otimes x_{(2)}$.

Theorem 27 *For all $x, y \in \mathcal{BT}$, we set $\langle x, y \rangle = \Phi(x)(y)$. Then:*

1. $\langle 1, x \rangle = \varepsilon(x)$ for all $x \in \mathcal{BT}$.
2. $\langle xy, z \rangle = \sum_z \langle x, z^{(1)} \rangle \langle y, z^{(2)} \rangle$ for all $x, y, z \in \mathcal{BT}$.
3. $\langle B(x \otimes y), z \rangle = \sum_z \langle x, z_{(1)} \rangle \langle y, z_{(2)} \rangle$ for all $x, y, z \in \mathcal{BT}$.

Moreover:

4. $\langle -, - \rangle$ is symmetric.
5. If x and y are homogeneous of different degrees, $\langle x, y \rangle = 0$.
6. $\langle S(x), y \rangle = \langle x, S(y) \rangle$ for all $x, y \in \mathcal{BT}$.

Remark. Assertions 1-3 entirely determine $\langle F, G \rangle$ for $F, G \in \mathbb{F}$ by induction on the degree. Therefore $\langle -, - \rangle$ is the unique pairing satisfying the assertions 1-3.

Proof. First, as Φ is a Hopf algebra morphism, we have for all $x, y, z \in \mathcal{BT}$,

$$\langle 1, x \rangle = \Phi(1)(x) = \varepsilon(x) = \varepsilon \circ \Phi(x) = \Phi(x)(1) = \langle x, 1 \rangle,$$

$$\begin{aligned} \sum_z \langle x, z^{(1)} \rangle \langle y, z^{(2)} \rangle &= \sum_z \Phi(x)(z^{(1)}) \Phi(y)(z^{(2)}) \\ &= (\Phi(x) \Phi(y))(z) \\ &= \Phi(xy)(z) \\ &= \langle xy, z \rangle, \\ \sum_x \langle x^{(1)}, y \rangle \langle x^{(2)}, z \rangle &= \sum_x \Phi(x^{(1)})(y) \Phi(x^{(2)})(z) \\ &= \sum_x \Phi(x)^{(1)}(y) \Phi(x)^{(2)}(z) \\ &= \Phi(x)(yz) \\ &= \langle x, yz \rangle \end{aligned}$$

and

$$\langle S(x), y \rangle = \Phi(S(x))(y) = S^*(\Phi(x))(y) = \Phi(x)(S(y)) = \langle x, S(y) \rangle.$$

We obtain points 1, 2 and 6.

As Φ is homogeneous of degree 0, this implies point 5.

Let $x, y, z \in \mathcal{BT}$. Then:

$$\begin{aligned} \langle B(x \otimes y), z \rangle &= \Phi \circ B(x \otimes y)(z) \\ &= \Gamma^* \circ (\Phi(x) \otimes \Phi(y))(z) \\ &= (\Phi(x) \otimes \Phi(y))(\Gamma(z)) \\ &= \sum_z \langle x, z_{(1)} \rangle \langle y, z_{(2)} \rangle. \end{aligned}$$

We obtain point 3.

It remains to prove point 4, that is to say that $\langle -, - \rangle$ is symmetric. First we show that for all $x, y, z \in \mathcal{BT}$,

$$\sum_x \langle x_{(1)}, y \rangle \langle x_{(2)}, z \rangle = \langle x, B(y \otimes z) \rangle.$$

We can suppose by bilinearity that x, y and z are three forests. By induction on the degree n of x . If $n = 0$, then $x = 1$ and

$$\begin{aligned} \sum_x \langle x_{(1)}, y \rangle \langle x_{(2)}, z \rangle &= \langle 0, y \rangle \langle 0, z \rangle = 0, \\ \langle x, B(y \otimes z) \rangle &= \varepsilon(B(y \otimes z)) = 0. \end{aligned}$$

If $n = 1$, then $x = \bullet$ and $\Gamma(\bullet) = \gamma(\bullet) = 1 \otimes 1$. Then

$$\sum_x \langle x_{(1)}, y \rangle \langle x_{(2)}, z \rangle = \langle 1, y \rangle \langle 1, z \rangle = \delta_{y,1} \delta_{z,1}$$

$$\begin{aligned} \langle x, B(y \otimes z) \rangle &= \langle B(1 \otimes 1), B(y \otimes z) \rangle \\ &= \sum_{B(y \otimes z)} \langle 1, B(y \otimes z)_{(1)} \rangle \langle 1, B(y \otimes z)_{(2)} \rangle \\ &= \delta_{B(y \otimes z), \bullet} = \delta_{y,1} \delta_{z,1}. \end{aligned}$$

Suppose the result true for every forest x of degree $< n$. Two cases are possible:

1. x is a tree, $x = B(x' \otimes x'')$. First, $\Gamma(x) = (-1)^{|x''|} x' \otimes x''$ so

$$\begin{aligned} \sum_x \langle x_{(1)}, y \rangle \langle x_{(2)}, z \rangle &= (-1)^{|x''|} \langle x', y \rangle \langle x'', z \rangle \\ &= \begin{cases} 0 & \text{if } \deg(x'') \neq \deg(z), \\ (-1)^d \langle x', y \rangle \langle x'', z \rangle & \text{if } d = \deg(x'') = \deg(z). \end{cases} \end{aligned}$$

Moreover,

$$\begin{aligned} \langle x, B(y \otimes z) \rangle &= \langle B(x' \otimes x''), B(y \otimes z) \rangle \\ &= \sum_{B(y \otimes z)} \langle x', B(y \otimes z)_{(1)} \rangle \langle x'', B(y \otimes z)_{(2)} \rangle \\ &= (-1)^{|z|} \langle x', y \rangle \langle x'', z \rangle \\ &= \begin{cases} 0 & \text{if } \deg(x'') \neq \deg(z), \\ (-1)^d \langle x', y \rangle \langle x'', z \rangle & \text{if } d = \deg(x'') = \deg(z). \end{cases} \end{aligned}$$

2. x is a forest with at least two trees. x can be written $x = x'x''$, with the induction hypothesis available for x' and x'' . Then

$$\begin{aligned}
\langle x, B(y \otimes z) \rangle &= \langle x'x'', B(y \otimes z) \rangle \\
&= \sum_{B(y \otimes z)} \langle x', B(y \otimes z)^{(1)} \rangle \langle x'', B(y \otimes z)^{(2)} \rangle \\
&= \langle x', B(y \otimes z) \rangle \langle x'', 1 \rangle + \sum_{y, z} \langle x', y^{(1)} z^{(1)} \rangle \langle x'', B(y^{(2)} \otimes z^{(2)}) \rangle \\
&= \langle x', B(y \otimes z) \rangle \varepsilon(x'') \\
&\quad + \sum_{y, z} \left(\sum_{x'} \langle x'^{(1)}, y^{(1)} \rangle \langle x'^{(2)}, z^{(1)} \rangle \right) \left(\sum_{x''} \langle x''^{(1)}, y^{(2)} \rangle \langle x''^{(2)}, z^{(2)} \rangle \right) \\
&= \langle x', B(y \otimes z) \rangle \varepsilon(x'') \\
&\quad + \sum_{x', x''} \left(\sum_y \langle x'^{(1)}, y^{(1)} \rangle \langle x''^{(1)}, y^{(2)} \rangle \right) \left(\sum_z \langle x'^{(2)}, z^{(1)} \rangle \langle x''^{(2)}, z^{(2)} \rangle \right) \\
&= \sum_{x'} \langle x'_{(1)}, y \rangle \langle x'_{(2)}, z \rangle \varepsilon(x'') + \sum_{x', x''} \langle x'^{(1)} x''_{(1)}, y \rangle \langle x'^{(2)} x''_{(2)}, z \rangle \\
&= \sum_x \langle x_{(1)}, y \rangle \langle x_{(2)}, z \rangle,
\end{aligned}$$

by point 2 of Lemma 24 for the seventh equality.

So for all $x, y, z \in \mathcal{BT}$,

$$\sum_x \langle x_{(1)}, y \rangle \langle x_{(2)}, z \rangle = \langle x, B(y \otimes z) \rangle.$$

Let us prove that $\langle -, - \rangle$ is symmetric. By induction on the degree n of x . If $n = 0$, then $x = 1$ and

$$\langle x, y \rangle = \langle 1, y \rangle = \varepsilon(y) = \langle y, 1 \rangle = \langle y, x \rangle.$$

If $\deg(x) \geq 1$, two cases are possible:

1. x is a tree, $x = B(x' \otimes x'')$. Then:

$$\begin{aligned}
\langle x, y \rangle &= \langle B(x' \otimes x''), y \rangle \\
&= \sum_y \langle x', y_{(1)} \rangle \langle x'', y_{(2)} \rangle \\
&= \sum_y \langle y_{(1)}, x' \rangle \langle y_{(2)}, x'' \rangle \\
&= \langle y, B(x' \otimes x'') \rangle \\
&= \langle y, x \rangle.
\end{aligned}$$

2. x is a forest with at least two trees. x can be written $x = x'x''$ with $\deg(x'), \deg(x'') < \deg(x)$. Then:

$$\begin{aligned}
\langle x, y \rangle &= \langle x'x'', y \rangle \\
&= \sum_y \langle x', y^{(1)} \rangle \langle x'', y^{(2)} \rangle \\
&= \sum_y \langle y^{(1)}, x' \rangle \langle y^{(2)}, x'' \rangle \\
&= \langle y, x'x'' \rangle \\
&= \langle y, x \rangle.
\end{aligned}$$

Definition 29 \mathcal{BG} is the operad $\mathcal{P}(E, R)$, where E is concentrated in degree 2, with $E(2) = \mathbb{K}m \oplus \mathbb{K} \succ \oplus \mathbb{K} \prec$, and R is concentrated in degree 3, with $R(3) \subseteq \mathcal{P}_E(3)$ being the subspace generated by

$$\begin{cases} \succ \circ (m, I) - \succ \circ (I, \succ), \\ m \circ (\succ, I) - \succ \circ (I, m), \\ \prec \circ (\prec, I) - \prec \circ (I, m), \\ \prec \circ (m, I) - m \circ (I, \prec), \\ \prec \circ (\succ, I) - \succ \circ (I, \prec), \\ m \circ (m, I) - m \circ (I, m). \end{cases}$$

Remark. Graphically, the relations defining \mathcal{BG} can be written in the following way:

Notation. We denote $\widetilde{\mathcal{BG}}$ the nonsymmetric operad associated with the regular operad \mathcal{BG} .

3.2 Free \mathcal{BG} -algebra and dimension of \mathcal{BG}

Let $F, G \in \mathbb{F} \setminus \{1\}$. We put $F = F_1 \dots F_n$, $G = G_1 \dots G_m$, $F_1 = B(F_1^1 \otimes F_1^2)$ and $G_m = B(G_m^1 \otimes G_m^2)$. We define :

$$\begin{aligned} G \succ F &= B(GF_1^1 \otimes F_1^2)F_2 \dots F_n, \\ G \prec F &= G_1 \dots G_{m-1}B(G_m^1 \otimes G_m^2 F). \end{aligned}$$

In other terms, in the first case, G is grafted on the root of the first tree of F on the left with an edge decorated by l . In the second case, F is grafted on the root of the last tree of G on the right with an edge decorated by r . In particular, $F \succ \bullet = B(F \otimes 1)$ and $\bullet \prec F = B(1 \otimes F)$.

Examples.

$\bullet \bullet \succ \bullet \mathcal{V}_r = \mathcal{V}_l \mathcal{V}_r$	$\mathcal{V}_r \succ \bullet = \mathcal{V}_l^r$	$\mathcal{V}_r \succ \mathcal{V}_r = \mathcal{V}_l^r \mathcal{V}_r$
$\bullet \succ \mathcal{V}_l^r \mathcal{V}_r = \mathcal{V}_l^r \mathcal{V}_r$	$\bullet \bullet \succ \mathcal{V}_r = \mathcal{V}_l^r \mathcal{V}_r$	$\mathcal{V}_r \succ \bullet \bullet = \mathcal{V}_l^r \mathcal{V}_r$
$\bullet \bullet \prec \bullet \bullet = \bullet \mathcal{V}_r$	$\mathcal{V}_r \mathcal{V}_r \prec \mathcal{V}_r = \mathcal{V}_r \mathcal{V}_l^r$	$\bullet \bullet \prec \mathcal{V}_l^r = \bullet \mathcal{V}_l^r$
$\mathcal{V}_r \prec \bullet = \mathcal{V}_l^r \mathcal{V}_r$	$\mathcal{V}_r \mathcal{V}_l^r \prec \bullet = \mathcal{V}_r \mathcal{V}_l^r$	$\bullet \prec \mathcal{V}_r = \mathcal{V}_l^r \mathcal{V}_r$

We denote by \mathcal{M} the augmentation ideal of \mathcal{BT} , that is to say the vector space generated by the nonempty forests of \mathcal{BT} . We extend $\succ, \prec: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ by linearity.

Remark. \succ and \prec are not associative:

$$\begin{aligned} \bullet \succ (\bullet \succ \bullet) &= \mathcal{V}_l \neq (\bullet \succ \bullet) \succ \bullet = \mathcal{V}_l^l, \\ \bullet \prec (\bullet \prec \bullet) &= \mathcal{V}_r^r \neq (\bullet \prec \bullet) \prec \bullet = \mathcal{V}_r. \end{aligned}$$

Proposition 30 *For all forests $F, G \in \mathcal{M}$, $(G \succ F)^\dagger = F^\dagger \prec G^\dagger$ and $(G \prec F)^\dagger = F^\dagger \succ G^\dagger$.*

Proof. Let $F, G \in \mathcal{M}$ be two forests. We put $F = F_1 \dots F_n$, $G = G_1 \dots G_m$, $F_1 = B(F_1^1 \otimes F_1^2)$ and $G_m = B(G_m^1 \otimes G_m^2)$. Then

$$\begin{aligned} (G \succ F)^\dagger &= (B(GF_1^1 \otimes F_1^2)F_2 \dots F_n)^\dagger \\ &= F_n^\dagger \dots F_2^\dagger B((F_1^2)^\dagger \otimes (GF_1^1)^\dagger) \\ &= F_n^\dagger \dots F_2^\dagger B((F_1^2)^\dagger \otimes (F_1^1)^\dagger G^\dagger) \\ &= \left(F_n^\dagger \dots F_2^\dagger B((F_1^2)^\dagger \otimes (F_1^1)^\dagger) \right) \prec G^\dagger \\ &= F^\dagger \prec G^\dagger. \end{aligned}$$

Therefore $(G \succ F)^\dagger = F^\dagger \prec G^\dagger$ or all forests $F, G \in \mathcal{M}$. Moreover, as \dagger is an involution over \mathcal{M} (proposition 18), $(G \prec F)^\dagger = ((G^\dagger)^\dagger \prec (F^\dagger)^\dagger)^\dagger = ((F^\dagger \succ G^\dagger)^\dagger)^\dagger = F^\dagger \succ G^\dagger$. \square

Proposition 31 *For all $x, y, z \in \mathcal{M}$:*

$$(xy) \succ z = x \succ (y \succ z), \quad (8)$$

$$(x \succ y)z = x \succ (yz), \quad (9)$$

$$(x \prec y) \prec z = x \prec (yz), \quad (10)$$

$$(xy) \prec z = x(y \prec z), \quad (11)$$

$$(x \succ y) \prec z = x \succ (y \prec z). \quad (12)$$

Proof. We can restrict ourselves to $x, y, z \in \mathbb{F} \setminus \{1\}$. (9) and (11) are immediate. We put $x = x_1 B(x_2 \otimes x_3)$ and $z = B(z_1 \otimes z_2)z_3$ with $x_i, z_i \in \mathbb{F}$. Then

$$\begin{aligned} x \succ (y \succ z) &= x \succ (B(yz_1 \otimes z_2)z_3) = B(xy z_1 \otimes z_2)z_3 = (xy) \succ (B(z_1 \otimes z_2)z_3) = (xy) \succ z, \\ (x \prec y) \prec z &= (x_1 B(x_2 \otimes x_3 y)) \prec z = x_1 B(x_2 \otimes x_3 yz) = (x_1 B(x_2 \otimes x_3)) \prec (yz) = x \prec (yz). \end{aligned}$$

So we have proved (8) and (10). In order to prove (12), we must study two cases :

1. $y = B(y_1 \otimes y_2)$ is a tree,

$$(x \succ y) \prec z = B(xy_1 \otimes y_2) \prec z = B(xy_1 \otimes y_2 z) = x \succ B(y_1 \otimes y_2 z) = x \succ (y \prec z).$$

2. $y = y_1 y_2$ with $y_1, y_2 \in \mathbb{F} \setminus \{1\}$,

$$(x \succ y) \prec z = ((x \succ y_1)y_2) \prec z = (x \succ y_1)(y_2 \prec z) = x \succ (y_1(y_2 \prec z)) = x \succ (y \prec z),$$

by (9) and (11). \square

As an immediate consequence, we obtain

Corollary 32 *\mathcal{M} is given a graded \mathcal{BG} -algebra structure by its products m (the concatenation), \succ and \prec .*

We now prove that \mathcal{M} is the free \mathcal{BG} -algebra generated by \bullet and deduce the dimension of $\mathcal{BG}(n)$ for all n .

Theorem 33 *$(\mathcal{M}, m, \succ, \prec)$ is the free \mathcal{BG} -algebra generated by \bullet .*

Proof. Let A be a \mathcal{BG} -algebra and let $a \in A$. Let us prove that there exists a unique morphism of \mathcal{BG} -algebras $\phi : \mathcal{M} \rightarrow A$, such that $\phi(\cdot) = a$. We define $\phi(F)$ for any nonempty forest $F \in \mathcal{M}$ inductively on the degree of F by:

$$\left\{ \begin{array}{ll} \phi(\cdot) &= a, \\ \phi(F_1 \dots F_k) &= \phi(F_1) \dots \phi(F_k) \text{ if } k \geq 2, \\ \phi(F) &= (\phi(F^1) \succ a) \prec \phi(F^2) \text{ if } F = B(F^1 \otimes F^2) \text{ with } F^1, F^2 \in \mathbb{F}. \end{array} \right.$$

As the product of A is associative, this is perfectly defined. This map is linearly extended into a map $\phi : \mathcal{M} \rightarrow A$. Let us show that it is a morphism of \mathcal{BG} -algebras. By the second point, $\phi(xy) = \phi(x)\phi(y)$ for any $x, y \in \mathcal{M}$. Let F, G be two nonempty trees. Let us prove that $\phi(F \succ G) = \phi(F) \succ \phi(G)$ and $\phi(F \prec G) = \phi(F) \prec \phi(G)$. Denote $F = B(F^1 \otimes F^2)$, $G = B(G^1 \otimes G^2)$ with F^1, F^2 and G^1, G^2 in \mathbb{F} . Then:

1. For $\phi(F \succ G) = \phi(F) \succ \phi(G)$,

$$\begin{aligned} \phi(F \succ G) &= \phi(B(FG^1 \otimes G^2)) \\ &= (\phi(FG^1) \succ a) \prec \phi(G^2) \\ &= (\phi(F) \succ (\phi(G^1) \succ a)) \prec \phi(G^2) \\ &= \phi(F) \succ ((\phi(G^1) \succ a) \prec \phi(G^2)) \\ &= \phi(F) \succ \phi(G). \end{aligned}$$

2. For $\phi(F \prec G) = \phi(F) \prec \phi(G)$,

$$\begin{aligned} \phi(F \prec G) &= \phi(B(F^1 \otimes F^2G)) \\ &= (\phi(F^1) \succ a) \prec \phi(F^2G) \\ &= \phi(F^1) \succ (a \prec \phi(F^2G)) \\ &= \phi(F^1) \succ ((a \prec \phi(F^2)) \prec \phi(G)) \\ &= ((\phi(F^1) \succ a) \prec \phi(F^2)) \prec \phi(G) \\ &= \phi(F) \prec \phi(G). \end{aligned}$$

If F, G are two nonempty forests, $F = F_1 \dots F_n$, $G = G_1 \dots G_m$. Then:

$$\begin{aligned} \phi(F \succ G) &= \phi((F_1 \succ (\dots F_{n-1} \succ (F_n \succ G_1) \dots))G_2 \dots G_m) \\ &= \phi(F_1 \succ (\dots F_{n-1} \succ (F_n \succ G_1) \dots))\phi(G_2 \dots G_m) \\ &= (\phi(F_1) \succ (\dots \phi(F_{n-1}) \succ (\phi(F_n) \succ \phi(G_1)) \dots))\phi(G_2 \dots G_m) \\ &= (\phi(F_1 \dots F_n) \succ \phi(G_1))\phi(G_2 \dots G_m) \\ &= \phi(F) \succ \phi(G), \end{aligned}$$

and

$$\begin{aligned} \phi(F \prec G) &= \phi(F_1 \dots F_{n-1}(\dots (F_n \prec G_1) \prec G_2 \dots) \prec G_m) \\ &= \phi(F_1 \dots F_{n-1})\phi((\dots (F_n \prec G_1) \prec G_2 \dots) \prec G_m) \\ &= \phi(F_1 \dots F_{n-1})((\dots (\phi(F_n) \prec \phi(G_1)) \prec \phi(G_2) \dots) \prec \phi(G_m)) \\ &= \phi(F_1 \dots F_{n-1})(\phi(F_n) \prec \phi(G_1 \dots G_m)) \\ &= \phi(F) \prec \phi(G). \end{aligned}$$

So ϕ is a morphism of \mathcal{BG} -algebras.

Let $\phi' : \mathcal{M} \rightarrow A$ be another morphism of \mathcal{BG} -algebras such that $\phi'(\cdot) = a$. Then for any planar trees F_1, \dots, F_k , $\phi'(F_1 \dots F_k) = \phi'(F_1) \dots \phi'(F_k)$. For any forests $F^1, F^2 \in \mathcal{M}$,

$$\begin{aligned} \phi'(B(F^1 \otimes F^2)) &= \phi'((F^1 \succ \cdot) \prec F^2) \\ &= (\phi'(F^1) \succ \phi'(\cdot)) \prec \phi'(F^2) \\ &= (\phi'(F^1) \succ a) \prec \phi'(F^2). \end{aligned}$$

So $\phi = \phi'$. □

Corollary 34 *For all $n \in \mathbb{N}^*$, $\dim(\widetilde{\mathcal{BG}}(n)) = f_n^{\mathcal{BT}}$ and the following map is bijective:*

$$\Psi : \begin{cases} \widetilde{\mathcal{BG}}(n) & \rightarrow \text{Vect}(\text{planar forests} \in \mathcal{BT} \text{ of degree } n) \subseteq \mathcal{M} \\ p & \rightarrow p(\cdot, \dots, \cdot). \end{cases} \quad (13)$$

Proof. It suffices to show that Ψ is bijective. $(\mathcal{M}, m, \succ, \prec)$ is generated by \cdot as \mathcal{BG} -algebra (with theorem 33) therefore Ψ is surjective. Ψ is injective by the freedom in theorem 33. □

The free \mathcal{BG} -algebra over a vector space V is the \mathcal{BG} -algebra $BG(V)$ such that any map from V to a \mathcal{BG} -algebra A has a natural extension as a \mathcal{BG} -morphism $BG(V) \rightarrow A$. In other words the functor $BG(-)$ is the left adjoint to the forgetful functor from \mathcal{BG} -algebras to vector spaces. Because the operad \mathcal{BG} is regular, we get the following result :

Proposition 35 *Let V be a \mathbb{K} -vector space. Then the free \mathcal{BG} -algebra on V is*

$$BG(V) = \bigoplus_{n \geq 1} \mathbb{K}[\mathbb{F}(n)] \otimes V^{\otimes n},$$

equipped with the following binary operations : for all $F \in \mathbb{F}(n)$, $G \in \mathbb{F}(m)$, $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \dots \otimes w_m \in V^{\otimes m}$,

$$\begin{aligned} (F \otimes v_1 \otimes \dots \otimes v_n) * (G \otimes w_1 \otimes \dots \otimes w_m) &= (FG \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m), \\ (F \otimes v_1 \otimes \dots \otimes v_n) \succ (G \otimes w_1 \otimes \dots \otimes w_m) &= (F \succ G \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m), \\ (F \otimes v_1 \otimes \dots \otimes v_n) \prec (G \otimes w_1 \otimes \dots \otimes w_m) &= (F \prec G \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m). \end{aligned}$$

3.3 A combinatorial description of the composition

We identify \mathcal{BG} with the vector space of nonempty planar forests via corollary 34. In other terms, we identify $F \in \mathbb{F}(n)$ and $\Psi^{-1}(F) \in \mathcal{BG}(n)$.

Notations. In order to distinguish the composition in \mathcal{BG} and the action of the operad \mathcal{BG} on \mathcal{M} , we now denote by

1. $F \circ (F_1, \dots, F_n)$ the composition of \mathcal{BG} .
2. $F \bullet (F_1, \dots, F_n)$ the action of \mathcal{BG} on \mathcal{M} .

In the following, we will describe the composition of \mathcal{BG} in term of forests.

Proposition 36 *1. \cdot is the unit element of \mathcal{BG} .*

2. $\cdot \cdot = m$ in $\mathcal{BG}(2)$. Consequently, in \mathcal{BG} $\cdot \cdot \circ (F, G) = FG$ for all $F, G \in \mathbb{F} \setminus \{1\}$.

3. $\mathfrak{u} = \succ$ in $\mathcal{BG}(2)$. Consequently, in \mathcal{BG} $\mathfrak{u} \circ (F, G) = F \succ G$ for all $F, G \in \mathbb{F} \setminus \{1\}$.

4. $\mathfrak{r} = \prec$ in $\mathcal{BG}(2)$. Consequently, in \mathcal{BG} $\mathfrak{r} \circ (F, G) = F \prec G$ for all $F, G \in \mathbb{F} \setminus \{1\}$.

Proof.

1. Indeed, $\Psi(\cdot) = \cdot = \Psi(I)$. Hence, $\cdot = I$.
2. By definition, $\Psi(\cdot\cdot) = \cdot\cdot = \Psi(m)$. So $\cdot\cdot = m$ in $\mathcal{BG}(2)$. Moreover, for all $F, G \in \mathbb{F} \setminus \{1\}$,

$$\begin{aligned}
 \Psi(FG) &= FG \\
 &= m \bullet (F, G) \\
 &= m \bullet (F \bullet (\cdot, \dots, \cdot), G \bullet (\cdot, \dots, \cdot)) \\
 &= (m \circ (F, G)) \bullet (\cdot, \dots, \cdot) \\
 &= \Psi(m \circ (F, G)).
 \end{aligned}$$

So $FG = m \circ (F, G) = \cdot\cdot \circ (F, G)$.

3. Indeed, $\Psi(\mathfrak{r}) = \cdot \succ \cdot = \Psi(\succ)$. So $\mathfrak{r} = \succ$ in $\mathcal{BG}(2)$. Moreover, for all $F, G \in \mathbb{F} \setminus \{1\}$,

$$\begin{aligned}
 \Psi(F \succ G) &= F \succ G \\
 &= \succ \bullet (F, G) \\
 &= \succ \bullet (F \bullet (\cdot, \dots, \cdot), G \bullet (\cdot, \dots, \cdot)) \\
 &= (\succ \circ (F, G)) \bullet (\cdot, \dots, \cdot) \\
 &= \Psi(\succ \circ (F, G)).
 \end{aligned}$$

So $F \succ G = \succ \circ (F, G) = \mathfrak{r} \circ (F, G)$.

4. $\Psi(\mathfrak{r}) = \cdot \prec \cdot = \Psi(\prec)$. So $\mathfrak{r} = \prec$ in $\mathcal{BG}(2)$. Moreover, for all $F, G \in \mathbb{F} \setminus \{1\}$,

$$\begin{aligned}
 \Psi(F \prec G) &= F \prec G \\
 &= \prec \bullet (F, G) \\
 &= \prec \bullet (F \bullet (\cdot, \dots, \cdot), G \bullet (\cdot, \dots, \cdot)) \\
 &= (\prec \circ (F, G)) \bullet (\cdot, \dots, \cdot) \\
 &= \Psi(\prec \circ (F, G)).
 \end{aligned}$$

So $F \prec G = \prec \circ (F, G) = \mathfrak{r} \circ (F, G)$.

□

Proposition 37 *Let $F \in \mathbb{F}(m)$ and $G \in \mathbb{F}(n)$ with $m, n \geq 1$. Let $H_1, \dots, H_{m+n+1} \in \mathbb{F} \setminus \{1\}$. Then in \mathcal{BG} ,*

$$\begin{aligned}
 (FG) \circ (H_1, \dots, H_{m+n}) &= F \circ (H_1, \dots, H_m) G \circ (H_{m+1}, \dots, H_{m+n}) \\
 B(F \otimes G) \circ (H_1, \dots, H_{m+n+1}) &= ((F \circ (H_1, \dots, H_m)) \succ H_{m+1}) \\
 &\quad \prec (G \circ (H_{m+2}, \dots, H_{m+n+1})).
 \end{aligned}$$

Proof. Indeed, in \mathcal{BG} ,

$$\begin{aligned}
 (FG) \circ (H_1, \dots, H_{m+n}) &= (m \circ (F, G)) \circ (H_1, \dots, H_{m+n}) \\
 &= m \circ (F \circ (H_1, \dots, H_m), G \circ (H_{m+1}, \dots, H_{m+n})) \\
 &= F \circ (H_1, \dots, H_m) G \circ (H_{m+1}, \dots, H_{m+n}),
 \end{aligned}$$

and

$$\begin{aligned}
& B(F \otimes G) \circ (H_1, \dots, H_{m+n+1}) \\
&= ((F \succ \bullet) \prec G) \circ (H_1, \dots, H_{m+n+1}) \\
&= (\mathfrak{l} \circ (\mathfrak{l} \circ (F, \bullet), G)) \circ (H_1, \dots, H_{m+n+1}) \\
&= \mathfrak{l} \circ ((\mathfrak{l} \circ (F, \bullet)) \circ (H_1, \dots, H_{m+1}), G \circ (H_{m+2}, \dots, H_{m+n+1})) \\
&= \mathfrak{l} \circ (\mathfrak{l} \circ (F \circ (H_1, \dots, H_m), \bullet \circ H_{m+1}), G \circ (H_{m+2}, \dots, H_{m+n+1})) \\
&= \mathfrak{l} \circ ((F \circ (H_1, \dots, H_m)) \succ H_{m+1}, G \circ (H_{m+2}, \dots, H_{m+n+1})) \\
&= ((F \circ (H_1, \dots, H_m)) \succ H_{m+1}) \prec (G \circ (H_{m+2}, \dots, H_{m+n+1})).
\end{aligned}$$

□

Combining propositions 36 and 37, we obtain the following theorem :

Theorem 38 *The composition of \mathcal{BG} in the basis of forests belong to $\mathbb{F} \setminus \{1\}$ can be inductively defined in this way:*

$$\left\{ \begin{array}{ll} \bullet \circ (H) &= H, \\ FG \circ (H_1, \dots, H_{|F|+|G|}) &= F \circ (H_1, \dots, H_{|F|}) G \circ (H_{|F|+1}, \dots, H_{|F|+|G|}), \\ B(F \otimes G) \circ (H_1, \dots, H_{|F|+|G|+1}) &= ((F \circ (H_1, \dots, H_{|F|})) \succ H_{|F|+1}) \\ &\prec (G \circ (H_{|F|+2}, \dots, H_{|F|+|G|+1})). \end{array} \right.$$

Examples. Let $F_1, F_2, F_3 \in \mathbb{F} \setminus \{1\}$.

$\bullet \circ (F_1, F_2) = F_1 F_2$	$\mathfrak{l} \circ (F_1, F_2, F_3) = (F_1 \succ F_2) \prec F_3$
$\mathfrak{l} \circ (F_1, F_2) = F_1 \succ F_2$	$\mathfrak{l} \circ (F_1, F_2, F_3) = F_1 \prec (F_2 F_3)$
$\mathfrak{l} \circ (F_1, F_2) = F_1 \prec F_2$	$\mathfrak{l} \circ (F_1, F_2, F_3) = (F_1 F_2) \succ F_3$
$\bullet \mathfrak{l} \circ (F_1, F_2, F_3) = F_1 (F_2 \succ F_3)$	$\mathfrak{l} \circ (F_1, F_2, F_3) = (F_1 \prec F_2) \succ F_3$
$\mathfrak{l} \bullet \circ (F_1, F_2, F_3) = (F_1 \prec F_2) F_3$	$\mathfrak{l} \circ (F_1, F_2, F_3) = F_1 \prec (F_2 \succ F_3)$
$\dots \circ (F_1, F_2, F_3) = F_1 F_2 F_3$	$\mathfrak{l} \circ (F_1, F_2, F_3) = (F_1 \succ F_2) \succ F_3$

3.4 Relations with the coproduct

We recall the definition of pre-dendriform algebra (see [Le1]):

Definition 39 *A pre-dendriform algebra is a \mathbb{K} -vector space A equipped with three binary operations denoted $*$, \succ and \prec satisfying the four relations : for all $x, y, z \in A$,*

$$\begin{aligned}
(x * y) * z &= x * (y * z), \\
(x * y) \succ y &= x \succ (y \succ z), \\
(x \prec y) \prec z &= x \prec (y * z), \\
(x \succ y) \prec z &= x \succ (y \prec z).
\end{aligned}$$

*In other words, $(A, *, \succ)$ and $(A, *, \prec)$ are dipterous algebras with the entanglement relation $(x \succ y) \prec z = x \succ (y \prec z)$.*

Remark. We can define *PreDend*-alg the category of pre-dendriform algebras. A \mathcal{BG} -algebra is also a pre-dendriform algebra. We get the following canonical functor: $\mathcal{BG}\text{-alg} \rightarrow \text{PreDend}\text{-alg}$.

As bigraft algebras are not objects with unit, we consider again the extended tensor product $A \overline{\otimes} B$ for A, B two \mathcal{BG} -algebras. If A is a bigraft algebra, we extend $\succ, \prec: A \otimes A \rightarrow A$ into maps $\succ, \prec: A \overline{\otimes} A \rightarrow A$ in the following way : for all $a \in A$,

$$a \succ 1 = 0, \quad a \prec 1 = a, \quad 1 \succ a = a, \quad 1 \prec a = 0.$$

Moreover, we extend the product of A into a map from $(A \oplus \mathbb{K}) \otimes (A \oplus \mathbb{K})$ to $A \oplus \mathbb{K}$ by putting $1.a = a.1 = a$ for all $a \in A$ and $1.1 = 1$. Note that the relations (7) are now satisfied on $A \overline{\otimes} A \overline{\otimes} A$.

Proposition 40 *Let A and B be two bigraft algebras. Then $A \overline{\otimes} B$ is given a structure of pre-dendriform algebra in the following way : for $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,*

$$\begin{aligned} (a \otimes b) * (a' \otimes b') &= (a * a') \otimes (b * b'), \\ (a \otimes b) \succ (a' \otimes b') &= (a * a') \otimes (b \succ b'), \text{ if } b \text{ or } b' \in B, \\ (a \otimes 1) \succ (a' \otimes 1) &= (a \succ a') \otimes 1, \\ (a \otimes b) \prec (a' \otimes b') &= (a * a') \otimes (b \prec b'), \text{ if } b \text{ or } b' \in B, \\ (a \otimes 1) \prec (a' \otimes 1) &= (a \prec a') \otimes 1. \end{aligned}$$

Proof. With proposition 6, $(A \overline{\otimes} B, *, \prec)$ is a \mathcal{RG} -algebra. In the same way, $(A \overline{\otimes} B, *, \succ)$ is a \mathcal{LG} -algebra. It remains to show the entanglement relation: for all $a, a', a'' \in A$ and $b, b', b'' \in B$,

$$\begin{aligned} ((a \otimes b) \succ (a' \otimes b')) \prec (a'' \otimes b'') &= ((a * a') * a'') \otimes ((b \succ b') \prec b'') \\ &= (a * (a' * a'')) \otimes (b \succ (b' \prec b'')) \\ &= (a \otimes b) \succ ((a' \otimes b') \prec (a'' \otimes b'')). \end{aligned}$$

This calculation is still true if b, b' or b'' is equal to 1 or if $b' = b'' = 1$ and $b \in B$. If $b = b' = 1$ and $b'' \in B$,

$$\begin{aligned} ((a \otimes 1) \succ (a' \otimes 1)) \prec (a'' \otimes b'') &= ((a \succ a') * a'') \otimes (1 \prec b'') \\ &= 0, \\ (a \otimes 1) \succ ((a' \otimes 1) \prec (a'' \otimes b'')) &= (a \otimes 1) \succ ((a' * a'') \otimes (1 \prec b'')) \\ &= 0. \end{aligned}$$

If $b' = b'' = 1$ and $b \in B$, it is the same calculation as previously. Finally if $b = b' = b'' = 1$,

$$\begin{aligned} ((a \otimes 1) \succ (a' \otimes 1)) \prec (a'' \otimes 1) &= ((a \succ a') \prec a'') \otimes 1 \\ &= (a \succ (a' \prec a'')) \otimes 1 \\ &= (a \otimes 1) \succ ((a' \otimes 1) \prec (a'' \otimes 1)). \end{aligned}$$

□

Proposition 41 *For all forests $F, G \in \mathcal{M}$ and for all tree $T \in \mathcal{M}$,*

$$\Delta(F \succ T \prec G) = \Delta(F) \succ \Delta(T) \prec \Delta(G). \quad (14)$$

Proof. With proposition 7, we have $\Delta(T \prec G) = \Delta(T) \prec \Delta(G)$. Moreover,

$$\begin{aligned} \Delta(F \succ T) &= \Delta((F^\dagger)^\dagger \succ (T^\dagger)^\dagger) \\ &= (\dagger \otimes \dagger) \circ \Delta(T^\dagger \prec F^\dagger) \\ &= (\dagger \otimes \dagger) \circ (\Delta(T^\dagger) \prec \Delta(F^\dagger)) \\ &= (\dagger \otimes \dagger) \circ (((\dagger \otimes \dagger) \circ \Delta(T)) \prec ((\dagger \otimes \dagger) \circ \Delta(F))) \\ &= (\dagger \otimes \dagger) \circ (\dagger \otimes \dagger) \circ (\Delta(F) \succ \Delta(T)) \\ &= \Delta(F) \succ \Delta(T), \end{aligned}$$

using propositions 21 and 30. As the entanglement relation is satisfied in \mathcal{M} and $\mathcal{M} \overline{\otimes} \mathcal{M}$, any parenthesizing of (14) gives the same relation. \square

Remark. As $\mathcal{M} \overline{\otimes} \mathcal{M}$ is not a \mathcal{BG} -algebra, the relation (14) is not true if T is a forest in general.

4 Koszul duality

4.1 Koszul dual of the bigraft operad

4.1.1 Presentation

As \mathcal{BG} is a binary and quadratic operad, it admits a dual, in the sense of V. Ginzburg and M. Kapranov (see [GK]). We denote by $\mathcal{BG}^!$ the Koszul dual of \mathcal{BG} . We prove that $\mathcal{BG}^!$ is defined as follows:

Theorem 42 *The operad $\mathcal{BG}^!$, dual of \mathcal{BG} , is the quadratic operad $\mathcal{P}(E, R)$, where E is concentrated in degree 2, with $E(2)$ the free S_2 -module $\mathbb{K}m \oplus \mathbb{K}\succ \oplus \mathbb{K}\prec$ and R is concentrated in degree 3, with $R(3) \subseteq \mathcal{P}_E(3)$ the sub- S_3 -module generated by*

$$\left\{ \begin{array}{l} r_1 = \succ \circ (m, I) - \succ \circ (I, \succ), \\ r_2 = m \circ (\succ, I) - \succ \circ (I, m), \\ r_3 = \prec \circ (\prec, I) - \prec \circ (I, m), \\ r_4 = \prec \circ (m, I) - m \circ (I, \prec), \\ r_5 = \prec \circ (\succ, I) - \succ \circ (I, \prec), \\ r_6 = m \circ (m, I) - m \circ (I, m), \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} r_7 = \succ \circ (\succ, I), \\ r_8 = \succ \circ (\prec, I), \\ r_9 = m \circ (\prec, I), \\ r_{10} = \prec \circ (I, \prec), \\ r_{11} = \prec \circ (I, \succ), \\ r_{12} = m \circ (I, \succ). \end{array} \right. \quad (15)$$

The notations r_1 to r_{12} of the relations are introduced for future references.

Proof. Recall some notations. For any right S_n -module V , we denote by $V^!$ the right S_n -module $V^* \otimes (\varepsilon)$, where (ε) is the one-dimensional signature representation. Explicitly, the action of S_n on $V^!$ is defined by $f^\sigma(x) = \varepsilon(\sigma)f(x^{\sigma^{-1}})$ for all $\sigma \in S_n$, $f \in V^!$, $x \in V$. The pairing between $V^!$ and V is given by: if $f \in V^!$, $x \in V$,

$$\langle -, - \rangle : V^! \otimes V \longrightarrow \mathbb{K}, \langle f, x \rangle = f(x), \quad (16)$$

and if $\sigma \in S_n$, $\langle f^\sigma, x^\sigma \rangle = \varepsilon(\sigma) \langle f, x \rangle$.

Moreover, $\mathcal{BG} = \mathcal{P}(E, R)$, where E is the free S_2 -module generated by m , \succ and \prec and R the sub- S_3 -module of $\mathcal{P}_E(3)$ generated by r_1, \dots, r_6 . Note that $\dim(E) = 6$, $\dim(\mathcal{P}_E(3)) = 108$ and $\dim(R) = 36$. $\mathcal{BG}^! = \mathcal{P}(E^!, R^\perp)$ where $E^! = (E(n^!))_{n \in \mathbb{N}}$ and R^\perp is the annihilator of R for the pairing (16). So $\dim(R^\perp) = \dim(\mathcal{P}_E(3)) - \dim(R) = 108 - 36 = 72$. We then verify that the given relations (15) for $\mathcal{BG}^!$ are indeed in R^\perp , that each of them generates a free S_3 -module, and finally that these S_3 -modules are in direct sum. So these relations entirely generate R^\perp . \square

Remarks.

1. $\mathcal{BG}^!$ is a quotient of \mathcal{BG} .
2. $\mathcal{BG}^!$ is the symmetrization of the nonsymmetric operad $\widetilde{\mathcal{BG}}^!$ generated by m , \succ and \prec and satisfying the relations r_1 to r_{12} .
3. Graphically, the relations defining $\mathcal{BG}^!$ can be written in the following way :

$$\begin{array}{c}
\begin{array}{ccc}
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ m \quad 3 \\ \diagup \quad \diagdown \\ \neg \quad | \end{array} & - & \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad m \\ \diagup \quad \diagdown \\ \neg \quad | \end{array} \\
\end{array}, \quad
\begin{array}{ccc}
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \neg \quad 3 \\ \diagup \quad \diagdown \\ \neg \quad | \end{array} & - & \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad \neg \\ \diagup \quad \diagdown \\ \neg \quad | \end{array} \\
\end{array}, \quad
\begin{array}{ccc}
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ m \quad 3 \\ \diagup \quad \diagdown \\ m \quad | \end{array} & - & \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad m \\ \diagup \quad \diagdown \\ m \quad | \end{array} \\
\end{array}, \\
\\
\begin{array}{c}
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \neg \quad 3 \\ \diagup \quad \diagdown \\ \neg \quad | \end{array}, \quad
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \neg \quad 3 \\ \diagup \quad \diagdown \\ \neg \quad | \end{array}, \quad
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \neg \quad 3 \\ \diagup \quad \diagdown \\ m \quad | \end{array}, \quad
\begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad \neg \\ \diagup \quad \diagdown \\ \neg \quad | \end{array}, \quad
\begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad \neg \\ \diagup \quad \diagdown \\ \neg \quad | \end{array}, \quad
\begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad \neg \\ \diagup \quad \diagdown \\ m \quad | \end{array}.
\end{array}$$

Proposition 43 1. $\mathcal{BG}^!(n)$ is generated, as a S_n -module, by the following trees:

$$\begin{array}{c}
1 \quad 2 \\
\diagdown \quad \diagup \\
m \quad \dots \quad l-1 \\
\diagdown \quad \diagup \\
m \quad l \\
\diagdown \quad \diagup \\
\neg \quad l+1 \\
\diagdown \quad \diagup \\
m \quad \dots \quad k \\
\diagdown \quad \diagup \\
m \quad k+1 \\
\diagdown \quad \diagup \\
\neg \quad \dots \quad n
\end{array} \tag{17}$$

where $1 \leq l \leq k \leq n$.

2. For all $n \in \mathbb{N}$, $\dim(\mathcal{BG}^!(n)) \leq \frac{n(n+1)!}{2}$.

Proof. From relations r_1, \dots, r_6 and r_{10}, r_{11}, r_{12} , we obtain that $\mathcal{BG}^!(3)$ is generated by the trees of the following form:

$$\begin{array}{c}
1 \quad 2 \\
\diagdown \quad \diagup \\
b \quad 3 \\
\diagup \quad \diagdown \\
a \quad |
\end{array},$$

with $a, b \in \{\neg, \succ, m\}$. Using relations r_7, r_8, r_9 ,

$$\begin{array}{ccc}
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \neg \quad 3 \\ \diagup \quad \diagdown \\ \neg \quad | \end{array}, &
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \neg \quad 3 \\ \diagup \quad \diagdown \\ \neg \quad | \end{array}, &
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \neg \quad 3 \\ \diagup \quad \diagdown \\ m \quad | \end{array}
\end{array}$$

are eliminated. We deduce that $\mathcal{BG}^!(3)$ is generated by the trees of the following form:

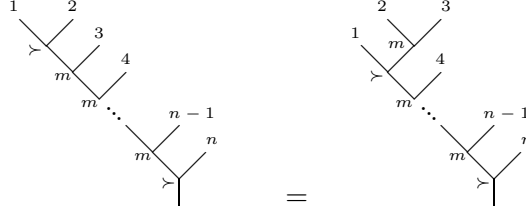
$$\begin{array}{c}
1 \quad 2 \\
\diagdown \quad \diagup \\
b \quad 3 \\
\diagup \quad \diagdown \\
a \quad |
\end{array},$$

with $(a, b) \in \{(\succ, m), (m, \succ), (m, m), (\neg, \succ), (\neg, m), (\neg, \neg)\}$.

Moreover, let us prove that for all $n \geq 3$, the following tree is zero :

$$\begin{array}{c}
1 \quad 2 \\
\diagdown \quad \diagup \\
\neg \quad 3 \\
\diagup \quad \diagdown \\
m \quad \dots \quad n-1 \\
\diagdown \quad \diagup \\
m \quad n \\
\diagup \quad \diagdown \\
\neg \quad |
\end{array}$$

By induction on n . If $n = 3$ this tree is zero with the relation r_7 . If $n \geq 4$,



with the relation r_2 and this tree is zero by induction hypothesis.

So $\mathcal{BG}^!(n)$ is generated, as a S_n -module, by the $\frac{n(n+1)}{2}$ trees given by the formula (17). We immediately deduce the second assertion. \square

4.1.2 Free $\mathcal{BG}^!$ -algebra and dimension of $\mathcal{BG}^!$

Let $\mathbb{F}^!$ be the subset of forests of \mathbb{F} containing the empty tree 1 and satisfying the following conditions : if $F_1 \dots F_n \in \mathbb{F}^!$ with the F_i 's nonempty trees, then

1. F_1, \dots, F_n are corollas $\in \mathbb{F}$,
2. if $\exists e \in E(F_i)$ decorated by l , then $i = 1$,
3. if $\exists e \in E(F_i)$ decorated by r , then $i = n$.

We set $\mathbb{G} = \mathbb{F} \setminus \mathbb{F}^!$.

Remark. If $F_1 \dots F_n \in \mathbb{F}^!$ with $n \geq 2$, then $F_2, \dots, F_{n-1} = \bullet$, every $e \in E(F_1)$ is decorated by l and every $e \in E(F_n)$ is decorated by r . In particular, we have $\dagger(\mathbb{F}^!) = \mathbb{F}^!$.

Examples. Forests of $\mathbb{F}^!$:

- In degree 1 : ..
- In degree 2 : .., \mathfrak{l} , \mathfrak{r} .
- In degree 3 : ..., \mathfrak{l} .., \mathfrak{r} , $\mathfrak{l}\mathfrak{l}$, $\mathfrak{l}\mathfrak{r}$, $\mathfrak{r}\mathfrak{l}$, $\mathfrak{r}\mathfrak{r}$.
- In degree 4 :, \mathfrak{l} , \mathfrak{r} , \mathfrak{l} \mathfrak{r} , $\mathfrak{l}\mathfrak{l}$.., $\mathfrak{r}\mathfrak{r}$, $\mathfrak{l}\mathfrak{l}\mathfrak{l}$, $\mathfrak{l}\mathfrak{l}\mathfrak{r}$, $\mathfrak{l}\mathfrak{r}\mathfrak{l}$, $\mathfrak{l}\mathfrak{r}\mathfrak{r}$, $\mathfrak{r}\mathfrak{l}\mathfrak{l}$, $\mathfrak{r}\mathfrak{l}\mathfrak{r}$, $\mathfrak{r}\mathfrak{r}\mathfrak{l}$, $\mathfrak{r}\mathfrak{r}\mathfrak{r}$.

Let $\mathcal{BT}^!$ be the \mathbb{K} -vector space generated by $\mathbb{F}^!$ and $\mathcal{M}^!$ the \mathbb{K} -vector space generated by $\mathbb{F}^! \setminus \{1\}$. We denote by $t_n^{\mathcal{BT}^!}$ the number of trees of degree n in $\mathcal{BT}^!$ and $f_n^{\mathcal{BT}^!}$ the number of forests of degree n in $\mathcal{BT}^!$. We put $T_{\mathcal{BT}^!}(x) = \sum_{n \geq 1} t_n^{\mathcal{BT}^!} x^n$ and $F_{\mathcal{BT}^!}(x) = \sum_{n \geq 1} f_n^{\mathcal{BT}^!} x^n$.

Proposition 44 *The formal series of $\mathcal{BT}^!$ are given by :*

$$T_{\mathcal{BT}^!}(x) = \frac{x}{(1-x)^2} \quad \text{and} \quad F_{\mathcal{BT}^!}(x) = \frac{x}{(1-x)^3}.$$

Proof. We have $t_1^{\mathcal{BT}^!} = 1, f_1^{\mathcal{BT}^!} = 1$. For all $n \geq 2$, $t_n^{\mathcal{BT}^!} = n$ because the corollas of degree n have $n-1$ edges with n possible different decorations. Moreover, for all $n \geq 2$,

$$f_n^{\mathcal{BT}^!} = f_{n-1}^{\mathcal{BT}^!} - t_{n-1}^{\mathcal{BT}^!} + t_n^{\mathcal{BT}^!} + (n-1)$$

where the term :

- $f_{n-1}^{\mathcal{BT}^!} - t_{n-1}^{\mathcal{BT}^!}$ corresponds to forests of degree n and of length ≥ 3 obtained from the forests of degree $n-1$ and of length ≥ 2 by adding \bullet in the middle. For example, in degree 4, these are the forests $\bullet \bullet \bullet \bullet, \bullet \bullet \bullet \bullet, \bullet \bullet \bullet \bullet$.
- $t_n^{\mathcal{BT}^!}$ corresponds to trees of degree n . In degree 4, these are the trees $\bullet \bullet \bullet \bullet, \bullet \bullet \bullet \bullet, \bullet \bullet \bullet \bullet, \bullet \bullet \bullet \bullet$.
- $n-1$ corresponds to forests of degree n and of length 2 obtained by product of a tree of degree k and a tree of degree $n-k$, $1 \leq k \leq n-1$. For example, in degree 4, these are the forests $\bullet \bullet \bullet \bullet, \bullet \bullet \bullet \bullet, \bullet \bullet \bullet \bullet$.

So $f_n^{\mathcal{BT}^!} = f_{n-1}^{\mathcal{BT}^!} + n = \frac{n(n+1)}{2}$. We deduce that $T_{\mathcal{BT}^!}(x) = \frac{x}{(1-x)^2}$ and $F_{\mathcal{BT}^!}(x) = \frac{x}{(1-x)^3}$. \square

Lemma 45 *Let $F, G \in \mathbb{F}$ be two nonempty forests such that $F \in \mathbb{G}$ or $G \in \mathbb{G}$. Then $FG, F \succ G$ and $F \prec G \in \mathbb{G}$.*

Proof. Suppose that $F \notin \mathbb{F}^!$. We have two cases :

1. If F is not a monomial of corollas. Then $h(F) \geq 2$. So $h(F \bullet G), h(G \bullet F) \geq 2$ and $F \bullet G \in \mathbb{G}, G \bullet F \in \mathbb{G}$ for all $\bullet \in \{*, \succ, \prec\}$.
2. If F is a monomial of corollas. As $F \notin \mathbb{F}^!$, $h(F) \geq 1$ and $F = F_1 F_2$ with F_1, F_2 nonempty such that $\exists e \in E(F_2)$ decorated by l (this is the same argument with r). Then $FG, GF \in \mathbb{G}$, $G \succ F = (G \succ F_1) F_2 \in \mathbb{G}$ and $F \prec G = F_1 (F_2 \prec G) \in \mathbb{G}$. Moreover $h(F \succ G), h(G \prec F) \geq 2$ and $F \succ G, G \prec F \in \mathbb{G}$.

\square

In other words, $\mathbb{K}[\mathbb{G}]$ is a \mathcal{BG} -ideal of the \mathcal{BG} -algebra \mathcal{M} . So the quotient vector space $\mathcal{M}^! = \mathcal{M}/\mathbb{K}[\mathbb{G}]$ is a \mathcal{BG} -algebra. In the sequel, we shall identify a forest $F \in \mathcal{M}$ and its class in $\mathcal{M}^!$.

Proposition 46 *$(\mathcal{M}^!, *, \succ, \prec)$ is a $\mathcal{BG}^!$ -algebra generated by \bullet .*

Proof. We already have that $\mathcal{M}^!$ is a \mathcal{BG} -algebra. Let us prove relations $r7$ to $r12$. Let $F, G, H \in \mathcal{M}$ be three nonempty forests. $h((F \succ G) \succ H), h((F \prec G) \succ H), h(F \prec (G \prec H))$ and $h(F \prec (G \succ H)) \geq 3$. Therefore $(F \succ G) \succ H, (F \prec G) \succ H, F \prec (G \prec H), F \prec (G \succ H) \in \mathbb{G}$ and the relations $r7, r8, r10, r11$ are satisfied in $\mathcal{M}^!$. Moreover $(F \prec G)H, F(G \succ H) \in \mathbb{G}$ by considering the decorations of the edges and $r9, r12$ are true in $\mathcal{M}^!$. So $(\mathcal{M}^!, *, \succ, \prec)$ is a $\mathcal{BG}^!$ -algebra. As \mathcal{M} is generated as \mathcal{BG} -algebra by \bullet (with theorem 33), $\mathcal{M}^!$ is also generated by \bullet as \mathcal{BG} -algebra and therefore as $\mathcal{BG}^!$ -algebra. \square

Theorem 47 *1. $\mathcal{M}^!$ is the free $\mathcal{BG}^!$ -algebra generated by \bullet .*

2. For all $n \in \mathbb{N}^$, $\mathcal{BG}^!(n) = \frac{n(n+1)!}{2}$.*

3. For all $n \in \mathbb{N}^$, $\mathcal{BG}^!(n)$ is freely generated, as a S_n -module, by the trees given by the formula (17).*

Proof. Let $n \in \mathbb{N}^*$. Consider the following application :

$$\Omega : \begin{cases} \widetilde{\mathcal{BG}}^!(n) & \rightarrow \text{Vect}(\text{planar forests} \in \mathcal{BT}^! \text{ of degree } n) \subseteq \mathcal{M}^! \\ p & \rightarrow p(\bullet, \dots, \bullet). \end{cases}$$

By proposition 46, Ω is surjective. So $\dim(\mathcal{BG}^!(n)) \geq f_n^{\mathcal{BT}^!} n! = \frac{n(n+1)!}{2}$. From proposition 43, we deduce that $\dim(\mathcal{BG}^!(n)) = \frac{n(n+1)!}{2}$. So Ω is bijective and $\mathcal{M}^!$ is the free $\mathcal{BG}^!$ -algebra generated by \bullet . Moreover $\mathcal{BG}^!(n)$ is freely generated, as a S_n -module, by the trees given by the formula

(17), by equality of dimensions. \square

We give some numerical values:

n	1	2	3	4	5	6	7	8	9	10
$\dim(\widetilde{\mathcal{BG}}^!(n))$	1	3	6	10	15	21	28	36	45	55
$\dim(\mathcal{BG}^!(n))$	1	6	36	240	1800	15120	141120	1451520	16329600	199584000

These are the sequences A000217 and A001286 in [Sl].

Remark. We denote $F_{\mathcal{BG}}(x)$ and $F_{\mathcal{BG}^!}(x)$ the formal series associated to operads \mathcal{BG} and $\mathcal{BG}^!$. Using proposition 44, $F_{\mathcal{BG}^!}(x) = \frac{x}{(1-x)^3}$. Moreover, by proposition 17 (see the proof), $F_{\mathcal{BG}}(x) = \frac{T(x)}{1-T(x)}$ where $T(x) = (x - 2x^2 + x^3)^{-1}$. So $F_{\mathcal{BG}}^{-1}(x) = \frac{x}{(1+x)^3}$ and we have :

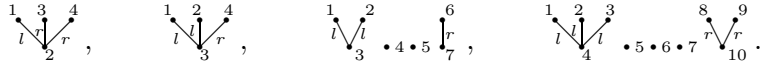
$$F_{\mathcal{BG}}(-F_{\mathcal{BG}^!}(-x)) = x.$$

This result is also a consequence of theorem 50.

To conclude this section, we introduce an order relation on the set of the vertices of a forest $\in \mathbb{F}^!$. Let $F = F_1 \dots F_n \in \mathbb{F}^! \setminus \{1\}$ and s, s' be two vertices of F . Then $s \leq s'$ if one of these assertions is satisfied :

1. $s \in V(F_i)$, $s' \in V(F_j)$ and $i < j$.
2. $s, s' \in V(F_i)$ with $F_i = B(G_1 \dots G_p \otimes H_1 \dots H_q)$ (the G_k 's and the H_k 's are equal to \bullet) and :
 - (a) $s \in V(G_k)$, $s' \in V(G_l)$ and $k < l$.
 - (b) $s \in V(G_k)$ and s' is the root of F_i .
 - (c) s is the root of F_i and $s' \in V(H_k)$.
 - (d) $s \in V(H_k)$, $s' \in V(H_l)$ and $k < l$.

Examples. We give the order relation on the vertices for the following forests:



4.2 Homology of a \mathcal{BG} -algebra

We denote by $BG^!(V)$ the free $\mathcal{BG}^!$ -algebra over a vector space V . Because the operad $\mathcal{BG}^!$ is regular, we get the following result :

Proposition 48 *Let V be a \mathbb{K} -vector space. Then the free $\mathcal{BG}^!$ -algebra on V is*

$$BG^!(V) = \bigoplus_{n \geq 1} \mathbb{K}[\mathbb{F}^!(n)] \otimes V^{\otimes n},$$

equipped with the following binary operations : for all $F \in \mathbb{F}^!(n)$, $G \in \mathbb{F}^!(m)$, $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \dots \otimes w_m \in V^{\otimes m}$,

$$\begin{aligned} (F \otimes v_1 \otimes \dots \otimes v_n) * (G \otimes w_1 \otimes \dots \otimes w_m) &= (FG \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m), \\ (F \otimes v_1 \otimes \dots \otimes v_n) \succ (G \otimes w_1 \otimes \dots \otimes w_m) &= (F \succ G \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m), \\ (F \otimes v_1 \otimes \dots \otimes v_n) \prec (G \otimes w_1 \otimes \dots \otimes w_m) &= (F \prec G \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m). \end{aligned}$$

Remark. We can see an element $(F \otimes v_1 \otimes \dots \otimes v_n) \in BG^!(V)$ as the forest F where the vertex i (for the order relation introduced at the end of the section 4.1.2) is decorated by v_i .

By taking the elements of V homogenous of degree 1, $BG^!(V)$ is naturally graduated. We define by duality on $BG^!(V)$ three coproducts given in the following way: if $F = F_1 \dots F_k \in \mathbb{F}^!(n)$ and $v = v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$, with

- if $k = 1$,

$$F = B(\underbrace{\bullet \dots \bullet}_{p \times} \otimes \underbrace{\bullet \dots \bullet}_{q \times}),$$

- if $k \geq 2$,

$$F_1 = B(\underbrace{\bullet \dots \bullet}_{p \times} \otimes 1), F_k = B(1 \otimes \underbrace{\bullet \dots \bullet}_{q \times}) \text{ and for all } 2 \leq i \leq k-1, F_i = \bullet,$$

($p + q + k = n$) then

$$\begin{aligned} \Delta(F \otimes v) &= \sum_{i=0}^k (F_1 \dots F_i \otimes v_1 \otimes \dots \otimes v_{p+i}) \otimes (F_{i+1} \dots F_k \otimes v_{p+i+1} \otimes \dots \otimes v_n), \\ \Delta_{\succ}(F \otimes v) &= \sum_{i=0}^p (\underbrace{\bullet \dots \bullet}_{i \times} \otimes v_1 \otimes \dots \otimes v_i) \otimes (B(\underbrace{\bullet \dots \bullet}_{p-i \times} \otimes 1) F_2 \dots F_k \otimes v_{i+1} \otimes \dots \otimes v_n), \\ \Delta_{\prec}(F \otimes v) &= \sum_{i=0}^q (F_1 \dots F_{k-1} B(1 \otimes \underbrace{\bullet \dots \bullet}_{q-i \times}) \otimes v_1 \otimes \dots \otimes v_{n-i}) \otimes (\underbrace{\bullet \dots \bullet}_{i \times} \otimes v_{n-i+1} \otimes \dots \otimes v_n). \end{aligned}$$

Let $(A, *, \succ, \prec)$ be a \mathcal{BG} -algebra. We define a differential $d : BG^!(A)(n) \rightarrow BG^!(A)(n-1)$ uniquely determined by the following conditions :

1. For all $a \in A$, $d(\bullet \otimes a) = 0$.
2. For all $a, b \in A$, $d(\bullet \otimes a \otimes b) = a * b$, $d(\mathbf{r} \otimes a \otimes b) = a \succ b$ and $d(\mathbf{l} \otimes a \otimes b) = a \prec b$.
3. Let $\theta : BG^!(A) \rightarrow BG^!(A)$ be the following map:

$$\theta : \begin{cases} BG^!(A) & \rightarrow BG^!(A) \\ x & \rightarrow (-1)^{\text{degree}(x)} x \text{ for all homogeneous } x. \end{cases}$$

Then d is a θ -coderivation : for all $x \in BG^!(A)$,

$$\begin{aligned} \Delta(d(x)) &= (d \otimes id + \theta \otimes d) \circ \Delta(x), \\ \Delta_{\succ}(d(x)) &= (d \otimes id + \theta \otimes d) \circ \Delta_{\succ}(x), \\ \Delta_{\prec}(d(x)) &= (d \otimes id + \theta \otimes d) \circ \Delta_{\prec}(x). \end{aligned}$$

So, d is the map which sends the element $(B(\underbrace{\bullet \dots \bullet}_{p \times} \otimes 1) \underbrace{\bullet \dots \bullet}_{k-2 \times} B(1 \otimes \underbrace{\bullet \dots \bullet}_{q \times}) \otimes v_1 \otimes \dots \otimes v_n)$, where $p, q, k \in \mathbb{N}$, $k \geq 1$ and $p + q + k = n$ (if $k = 1$, the element is $(B(\underbrace{\bullet \dots \bullet}_{p \times} \otimes \underbrace{\bullet \dots \bullet}_{q \times}) \otimes v_1 \otimes \dots \otimes v_n)$),

$\dots \otimes v_n))$, to

$$\begin{aligned}
& \sum_{i=1}^{p-1} (-1)^{i-1} (B(\underbrace{\dots}_{p-1 \times} \otimes 1) \underbrace{\dots}_{k-2 \times} B(1 \otimes \underbrace{\dots}_{q \times}) \otimes v_1 \otimes \dots \otimes v_i * v_{i+1} \otimes \dots \otimes v_n) \\
& + (-1)^{p-1} (B(\underbrace{\dots}_{p-1 \times} \otimes 1) \underbrace{\dots}_{k-2 \times} B(1 \otimes \underbrace{\dots}_{q \times}) \otimes v_1 \otimes \dots \otimes v_p \succ v_{p+1} \otimes \dots \otimes v_n) \\
& + \sum_{i=p+1}^{p+k-1} (-1)^{i-1} (B(\underbrace{\dots}_{p \times} \otimes 1) \underbrace{\dots}_{k-3 \times} B(1 \otimes \underbrace{\dots}_{q \times}) \otimes v_1 \otimes \dots \otimes v_i * v_{i+1} \otimes \dots \otimes v_n) \\
& + (-1)^{p+k-1} (B(\underbrace{\dots}_{p \times} \otimes 1) \underbrace{\dots}_{k-2 \times} B(1 \otimes \underbrace{\dots}_{q-1 \times}) \otimes v_1 \otimes \dots \otimes v_{p+k} \prec v_{p+k+1} \otimes \dots \otimes v_n) \\
& + \sum_{i=p+k+1}^{n-1} (-1)^{i-1} (B(\underbrace{\dots}_{p \times} \otimes 1) \underbrace{\dots}_{k-2 \times} B(1 \otimes \underbrace{\dots}_{q-1 \times}) \otimes v_1 \otimes \dots \otimes v_i * v_{i+1} \otimes \dots \otimes v_n).
\end{aligned}$$

The homology of this complex will be denoted by $H_*(A)$. More clearly, for all $n \in \mathbb{N}$:

$$H_n(A) = \frac{\text{Ker} \left(d|_{BG^1(A)(n+1)} \right)}{\text{Im} \left(d|_{BG^1(A)(n+2)} \right)}$$

Examples. Let $a, b, c \in A$. Then $d(\cdot \otimes a) = 0$, $d(\cdot \otimes a \otimes b) = a * b$, $d(\mathfrak{l} \otimes a \otimes b) = a \succ b$ and $d(\mathfrak{r} \otimes a \otimes b) = a \prec b$. In degree 3,

$$\begin{aligned}
d(\dots \otimes a \otimes b \otimes c) &= -(\cdot \otimes a \otimes (b * c)) + (\cdot \otimes (a * b) \otimes c) \\
d(\mathfrak{l} \cdot \otimes a \otimes b \otimes c) &= (\cdot \otimes (a \succ b) \otimes c) - (\mathfrak{l} \otimes a \otimes (b * c)) \\
d(\cdot \mathfrak{r} \otimes a \otimes b \otimes c) &= (\mathfrak{r} \otimes (a * b) \otimes c) - (\cdot \otimes a \otimes (b \prec c)) \\
d(\mathfrak{l} \mathfrak{l} \otimes a \otimes b \otimes c) &= -(\mathfrak{l} \otimes a \otimes (b \succ c)) + (\mathfrak{l} \otimes (a * b) \otimes c) \\
d(\mathfrak{l} \mathfrak{r} \otimes a \otimes b \otimes c) &= -(\mathfrak{l} \otimes a \otimes (b \prec c)) + (\mathfrak{r} \otimes (a \succ b) \otimes c) \\
d(\mathfrak{r} \mathfrak{r} \otimes a \otimes b \otimes c) &= -(\mathfrak{r} \otimes a \otimes (b * c)) + (\mathfrak{r} \otimes (a \prec b) \otimes c)
\end{aligned}$$

So, we obtain

$$\begin{aligned}
d^2(\dots \otimes a \otimes b \otimes c) &= -a * (b * c) + (a * b) * c \\
d^2(\mathfrak{l} \cdot \otimes a \otimes b \otimes c) &= (a \succ b) * c - a \succ (b * c) \\
d^2(\cdot \mathfrak{r} \otimes a \otimes b \otimes c) &= (a * b) \prec c - a * (b \prec c) \\
d^2(\mathfrak{l} \mathfrak{l} \otimes a \otimes b \otimes c) &= -a \succ (b \succ c) + (a * b) \succ c \\
d^2(\mathfrak{l} \mathfrak{r} \otimes a \otimes b \otimes c) &= -a \succ (b \prec c) + (a \succ c) \prec c \\
d^2(\mathfrak{r} \mathfrak{r} \otimes a \otimes b \otimes c) &= -a \prec (b * c) + (a \prec b) \prec c
\end{aligned}$$

Hence, the nullity of d^2 is equivalent to the six relations defining a \mathcal{BG} -algebra (see formula 7). In particular :

$$H_0(A) = \frac{A}{A * A + A \succ A + A \prec A}$$

4.3 The bigraft operad is Koszul

In this section, we use the rewriting method described in [LV], see also [DK, Hof].

We consider $\widetilde{\mathcal{BG}}^! = \mathcal{P}(E, R)$ the nonsymmetric operad associated to $\mathcal{BG}^!$, where E is concentrated in degree 2 with $E(2) = \mathbb{K}m \oplus \mathbb{K}\succ \oplus \mathbb{K}\prec$ and R is concentrated in degree 3 with $R(3) \subseteq \mathcal{P}_E(3)$ the subspace generated by r_i , for all $i \in \{1, \dots, 12\}$, defined in formula (15).

We use the lexicographical order and we set $\succ < m < \prec$. So, we can give the leading term for each r_i :

r_1	r_2	r_3	r_4	r_5	r_6
$\succ \circ(m, I)$	$m \circ(\succ, I)$	$\prec \circ(\prec, I)$	$\prec \circ(m, I)$	$\prec \circ(\succ, I)$	$m \circ(m, I)$
r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
$\succ \circ(\succ, I)$	$\succ \circ(\prec, I)$	$m \circ(\prec, I)$	$\prec \circ(I, \prec)$	$\prec \circ(I, \succ)$	$m \circ(I, \succ)$

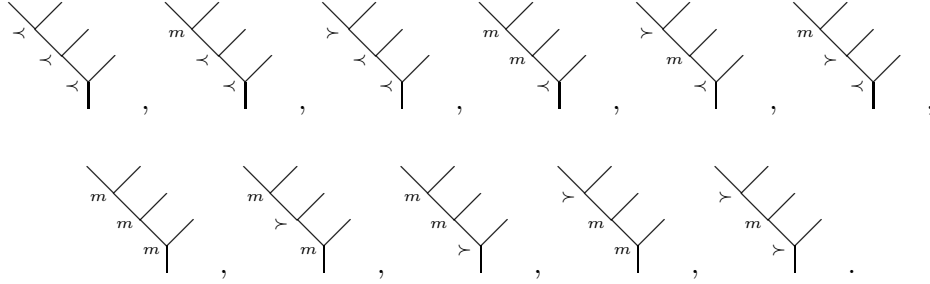
Observe that such a relation gives rise to a rewriting rule in the operad $\widetilde{\mathcal{BG}}^!$:

$\succ \circ(m, I) \mapsto \succ \circ(I, \succ)$	$\prec \circ(m, I) \mapsto m \circ(I, \prec)$	$\succ \circ(\succ, I) \mapsto 0$	$\prec \circ(I, \prec) \mapsto 0$
$m \circ(\succ, I) \mapsto \succ \circ(I, m)$	$\prec \circ(\succ, I) \mapsto \succ \circ(I, \prec)$	$\succ \circ(\prec, I) \mapsto 0$	$\prec \circ(I, \succ) \mapsto 0$
$\prec \circ(\prec, I) \mapsto \prec \circ(I, m)$	$m \circ(m, I) \mapsto m \circ(I, m)$	$m \circ(\prec, I) \mapsto 0$	$m \circ(I, \succ) \mapsto 0$

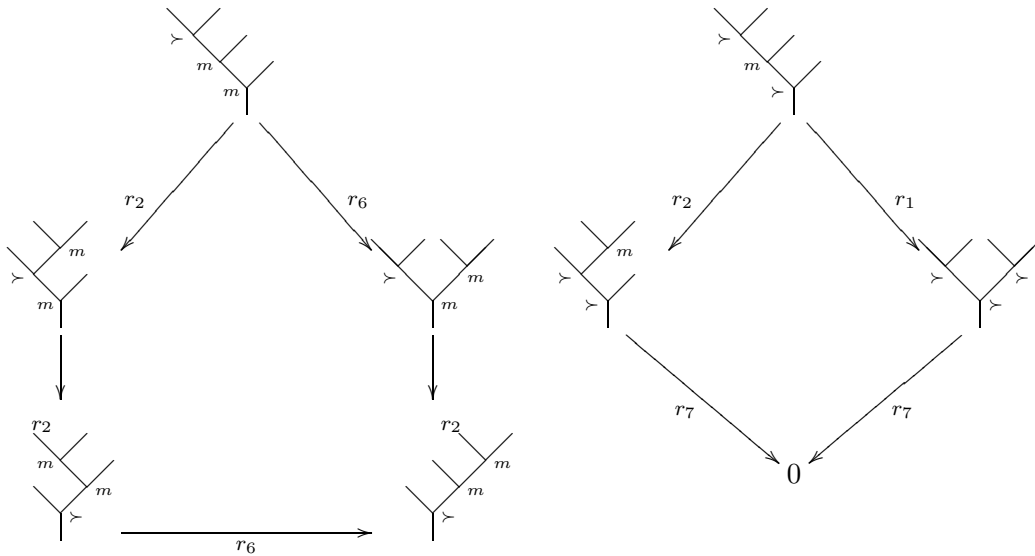
Then the following proposition holds:

Proposition 49 *All critical monomials are confluent in the nonsymmetric operad $\widetilde{\mathcal{BG}}^!$.*

Proof. We show that every critical monomial is confluent. We have 11 criticals monomials:

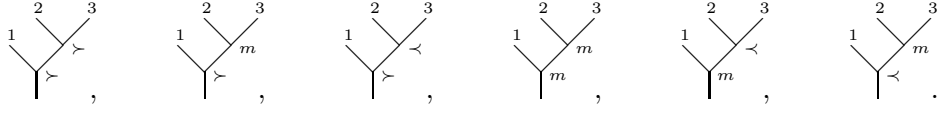


We present for example the confluent graph for the last two criticals monomials:

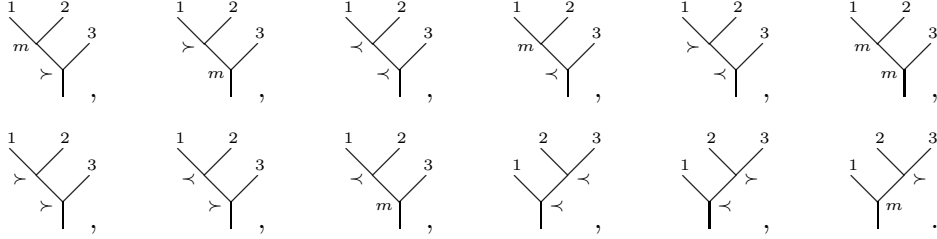


We can construct in the same way the confluent graph for each of the nine other criticals monomials. \square

We can give the quadratic part of a PBW basis of $\mathcal{BG}^!$:



The quadratic part of a PBW basis of its dual $(\mathcal{BG}^!)^! = \mathcal{BG}$ is:



We immediately deduce the following theorem:

Theorem 50 \mathcal{BG} is Koszul.

Recall that $H_*(.)$ is the homology of the complex defined in the section 4.2. Then we deduce of the previous theorem the following result:

Corollary 51 Let $N \geq 1$ and $(A, *, \succ, \prec)$ be the free \mathcal{BG} -algebra generated by N elements. Then $H_0(A)$ is N -dimensional and if $n \geq 1$, $H_n(A) = (0)$.

5 A rigidity theorem

5.1 \mathcal{BG} -infinitesimal bialgebra

Definition 52 A 2-dipterous algebra is a \mathbb{K} -vector space A equipped with three binary operations denoted $*$, \succ and \prec satisfying the following relations : for all $x, y, z \in A$,

$$\begin{aligned} (x * y) * z &= x * (y * z), \\ (x * y) \succ y &= x \succ (y \succ z), \\ (x \prec y) \prec z &= x \prec (y * z). \end{aligned}$$

In other words, $(A, *, \succ)$ and $(A, *, \prec)$ are respectively left and right dipterous algebras.

Remark. We can define $2Dipt\text{-alg}$ the category of 2-dipterous algebras. A pre-dendriform algebra is also a 2-dipterous algebra. We get the following canonical functors: $\mathcal{BG}\text{-alg} \rightarrow PreDend\text{-alg} \rightarrow 2Dipt\text{-alg}$.

We give in the following proposition another definition of \succ and \prec , to be compared with proposition 40 (and we will use this definition in the following) :

Proposition 53 Let A and B be two bigraft algebras. Then $A \overline{\otimes} B$ is given a structure of 2-dipterous algebra in the following way : for $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,

$$\begin{aligned} (a \otimes b) * (a' \otimes b') &= (a * a') \otimes (b * b'), \\ (a \otimes b) \succ (a' \otimes b') &= (a \succ a') \otimes (b * b'), \text{ if } a \text{ or } a' \in A, \\ (1 \otimes b) \succ (1 \otimes b') &= 1 \otimes (b \succ b'), \\ (a \otimes b) \prec (a' \otimes b') &= (a * a') \otimes (b \prec b'), \text{ if } b \text{ or } b' \in B, \\ (a \otimes 1) \prec (a' \otimes 1) &= (a \prec a') \otimes 1. \end{aligned}$$

Proof. With proposition 6, $(A \overline{\otimes} B, *, \prec)$ is a right dipterous algebra. Applying the same reasoning to $(A \overline{\otimes} B, *, \succ)$, we prove that this is a left dipterous algebra. So $(A \overline{\otimes} B, *, \succ, \prec)$ is a 2-dipterous algebra. \square

Remark. The entanglement relation is not true in general : for $a, a'' \in A$ and $b, b', b'' \in B$,

$$\begin{aligned} ((a \otimes b) \succ (1 \otimes b')) \prec (a'' \otimes b'') &= ((a \succ 1) \otimes (b * b')) \prec (a'' \otimes b'') = 0 \\ (a \otimes b) \succ ((1 \otimes b') \prec (a'' \otimes b'')) &= (a \succ a'') \otimes (b * (b' \prec b'')). \end{aligned}$$

We define another coproduct $\Delta_{\mathcal{A}ss}$ on \mathcal{BT} (the deconcatenation) in the following way: for all $F \in \mathbb{F}$,

$$\Delta_{\mathcal{A}ss}(F) = \sum_{F_1, F_2 \in \mathbb{F}, F_1 F_2 = F} F_1 \otimes F_2.$$

Remark that $\Delta_{\mathcal{A}ss}(F^\dagger) = \tau \circ (\dagger \otimes \dagger) \circ \Delta_{\mathcal{A}ss}(F)$ for all $F \in \mathbb{F}$.

We now have defined three products, namely m , \succ and \prec and one coproduct, namely $\tilde{\Delta}_{\mathcal{A}ss}$, on \mathcal{M} , obtained from $\Delta_{\mathcal{A}ss}$ by subtracting its primitive parts. The following properties sum up the different compatibilities.

Proposition 54 *For all $x, y \in \mathcal{M}$;*

$$\begin{cases} \tilde{\Delta}_{\mathcal{A}ss}(xy) &= (x \otimes 1) \tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x)(1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \succ y) &= (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y), \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) &= \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y). \end{cases} \quad (18)$$

Proof. We can restrict ourselves to $F, G \in \mathbb{F} \setminus \{1\}$. We put $F = F_1 \dots F_n$, $G = G_1 \dots G_m$ where the F_i 's and the G_i 's are trees and $G_1 = B(G_1^1 \otimes G_1^2)$. Hence :

$$\begin{aligned} \tilde{\Delta}_{\mathcal{A}ss}(FG) &= \sum_{H_1, H_2 \in \mathbb{F} \setminus \{1\}, H_1 H_2 = FG} H_1 \otimes H_2 \\ &= \sum_{H_1, H_2 \in \mathbb{F} \setminus \{1\}, H_1 H_2 = G} FH_1 \otimes H_2 + \sum_{H_1, H_2 \in \mathbb{F} \setminus \{1\}, H_1 H_2 = F} H_1 \otimes H_2 G + F \otimes G \\ &= (F \otimes 1) \tilde{\Delta}_{\mathcal{A}ss}(G) + \tilde{\Delta}_{\mathcal{A}ss}(F)(1 \otimes G) + F \otimes G, \end{aligned}$$

$$\begin{aligned} \tilde{\Delta}_{\mathcal{A}ss}(F \succ G) &= \tilde{\Delta}_{\mathcal{A}ss}(B(FG_1^1 \otimes G_1^2)G_2 \dots G_m) \\ &= B(FG_1^1 \otimes G_1^2) \otimes G_2 \dots G_m + \sum_{i=2}^{m-1} B(FG_1^1 \otimes G_1^2)G_2 \dots G_i \otimes G_{i+1} \dots G_m \\ &= F \succ G_1 \otimes G_2 \dots G_m + \sum_{i=2}^{m-1} F \succ G_1 G_2 \dots G_i \otimes G_{i+1} \dots G_m \\ &= (F \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(G). \end{aligned}$$

We deduce the third relation of (18) from the second one in this way :

$$\begin{aligned} \tilde{\Delta}_{\mathcal{A}ss}(F \prec G) &= \tilde{\Delta}_{\mathcal{A}ss}((G^\dagger \succ F^\dagger)^\dagger) \\ &= \tau \circ (\dagger \otimes \dagger) \circ \tilde{\Delta}_{\mathcal{A}ss}(G^\dagger \succ F^\dagger) \\ &= \tau \circ (\dagger \otimes \dagger) ((G^\dagger \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(F^\dagger)) \\ &= \tilde{\Delta}_{\mathcal{A}ss}(F^\dagger) \prec (1 \otimes G). \end{aligned}$$

\square

This justifies the following definition :

Definition 55 A \mathcal{BG} -infinitesimal bialgebra is a family $(A, m, \succ, \prec, \tilde{\Delta}_{Ass})$ where $m, \succ, \prec: A \otimes A \rightarrow A$, $\tilde{\Delta}_{Ass}: A \rightarrow A \otimes A$, with the following compatibilities :

1. (A, m, \succ, \prec) is a \mathcal{BG} -algebra.

2. For all $x, y \in A$:

$$\begin{cases} \tilde{\Delta}_{Ass}(xy) &= (x \otimes 1)\tilde{\Delta}_{Ass}(y) + \tilde{\Delta}_{Ass}(x)(1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{Ass}(x \succ y) &= (x \otimes 1) \succ \tilde{\Delta}_{Ass}(y), \\ \tilde{\Delta}_{Ass}(x \prec y) &= \tilde{\Delta}_{Ass}(x) \prec (1 \otimes y). \end{cases} \quad (19)$$

With proposition 54 and corollary 32, we have immediately :

Proposition 56 $(\mathcal{M}, m, \succ, \prec, \tilde{\Delta}_{Ass})$ is a \mathcal{BG} -infinitesimal bialgebra.

Notations. If A is a \mathcal{BG} -infinitesimal bialgebra, we denote $Prim(A) = Ker(\tilde{\Delta}_{Ass})$. In the \mathcal{BG} -infinitesimal bialgebra \mathcal{M} , $Prim(\mathcal{M}) = \mathbb{K}[\mathbb{T} \setminus \{1\}]$ and we denote by \mathcal{P} the primitive part of \mathcal{M} .

Recall the definition of a \mathcal{L} -algebra introduced by P. Leroux in [Le2] :

Definition 57 A \mathcal{L} -algebra is a \mathbb{K} -vector space A equipped with two binary operations $\succ, \prec: A \otimes A \rightarrow A$ verifying the entanglement relation:

$$(x \succ y) \prec z = x \succ (y \prec z),$$

for all $x, y, z \in A$.

The operad \mathcal{L} is binary, quadratic, regular and set-theoretic. We denote by $\tilde{\mathcal{L}}$ the nonsymmetric operad associated with the regular operad \mathcal{L} . We do not suppose that \mathcal{L} -algebras have unit for \succ or \prec . If A and B are two \mathcal{L} -algebras, a \mathcal{L} -morphism from A to B is a \mathbb{K} -linear map $f: A \rightarrow B$ such that $f(x \succ y) = f(x) \succ f(y)$ and $f(x \prec y) = f(x) \prec f(y)$ for all $x, y \in A$. We denote by $\mathcal{L}\text{-alg}$ the category of \mathcal{L} -algebras.

Proposition 58 For any \mathcal{BG} -infinitesimal bialgebra, its primitive part is a \mathcal{L} -algebra.

Proof. Let A be a \mathcal{BG} -infinitesimal bialgebra. We put $x, y \in Prim(A)$. Then with (19)

$$\begin{aligned} \tilde{\Delta}_{Ass}(x \succ y) &= (x \otimes 1) \succ \tilde{\Delta}_{Ass}(y) = 0, \\ \tilde{\Delta}_{Ass}(x \prec y) &= \tilde{\Delta}_{Ass}(x) \prec (1 \otimes y) = 0. \end{aligned}$$

Therefore, $x \succ y, x \prec y \in Prim(A)$. As we always have the relation $(x \succ y) \prec z = x \succ (y \prec z)$ for all $x, y, z \in Prim(A)$, $Prim(A)$ is a \mathcal{L} -algebra. \square

Corollary 59 $(\mathcal{P}, \succ, \prec)$ is a \mathcal{L} -algebra.

We have even more than this :

Theorem 60 $(\mathcal{P}, \succ, \prec)$ is the free \mathcal{L} -algebra generated by \bullet .

Proof. Let A be a \mathcal{L} -algebra and $a \in A$. Let us prove that there exists a unique morphism of \mathcal{L} -algebras $\phi: \mathcal{P} \rightarrow A$ such that $\phi(\bullet) = a$. We define $\phi(F)$ for any nonempty tree $F \in \mathcal{P}$ inductively on the degree of F by:

$$\begin{cases} \phi(\bullet) &= a, \\ \phi(F) &= (\dots ((\phi(F_1^1) \succ (\dots (\phi(F_{p-1}^1) \succ (\phi(F_p^1) \succ a)) \dots)) \prec \phi(F_1^2)) \dots) \prec \phi(F_q^2) \\ &\quad \text{if } F = B(F_1^1 \dots F_p^1 \otimes F_1^2 \dots F_q^2) \text{ with the } F_i^1\text{'s and the } F_j^2\text{'s in } \mathbb{T}. \end{cases}$$

This map is linearly extended into a map $\phi: \mathcal{P} \rightarrow A$. Let us show that it is a morphism of \mathcal{L} -algebras, that is to say $\phi(F \succ G) = \phi(F) \succ \phi(G)$ and $\phi(F \prec G) = \phi(F) \prec \phi(G)$ for all $F, G \in \mathbb{T} \setminus \{1\}$. Note $F = B(F_1^1 \dots F_p^1 \otimes F_1^2 \dots F_q^2)$, $G = B(G_1^1 \dots G_r^1 \otimes G_1^2 \dots G_s^2)$ with $F_1^1, \dots, F_p^1, F_1^2, \dots, F_q^2$ and $G_1^1, \dots, G_r^1, G_1^2, \dots, G_s^2$ in \mathbb{T} . Then:

1. For $\phi(F \succ G) = \phi(F) \succ \phi(G)$,

$$\begin{aligned} \phi(F \succ G) &= \phi(B(FG_1^1 \dots G_r^1 \otimes G_1^2 \dots G_s^2)) \\ &= (\dots ((\phi(F) \succ (\phi(G_1^1) \succ (\dots (\phi(G_r^1) \succ a) \dots))) \prec \phi(G_1^2)) \dots) \prec \phi(G_s^2) \\ &= \phi(F) \succ ((\dots ((\phi(G_1^1) \succ (\dots (\phi(G_r^1) \succ a) \dots))) \prec \phi(G_1^2)) \dots) \prec \phi(G_s^2) \\ &= \phi(F) \succ \phi(G). \end{aligned}$$

2. For $\phi(F \prec G) = \phi(F) \prec \phi(G)$,

$$\begin{aligned} \phi(F \prec G) &= \phi(B(F_1^1 \dots F_p^1 \otimes F_1^2 \dots F_q^2 G)) \\ &= ((\dots ((\phi(F_1^1) \succ (\dots (\phi(F_p^1) \succ a) \dots))) \prec \phi(F_1^2)) \dots) \prec \phi(F_q^2) \prec \phi(G) \\ &= \phi(F) \prec \phi(G). \end{aligned}$$

So ϕ is a morphism of \mathcal{L} -algebras.

Let $\phi' : \mathcal{P} \rightarrow A$ be another morphism of \mathcal{L} -algebras such that $\phi'(\cdot) = a$. For any forest $F_1^1 \dots F_p^1$ and $F_1^2 \dots F_q^2 \in \mathbb{F}$,

$$\begin{aligned} &\phi'(B(F_1^1 \dots F_p^1 \otimes F_1^2 \dots F_q^2)) \\ &= \phi'((\dots ((F_1^1 \succ (\dots (F_{p-1}^1 \succ (F_p^1 \succ \cdot)) \dots)) \prec F_1^2) \dots) \prec F_q^2) \\ &= (\dots ((\phi'(F_1^1) \succ (\dots (\phi'(F_{p-1}^1) \succ (\phi'(F_p^1) \succ \phi'(\cdot))) \dots)) \prec \phi'(F_1^2)) \dots) \prec \phi'(F_q^2) \\ &= (\dots ((\phi'(F_1^1) \succ (\dots (\phi'(F_{p-1}^1) \succ (\phi'(F_p^1) \succ a)) \dots)) \prec \phi'(F_1^2)) \dots) \prec \phi'(F_q^2). \end{aligned}$$

So $\phi = \phi'$. □

Remark. We deduce from theorem 60 that $\dim(\tilde{\mathcal{L}}(n)) = t_n^{\mathcal{BG}}$ for all $n \in \mathbb{N}^*$ (with the same reasoning as in corollary 34). We find again the result already given in [Le2].

We denote by $L(V)$ the free \mathcal{L} -algebra over a \mathbb{K} -vector space V . The functor $L(-)$ is the left adjoint to the forgetful functor from \mathcal{L} -algebras to vector spaces. Because the operad \mathcal{L} is regular, we get the following result :

Proposition 61 *Let V be a \mathbb{K} -vector space. Then the free \mathcal{L} -algebra on V is*

$$L(V) = \bigoplus_{n \geq 1} \mathbb{K}[\mathbb{T}(n)] \otimes V^{\otimes n},$$

equipped with the following binary operations : for all $F \in \mathbb{T}(n)$, $G \in \mathbb{T}(m)$, $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \dots \otimes w_m \in V^{\otimes m}$,

$$\begin{aligned} (F \otimes v_1 \otimes \dots \otimes v_n) \succ (G \otimes w_1 \otimes \dots \otimes w_m) &= (F \succ G \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m), \\ (F \otimes v_1 \otimes \dots \otimes v_n) \prec (G \otimes w_1 \otimes \dots \otimes w_m) &= (F \prec G \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m). \end{aligned}$$

5.2 Universal enveloping bigraft algebra

The functor $(-)_\mathcal{L} : \{\mathcal{BG}\text{-alg}\} \rightarrow \{\mathcal{L}\text{-alg}\}$ associates to a \mathcal{BG} -algebra (A, m, \succ, \prec) the \mathcal{L} -algebra (A, \succ, \prec) .

Reciprocally, we define the universal enveloping bigraft algebra of a \mathcal{L} -algebra (A, \succ, \prec) , denoted $U_{\mathcal{BG}}(A)$, as the augmentation ideal $\bar{T}(A)$ of tensor algebra $T(A)$ over the \mathbb{K} -vector space

A equipped with two operations also denoted by \succ and \prec and defined by : for all $p, q \in \mathbb{N}^*$ and $a_i, b_j \in A$, $1 \leq i \leq p$ and $1 \leq j \leq q$,

$$\begin{aligned} (a_1 \dots a_p) \succ (b_1 \dots b_q) &:= (a_1 \succ (\dots (a_{p-1} \succ (a_p \succ b_1)) \dots)) b_2 \dots b_q, \\ (a_1 \dots a_p) \prec (b_1 \dots b_q) &:= a_1 \dots a_{p-1} ((\dots ((a_p \prec b_1) \prec b_2) \dots) \prec b_q). \end{aligned} \quad (20)$$

Then we have immediately that $(\overline{T}(A), m, \succ, \prec)$ is a nonunitary bigraft algebra, where m is the concatenation.

Lemma 62 *The functor $U_{\mathcal{BG}} : \{\mathcal{L} - \text{alg}\} \rightarrow \{\mathcal{BG} - \text{alg}\}$ is the left adjoint to $(-)_\mathcal{L} : \{\mathcal{BG} - \text{alg}\} \rightarrow \{\mathcal{L} - \text{alg}\}$.*

Proof. We put A a \mathcal{L} -algebra and B a \mathcal{BG} -algebra. Let $f : A \rightarrow (B)_\mathcal{L}$ be a morphism of \mathcal{L} -algebras. It determines uniquely a morphism of algebras $\tilde{f} : \overline{T}(A) \rightarrow B$ because (B, m) is an associative algebra. We endow $\overline{T}(A)$ with a \mathcal{BG} -algebra structure defined by (20). Then $\tilde{f} : U_{\mathcal{BG}}(A) \rightarrow B$ is a morphism of \mathcal{BG} -algebras : for all $p, q \in \mathbb{N}^*$ and $a_i, b_j \in A$, $1 \leq i \leq p$ and $1 \leq j \leq q$,

$$\begin{aligned} \tilde{f}((a_1 \dots a_p) \succ (b_1 \dots b_q)) &= \tilde{f}((a_1 \succ (\dots (a_{p-1} \succ (a_p \succ b_1)) \dots)) b_2 \dots b_q) \\ &= (\tilde{f}(a_1) \succ (\dots (\tilde{f}(a_{p-1}) \succ (\tilde{f}(a_p) \succ \tilde{f}(b_1))) \dots)) \tilde{f}(b_2) \dots \tilde{f}(b_q) \\ &= (\tilde{f}(a_1) \dots \tilde{f}(a_p)) \succ (\tilde{f}(b_1) \dots \tilde{f}(b_q)) \\ &= \tilde{f}(a_1 \dots a_p) \succ \tilde{f}(b_1 \dots b_q), \\ \tilde{f}((a_1 \dots a_p) \prec (b_1 \dots b_q)) &= \tilde{f}(a_1 \dots a_{p-1} ((\dots ((a_p \prec b_1) \prec b_2) \dots) \prec b_q)) \\ &= \tilde{f}(a_1) \dots \tilde{f}(a_{p-1}) ((\dots ((\tilde{f}(a_p) \prec \tilde{f}(b_1)) \prec \tilde{f}(b_2)) \dots) \prec \tilde{f}(b_q)) \\ &= (\tilde{f}(a_1) \dots \tilde{f}(a_p)) \prec (\tilde{f}(b_1) \dots \tilde{f}(b_q)) \\ &= \tilde{f}(a_1 \dots a_p) \prec \tilde{f}(b_1 \dots b_q). \end{aligned}$$

On the other hand, let $g : U_{\mathcal{BG}}(A) \rightarrow B$ be a morphism of \mathcal{BG} -algebras. From the construction of $U_{\mathcal{BG}}(A)$ it follows that the map $A \rightarrow U_{\mathcal{BG}}(A)$ is a \mathcal{L} -algebra morphism. Hence the composition \tilde{g} with g gives a \mathcal{L} -algebra morphism $A \rightarrow B$.

These two constructions are inverse of each other, and therefore $U_{\mathcal{BG}}$ is the left adjoint to $(-)_\mathcal{L}$. \square

Corollary 63 *The universal enveloping bigraft algebra of the free \mathcal{L} -algebra is canonically isomorphic to the free bigraft algebra :*

$$U_{\mathcal{BG}}(L(V)) \cong BG(V).$$

Proof. $U_{\mathcal{BG}}$ is left adjoint to $(-)_\mathcal{L}$ and L is left adjoint to the forgetful functor. The composite is the left adjoint to the forgetful functor from \mathcal{BG} -algebras to vector spaces. Hence it is the functor BG . \square

Theorem 64 *For any \mathcal{BG} -infinitesimal bialgebra A over a field \mathbb{K} , the following are equivalent:*

1. A is a connected \mathcal{BG} -infinitesimal bialgebra,
2. A is cofree among the connected coalgebras,
3. A is isomorphic to $U_{\mathcal{BG}}(\text{Prim}(A))$ as a \mathcal{BG} -infinitesimal bialgebra.

Proof. We prove the following implications $1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 1.$

1. \Rightarrow 2. If A is a connected \mathcal{BG} -infinitesimal bialgebra then A is isomorphic to $(\bar{T}(Prim(A)), m, \Delta)$ as an infinitesimal bialgebra, where $\bar{T}(Prim(A))$ is the augmentation ideal of the tensor algebra over $Prim(A)$, m is the concatenation and Δ is the deconcatenation (see [LR] for a proof). Therefore A is cofree.

2. \Rightarrow 3. If A is cofree, then it is isomorphic as an infinitesimal bialgebra to $(\bar{T}(Prim(A)), m, \Delta)$ and $Prim(A)$ is a \mathcal{L} -algebra with proposition 58. $\bar{T}(Prim(A))$ is a \mathcal{BG} -algebra with the two operations \succ and \prec defined as in (20) and this is exactly $U_{\mathcal{BG}}(Prim(A))$. So A is isomorphic as an infinitesimal bialgebra to $U_{\mathcal{BG}}(Prim(A))$ and it is a \mathcal{BG} -morphism by using (7) and (20).

3. \Rightarrow 1. By construction, $U_{\mathcal{BG}}(Prim(A))$ is isomorphic to $\bar{T}(Prim(A))$ as a bialgebra. Therefore A is isomorphic to $\bar{T}(Prim(A))$ as a bialgebra. As $\bar{T}(Prim(A))$ is connected, A is connected. \square

In other terms :

Theorem 65 *The triple of operads $(Ass, \mathcal{BG}, \mathcal{L})$ is a good triple of operads.*

Remark. Note that if A is a \mathcal{BG} -infinitesimal bialgebra, then $(A, m, \tilde{\Delta}_{Ass})$ is a nonunitary infinitesimal bialgebra. Hence, if $(\mathbb{K} \oplus A, m, \Delta_{Ass})$ has an antipode S , then $-S$ is an eulerian idempotent for A and we have :

$$S(a) = \begin{cases} -a & \text{if } a \in Prim(A), \\ 0 & \text{if } a \in A^2. \end{cases}$$

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