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# Return Probabilities for the Reflected Random Walk on $\mathbb{N}_0$

Rim Essifi & Marc Peigné

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## Abstract

Let  $(Y_n)$  be a sequence of i.i.d.  $\mathbb{Z}$ -valued random variables with law  $\mu$ . The reflected random walk  $(X_n)$  is defined recursively by  $X_0 = x \in \mathbb{N}_0$ ,  $X_{n+1} = |X_n + Y_{n+1}|$ . Under mild hypotheses on the law  $\mu$ , it is proved that, for any  $y \in \mathbb{N}_0$ , as  $n \rightarrow +\infty$ , one gets  $\mathbb{P}_x[X_n = y] \sim C_{x,y} R^{-n} n^{-3/2}$  when  $\sum_{k \in \mathbb{Z}} k \mu(k) > 0$  and  $\mathbb{P}_x[X_n = y] \sim C_y n^{-1/2}$  when  $\sum_{k \in \mathbb{Z}} k \mu(k) = 0$ , for some constants  $R, C_{x,y}$  and  $C_y > 0$ .

## 1 Introduction

We consider a sequence  $(Y_i)_{i \geq 1}$  of  $\mathbb{Z}$ -valued independent and identically distributed random variables, with common law  $\mu$ , defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

We denote by  $(S_n)_{n \geq 0}$  the classical random walk with law  $\mu$  on  $\mathbb{Z}$ , defined by  $S_0 = 0$  and  $S_n = Y_1 + \dots + Y_n$ ; the canonical filtration associated with the sequence  $(Y_i)_{i \geq 1}$  is denoted  $(\mathcal{T}_n)_{n \geq 1}$ . The **reflected random walk** on  $\mathbb{N}_0$  is defined by: for  $X_0$  given and  $\mathbb{N}_0$  valued, one sets

$$\forall n \geq 0 \quad X_{n+1} = |X_n + Y_{n+1}|.$$

The process  $(X_n)_{n \geq 0}$  is a Markov chain on  $\mathbb{N}_0$  with initial law  $\mathcal{L}(X_0)$  and transition matrix  $Q = (Q_{x,y})_{x,y \in \mathbb{N}_0}$  given by

$$\forall x, y \geq 0 \quad q(x, y) = \begin{cases} \mu(y-x) + \mu(y+x) & \text{if } y \neq 0 \\ \mu(-x) & \text{if } y = 0 \end{cases}.$$

When  $X_0 = x$   $\mathbb{P}$ -a.s., with  $x \in \mathbb{N}_0$  fixed, the random walk  $(X_n)_{n \geq 0}$  is denoted  $(X_n^x)_{n \geq 0}$ ; the probability measure on  $(\Omega, \mathcal{T})$  conditioned to the event  $[X_0 = x]$  will be denoted  $\mathbb{P}_x$  and the corresponding expectation  $\mathbb{E}_x$ .

We are interested with the behavior of the probabilities  $\mathbb{P}_x[X_n = y]$ ,  $x, y \in \mathbb{N}_0$  as  $n \rightarrow +\infty$ ; it is thus natural to consider the following generating function  $G$  associated with  $(X_n)_{n \geq 0}$  and defined formally as follows:

$$\forall x, y \in \mathbb{N}_0, \forall s \in \mathbb{C} \quad \mathfrak{G}(s|x, y) := \sum_{n \geq 0} \mathbb{P}_x[X_n = y] s^n.$$

The radius of convergence  $R$  of this series is of course  $\geq 1$ . The reflected random walk is positive recurrent when  $\mathbb{E}[|Y_i|] < +\infty$  and  $\mathbb{E}[Y_i] < 0$  (see [7] for instance and references therein) and consequently  $R = 1$ ; it is also the case when the  $Y_i$  are centered, under the stronger assumption  $\mathbb{E}[|Y_i|^{3/2}] < +\infty$ . A contrario, when  $\mathbb{E}[|Y_i|] < +\infty$  et  $\mathbb{E}[Y_i] > 0$ , as in the case of the classical

random walk on  $\mathbb{Z}$ , it is natural to assume that  $\mu$  has exponential moments<sup>(1)</sup> and, under this additional assumption, we will see that  $R > 1$ .

The generating functions are of interest since we can often recover information about the asymptotic behavior of probabilities, for instance by resorting a Tauberian theorem, e.g. that of Karamata; unfortunately, in this situation we have no way of obtaining the necessary information about these probabilities to apply such a Tauberian theorem (usually, we require that the sequence  $p_n$  is monotone, which is far to be right in our situation) and we will employ the following theorem of Darboux: it requires more regularity of the generating function in a neighborhood of the singular point  $z = R$  than does Karamata's theorem but no monotony type assumption:

**Theorem 1.0.1** *Let  $\mathfrak{G}(s) = \sum_{n=0}^{+\infty} g_n s^n$  be a power series with nonnegative coefficients  $p_n$  and radius of convergence  $R > 0$ . We assume that  $\mathfrak{G}$  has no singularities in the closed disk  $\{s \in \mathbb{C}/|s| \leq R\}$  except  $s = R$  (in other words,  $\mathfrak{G}$  has an analytic continuation to an open neighborhood of the set  $\{s \in \mathbb{C}/|s| \leq R\} \setminus \{R\}$ ) and that in a neighborhood of  $s = R$*

$$\mathfrak{G}(z) = \mathfrak{A}(s)(R - s)^\alpha + \mathfrak{B}(s) \quad (1)$$

where  $\mathfrak{A}$  and  $\mathfrak{B}$  are analytic functions<sup>(2)</sup>. Then

$$g_n \sim \frac{\mathfrak{A}(R)R^{1-n}}{\Gamma(-\alpha)n^{1+\alpha}} \quad \text{as } n \rightarrow +\infty. \quad (2)$$

This approach has been yet developed by S. Lalley in the general context of *random walk with a finite reflecting zone*; the transitions  $q(x, \cdot)$  of Markov chains of this class are the ones of a classical random walk on  $\mathbb{N}_0$  whenever  $x \geq K$  for some  $K \geq 0$ . In our context of the reflected random walk on  $\mathbb{N}_0$ , it means that the support of  $\mu$  is bounded from below (namely by  $-K$ ); we will not assume this in the sequel and will thus not follow the same strategy than S. Lalley. The methods required for the analysis of random walks with non localized reflections are more delicate, this is the aim of the present work for a particular such a process.

The reflected random walk on  $\mathbb{N}_0$  is characterized by the existence of reflection times. We have to consider the sequence  $(\mathbf{r}_k)_{k \geq 0}$  of successive reflection times; this is a sequence of waiting time with respect to the filtration  $(\mathcal{T}_n)_{n \geq 0}$ , defined by

$$\mathbf{r}_0 = 0 \quad \text{and} \quad \mathbf{r}_{k+1} := \inf\{n > \mathbf{r}_k : X_{\mathbf{r}_k} + Y_{\mathbf{r}_k+1} + \dots + Y_n < 0\} \quad \text{for all } k \geq 0.$$

In the sequel we will often omit the index for  $\mathbf{r}$  and denote the first reflection time  $\mathbf{r}$ . If one assume  $\mathbb{E}[|Y_i|] < +\infty$  and  $\mathbb{E}[Y_i] \leq 0$ , one gets  $\mathbb{P}_x[\mathbf{r}_k < +\infty] = 1$  for all  $x \in \mathbb{N}_0$  et  $k \geq 0$ ; on the contrary, when  $\mathbb{E}[|Y_i|] < +\infty$  and  $\mathbb{E}[Y_i] > 0$ , one gets  $\mathbb{P}_x[\mathbf{r}_k < +\infty] < 1$  and in order to have  $\mathbb{P}_x[\mathbf{r}_k < +\infty] > 0$  it is necessary to assume that  $\mu(\mathbb{Z}^{*-}) > 0$ .

The following identity will be essential in this work, it can be stated in an elementary way :

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<sup>1</sup>namely we will assume that  $\sum_{k \in \mathbb{Z}} r^k \mu(k) < +\infty$  for any  $r > 0$

<sup>2</sup>in equation 1, this is the positive branch  $s^\alpha$  which is meant, which implies that the branch cut is along the negative axis; so the branch cut for the function  $\mathfrak{G}(s)$  is along the halfline  $[R, +\infty]$

**Proposition 1.0.2** For all  $x, y \in \mathbb{N}_0$  and  $s \in \mathbb{C}$ , one gets

$$\mathfrak{G}(s|x, y) = \mathfrak{E}(s|x, y) + \sum_{w \in \mathbb{N}^*} \mathfrak{R}(s|x, w) \mathfrak{G}(s|w, y), \quad (3)$$

with

- for all  $x, y \geq 0$

$$\mathfrak{E}(s|x, y) := \sum_{n=0}^{+\infty} s^n \mathbb{P}_x[X_n = y, \mathbf{r} > n],$$

- for all  $x \geq 0$  and  $w \geq 1$

$$\begin{aligned} \mathfrak{R}(s|x, w) &:= \mathbb{E}_x[1_{[\mathbf{r} < +\infty, X_{\mathbf{r}}=w]} s^{\mathbf{r}}] \\ &= \sum_{n \geq 0} s^n \mathbb{P}[x + S_1 \geq 0, \dots, x + S_{n-1} \geq 0, x + S_n = -w]. \end{aligned}$$

The generating function  $\mathfrak{E}$  concerns the excursion of the Markov chain  $(X_n)_{n \geq 0}$  before its first reflection and  $\mathfrak{R}$  is related to the process of reflection  $(X_{\mathbf{r}_k})_{k \geq 0}$ .

Proof. Let us decompose  $\mathfrak{G}(s|x, y)$  into  $\mathfrak{G}_1(s|x, y) + \mathfrak{G}_2(s|x, y)$  with

$$\mathfrak{G}_1(s|x, y) := \mathbb{E}_x \left[ \sum_{n=0}^{\mathbf{r}-1} 1_{\{y\}}(X_n) s^n \right] \quad \text{and} \quad \mathfrak{G}_2(s|x, y) := \mathbb{E}_x \left[ 1_{[\mathbf{r} < +\infty]} \sum_{n=\mathbf{r}}^{+\infty} 1_{\{y\}}(X_n) s^n \right].$$

One gets  $\mathfrak{G}_1(s|x, y) = \mathfrak{E}(s|x, y)$  and, on the other side, by the strong Markov property,

$$\begin{aligned} \mathfrak{G}_2(s|x, y) &= \sum_{k \geq 0} \mathbb{E}_x \left[ 1_{[\mathbf{r} < +\infty]} 1_{[X_{\mathbf{r}+k}=y]} s^{\mathbf{r}+k} \right] \\ &= \sum_{k \geq 0} \sum_{w \in \mathbb{N}^*} \mathbb{E}_x \left[ 1_{[\mathbf{r} < +\infty, X_{\mathbf{r}}=w]} s^{\mathbf{r}} \mathbb{P}_w[X_k = y] s^k \right] \\ &= \sum_{w \in \mathbb{N}^*} \mathbb{E}_x \left[ 1_{[\mathbf{r} < +\infty, X_{\mathbf{r}}=w]} s^{\mathbf{r}} \right] \times \sum_{k \geq 0} \mathbb{P}_w[X_k = y] s^k \\ &= \sum_{w \in \mathbb{N}^*} \mathfrak{R}(s|x, w) \mathfrak{G}(s|w, y). \end{aligned}$$

□

By (3), one easily sees that, to precise the asymptotic behavior of the  $\mathbb{P}_x[X_n = y]$ , it is necessary to control the excursions of the walk between two successive reflection times. Note that this interrelationship among the Green's functions  $G, F$  and  $H$  may be written as a single matrix equation involving matrix-valued generating functions. For  $s \in \mathbb{C}$ , let us denote  $\mathcal{G}_s, \mathcal{E}_s$  and  $\mathcal{R}_s$  the following infinite matrices

- $\mathcal{G}_s = (\mathcal{G}_s(x, y))_{x, y \in \mathbb{N}_0}$  with  $\mathcal{G}_s(x, y) = \mathfrak{G}(s|x, y)$  for all  $x, y \in \mathbb{N}_0$ ,
- $\mathcal{E}_s = (\mathcal{E}_s(x, y))_{x, y \in \mathbb{N}_0}$  with  $\mathcal{E}_s(x, y) = \mathfrak{E}(s|x, y)$  for all  $x, y \in \mathbb{N}_0$ ,
- $\mathcal{R}_s = (\mathcal{R}_s(x, y))_{x \in \mathbb{N}_0, y \in \mathbb{N}^*}$  with  $\mathcal{R}_s(x, y) = \mathfrak{R}(s|x, y)$ .

Thus for all  $x, y \in \mathbb{N}_0$  and  $s \in \mathbb{C}$ , one gets

$$\mathcal{G}_s = \mathcal{E}_s + \mathcal{R}_s \mathcal{G}_s. \quad (4)$$

This shows that the Green functions  $\mathfrak{G}(\cdot|x, y)$  may be computed when  $I - \mathcal{R}_s$  is invertible, in which case one may write

$$\mathcal{G}_s = (I - \mathcal{R}_s)^{-1} \mathcal{E}_s.$$

Let us now introduce some general assumptions:

**Hypotheses H:**

**H1:** the measure  $\mu$  is **adapted** on  $\mathbb{Z}$  (i.e. the group generated by its support  $S_\mu$  is equal to  $\mathbb{Z}$ ) and **aperiodic** (i.e. the group generated by  $S_\mu - S_\mu$  is equal to  $\mathbb{Z}$ )

**H2:** the measure  $\mu$  has exponential moment of any order (i.e.  $\sum_{n \in \mathbb{Z}} r^n \mu(n) < +\infty$  for any  $r \in ]0, +\infty[$ ) and  $\sum_{n \in \mathbb{Z}} n \mu(n) \geq 0$ . <sup>(3)</sup>

We now state the main result of this paper, which extends [5] in our situation:

**Theorem 1.0.3** *Let  $(Y_n)_{n \geq 1}$  be a sequence of  $\mathbb{Z}$ -valued independent and identically distributed random variables with law  $\mu$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Assume that  $\mu$  satisfies Hypotheses **H** and let  $(X_n)_{n \geq 0}$  be the reflected random walk defined inductively by*

$$X_0 = x \quad \text{and} \quad X_{n+1} = |X_n + Y_{n+1}| \quad \text{for } n \geq 0.$$

• If  $\mathbb{E}[Y_1] = \sum_{k \in \mathbb{Z}} k \mu(k) = 0$ , then for any  $y \in \mathbb{N}_0$ , there exists a constant  $C_y \in \mathbb{R}^{*+}$  such that, for any starting point  $x \in \mathbb{N}_0$ , one gets

$$\mathbb{P}_x[X_n = y] \sim \frac{C_y}{\sqrt{n}} \quad \text{as } n \rightarrow +\infty,$$

• If  $\mathbb{E}[Y_1] = \sum_{k \in \mathbb{Z}} k \mu(k) > 0$  then, for any  $x, y \in \mathbb{N}_0$ , there exists a constant  $C_{x,y} \in \mathbb{R}^{*+}$  such that

$$\mathbb{P}_x[X_n = y] \sim C_{x,y} \frac{\rho^n}{n^{3/2}}$$

for some  $\rho = \rho(\mu) \in ]0, 1[$ .

The constant  $\rho(\mu)$  which appears in this statement is the infimum over  $\mathbb{R}$  of the generating function of  $\mu$ . We also know the exact value of the constants  $C_y$  and  $C_{x,y}, x, y \in \mathbb{N}_0$  which appear in the previous statement : see formulae (37) and (41).

## 2 Decomposition of the trajectories and factorizations

In this section, we will consider the subprocess of reflections  $(X_{\mathbf{r}_k})_{k \geq 0}$  in order to decompose the trajectories of the reflected random walk in several parts which can be analyzed.

We first introduce some notations which appear classically in the fluctuation theory of 1-dimensional random walks.

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<sup>3</sup>we can in fact consider weaker assumptions: there exist  $0 < r_- < 1 < r^+$  such that  $\hat{\mu}(r) := \sum_{n \in \mathbb{Z}} r^n \mu(n) < +\infty$  for any  $r \in ]r_-, r^+[$  and  $\mu_r$  reaches its minimum on this interval at a (unique)  $r_0 \in ]r_-, 1[$ . We thus need much more notations at the beginning, this complicates in fact the understanding of the proof and is not really of interest.

## 2.1 On the fluctuations of a classical random walk on $\mathbb{Z}$

Let  $\tau^{*-}$  the first strict descending time of the random walk  $(S_n)_{n \geq 0}$ :

$$\tau^{*-} := \inf\{n \geq 1/S_n < 0\}$$

(with the convention  $\inf \emptyset = +\infty$ ). The variable  $\tau^{*-}$  is a stopping time with respect to the filtration  $(\mathcal{T}_n)_{n \geq 0}$ .

We denote by  $(T_n^{*-})_{n \geq 0}$  the sequence of successive ladder descending epoch of the random walk  $(S_n)_{n \geq 0}$  defined by  $T_0^{*-} = 0$  and  $T_{n+1}^{*-} = \inf\{k > T_n^{*-}/S_k < S_{T_n^{*-}}\}$  for  $n \geq 0$ . One gets in particular  $T_1^{*-} = \tau^{*-}$ ; furthermore, setting  $\tau_n^{*-} := T_n^{*-} - T_{n-1}^{*-}$  for any  $n \geq 1$ , one may write  $T_n^{*-} = \tau_1^{*-} + \dots + \tau_n^{*-}$  where  $(\tau_n^{*-})_{n \geq 1}$  is a sequence of independent and identically random variables with law  $\mu^{*-} := \mathcal{L}(S_{\tau^{*-}})$ . The potential associated to  $\mu^{*-}$  is denoted by  $U^{*-}$ ; one gets

$$U^{*-}(\cdot) := \sum_{n=0}^{+\infty} (\mu^{*-})^{*n}(\cdot) = \sum_{n=0}^{+\infty} \mathbb{E}[\delta_{S_{T_n^{*-}}}(\cdot)].$$

Similarly, we can introduce the first ascending time  $\tau^+ := \inf\{n \geq 1/S_n \geq 0\}$  of the random walk  $(S_n)_{n \geq 0}$  (with the convention  $\inf \emptyset = +\infty$ ) and the the sequence  $(T_n^+)_{n \geq 0}$  of successive ladder ascending epoch of  $(S_n)_{n \geq 0}$  defined by  $T_0^+ = 0$  and  $T_{n+1}^+ = \inf\{k > T_n^+/S_k \geq S_{T_n^+}\}$  for  $n \geq 0$ ; as above, one may write  $T_n^+ = \tau_1^+ + \dots + \tau_n^+$  where  $(\tau_n^+)_{n \geq 1}$  is a sequence of i.i.d. random variables with law  $\mu^+ := \mathcal{L}(S_{\tau^+})$ . The potential associated with  $\mu^+$  is denoted by  $U^+$ ; one gets

$$U^+(\cdot) := \sum_{n=0}^{+\infty} (\mu^+)^{*n}(\cdot) = \sum_{n=0}^{+\infty} \mathbb{E}[\delta_{S_{T_n^+}}(\cdot)].$$

We need to control the law of the couple  $(\tau^{*-}, S_{\tau^{*-}})$  and thus introduce the ‘‘characteristic’’ function  $\varphi^{*-}$  defined formally by

$$\varphi^{*-} : (s, z) \mapsto \sum_{n \geq 1} s^n \mathbb{E}[1_{[\tau^{*-}=n]} z^{S_n}]$$

for  $s, z \in \mathbb{C}$ . In other words, one gets

$$\varphi^{*-}(s, z) = \mathbb{E}[1_{[\tau^{*-} < +\infty]} s^{\tau^{*-}} z^{S_{\tau^{*-}}}]$$

when the  $Y_i$  are centered, we know that  $\tau^{*-}$  is a.s. finite and the indicator function will be omitted in the sequel, otherwise we will modify suitably the choice of the law of the  $Y_i$  and will pull back the study of  $\varphi^{*-}$  in the centered case.

We also introduce the characteristic function associated to the potential of  $(\tau^{*-}, S_{\tau^{*-}})$ , defined formally by

$$\Phi^{*-}(s, z) = \sum_{k \geq 0} \mathbb{E} \left[ s^{T_k^{*-}} z^{S_{T_k^{*-}}} \right] = \sum_{k \geq 0} \varphi^{*-}(s, z)^k = \frac{1}{1 - \varphi^{*-}(s, z)}.$$

There is be a natural duality between the open half-line  $\mathbb{R}^{*-}$  and its complementary set  $\mathbb{R}^+$ ; as above, we associate to the couple  $(\tau^+, S_{\tau^+})$  the function  $\varphi^+$  defined by

$$\varphi^+ : (s, z) \mapsto \mathbb{E}[s^{\tau^+} z^{S_{\tau^+}}],$$

for  $s, z \in \mathbb{C}$  with modulus  $\leq 1$ . In fact, in a natural way will appear the “potential” associated with  $(\tau^+, S_{\tau^+})$  and whose “characteristic” function  $(s, z) \mapsto \Phi^+(s, z)$  is given by

$$\Phi^+(s, z) := \sum_{k \geq 0} \mathbb{E} \left[ s^{T_k^+} z^{S_{T_k^+}} \right] = \sum_{k \geq 0} \varphi^+(s, z)^k = \frac{1}{1 - \varphi^+(s, z)}$$

for complex numbers  $s, z$  with modulus  $< 1$  (since in this case  $|\varphi^+(s, z)| < 1$ ). Notice that, by a straightforward argument, called *duality lemma* in the book by Feller [4], one also gets

$$\Phi^+(s, z) = \sum_{n \geq 0} s^n \mathbb{E} [\tau^{*-} > n, z^{S_n}]. \quad (5)$$

We now introduce the corresponding generating functions  $\mathfrak{T}^{*-}, \mathfrak{U}^{*-}$  and  $\mathfrak{U}^+$  defined by, for any  $s \in \mathbb{C}, |s| \leq 1$  and  $x \in \mathbb{Z}$

$$\begin{aligned} \mathfrak{T}^{*-}(s|x) &= \mathbb{E} \left[ s^{\tau^{*-}} 1_{\{x\}}(S_{\tau^{*-}}) \right] = \sum_{n \geq 1} s^n \mathbb{P} [\tau^{*-} = n, S_n = x], \\ \mathfrak{T}^+(s|x) &= \mathbb{E} \left[ s^{\tau^+} 1_{\{x\}}(S_{\tau^+}) \right] = \sum_{n \geq 1} s^n \mathbb{P} [\tau^+ = n, S_n = x], \\ \mathfrak{U}^{*-}(s|x) &= \sum_{k \geq 0} \mathbb{E} \left[ s^{T_k^{*-}} 1_{\{x\}}(S_{T_k^{*-}}) \right] = \sum_{n \geq 0} s^n \mathbb{P} [\tau^+ > n, S_n = x], \\ \mathfrak{U}^+(s|x) &= \sum_{k \geq 0} \mathbb{E} \left[ s^{T_k^+} 1_{\{x\}}(S_{T_k^+}) \right] = \sum_{n \geq 0} s^n \mathbb{P} [\tau^{*-} > n, S_n = x]. \end{aligned}$$

Note that  $\mathfrak{U}^{*-}(s|x) = 0$  when  $x \geq 0$  and  $\mathfrak{U}^+(s|x) = 0$  when  $x \leq -1$ .

We will first study the regularity of the Fourier transforms  $\varphi^{*-}$  and  $\varphi^+$  to describe the one of the functions  $\mathfrak{T}^{*-}(\cdot|x)$  and  $\mathfrak{T}^+(\cdot|x)$ ; to do this we will use the Wiener-Hopf factorization theory, in a quite strong version, in order to obtain some uniformity in the estimations we will need. We could adapt the same approach for the functions  $\mathfrak{U}^{*-}(\cdot|x)$  and  $\mathfrak{U}^+(\cdot|x)$ , but it is more difficult to control the behavior near  $s = 1$  of their respective Fourier transforms  $\Phi^{*-}$  and  $\Phi^+$ . We will thus prefer to note that, for any  $x \in \mathbb{Z}^{*-}$ , the function  $\mathfrak{U}^{*-}(\cdot|x)$  is equal to the finite sum  $\sum_{k=0}^{|x|} \mathbb{E} \left[ s^{T_k^{*-}} 1_{\{x\}}(S_{T_k^{*-}}) \right]$ , since  $T_k^{*-} \geq k$  a.s; the same remark does not hold for  $\mathfrak{U}^+(\cdot|x)$  since  $\mathbb{P}[S_{\tau^+} = 0] > 0$  but we will see that the series  $\sum_{k=0}^{+\infty} \mathbb{E} \left[ s^{T_k^+} 1_{\{x\}}(S_{T_k^+}) \right]$  converges exponentially fast and a similar approach will be developed.

It will be of interest to consider the following square infinite matrices

- $\mathcal{T}_s^{*-} = \left( \mathcal{T}_s^{*-}(x, y) \right)_{x, y \in \mathbb{Z}^-}$  with  $\mathcal{T}_s^{*-}(x, y) := \mathfrak{T}^{*-}(s|y - x)$  for any  $x, y \in \mathbb{Z}^-$ ,
- $\mathcal{U}_s^{*-} = \left( \mathcal{U}_s^{*-}(x, y) \right)_{x, y \in \mathbb{Z}^-}$  with  $\mathcal{U}_s^{*-}(x, y) := \mathfrak{U}^{*-}(s|y - x)$  for any  $x, y \in \mathbb{Z}^-$ .

The element of  $\mathbb{Z}^-$  are labelled here in the decreasing order. Notice that the matrix  $\mathcal{T}_s^{*-}$  is strictly upper triangular; so for any  $x, y \in \mathbb{Z}^-$  one gets  $\mathcal{U}_s^{*-}(x, y) = \sum_{k=0}^{|x-y|} (\mathcal{T}_s^{*-})^k(x, y)$ .

- $\mathcal{T}_s^+ = \left( \mathcal{T}_s^+(x, y) \right)_{x, y \in \mathbb{N}_0}$  with  $\mathcal{T}_s^+(x, y) := \mathfrak{T}^+(s|y - x)$  for any  $x, y \in \mathbb{N}_0$ ,

- $\mathcal{U}_s^+ = \left( \mathcal{U}_s^+(x, y) \right)_{x, y \in \mathbb{N}_0}$  with  $\mathcal{U}_s^+(x, y) := \mathfrak{U}^+(s|y - x)$  for any  $x, y \in \mathbb{N}_0$ .

We will also have  $\mathcal{U}_s^+(x, y) = \sum_{k \geq 0} (\mathcal{T}_s^+)^k(x, y)$  for any  $x, y \in \mathbb{N}_0$ , the number of terms in the sum will not be finite in this case but it will not be difficult to derive the regularity of the function  $s \mapsto \mathcal{U}_s^+(x, y)$  from the one of each term  $s \mapsto \mathcal{T}_s^+(x, y)$ .

In the sequel, we will consider the matrices  $\mathcal{T}_s^{*-}$  and  $\mathcal{T}_s^+$  as operators acting on  $(\mathbb{C}^{\mathbb{N}_0}, |\cdot|_\infty)$ ; it will not be possible to give sense to the above inversion formula on the Banach space of linear continuous operators acting on  $(\mathbb{C}^{\mathbb{N}_0}, |\cdot|_\infty)$  and we will have to consider the action of these matrix and a larger space of  $\mathbb{C}$ -valued sequences.

In the following subsections, we decompose both the excursion of  $(X_n)_{n \geq 0}$  before the first reflection and the process of reflections  $(X_{\mathbf{r}_k})_{k \geq 0}$  in terms of quantities introduced here.

## 2.2 The approach process and the matrices $\mathcal{T}_s$

The trajectories of the reflected random walk are governed by the strict descending ladder epoch of the corresponding classical random walk on  $\mathbb{Z}$ , and the generating function  $\mathfrak{T}^{*-}$  introduced in the previous section will be essential in the sequel. Since the starting point may be any  $x \in \mathbb{N}_0$ , we have to consider the first time at which the random walk  $(X_n)_{n \geq 0}$  goes on the "left" on the initial point (with eventually a reflexion at this time, in which case the arrival point may be  $> x$ ), that is the strict descending ladder epoch  $\tau^{*-}$  of the random walk  $(S_n)_{n \geq 0}$ . We thus introduce the matrices  $\mathcal{T}_s$  which contains a lot of information for the reflected random walk, defined by  $\mathcal{T}_s = \left( \mathcal{T}_s(x, y) \right)_{x, y \in \mathbb{N}_0}$  with

$$\forall x, y \in \mathbb{N}_0 \quad \mathcal{T}_s(x, y) := \mathfrak{T}^{*-}(s|y - x). \quad (6)$$

Notice that the matrices  $\mathcal{T}_s$  are strictly lower triangular.

## 2.3 The excursion before the first reflection

Recall that the function  $\mathfrak{E}$  is defined by

$$\forall x, y \in \mathbb{N}_0, \forall s \in \mathbb{C} \quad \mathfrak{E}(s|x, y) := \sum_{n \geq 0} s^n \mathbb{P}_x[\mathbf{r} > n, X_n = y].$$

We have the following identity: for all  $s \in \mathbb{C}$  and  $x, y \in \mathbb{N}_0$

$$\mathfrak{E}(s|x, y) = \mathfrak{U}^+(s|y - x) + \sum_{w=0}^{x-1} \mathfrak{T}^{*-}(s|w - x) \mathfrak{E}(s|w, y).$$

As above, we introduce the square infinite matrices  $\mathcal{E}_s = \left( \mathcal{E}_s(x, y) \right)_{x, y \in \mathbb{N}_0}$ , with  $\mathcal{E}_s(x, y) := \mathfrak{E}(s|x, y)$  for any  $x, y \in \mathbb{N}_0$ , and rewrite this identity as follows

$$\mathcal{E}_s = \mathcal{U}_s^+ + \mathcal{T}_s \mathcal{E}_s.$$

Since  $\mathcal{T}_s$  is strictly lower triangular, the matrix  $I - \mathcal{T}_s$  will be invertible (in a suitable space to be precised) and one will get

$$\mathcal{E}_s = \left( I - \mathcal{T}_s \right)^{-1} \mathcal{U}_s^+. \quad (7)$$

In the following sections, we will give sense to this inversion formula and describe the regularity in  $s$  of the matrix-valued function  $s \mapsto \mathcal{E}_s$ .



## 2.4 The process of reflections

Under the hypothesis  $\mathbb{P}[\tau^{*-} < +\infty] = 1$  <sup>(4)</sup>, the distribution law of the variable  $S_{\tau^{*-}}$  is denoted  $\mu^{*-}$  and its potential  $U^{*-} := \sum_{n \geq 0} (\mu^{*-})^{*n}$ ; all the waiting times  $T_n^{*-}$  are thus a.s. finite and one gets

$(\mu^{*-})^{*n} = \mathcal{L}(S_{T_n^{*-}})$ , furthermore, for any  $x \in \mathbb{N}_0$  the successive reflection times  $\mathbf{r}_k, k \geq 0$ , are also a.s. finite. The process  $(X_{\mathbf{r}_k})_{k \geq 0}$  appears in a crucial way in [7] to study the recurrence/transience properties of the reflected walk; indeed, we have the

**Fact 2.4.1** [7] *Under the hypothesis  $\mathbb{P}[\tau^{*-} < +\infty] = 1$ , the process of reflections  $(X_{\mathbf{r}_k})_{k \geq 0}$  is a Markov chain on  $\mathbb{N}_0$  with transition probability  $\mathcal{R}$  given by*

$$\forall x \in \mathbb{N}_0, \forall y \in \mathbb{N}_0 \quad \mathcal{R}(x, y) = \begin{cases} 0 & \text{if } y = 0 \\ \sum_0^x U^{*-}(-w) \mu^{*-}(w - x - y) & \text{if } y \geq 1 \end{cases} \quad (8)$$

Furthermore, the measure  $\nu_{\mathbf{r}}$  on  $\mathbb{N}^*$  defined by

$$\forall x \in \mathbb{N}^* \quad \nu_{\mathbf{r}}(x) := \sum_{y=1}^{+\infty} \left( \frac{\mu^{*-}(-x)}{2} + \mu^{*-}(\lceil -x - y, -x \rceil) + \frac{\mu^{*-}(-x - y)}{2} \right) \mu^{*-}(-y) \quad (9)$$

is stationary for  $(X_{\mathbf{r}_k})_{k \geq 0}$  and is unique up to a multiplicative constant; it is finite as soon as  $\mathbb{E}[|S_{\tau^{*-}}|] = \sum_{k \geq 1} k \mu^{*-}(-k) < +\infty$ .

This statement is a bit different from the one in [7] since we assume here that at the reflection time the process  $(X_n)_{n \geq 0}$  belongs to  $\mathbb{N}^*$ ; nevertheless, the proof goes exactly along the same lines. This result is crucial in the sequel in order to control the spectrum of the stochastic infinite matrix  $\mathcal{R} = (\mathcal{R}(x, y))_{x, y \in \mathbb{N}_0}$ ; namely, we have the

**Property 2.4.2** *There exists a constant  $\kappa \in ]0, 1[$  such that, for any  $x \in \mathbb{N}_0$  and  $y \in \mathbb{N}^*$  one gets*

$$\mathcal{R}(x, y) \geq \kappa \mu^{*-}(-y).$$

In particular, the operator  $\mathcal{R}$  acting on  $(\mathbb{C}^{\mathbb{N}_0}, |\cdot|_{\infty})$  is quasi-compact : more precisely, the eigenvalue 1 is simple, with associated eigenvector  $h = (1)_{n \in \mathbb{N}_0}$  and the rest of the spectrum is included in a disk of radius  $\leq 1 - \kappa$ .

Furthermore, for any  $K > 1$ , the operator  $\mathcal{R}$  acts also on the Banach space  $(\mathbb{C}^{\mathbb{N}_0}, |\cdot|_K)$ , where  $|\cdot|_K$  denotes the norm defined by

$$\forall \mathbf{a} = (a_x)_{x \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} \quad |\mathbf{a}|_K := \sup_{x \in \mathbb{N}_0} \frac{|a_x|}{K^x}, \quad (10)$$

the eigenvalue 1 is simple with associated eigenvector  $h$  and the rest of the spectrum of  $\mathcal{R}$  acting on  $(\mathbb{C}^{\mathbb{N}_0}, |\cdot|_K)$  is included in a disk of radius  $\leq 1 - \kappa$ .

---

<sup>4</sup>this condition is satisfied for instance when  $\mathbb{E}[|Y_i|] < +\infty$  and  $\mathbb{E}[Y_i] \leq 0$ .

Proof. Let  $N_\mu := \inf\{k \leq -1/\mu\{k\} > 0\}$  (with  $N = -\infty$  if the support of  $\mu$  is not bounded from below). Since  $\mu$  is adapted, one gets  $\mu^{*-}(k) > 0$  for any  $k \in \{-N_\mu, \dots, -1\}$  (and any  $k \in \mathbb{Z}^{*-}$  when  $N_\mu = -\infty$ ); as a direct consequence, one gets  $U^{*-}(k) > 0$  for any  $k \in \mathbb{Z}^{*-}$ . In fact, by the 1-dimensional renewal theorem, one knows that  $\lim_{k \rightarrow -\infty} U^{*-}(k) = \frac{1}{-\mathbb{E}[S_{\tau^{*-}}]} > 0$  since  $\mathbb{E}[S_{\tau^{*-}}] > -\infty$  when  $\mu$  has exponential moments; it readily follows that  $\kappa := \inf_{k \in \mathbb{Z}^-} U^{*-}(k) > 0$ . Using (8), one may thus write, for any  $x \in \mathbb{N}_0$  and  $y \in \mathbb{N}^*$

$$\mathcal{R}(x, y) \geq U^{*-}(x)\mu^{*-}(-y) \geq \kappa\mu^{*-}(-y).$$

The matrix  $(\mathcal{R}(x, y)_{x, y \in \mathbb{N}_0})$  thus satisfies the so-called ‘‘Doeblin condition’’ and it is quasi-compact on  $(\mathbb{C}^{\mathbb{N}_0}, |\cdot|_\infty)$  (see for instance [1] for a precise statement).

The same spectral property holds on  $(\mathbb{C}^{\mathbb{N}_0}, |\cdot|_K)$  since  $\mu^{*-}$  has exponential moment of any order, which allows to check by a straightforward computation that

$$\sup_{x \in \mathbb{N}_0} \sum_{y \in \mathbb{N}_0} \mathcal{R}(x, y)K^y < +\infty.$$

□

**For technical reasons which will appear in Section 4**, we will replace the function  $x \mapsto K^x$  by a function denoted also  $K$  which satisfies the following conditions

$$\forall x \in \mathbb{N}_0 \quad K(x) \geq 1, \quad \mathcal{R}K(x) \leq 1 \quad \text{and} \quad K(x) \sim K^x. \quad (11)$$

It suffices to consider the function  $x \mapsto \left(1 \vee \frac{K(x)}{M}\right)$  with  $M := \sup_{x \in \mathbb{N}_0} \sum_{y \in \mathbb{N}^*} \mathcal{R}(x, y)K^y$  (we now that

$M < +\infty$  by proof of Property 2.4.2). **The set of fonctions which satisfy the conditions (11) will be denoted  $\mathcal{K}(K)$ .**

We now explicit the connection between  $\mathcal{R}_s$  and the matrix  $\mathcal{T}_s$  introduced above; namely, there exists a similar factorization identity than (3) for the process of reflection. Using the fact that the first reflection time may appear or not at time  $\tau^{*-}$ , one may write: for all  $s \in \mathbb{C}$  and  $x \in \mathbb{N}_0$  and  $y \in \mathbb{N}^*$

$$\mathfrak{R}(s|x, y) = \mathfrak{T}(s|x - x - y) + \sum_{w=0}^{x-1} \mathfrak{T}(s|w - x)\mathfrak{R}(s|w, y), \quad (12)$$

which leads to the following equality:

$$\mathcal{R}_s = \left(I - \mathcal{T}_s\right)^{-1} \mathcal{V}_s \quad (13)$$

where we have set  $\mathcal{V}_s = \left(\mathcal{V}_s(x, y)\right)_{x, y \in \mathbb{N}_0}$  with

$$\mathcal{V}_s(x, y) := \begin{cases} 0 & \text{if } y = 0 \\ \mathfrak{T}^{*-}(s|x - x - y) & \text{if } y \in \mathbb{N}^*. \end{cases} \quad (14)$$

The crucial point in the sequel will be thus to describe the regularity of the maps  $s \mapsto \mathcal{T}_s$ ,  $s \mapsto \mathcal{V}_s$  and  $s \mapsto \mathcal{U}_s^+$  near the point  $s = 1$ . We will first detail the centered case; the main ingredient is the classical Wiener-Hopf factorization which permits to control both functions  $\varphi^{*-}$  and  $\varphi^+$ .

Another essential point will be to describe the one of the maps  $(I - \mathcal{T}_s)^{-1}$  and  $(I - \mathcal{R}_s)^{-1}$  and this question is related to the description of the spectrum of the operators  $\mathcal{T}_s$  and  $\mathcal{R}_s$  when  $s$  is closed to 1: this is not difficult for  $\mathcal{T}_s$  since it is a strictly lower triangular matrix but more delicate for  $\mathcal{R}_s$  in the centered case where  $\mathcal{R} = \mathcal{R}_1$  is a Markov operator.

### 3 A strong version of the Wiener-Hopf factorization and its applications to classical random walks

#### 3.1 Introduction and notations

The Wiener-Hopf factorization proposes a decomposition of the space-time characteristic function  $(s, z) \mapsto 1 - s\mathbb{E}[z^{Y_1}] = 1 - s\hat{\mu}(z)$  in terms of  $\varphi^{*-}$  and  $\varphi^+$ ; namely, one gets; for all  $s, z \in \mathbb{C}$  with modulus  $< 1$

$$1 - s\hat{\mu}(z) = \left(1 - \varphi^{*-}(s, z)\right) \left(1 - \varphi^+(s, z)\right). \quad (15)$$

In [3], we already use this factorization in order to state local limit theorems for fluctuations of the random walk  $(S_n)_{n \geq 0}$ ; we first propose another such a decomposition, and, by identification of the corresponding factors, we obtain another expression for each of the functions  $\varphi^{*-}$  and  $\varphi^+$ . This new expression allows us to use elementary arguments coming from entire functions theory in order to describe for instance the asymptotic behavior of the sequences  $\left(\mathbb{P}[S_n = x, \tau^{*-} = n]\right)_{n \geq 1}$  and  $\left(\mathbb{P}[S_n = y, \tau^{*-} > n]\right)_{n \geq 1}$  for any  $x \in \mathbb{Z}^{*-}$  and  $y \in \mathbb{Z}^+$ .

In the present situation, we need first to obtain similar results than in [3] but in terms of regularity with respect to the variable  $s$  of the functions  $\varphi^{*-}$  and  $\varphi^+$  around the unit circle, with a precise description of their singularity near the point  $s = 1$ ; by the identity (3) we will show that these properties spread to the function  $G(s|x, y)$ , which allows us to conclude, using the classical Darboux's method for entire functions.

We will assume that the law  $\mu$  as exponential moment of any order, i.e.  $\sum_{n \in \mathbb{Z}} r^n \mu(n) < +\infty$  for any  $r \in \mathbb{R}^{*+}$ ; it readily implies that its generating function  $\hat{\mu} : z \mapsto \sum_{n \in \mathbb{Z}} z^n \mu(n)$  is analytic on  $\mathbb{C}^*$ ; furthermore, its restriction to  $]0, +\infty[$  is strictly convex and one gets  $\lim_{r \rightarrow +\infty} \hat{\mu}(r) = \lim_{r \rightarrow < 0} \hat{\mu}(r) = +\infty$  as soon as  $\mu$  charges  $\mathbb{Z}^{*+}$  and  $\mathbb{Z}^{*-}$ . In particular, under these conditions, there exists a unique  $r_0 > 0$  such that  $\hat{\mu}(r_0) = \inf_{r > 0} \hat{\mu}(r)$ ; one gets  $\hat{\mu}'(r_0) = 0, \hat{\mu}''(r_0) > 0$  and sets  $\rho_0 := \hat{\mu}(r_0)$ . Note that  $\rho_0 = 1$  when  $\mu$  is centered and  $\rho_0 \in ]0, 1[$  otherwise; we will set  $R_\circ := \frac{1}{\rho_0}$ .

We now fix  $0 < r_- < r_0 < r_+ < +\infty$  and will denote by  $\mathbf{L} = \mathbf{L}[r_-, r_+]$  the space of functions  $F : \mathbb{C}^* \rightarrow \mathbb{C}$  of the form  $F(z) := \sum_{n \in \mathbb{Z}} a_n z^n$  for some (bilateral)-sequence  $(a_n)_{n \in \mathbb{Z}}$  such that  $\sum_{n \leq 0} |a_n| r_-^n + \sum_{n \geq 0} |a_n| r_+^n < +\infty$ ; the elements of  $\mathbf{L}$  are called **Laurent functions** on the annulus  $[r_-, r_+] := \{r_- \leq |z| \leq r_+\}$  and the Banach space  $(\mathbf{L}, |\cdot|_\infty)$  <sup>(5)</sup> contains the function  $\hat{\mu}$  defined above.

#### 3.2 The centered case

Lets us first consider the centered case:  $\mathbb{E}[Y_i] = \hat{\mu}'(1) = 0$ ; we thus have  $r_0 = 1$  and  $\rho_0 = R_\circ = 1$ . Under the aperiodicity condition on  $\mu$ , one gets  $|1 - s\hat{\mu}(z)| > 0$  for any  $z \in \mathbb{C}^*, |z| = 1$ , and  $s$  such that  $|s| \leq 1$ , excepted  $s = 1$ ; it follows that for any  $z \in \mathbb{C}^*, |z| = 1$ , the function  $s \mapsto \frac{1}{1 - s\hat{\mu}(z)}$  may be analytically extended on the set  $\{s \in \mathbb{C}/|s| \leq 1 + \delta\} \setminus [1, 1 + \delta[$  for some  $\delta > 0$ . On the other

<sup>5</sup>where  $|\cdot|_\infty$  denotes the norm of uniform convergence on the annulus  $\{r_- \leq |z| \leq r_+\}$

hand, setting  $\sigma^2 := \mathbb{E}[Y_i^2]$ , one gets  $\hat{\mu}''(1) = \sigma^2 > 0$ . One thus gets, setting  $\Psi(s, z) := 1 - s\hat{\mu}(z)$ ,

$$\frac{\partial}{\partial z}\Psi(1, 1) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial z^2}\Psi(1, 1) = \sigma^2 > 0.$$

The Weierstrass preparation lemma thus implies that, on a neighborhood of  $(1, 1)$  one may write

$$1 - s\hat{\mu}(z) = (z^2 + b(s)z + c(s))\Psi(s, z)$$

with  $\Psi$  analytic on  $\mathbb{C} \times \mathbb{C}^*$  and  $\Psi \neq 0$  on a neighborhood of  $(1, 1)$ . One gets

$$z^2 + b(s)z + c(s) = (z - z_-(s))(z - z_+(s)),$$

with  $z_-(s) < 1 < z_+(s)$  when  $s \in [0, 1[$  and  $z_-(1) = z_+(1) = 1$ .

In order to solve this last equation, we fix the principal determination of the function  $Z \mapsto \sqrt{Z}$  <sup>(6)</sup> in such a way  $s \mapsto \sqrt{1-s}$  is well defined on the set  $\mathcal{O}_\delta(1) := B(1, \delta) \setminus [1, 1 + \delta[$ . It follows that the functions  $z_\pm$  admit the analytic expansion  $z_\pm(s) = 1 + \sum_{n \geq 1} (\pm 1)^n \alpha_n (1-s)^{n/2}$  on  $\mathcal{O}_\delta(1)$  and the

equality  $\hat{\mu}(z_\pm(s)) = \frac{1}{s} = \sum_{n \geq 0} (1-s)^n$  valid for  $0 \leq s < 1$  leads to  $\alpha_1 = \frac{\sqrt{2}}{\sigma}$ .

This type of singularity of the functions  $z_\pm$  near  $s = 1$  is essential in the sequel because it contains the one of the functions  $\varphi^{*-}(s, z)$  and  $\varphi^+(s, z)$  near  $(1, 1)$ . The Wiener-Hopf factorization has several versions in the literature; we emphasize here that we need some kind of uniformity with respect to the parameter  $z$  in the local expansion of the function  $\varphi^{*-}$  near  $s = 1$ , this is why we consider the map  $s \mapsto \varphi^{*-}(s, \cdot)$  with values in  $\mathbf{L}[r_-, r_+]$ . It is proved in particular in [1] (see also [6] for a more precise statement, in the context of Markov walks) that there exists  $\delta > 0$  such that the function  $s \mapsto \left( z \mapsto \phi^{*-}(s, z) := \frac{1 - \varphi^{*-}(s, z)}{z - z_-(s)} \right)$  is analytic on the open ball  $B(1, \delta) \subset \mathbb{C}$ , with values in  $L[r_-, r_+]$ . Setting  $\phi^{*-}(s, \cdot) = \sum_{k \geq 0} \phi_{(k)}^{*-}(1-s)^k$  for  $|1-s| < \delta$  and  $\phi_{(k)}^{*-} \in \mathbf{L}[r_-, r_+]$  and using the local expansion  $z_-(s) = 1 - \frac{\sqrt{2}}{\sigma} \sqrt{1-s} + \dots$ , one thus gets for  $\delta$  small enough and  $s \in \mathcal{O}_\delta(1)$

$$\varphi^{*-}(s, \cdot) = \varphi^{*-}(1, \cdot) + \sum_{k \geq 1} \varphi_{(k)}^{*-}(1-s)^{k/2}$$

with  $\sum_{k \geq 0} |\varphi_{(k)}^{*-}|_\infty \delta^k < +\infty$  and  $\varphi_{(1)}^{*-} : z \mapsto \frac{\sqrt{2}}{\sigma} \times \frac{1 - \mathbb{E}[z^{S_{\tau^{*-}}}]}{1-z}$ .

We summarize the informations we will need in the following

**Proposition 3.2.1** *For any  $r_- < 1 < r_+$ , the function  $s \mapsto \varphi^{*-}(s, \cdot)$  has an analytic continuation to an open neighborhood of  $\overline{B(0, 1)} \setminus \{1\}$  with values in  $\mathbf{L}[r_-, r_+]$ ; furthermore, for  $\delta > 0$ , this function is analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1) = B(1, \delta) \setminus [1, 1 + \delta[$  and its local expansion of order 1 in  $\mathbf{L}[r_-, r_+]$  is given by*

$$\varphi^{*-}(s, \cdot) = \varphi^{*-}(1, \cdot) + \sqrt{1-s} \varphi_{(1)}^{*-}(\cdot) + \mathbf{O}(s, \cdot) \tag{16}$$

with  $\varphi_{(1)}^{*-} : z \mapsto \frac{\sqrt{2}}{\sigma} \times \frac{1 - \mathbb{E}[z^{S_{\tau^{*-}}}]}{1-z}$  and  $\mathbf{O}(s, \cdot)$  uniformly bounded in  $\mathbf{L}[r_-, r_+]$ .

---

<sup>6</sup>for  $Z$  in  $\mathbb{C} \setminus \mathbb{R}^{*-}$ , writing  $Z = |Z|e^{i\theta}$  for some  $-\pi < \theta < \pi$ , one set  $\sqrt{Z} = \sqrt{|Z|}e^{i\theta/2}$

A similar statement holds for the function  $\varphi^+$ ; in particular, the local expansion near  $s = 1$  follows from the one of the root  $z_+(s)$ , namely  $z_+(s) = 1 + \frac{\sqrt{2}}{\sigma}\sqrt{1-s} + \dots$ . We may thus state the

**Proposition 3.2.2** *For any  $\underline{r_-} < 1 < r_+$ , the function  $s \mapsto \varphi^+(s, \cdot)$  has an analytic continuation to an open neighborhood of  $\overline{B(0,1)} \setminus \{1\}$  with values in  $\mathbf{L}[r_-, r_+]$ ; furthermore, for  $\delta > 0$  small enough, this function is analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1) = B(1, \delta) \setminus [1, 1 + \delta[$  and its local expansion of order 1 in  $\mathbf{L}[r_-, r_+]$  is given by*

$$\varphi^+(s, \cdot) = \varphi^+(1, \cdot) + \sqrt{1-s} \varphi_{(1)}^+(\cdot) + \mathbf{O}(s, \cdot) \quad (17)$$

with  $\varphi_{(1)}^+ : z \mapsto -\frac{\sqrt{2}}{\sigma} \times \frac{1 - \mathbb{E}[z^{S_{\tau^+}}]}{1-z}$  and  $\mathbf{O}(s, \cdot)$  uniformly bounded in  $\mathbf{L}[r_-, r_+]$ .

### 3.3 The maps $s \mapsto \mathfrak{F}^{*-}(s|x)$ and $s \mapsto \mathfrak{F}^+(s|x)$ for $x \in \mathbb{Z}$

We use here the inverse Fourier's formula: for any  $x \in \mathbb{Z}^{*-}$  and  $s \in \mathbb{C}, |s| < 1$ , one gets, by a Fubini type argument,

$$\begin{aligned} \mathfrak{F}^{*-}(s|x) &= \mathbb{E} \left[ s^{\tau^{*-}} 1_{\{x\}}(S_{\tau^{*-}}) \right] \\ &= \mathbb{E} \left[ s^{\tau^{*-}} \frac{1}{2i\pi} \int_{\mathbb{T}} z^{S_{\tau^{*-}} - x - 1} dz \right] \\ &= \frac{1}{2i\pi} \int_{\mathbb{T}} z^{-x-1} \varphi^{*-}(s, z) dz. \end{aligned}$$

Similarly  $\mathfrak{F}^+(s|x) = \frac{1}{2i\pi} \int_{\mathbb{T}} z^{-x-1} \varphi^+(s, z) dz$  for any  $x \in \mathbb{N}_0$ . We will apply Propositions 3.2.2 and 3.2.2 and first identify the coefficients which appears in the local expansion as Fourier transforms of some known measures; let us denote

- $\delta_x$  the Dirac mass at  $x \in \mathbb{Z}$ ,
- $\lambda^{*-} = \sum_{x \leq -1} \delta_x$  the counting measures on  $\mathbb{Z}^{*-}$
- $\lambda^+ = \sum_{n \geq 0} \delta_n$  the counting measures on  $\mathbb{N}_0$ .

One easily checks that  $z \mapsto \frac{1 - \mathbb{E}[z^{S_{\tau^{*-}}}]}{z-1}$  and  $z \mapsto \frac{1 - \mathbb{E}[z^{S_{\tau^+}}]}{1-z}$  are the generating functions associated respectively with the measures  $(\delta_0 - \mu^{*-}) * \lambda^{*-}$  and  $(\delta_0 - \mu^+) * \lambda^+$ ; we may thus state the following

**Proposition 3.3.1** *There exists an open neighborhood  $\Omega$  of  $\overline{B(0,1)} \setminus \{1\}$  such that, for any  $x \in \mathbb{Z}$ , the functions  $s \mapsto \mathfrak{F}^{*-}(s|x) := \mathbb{E}[s^{\tau^{*-}} 1_{\{x\}}(S_{\tau^{*-}})]$  and  $s \mapsto \mathfrak{F}^+(s|x) := \mathbb{E}[s^{\tau^+} 1_{\{x\}}(S_{\tau^+})]$  have an analytic continuation to  $\Omega$ ; furthermore, for  $\delta > 0$  small enough, these functions are analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1)$  and their local expansions of order 1 are given by*

$$\mathfrak{F}^{*-}(s|x) = \mu^{*-}(x) - \sqrt{1-s} \frac{\sqrt{2}}{\sigma} \mu^{*-}([-\infty, x]) + (1-s) \mathbf{O}(s|x) \quad (18)$$

and

$$\mathfrak{F}^+(s|x) = \mu^+(x) - \sqrt{1-s} \frac{\sqrt{2}}{\sigma} \mu^+([x, +\infty[) + (1-s) \mathbf{O}(s|x) \quad (19)$$

with  $\mathbf{O}(s|x)$  analytic in the variable  $\sqrt{1-s}$  and uniformly bounded in  $s \in \mathcal{O}_\delta(1)$  and  $x \in \mathbb{Z}$ .

Furthermore, for any  $K > 1$ , there exists a constant  $\mathbf{O} > 0$  such that

$$K^{|x|} \left| \mathfrak{T}^{*-}(s|x) \right| \leq \mathbf{O}, \quad K^{|x|} \left| \mathfrak{T}^+(s|x) \right| \leq \mathbf{O} \quad \text{and} \quad K^{|x|} \left| \mathbf{O}(s|x) \right| \leq \mathbf{O}. \quad (20)$$

for any  $s \in \Omega \cup \mathcal{O}_\delta(1)$  and  $x \in \mathbb{Z}$ .

Proof. The analyticity property and the local expansions (18) and (19) are direct consequences of Propositions 3.2.2 and 3.2.2. To establish for instance the first inequality in (20), we use the fact that for  $s \in \Omega \cup \mathcal{O}_\delta(1)$ , the function  $z \mapsto \varphi^{*-}(s, z)$  is analytic on any annulus  $\{z \in \mathbb{C}/r_- < |z| < r_+\}$  with  $0 < r_- < 1 < r_+$  and so, for any  $K > 1$  and  $x \in \mathbb{Z}^{*-}$ , one gets

$$\mathfrak{T}^{*-}(s|x) = \frac{1}{2i\pi} \int_{\mathbb{T}} z^{-x-1} \varphi^{*-}(s, z) dz = \frac{1}{2i\pi} \int_{\{|z|=1/K\}} z^{-x-1} \varphi^{*-}(s, z) dz.$$

So  $\left| \mathfrak{T}^{*-}(s|x) \right| \leq \frac{K^{-|x|-1}}{2\pi} \times \sup_{\substack{s \in \Omega \cup \mathcal{O}_\delta(1) \\ |z|=1/K}} |\varphi^{*-}(s, z)|$ . The same argument holds for the quantities  $\mathfrak{T}^+(s|x)$

and  $\mathbf{O}(s|x)$ . □

### 3.4 The coefficient maps $s \mapsto \mathcal{T}_s^{*-}(x, y)$ and $s \mapsto \mathcal{T}_s^+(x, y)$ for $x, y \in \mathbb{Z}$

We first analyze here the consequences of the previous statement for the matrices coefficients  $\mathcal{T}_s^{*-}(x, y)$  and  $\mathcal{T}_s^+(x, y)$ . We have the

**Proposition 3.4.1** *There exists an open neighborhood  $\Omega$  of  $\overline{B(0, 1)} \setminus \{1\}$  such that for any  $x, y \in \mathbb{Z}$ , the functions  $s \mapsto \mathcal{T}_s^{*-}(x, y)$  and  $s \mapsto \mathcal{T}_s^+(x, y)$  have an analytic continuation to  $\Omega$ ; furthermore, for  $\delta > 0$  small enough, these functions are analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1)$  and their local expansions of order 1 are given by*

$$\mathcal{T}_s^{*-}(x, y) = \mathcal{T}^{*-}(x, y) + \sqrt{1-s} \tilde{\mathcal{T}}^{*-}(x, y) + (1-s) \mathbf{O}_s(x, y) \quad (21)$$

and

$$\mathcal{T}_s^+(x, y) = \mathcal{T}^+(x, y) + \sqrt{1-s} \tilde{\mathcal{T}}^+(x, y) + (1-s) \mathbf{O}_s(x, y) \quad (22)$$

where

- $\mathcal{T}^{*-}(x, y) = \mu^{*-}(y-x)$ ,
- $\tilde{\mathcal{T}}^{*-}(x, y) = -\frac{\sqrt{2}}{\sigma} \mu^{*-}([-\infty, y-x])$ ,
- $\mathcal{T}^+(x, y) = \mu^+(y-x)$ ,
- $\tilde{\mathcal{T}}^+(x, y) = -\frac{\sqrt{2}}{\sigma} \mu^+([y-x, +\infty])$ ,
- $\mathbf{O}_s(x, y)$  is analytic in the variable  $\sqrt{1-s}$  for  $s \in \mathcal{O}_\delta(1)$ .

Proof. We give the details for the maps  $s \mapsto \mathcal{T}_s^{*-}(x, y)$ , the proof goes along the same lines for  $s \mapsto \mathcal{T}_s^+(x, y)$ . Let  $\Omega$  be the open neighborhood of  $\overline{B(0, 1)} \setminus \{1\}$  given by Proposition 3.3.1 and

fix  $\delta > 0$  such that (18), (19) and (20) hold. In particular, we know that for any  $x, y \in \mathbb{Z}^-$ , the function  $s \mapsto \mathcal{T}_s^{*-}(x, y) = \mathfrak{T}^{*-}(s|y-x)$  is analytic on  $\Omega$  and has the local expansion, for  $s \in \mathcal{O}_\delta(1)$

$$\mathcal{T}_s^{*-}(x, y) = \mathcal{T}^{*-}(x, y) + \sqrt{1-s} \tilde{\mathcal{T}}^{*-}(x, y) + (1-s) \mathbf{O}_s(x, y)$$

whose coefficients are the ones given in the statement of the proposition and  $s \mapsto \mathbf{O}(x, y)$  is analytic in the variable  $\sqrt{1-s}$ ; furthermore, the quantities  $K^{|y-x|} \left| \mathcal{T}_s^+(x, y) \right|$  and  $K^{|y-x|} \left| \mathbf{O}_s(x, y) \right|$  are bounded, uniformly in  $x, y \in \mathbb{Z}^-$  and  $s \in \Omega \cup \mathcal{O}_\delta(1)$ .  $\square$

### 3.5 The coefficient maps $s \mapsto \mathcal{U}_s^{*-}(x, y)$ and $s \mapsto \mathcal{U}_s^+(x, y)$ for $x, y \in \mathbb{Z}$

We consider here the maps  $s \mapsto \mathcal{U}_s^{*-}(x, y)$  and  $s \mapsto \mathcal{U}_s^+(x, y)$ . Formally, the matrix  $U_s^{*-} = (\mathcal{U}_s^{*-}(x, y))_{x, y \in \mathbb{Z}}$  is the potential of  $\mathcal{T}_s^{*-} = (\mathcal{T}_s^{*-}(x, y))_{x, y \in \mathbb{Z}}$ ; since  $\mathcal{T}_s^{*-}$  is strictly upper triangular, each  $\mathcal{U}_s^{*-}(x, y)$  will be the combination by summations and products of finitely many coefficients  $\mathcal{T}_s^{*-}(i, j)$ ,  $i, j \in \mathbb{Z}$ , and their regularity will thus be a direct consequence of the previous statement. It will be a little more delicate for the coefficients of the matrix  $U_s^+ = \sum_{n \geq 0} (\mathcal{T}_s^+)^n$  since the matrix

$\mathcal{T}_s^+$  is upper triangular with non zero terms on the diagonal; we will mention the adjustments we need in this case. One gets the

**Proposition 3.5.1** *There exists an open neighborhood  $\Omega$  of  $\overline{B(0, 1)} \setminus \{1\}$  such that, for any  $x, y \in \mathbb{Z}^-$ , the functions  $s \mapsto \mathcal{U}_s^{*-}(x, y)$  have an analytic continuation to  $\Omega$ ; furthermore, for  $\delta > 0$  small enough, these functions are analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1)$  and their local expansions of order 1 are given by*

$$\mathcal{U}_s^{*-}(x, y) = \mathcal{U}^{*-}(x, y) + \sqrt{1-s} \tilde{\mathcal{U}}^{*-}(x, y) + (1-s) \mathbf{O}_s(x, y) \quad (23)$$

where

- $\mathcal{U}^{*-}(x, y) = U^{*-}(y-x)$
- $\tilde{\mathcal{U}}^{*-}(x, y) = -\frac{\sqrt{2}}{\sigma} U^{*-}(\lfloor y-x, 0 \rfloor)$
- $\mathbf{O}_s(x, y)$  is analytic in the variable  $\sqrt{1-s}$  and bounded for  $s \in \mathcal{O}_\delta(1)$ .

Similarly, for any  $x, y \in \mathbb{N}_0$ , the functions  $s \mapsto \mathcal{U}_s^+(x, y)$  have an analytic continuation to  $\Omega$  and these functions are analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1)$  with the local expansions of order 1 given by

$$\mathcal{U}_s^+(x, y) = \mathcal{U}^+(x, y) + \sqrt{1-s} \tilde{\mathcal{U}}^+(x, y) + (1-s) \mathbf{O}_s(x, y) \quad (24)$$

where

- $\mathcal{U}^+(x, y) = U^+(y-x)$
- $\tilde{\mathcal{U}}^+(x, y) = -\frac{\sqrt{2}}{\sigma} U^+(\lceil 0, y-x \rceil)$
- $\mathbf{O}_s(x, y)$  is analytic in the variable  $\sqrt{1-s}$  and bounded for  $s \in \mathcal{O}_\delta(1)$ .

Proof. Formally, one gets  $\mathcal{U}_s^{*-} = \sum_{n \geq 0} (\mathcal{T}_s^{*-})^n$ ; since the matrix is strictly upper triangular, for any  $x, y \in \mathbb{Z}^-$ , one gets  $(\mathcal{T}_s^{*-})^n(x, y) = 0$  for any  $n > |x - y|$ , so

$$\mathcal{U}_s^{*-}(x, y) = \sum_{n=0}^{|x-y|} (\mathcal{T}_s^{*-})^n(x, y). \quad (25)$$

The analyticity dependence, for fixed  $x, y \in \mathbb{Z}^-$ , of the coefficients  $\mathcal{U}_s^{*-}(x, y)$ , with respect to  $s \in \Omega$  and  $\sqrt{1-s}$  when  $s \in \mathcal{O}_\delta(1)$ , immediately follows from the previous Proposition.

Let us now establish the local expansion (23); for any fixed  $x, y \in \mathbb{Z}^-$ , one gets

$$\mathcal{U}_s^{*-}(x, y) = \sum_{n=0}^{|x-y|} \left( \mathcal{T}^{*-} + \sqrt{1-s} \tilde{\mathcal{T}}^{*-} + (1-s) \mathbf{O}_s \right)^n(x, y).$$

The constant term  $\mathcal{U}^{*-}(x, y)$  is thus equal to  $\sum_{n=0}^{|x-y|} (\mathcal{T}^{*-})^n(x, y) = \sum_{n=0}^{+\infty} (\mathcal{T}^{*-})^n(x, y)$ ; on the other hand, the coefficient corresponding to  $\sqrt{1-s}$  in this expansion is equal to

$$\tilde{\mathcal{U}}^{*-}(x, y) = \sum_{n=0}^{|x-y|} \sum_{k=0}^{n-1} (\mathcal{T}^{*-})^k \tilde{\mathcal{T}}^{*-} (\mathcal{T}^{*-})^{n-k-1}(x, y).$$

Inverting the order of summations and using the expression of  $\tilde{\mathcal{T}}^{*-}$  in Proposition 3.4.1, one gets

$$\begin{aligned} \tilde{\mathcal{U}}^{*-}(x, y) &= \mathcal{U}^{*-} \tilde{\mathcal{T}}^{*-} \mathcal{U}^{*-}(x, y) \\ &= -\frac{\sqrt{2}}{\sigma} \left( \mathcal{U}^{*-} * \left( \sum_{k \leq -1} \mu^{*-}(\cdot - \infty, k] \delta_k \right) * \mathcal{U}^{*-} \right) (y - x) \\ &= -\frac{\sqrt{2}}{\sigma} \mathcal{U}^{*-}(\cdot | y - x, 0] \end{aligned}$$

(to obtain the last equality, one compute the generating function of the measure

$$\mathcal{U}^{*-} * \left( \sum_{k \leq -1} \mu^{*-}(\cdot - \infty, k] \delta_k \right) * \mathcal{U}^{*-},$$

it is equal to the one of the measure  $\mathcal{U}^{*-} * \lambda^{*-}$ , and one concludes checking that

$$\mathcal{U}^{*-} * \lambda^{*-} = \sum_{k \leq -1} \mathcal{U}^{*-}(\cdot | k, 0] \delta_k.$$

The proof goes along the same lines for  $\mathcal{U}_s^+(x, y) = \sum_{n=0}^{+\infty} (\mathcal{T}_s^+)^n(x, y)$ . Nevertheless, since  $\mu^+(0) > 0$ , there are infinitely many terms in the sum; for  $s \in \Omega \cup \mathcal{O}_\delta(1)$ , one thus first sets  $\mathcal{T}_s^+ = \varepsilon_s I + T_s$



with  $\varepsilon_s := \mathbb{E} \left[ s^{\tau^+} 1_{\{0\}}(S_{\tau^+}) \right]$ . One gets  $\delta_1 = \mu^+(0) \in ]0, 1[$ , so  $|\varepsilon_s| < 1$  for  $\Omega$  and  $\delta$  small enough. Since  $I$  and  $T_s$  commute and  $T_s$  is strictly upper triangular, one may write, for any  $x, y \in \mathbb{N}_0$  and  $n \geq |x - y|$ ,

$$\begin{aligned} (\mathcal{T}_s^+)^n(x, y) &= \sum_{k=0}^n \binom{n}{k} \varepsilon_s^{n-k} T_s^k(x, y) \\ &= \sum_{k=0}^{|x-y|} \binom{n}{k} \varepsilon_s^{n-k} T_s^k(x, y) \end{aligned}$$

so that

$$\begin{aligned} \mathcal{U}_s^+(x, y) &= \sum_{n \geq 0} (\mathcal{T}_s^+)^n(x, y) \\ &= \sum_{n=0}^{|x-y|} (\mathcal{T}_s^+)^n(x, y) + \sum_{n > |x-y|} \sum_{k=0}^{|x-y|} \binom{n}{k} \varepsilon_s^{n-k} T_s^k(x, y) \\ &= \sum_{n=0}^{|x-y|} (\mathcal{T}_s^+)^n(x, y) + \sum_{k=0}^{|x-y|} \frac{1}{k!} \left( \sum_{n > |x-y|} n \cdots (n-k+1) \varepsilon_s^{n-k} \right) T_s^k(x, y) \end{aligned}$$

with  $s \mapsto \left( \sum_{n > |x-y|} n \cdots (n-k+1) \varepsilon_s^{n-k} \right)$  analytic on  $\Omega$  and analytic in  $\sqrt{1-s}$  on  $\mathcal{O}_\delta(1)$ . The analyticity of the map  $s \mapsto \mathcal{U}_s^+(x, y)$  follows immediately; the computation of the coefficients of the local expansion (24) goes along the same line than the ones of (23).  $\square$

## 4 The centered reflected random walk

Throughout this section, we will assume that hypotheses **H** hold and that  $\mu$  is centered. In this case, the radius of convergence of the generating functions  $\mathfrak{G}(\cdot|x, y)$ ,  $x, y \in \mathbb{N}_0$ , is equal to 1. By Darboux's theorem, the asymptotic behavior of the Taylor coefficients of these generating functions is related to the type of their singularity near  $s = 1$ ; in the following subsection, we state some preparatory results.

We denote by  $\mathcal{M}$  the space of infinite matrices  $M = (M(x, y))_{x, y \in \mathbb{N}_0}$ ; we will consider the elements of  $\mathcal{M}$  as operators acting on the Banach space  $(\mathbb{C}^{\mathbb{N}_0}, |\cdot|_\infty)$  and will thus endow  $\mathcal{M}$  with the norm  $\|\cdot\|_\infty$  defined by

$$\forall M \in \mathcal{M} \quad \|M\|_\infty = \sup_{x \in \mathbb{N}_0} \sum_{y \in \mathbb{N}_0} |M(x, y)|.$$

Notice that this is the norm of  $M$  considered as an operator acting on the Banach space  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_\infty)$  where  $|\cdot|_\infty$  denotes the norm of the supremum.

As we have already seen, we will also endow  $\mathcal{C}^{\mathbb{N}_0}$  with the norm  $|\cdot|_K$ , with  $K \in \mathcal{K}(1 + \eta)$  for some  $\eta > 0$ ; the corresponding operator norm  $\|\cdot\|_K$  on  $\mathcal{M}$  will thus be defined by

$$\forall M \in \mathcal{M} \quad \|M\|_K = \sup_{x \in \mathbb{N}_0} \sum_{y \in \mathbb{N}_0} \frac{K(y)}{K(x)} |M(x, y)|.$$

#### 4.1 The $\mathcal{M}$ -valued map $s \mapsto \mathcal{T}_s$ and its potential $\mathcal{U}_s$

Recall that the matrix  $\mathcal{T}_s$  is the lower triangular with coefficients  $\mathcal{T}_s(x, y), x, y \in \mathbb{N}_0$ , given by

$$\mathcal{T}_s(x, y) = \mathfrak{T}_s(y - x) = \mathbb{E} \left[ s^{\tau^{*-}} 1_{\{y-x\}}(S_{\tau^{*-}}) \right].$$

The following statement is thus a direct consequence of Proposition 3.3.1:

**Proposition 4.1.1** *There exists an open neighborhood  $\Omega$  of  $\overline{B(0, 1)} \setminus \{1\}$  such that the  $\mathcal{M}$ -valued function  $s \mapsto \mathcal{T}_s$  has an analytic continuation to  $\Omega$ ; furthermore, for  $\delta > 0$  small enough, this function is analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1)$  and its local expansions of order 1 in  $(\mathcal{M}, \|\cdot\|_\infty)$  is given by*

$$\mathcal{T}_s = \mathcal{T} + \sqrt{1-s} \tilde{\mathcal{T}} + (1-s) \mathbf{O}_s \quad (26)$$

where

- $\mathcal{T} = \left( \mathcal{T}(x, y) \right)_{x, y \in \mathbb{N}_0}$  with  $\mathcal{T}(x, y) = \begin{cases} \mu^{*-}(y-x) & \text{if } 0 \leq y \leq x-1 \\ 0 & \text{if } y \geq x \end{cases}$ ,
- $\tilde{\mathcal{T}} = \left( \tilde{\mathcal{T}}(x, y) \right)_{x, y \in \mathbb{N}_0}$  with  $\tilde{\mathcal{T}}(x, y) = \begin{cases} -\frac{\sqrt{2}}{\sigma} \mu^{*-}([\!-\infty, y-x]) & \text{if } 0 \leq y \leq x-1 \\ 0 & \text{if } y \geq x \end{cases}$ ,
- $\mathbf{O}_s$  is analytic in the variable  $\sqrt{1-s}$  and uniformly bounded in  $(\mathcal{M}, \|\cdot\|_\infty)$  for  $s \in \mathcal{O}_\delta(1)$ .

Proof. The regularity of each coefficient map  $s \mapsto \mathcal{T}_s(x, y)$  may be proved as in Proposition 3.4.1; we thus focus our attention on the analyticity of the  $\mathcal{M}$ -valued map  $s \mapsto \mathcal{T}_s$ . By a classical result in the theory of vector valued analytic functions of the complex variable (see for instance [2], Theorem 9.13), it suffices to check that this property is true for the functions  $s \mapsto \mathcal{T}_s(\mathbf{a})$  for any bounded sequence  $\mathbf{a} = (a_i)_{i \geq 0} \in \mathbb{C}_0^{\mathbb{N}}$ ; to check this, we will use the fact that any uniform limit on some open set of analytic functions is analytic on this set.

Fix  $N \geq 1$  and let  $\mathcal{T}_{s,N}$  be the “truncated” matrix defined by

$$\mathcal{T}_{s,N}(x, y) = \begin{cases} \mathcal{T}_s(x, y) & \text{if } \max(x-N, 0) \leq y \leq x-1 \\ 0 & \text{otherwise.} \end{cases}$$

One gets  $\mathcal{T}_{s,N}(\mathbf{a}) = \sum_1^N \mathfrak{T}_s^{*-}(-k) \mathbf{a}^{(k)}$  with  $\mathbf{a}^{(k)} := \underbrace{0, \dots, 0}_{k \text{ times}}, a_0, a_1, \dots$ , which implies that the

$\mathcal{M}$ -valued map  $s \rightarrow \mathcal{T}_{s,N}$  is analytic on  $\Omega$  and analytic in the variable  $\sqrt{1-s}$  on  $\mathcal{O}_\delta(1)$ . The same property holds for the map  $s \rightarrow \mathcal{T}_s$  since, by (20), one gets

$$\|\mathcal{T}_s - \mathcal{T}_{s,N}\|_\infty = \sup_{x \in \mathbb{N}_0} \sum_{|y-x| > N} |\mathcal{T}_s(x, y)| \leq \sum_{|y-x| > N} \frac{\mathbf{O}}{K^{|x-y|}} = \frac{\mathbf{O}}{(K-1)K^N} \xrightarrow{N \rightarrow +\infty} 0.$$

□

Let us now give sense to the matrix  $(I - \mathcal{T}_s)^{-1}$ ; formally one may write

$$(I - \mathcal{T}_s)^{-1} = \mathcal{U}_s := \sum_{k \geq 1} (\mathcal{T}_s)^k.$$

Since the matrices  $\mathcal{T}_s$  are strictly lower triangular, one gets  $\mathcal{T}_s^k(x, y) = 0$  for any  $x, y \in \mathbb{N}_0$  and  $k \geq |x - y| + 1$ ; it follows that, for any  $x, y \in \mathbb{N}_0$

$$(I - \mathcal{T}_s)^{-1}(x, y) = \mathcal{U}_s(x, y) = \sum_{k=0}^{|x-y|} (\mathcal{T}_s)^k(x, y). \quad (27)$$

The analyticity in the variable  $s$  (resp.  $\sqrt{1-s}$ ) on  $\Omega$  (resp. on  $\mathcal{O}_\delta(1)$ ) of each coefficient  $\mathcal{U}_s(x, y)$  follows by the previous fact and one may compute its local expansion near  $s = 1$ . Nevertheless, this property does not hold in the Banach space  $(\mathcal{M}, \|\cdot\|_\infty)$ , as can be seen easily in the following statement (clearly, the matrices  $\mathcal{U}$  and  $\tilde{\mathcal{U}}$  which appear in (28) do not belong to the Banach space  $(\mathcal{M}, \|\cdot\|_\infty)$ ), we have in fact to consider a norm of the type  $\|\cdot\|_K$  on  $\mathbb{C}^{\mathbb{N}_0}$  to obtain a similar statement. We may state the following

**Proposition 4.1.2** *Fix  $\eta > 0$  and  $K \in \mathcal{K}(1+\eta)$ . There exists an open neighborhood  $\Omega$  of  $\overline{B(0,1)} \setminus \{1\}$  such that the function  $s \mapsto \mathcal{U}_s$  has an analytic continuation to  $\Omega$ , with values in  $(\mathcal{M}, \|\cdot\|_K)$ ; furthermore, for  $\delta > 0$  small enough, this function is analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1)$  and its local expansion of order 1 in  $(\mathcal{M}, \|\cdot\|_K)$  is given by*

$$\mathcal{U}_s = \mathcal{U} + \sqrt{1-s} \tilde{\mathcal{U}} + (1-s) \mathbf{O}_s \quad (28)$$

where

- $\mathcal{U} = (\mathcal{U}(x, y))_{x, y \in \mathbb{N}_0}$  with  $\mathcal{U}(x, y) = \begin{cases} U^{*-}(y-x) & \text{if } 0 \leq y \leq x \\ 0 & \text{if } y > x, \end{cases}$ ,
- $\tilde{\mathcal{U}} = (\tilde{\mathcal{U}}(x, y))_{x, y \in \mathbb{N}_0}$  with  $\tilde{\mathcal{U}}(x, y) = \begin{cases} -\frac{\sqrt{2}}{\sigma} U^{*-}(\lfloor y-x, 0 \rfloor) & \text{if } 0 \leq y \leq x-1 \\ 0 & \text{if } y \geq x \end{cases}$ ,
- $\mathbf{O}_s = (\mathbf{O}_s(x, y))_{x, y \in \mathbb{N}_0}$  is analytic in the variable  $\sqrt{1-s}$  for  $s \in \mathcal{O}_\delta(1)$  and uniformly bounded in  $(\mathcal{M}, \|\cdot\|_K)$ .

Proof. Since  $\|\mathcal{T}\|_\infty = 1$ , one may choose  $\delta > 0$  in such a way  $\|\mathcal{T}_s\|_\infty \leq 1 + \frac{\eta}{2}$  for any  $s \in \mathcal{O}_\delta(1)$ ; it thus follows that, for such  $s$ , any  $x \in \mathbb{N}_0$  and  $y \in \{0, \dots, x-1\}$

$$|\mathcal{U}_s(x, y)| \leq \sum_{n=0}^{|x-y|} \|\mathcal{T}_s\|_\infty^n \leq (1 + 2/\eta) \left(1 + \frac{\eta}{2}\right)^{|x-y|}. \quad (29)$$

So,  $\|\mathcal{U}_s\|_K < +\infty$  when  $s \in \mathcal{O}_\delta(1)$  and  $K \in \mathcal{K}(1+\eta)$ . To prove the analyticity of the function  $s \mapsto \mathcal{U}_s$ , we consider as above the truncated matrix  $\mathcal{U}_{s,N}$  and check, first that for any  $\mathbf{a} \in \mathbb{C}^{\mathbb{N}_0}$  the maps  $s \mapsto \mathcal{U}_{s,N}(\mathbf{a})$  are analytic on  $\Omega$  and analytic in the variable  $\sqrt{1-s}$  on  $\mathcal{O}_\delta(1)$ , and second that the sequence  $(\mathcal{U}_{s,N})_{N \geq 1}$  converges to  $\mathcal{U}_s$  in  $(\mathcal{M}, \|\cdot\|_K)$ . The expansion (28) is a straightforward computation. □

**From now on, we fix a constant  $\eta > 0$  and a function  $K$  in  $\mathcal{K}(1+\eta)$ .**

## 4.2 The excursions $\mathcal{E}_s(\cdot, y)$ for $y \in \mathbb{N}_0$

The excursion  $\mathcal{E}_s$  before the first reflection has been defined formally in (7) as follows

$$\mathcal{E}_s = \left(I - \mathcal{T}_s\right)^{-1} \mathcal{U}_s^+ = \mathcal{U}_s \mathcal{U}_s^+.$$

The regularity with respect to the parameter  $s$  of the matrix coefficients  $\mathcal{U}_s^+(x, y)$  and the matrix  $\mathcal{U}_s = (I - \mathcal{T}_s)^{-1}$  is well described in Propositions 3.5.1 and 4.1.2. Each coefficient of  $\mathcal{E}_s$  is a finite sum of products of coefficients of  $\mathcal{U}_s$  and  $\mathcal{U}_s^+$  so the regularity of the map  $s \mapsto \mathcal{E}_s(x, y)$  will follow immediately; the number of terms in this sum is equal to  $\min(x, y)$ , it thus increases with  $x$  and  $y$  and it is not easy to obtain some kind of uniformity with respect to these parameters. In fact, it will be sufficient to fix the arrival site  $y$  and to describe the regularity of the  $\mathbb{C}_0^{\mathbb{N}}$ -valued map  $s \mapsto (\mathcal{E}_s(x, y))_{x \in \mathbb{N}_0}$ ; to do this, we endow the space  $\mathbb{C}_0^{\mathbb{N}}$  with the norm  $|\cdot|_K$  defined in (10).

We have the

**Proposition 4.2.1** *There exists an open neighborhood  $\Omega$  of  $\overline{B(0, 1)} \setminus \{1\}$  (depending on the function  $K$ ) such that, for any  $y \in \mathbb{N}_0$ , the functions  $s \mapsto \mathcal{E}_s(\cdot, y)$  have an analytic continuation on  $\Omega$  with values in the Banach space  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_K)$ ; furthermore, for  $\delta > 0$  small enough, these functions are analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1)$  and their local expansions of order 1 in  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_K)$  are given by*

$$\mathcal{E}_s(\cdot, y) = \mathcal{E}(\cdot, y) + \sqrt{1-s} \tilde{\mathcal{E}}(\cdot, y) + (1-s) \mathbf{O}_s(y) \quad (30)$$

where

- $\mathcal{E}(\cdot, y) = (I - \mathcal{T})^{-1} \mathcal{U}^+(\cdot, y) = \mathcal{U} \mathcal{U}^+(\cdot, y)$ ,
- $\tilde{\mathcal{E}}(\cdot, y) = \tilde{\mathcal{U}} \mathcal{U}^+(\cdot, y) + \mathcal{U} \tilde{\mathcal{U}}^+(\cdot, y)$ ,
- $\mathbf{O}_s(y)$  is analytic in the variable  $\sqrt{1-s}$  and uniformly bounded in  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_K)$  for  $s \in \mathcal{O}_\delta(1)$ .

Proof. Note that, for any  $x \in \mathbb{N}_0$ , one gets  $\mathcal{E}_s(x, y) = \sum_{z=0}^y \mathcal{U}_s(x, z) \mathcal{U}_s^+(z, y)$ . So, for  $x$  fixed, the conclusions above follows from Propositions 3.5.1 and ??; in particular, for any fixed  $N \geq 1$ , the  $\mathbb{C}_0^{\mathbb{N}}$ -valued map  $s \mapsto (\mathcal{E}_{s, N}(x, y))$  defined by  $\mathcal{E}_{s, N}(x, y) = \mathcal{E}_s(x, y)$  if  $0 \leq x \leq N$  and  $\mathcal{E}_{s, N}(x, y)$  otherwise, is analytic in  $s \in \Omega$  and  $\sqrt{1-s}$  when  $s \in \mathcal{O}_\delta(1)$ , with values in the Banach space  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_K)$ . It is sufficient to check that this sequence of vectors converges to  $\mathcal{E}_s(\cdot, y)$  in norm  $|\cdot|_K$  for some suitable choice of  $K > 1$ ; by (29), one gets

$$\left| \mathcal{E}_s(x, y) \right| \leq (y+1)(1+2/\eta) \left(1 + \frac{\eta}{2}\right)^x \times \max_{0 \leq z \leq y} |\mathcal{U}_s^+(z, y)|$$

so that  $\frac{|\mathcal{E}_s(x, y)|}{(1+\eta/2)^x} \leq \frac{C_y}{(1+\eta/2)^x}$ , for some constant  $C_y > 0$  depending only on  $y$ . Since  $K \in \mathcal{K}(1+\eta)$ , one gets  $\sup_{x \geq N} \frac{|\mathcal{E}_s(x, y)|}{K(x)} \rightarrow 0$  as  $N \rightarrow +\infty$ ; this proves that the sequence  $(\mathcal{E}_{s, N}(\cdot, y))_{N \geq 0}$  converges in  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_K)$  to  $\mathcal{E}(\cdot, y)$  as  $N \rightarrow +\infty$  and that  $s \mapsto \mathcal{E}_s(\cdot, y)$  is analytic. The local expansion (38) follows by a direct computation.  $\square$

### 4.3 On the $\mathcal{M}$ -valued map $s \mapsto \mathcal{R}_s$

The matrices  $\mathcal{R}_s$  which describe the dynamic of the space-time reflected process  $(\mathbf{r}_k, X_{\mathbf{r}_k})_{k \geq 0}$  is defined formally in Section 2:

$$\mathcal{R}_s = (I - \mathcal{T}_s)^{-1} \mathcal{V}_s = \mathcal{U}_s \mathcal{V}_s$$

with  $\mathcal{V}_s = \left( \mathcal{V}_s(x, y) \right)_{x, y \in \mathbb{N}_0}$  where  $\mathcal{V}_s(x, y) := \begin{cases} 0 & \text{if } y = 0 \\ \mathfrak{T}^{*-}(|s| - x - y) & \text{if } y \in \mathbb{N}^* \end{cases}$ . So, one first needs to control the regularity of the map  $s \mapsto \mathcal{V}_s$ ; as above, one gets the

**Fact 4.3.1** *The function  $s \mapsto \mathcal{V}_s$ , with values in the Banach space  $(\mathcal{M}, \|\cdot\|_K)$ , is analytic in  $s$  on  $\Omega$  and in the variable  $\sqrt{1-s}$  on  $\mathcal{O}_\delta(1)$ ; furthermore, it has the following local expansion of order 1 near  $s = 1$*

$$\mathcal{V}_s = \mathcal{V} + \sqrt{1-s} \tilde{\mathcal{V}} + (1-s) \mathbf{O}_s \quad (31)$$

where

- $\mathcal{V} = \left( \mathcal{V}(x, y) \right)_{x, y \in \mathbb{N}_0}$  with  $\mathcal{V}(x, y) := \begin{cases} 0 & \text{if } y = 0 \\ \mu^{*-}(-x - y) & \text{if } y \in \mathbb{N}^* \end{cases}$ .
- $\tilde{\mathcal{V}} = \left( \tilde{\mathcal{V}}(x, y) \right)_{x, y \in \mathbb{N}_0}$  with  $\tilde{\mathcal{V}}(x, y) := \begin{cases} 0 & \text{if } y = 0 \\ -\frac{\sqrt{2}}{\sigma} \mu^{*-}(\cdot - \infty, -x - y) & \text{if } y \in \mathbb{N}^* \end{cases}$ .
- $\mathbf{O}_s$  is analytic in the variable  $\sqrt{1-s}$  and uniformly bounded in  $(\mathcal{M}, \|\cdot\|_K)$  for  $s \in \mathcal{O}_\delta(1)$ .

We now may describe the regularity of the map  $s \mapsto \mathcal{R}_s$ :

**Proposition 4.3.2** *The function  $s \mapsto \mathcal{R}_s$  has an analytic continuation to an open neighborhood of  $\overline{B(0, 1)} \setminus \{1\}$  with values in the Banach space  $(\mathcal{M}, \|\cdot\|_K)$ ; furthermore, for  $\delta > 0$  small enough, this function is analytic in the variable  $\sqrt{1-s}$  on the set  $\mathcal{O}_\delta(1)$  and its local expansion of order 1 in  $(\mathcal{M}, \|\cdot\|_\infty)$  is given by*

$$\mathcal{R}_s = \mathcal{R} + \sqrt{1-s} \tilde{\mathcal{R}} + (1-s) \mathbf{O}_s \quad (32)$$

where

- $\tilde{\mathcal{R}} = \tilde{\mathcal{U}}\mathcal{V} + \mathcal{U}\tilde{\mathcal{V}}$ .
- $\mathbf{O}_s$  is analytic in the variable  $\sqrt{1-s}$  and uniformly bounded in  $(\mathcal{M}, \|\cdot\|_\infty)$  for  $s \in \mathcal{O}_\delta(1)$ .

*Proof.* The analyticity of this function with respect to the variables  $s$  or  $\sqrt{1-s}$  is clear by Proposition 4.1.2 and Fact 4.3.1 and one may write, for  $s \in \mathcal{O}_\delta(1)$ ,

$$\begin{aligned} \mathcal{R}_s &= \left( I - \mathcal{T}_s \right)^{-1} \mathcal{V}_s \\ &= \mathcal{U}_s \mathcal{V}_s \\ &= \left( \mathcal{U} + \sqrt{1-s} \tilde{\mathcal{U}} + (1-s) \mathbf{O}_s \right) \left( \mathcal{U} + \sqrt{1-s} \tilde{\mathcal{U}} + (1-s) \mathbf{O}_s \right) \\ &= \mathcal{U}\mathcal{V} + \sqrt{1-s} \left( \tilde{\mathcal{U}}\mathcal{V} + \mathcal{U}\tilde{\mathcal{V}} \right) + (1-s) \mathbf{O}_s. \end{aligned}$$

□

A direct computation gives in particular

$$\mathcal{E}(x, y) = \sum_{k=0}^{\min(x, y)} U^{*-}(k-x) U^+(y-k) \quad (33)$$

and

$$\tilde{\mathcal{R}}(x, y) = \mathcal{A}(x, y) + \mathcal{B}(x, y) \quad (34)$$

with

$$\mathcal{A}(x, y) := \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ -\frac{\sqrt{2}}{\sigma} \sum_{k=0}^{x-1} U^{*-}([k-x, 0]) \mu^{*-}(-k-y) & \text{otherwise,} \end{cases}$$

and

$$\mathcal{B}(x, y) := \begin{cases} 0 & \text{if } y = 0 \\ -\frac{\sqrt{2}}{\sigma} \sum_{k=0}^x U^{*-}(k-x) \mu^{*-}([-\infty, -k-y]) & \text{otherwise.} \end{cases}$$

#### 4.4 On the spectrum of $\mathcal{R}_s$ and its resolvent $(I - \mathcal{R}_s)^{-1}$

The question is more delicate in the centered case since the spectral radius of  $\mathcal{R}$  is equal to 1 (we will see in the next Section that it is  $< 1$  in the non centered case, which simplify this step).

##### 4.4.1 The spectrum of $\mathcal{R}_s$ for $|s| = 1$ and $s \neq 1$

Using Property 2.4.2, we first control the spectral radius of the  $\mathcal{R}_s$  for  $s \neq 1$ ; indeed, we may control the norm of  $\mathcal{R}_s^2$ :

**Fact 4.4.1** *For  $|s| = 1$  and  $s \neq 1$  one gets  $\|\mathcal{R}_s^2\|_K < 1$ ; in particular, the spectral radius of  $\mathcal{R}_s$  on  $(\mathbb{C}_0^{\mathbb{N}}, \|\cdot\|_K)$  is  $< 1$ .*

*Proof.* Fix  $s \in \mathbb{C} \setminus \{1\}$  of modulus 1; by strict convexity, for any  $w \in \mathbb{N}_0$  and  $y \in \mathbb{N}^*$ , there exists  $\rho_{w,y} \in ]0, 1[$ , depending also on  $s$ , such that  $|\mathcal{R}_s(w, y)| \leq \rho_{w,y} \mathcal{R}(w, y)$ ; on the other hand, by Property 2.4.2, we may choose  $\epsilon > 0$  and a finite set  $F \subset \mathbb{N}_0$  such that, for any  $x \in \mathbb{N}_0$ ,

$$\mathcal{R}(x, F) := \sum_{w \in F} \mathcal{R}(x, w) \geq \epsilon.$$

For any  $y \in \mathbb{N}_0$ , we set  $\rho_y := \max_{w \in F} \rho_{w,y}$ ; since  $F$  is finite, one gets  $\rho_y \in ]0, 1[$ .

Consequently, for any  $x \in \mathbb{N}_0$

$$\left| \mathcal{R}_s^2 K(x) \right| \leq \sum_{w \in \mathbb{N}^*} \sum_{y \in \mathbb{N}^*} \mathcal{R}(x, w) \times \left| \mathcal{R}_s(w, y) \right| K(y) \leq \mathcal{S}_1(s|x) + \mathcal{S}_2(s|x)$$

with

$$\begin{aligned} \mathcal{S}_1(s|x) &:= \sum_{w \in F} \sum_{y \in \mathbb{N}^*} \mathcal{R}(x, w) \times \left| \mathcal{R}_s(w, y) \right| K(y) \\ \mathcal{S}_2(s|x) &:= \sum_{w \notin F} \sum_{y \in \mathbb{N}^*} \mathcal{R}(x, w) \times \left| \mathcal{R}_s(w, y) \right| K(y). \end{aligned}$$

One gets

$$\mathcal{S}_1(s|x) \leq \sum_{w \in F} \mathcal{R}(x, w) \sum_{y \in \mathbb{N}^*} \rho_y \mathcal{R}(w, y) K(y) \leq \rho \mathcal{R}(x, F)$$

with  $\rho := \max_{w \in F} \sum_{y \in \mathbb{N}^*} \rho_y \mathcal{R}(w, y) K(y) \in ]0, 1[$ .

On the other hand  $\mathcal{S}_2(s|x) \leq \mathcal{R}(x, \mathbb{N}^* \setminus F) = 1 - \mathcal{R}(x, F)$ . Finally, since  $K \geq 1$ , one gets

$$\left| \frac{\mathcal{R}_s^2 K(x)}{K(x)} \right| \leq \left( \rho \mathcal{R}(x, F) + 1 - \mathcal{R}(x, F) \right) \leq 1 - (1 - \rho)\epsilon < 1,$$

and the lemma follows.  $\square$

Since the map  $s \mapsto \mathcal{R}_s$  is analytic on the set  $\{s \in \mathbb{C} / |s| < 1 + \delta\} \setminus [1, 1 + \delta[$ , the same property holds for the map  $s \mapsto (I - \mathcal{R}_s)^{-1}$  on a neighborhood of  $\{s \in \mathbb{C} / |s| \leq 1\} \setminus \{1\}$ .

#### 4.4.2 Perturbation theory and spectrum of $\mathcal{R}_s$ for $s$ closed to 1

We now focus our attention on  $s$  closed to 1. By Property 2.4.2, we know that the operator  $\mathcal{R}$  may be decomposed as follows on  $(\mathbb{C}^{\mathbb{N}_0}, \|\cdot\|_K)$

$$\mathcal{R} = \pi + \mathcal{Q}$$

where

- $\pi$  is the rank one projector, on the space  $\mathbb{C} \cdot \mathbf{h}$  generated by the sequence  $\mathbf{h} \in \mathbb{C}^{\mathbb{N}_0}$  whose terms are all equal to 1, defined by

$$\mathbf{a} = (a_k)_{k \geq 0} \mapsto \left( \sum_{i \geq 1} \nu_{\mathbf{r}}(k) a_k \right) \mathbf{h} \quad (7),$$

- $\mathcal{Q}$  is a bounded operator on  $(\mathbb{C}^{\mathbb{N}_0}, \|\cdot\|_K)$  with spectral radius  $< 1$ ,
- $\pi \circ \mathcal{Q} = \mathcal{Q} \circ \pi = 0$ .

Recall that the map  $s \mapsto \mathcal{R}_s$  is continuous on  $\mathcal{O}_\delta(1)$  and, more precisely, that  $s \mapsto \frac{\mathcal{R}_s - \mathcal{R}}{\sqrt{1-s}}$  is bounded on this set. By perturbation theory, for  $s \in \mathcal{O}_\delta(1)$  with  $\delta$  small enough, the operator  $\mathcal{R}_s$  admits a similar spectral decomposition as above ; namely, one gets

$$\forall s \in \mathcal{O}_\delta(1) \quad \mathcal{R}_s = \lambda_s \pi_s + \mathcal{Q}_s \quad (35)$$

with

- $\lambda_s$  is the dominant eigenvalue of  $\mathcal{R}_s$ , with corresponding eigenvector  $\mathbf{h}_s$ , normalized in such a way that  $\nu_{\mathbf{r}}(\mathbf{h}_s) = 1$ ,
- $\pi_s$  is a rank one projector on the space  $\mathbb{C} \cdot \mathbf{h}_s$ ,
- $\mathcal{Q}_s$  is a bounded operator on  $(\mathbb{C}^{\mathbb{N}_0}, \|\cdot\|_K)$  with spectral radius  $\leq \rho_\delta$  for some  $\rho_\delta < 1$ ,
- $\pi_s \circ \mathcal{Q}_s = \mathcal{Q}_s \circ \pi_s = 0$ .

Furthermore, the maps  $s \mapsto \frac{\lambda_s - 1}{\sqrt{1-s}}$ ,  $s \mapsto \frac{\pi_s - \pi}{\sqrt{1-s}}$ ,  $s \mapsto \frac{\mathbf{h}_s - \mathbf{h}}{\sqrt{1-s}}$  and  $s \mapsto \frac{\mathcal{Q}_s - \mathcal{Q}}{\sqrt{1-s}}$  are bounded on  $\mathcal{O}_\delta(1)$ . We may in fact precise the local behavior of the map  $s \mapsto \lambda_s$ ; by the above decomposition and Proposition 4.3.2, one gets, for  $s \in \mathcal{O}_\delta(1)$ ,

$$\begin{aligned} \lambda_s &= \nu_{\mathbf{r}}(\mathcal{R}_s \mathbf{h}) + \nu_{\mathbf{r}}\left((\mathcal{R}_s - \mathcal{R})(\mathbf{h}_s - \mathbf{h})\right) \\ &= 1 + \sqrt{1-s} \nu_{\mathbf{r}}(\tilde{\mathcal{R}} \mathbf{h}) + (1-s)O(s) \end{aligned}$$

---

<sup>7</sup>notice that  $\nu_{\mathbf{r}}(\mathbf{h}) = 1$  since  $\nu_{\mathbf{r}}$  is a probability measure on  $\mathbb{N}_0$

with  $O(s)$  bounded on  $\mathcal{O}_\delta(1)$ . Since  $\nu_r(\tilde{\mathcal{R}}\mathbf{h}) \neq 0$ , the operator  $I - \mathcal{R}_s$  is invertible when  $s \in \mathcal{O}_\delta(1)$  and  $\delta$  small enough, with inverse

$$(I - \mathcal{R}_s)^{-1} = \frac{1}{1 - \lambda_s} \pi_s + (I - \mathcal{Q}_s)^{-1}.$$

We have thus obtained the following

**Fact 4.4.2** *For  $\delta > 0$  small enough, the function  $s \mapsto (I - \mathcal{R}_s)^{-1}$  admits on  $\mathcal{O}_\delta(1)$  the following local expansion of order 1 with values in  $(\mathcal{M}, \|\cdot\|_K)$*

$$(I - \mathcal{R}_s)^{-1} = -\frac{1}{\sqrt{1-s} \times \nu_r(\tilde{\mathcal{R}}\mathbf{h})} \pi + \mathbf{O}_s \quad (36)$$

where  $\mathbf{O}_s$  is analytic in the variable  $\sqrt{1-s}$  and uniformly bounded in  $(\mathcal{M}, \|\cdot\|_K)$ .

#### 4.5 The return probabilities in the centered case: proof of the main theorem

We use here the identity  $\mathcal{G}_s = (I - \mathcal{R}_s)^{-1} \mathcal{E}_s$  given in the introduction. By Proposition 4.3.2 and Fact 4.4.1, for any fixed  $y \in \mathbb{N}_0$ , the function  $s \mapsto \mathcal{G}_s(\cdot, y)$  is analytic on a neighborhood of  $\overline{B(0,1)} \setminus \{1\}$ . Furthermore, for  $\delta > 0$  small enough and  $s \in \mathcal{O}_\delta(1)$ , one may write, using (38) and (39)

$$\mathcal{G}_s(\cdot, y) = -\frac{\nu_r(\mathcal{E}(\cdot, y))}{\nu_r(\tilde{\mathcal{R}}\mathbf{h})} \times \frac{1}{\sqrt{1-s}} + \mathbf{O}_s$$

with  $\nu_r, \mathcal{E}(\cdot, y)$  and  $\tilde{\mathcal{R}}$  given respectively by formulae (9), (33) and (34) and  $s \mapsto \mathbf{O}_s$  analytic on  $\mathcal{O}_\delta(1)$  in the variable  $\sqrt{1-s}$  and uniformly bounded in  $(\mathcal{M}, \|\cdot\|_K)$ .

We may thus apply Darboux's theorem 1.0.1 with  $R = 1, \alpha = -\frac{1}{2}$  (and so  $\Gamma(-\alpha) = \sqrt{\pi}$ ) and  $\mathfrak{A}(1) = -\frac{\nu_r(\mathcal{E}(\cdot, y))}{\nu_r(\tilde{\mathcal{R}}\mathbf{h})} > 0$ . One gets, for all  $x, y \in \mathbb{N}_0$

$$\mathbb{P}_x[X_n = y] \sim \frac{C_y}{\sqrt{n}} \quad \text{with} \quad C_y = -\frac{1}{\sqrt{\pi}} \times \frac{\nu_r(\mathcal{E}(\cdot, y))}{\nu_r(\tilde{\mathcal{R}}\mathbf{h})} > 0. \quad (37)$$

□

## 5 The non centered random walk

We assume here  $\mathbb{E}[Y_i] > 0$  and use a standard argument in probability theory, called sometimes "relativisation procedure", to reduce the question to the centered case.

### 5.1 The relativisation principle and its consequences

For any  $r > 0$ , we denote by  $\mu_r$  the probability measure defined on  $\mathbb{Z}$  by

$$\forall n \in \mathbb{Z} \quad \mu_r(n) = \frac{1}{\hat{\mu}(r)} r^n \mu(n).$$



Note that for any  $k \geq 0$  one gets  $(\mu^{*k})_r = (\mu_r)^{*k}$  and that the generating function  $\hat{\mu}_r$  is related to the one of  $\mu$  by the following identity  $\forall z \in \mathbb{C} \quad \hat{\mu}_r(z) := \frac{\hat{\mu}(rz)}{\hat{\mu}(r)}$ .

Notice that waiting times  $\tau^{*-}$  and  $\tau^+$  are defined on the space  $(\Omega, \mathcal{T})$ , with values in  $\mathbb{N}_0 \cup \{+\infty\}$ , independently on the measure  $\mu_r$  we choose; they are both a.s. finite if and only if  $\mu_r$  is centered, i.e.  $r = r_0$  (see Section 3.1 for the notations).

Throughout this section, we will denote  $\mathbb{P}^\circ$  the probability on  $(\Omega, \mathcal{T})$  which ensures that the  $Y_i$  are i.i.d. with law  $\mu_{r_0}$ ; the expectation with respect to  $\mathbb{P}^\circ$  is denoted  $\mathbb{E}^\circ$ . We set  $\rho_\circ = \hat{\mu}(r_0)$  and  $R_\circ = 1/\rho_\circ$ ; one gets  $R_\circ \in ]1, +\infty[$ . Notice that the variables  $Y_i$  have common law  $\mu_{r_0}$  under  $\mathbb{P}^\circ$ , they are in particular centered; we may thus apply the results of the previous section when we refer to this probability measure on  $(\Omega, \mathcal{T})$ .

We have the classical following

**Fact 5.1.1** *Let  $n \geq 1$  and  $\Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$  a bounded Borel function; then, one gets*

$$\mathbb{E} \left[ \Phi(S_0, S_1, \dots, S_n) \right] = \rho_\circ^n \times \mathbb{E}^\circ \left[ \Phi(S_0, S_1, \dots, S_n) r_0^{-S_n} \right].$$

As a direct consequence, for any  $x, y \in \mathbb{N}_0$  and  $s \in \mathbb{C}$ , one gets, at least formally

$$\mathcal{E}_s(x, y) = r_0^{x-y} \mathcal{E}_{\rho_\circ s}^\circ(x, y) \quad \text{and} \quad \mathcal{R}_s(x, y) = r_0^{x+y} \mathcal{R}_{\rho_\circ s}^\circ(x, y).$$

where we have set  $\mathcal{E}_s^\circ(x, y) := \sum_{n \geq 0} s^n \mathbb{E}_x^\circ[\mathbf{r} > n, X_n = y]$  and  $\mathcal{R}_s^\circ(x, y) := \sum_{n \geq 0} s^n \mathbb{E}_x^\circ[\mathbf{r} = n, X_n = y]$ .

We may thus introduce the diagonal matrice  $\Delta = (\Delta(x, y))_{x, y \in \mathbb{N}_0}$  defined by  $\Delta(x, y) = 0$  when  $x \neq y$  and  $\Delta(x, x) = r_0^x$  for any  $x \geq 0$ ; by the above, one gets formally

$$\mathcal{E}_s = \Delta \mathcal{E}_s^\circ \Delta^{-1} \quad \text{and} \quad \mathcal{R}_s = \Delta \mathcal{R}_s^\circ \Delta.$$

In the sequel, we will add the exponent  $\circ$  to the quantities  $\mathcal{U}^+, \mathcal{T}, \mathcal{U}, \mathcal{V}$  defined in the previous section when  $\mu$  was assume to be centered and considered here as variables defined on  $(\Omega, \mathcal{F}, \mathbb{P}^\circ)$ ; with these notations, we will have  $\mathcal{E}^\circ = \mathcal{U}^\circ \mathcal{U}^{\circ+}$ ,  $\tilde{\mathcal{E}}^\circ = \tilde{\mathcal{U}}^\circ \mathcal{U}^{\circ+} + \mathcal{U}^\circ \tilde{\mathcal{U}}^{\circ+}$ ,  $\mathcal{R}^\circ = \mathcal{U}^\circ \mathcal{V}^\circ$  and  $\tilde{\mathcal{R}}^\circ = \tilde{\mathcal{U}}^\circ \mathcal{V}^\circ + \mathcal{U}^\circ \tilde{\mathcal{V}}^\circ$ .

Combining Propositions 4.2.1 and 4.3.2, we may thus state the

**Proposition 5.1.2** *There exist a function  $K$ , an open neighborhood  $\Omega$  of  $\overline{B(0, R_\circ)} \setminus \{R_\circ\}$  and  $\delta > 0$  small enough such that, for any  $y \in \mathbb{N}_0$ , the functions  $s \mapsto (\mathcal{E}_s(x, y))_{x \in \mathbb{N}_0}$  have an analytic continuation on  $\Omega$  with values in the Banach space  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_K)$  and are analytic in the variable  $\sqrt{R_\circ - s}$  on the set  $\mathcal{O}_\delta(R_\circ) := B(R_\circ, \delta) \setminus [R_\circ, R_\circ + \delta)$ , with the following local expansion of order 1 in  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_K)$ :*

$$\mathcal{E}_s = \mathcal{E} + \sqrt{R_\circ - s} \tilde{\mathcal{E}} + (R_\circ - s) \mathbf{O}_s \tag{38}$$

where

- $\mathcal{E} = \Delta \mathcal{E}^\circ \Delta^{-1}$
- $\tilde{\mathcal{E}} = \sqrt{\rho_0} \Delta \tilde{\mathcal{E}}^\circ \Delta^{-1}$
- $\mathbf{O}_s = (\mathbf{O}_s(x, y))_{x \in \mathbb{N}_0}$  is analytic in the variable  $\sqrt{R_\circ - s}$  and uniformly bounded in  $(\mathbb{C}_0^{\mathbb{N}}, |\cdot|_K)$  for  $s \in \mathcal{O}_\delta(R_\circ)$ .

Similarly, the function  $s \mapsto \mathcal{R}_s = \left( \mathcal{R}_s(x, y) \right)_{x, y \in \mathbb{N}_0}$  has an analytic continuation to  $\Omega$ , with values in the Banach space  $(\mathcal{M}, \|\cdot\|_K)$ , and is analytic in the variable  $\sqrt{R_o - s}$  on the set  $\mathcal{O}_\delta(R_o)$  with the following local expansion of order 1 in  $(\mathcal{M}, \|\cdot\|_K)$ :

$$\mathcal{R}_s = \mathcal{R} + \sqrt{R_o - s} \tilde{\mathcal{R}} + (R_o - s) \mathbf{O}_s \quad (39)$$

where

- $\mathcal{R} = \Delta \mathcal{R}^\circ \Delta$
- $\tilde{\mathcal{R}} = \sqrt{\rho_0} \Delta \tilde{\mathcal{R}}^\circ \Delta$
- $\mathbf{O}_s = (\mathbf{O}_s(x, y))_{x \in \mathbb{N}_0}$  is analytic in the variable  $\sqrt{R_o - s}$  and uniformly bounded in  $(\mathbb{C}_0^{\mathbb{N}}, \|\cdot\|_K)$  for  $s \in \mathcal{O}_\delta(R_o)$ .

To prove the main theorem in the non centered case, we will thus apply the same strategy than in the previous section. The proof simplifies in this case since the operator  $I - \mathcal{R}_s$  becomes invertible; namely, one gets the

**Fact 5.1.3** For  $K$  suitably choosen,  $\delta > 0$  small enough and any  $s \in \mathcal{O}_\delta(R_o)$ , the spectral radius of the operator  $\mathcal{R}_s$  on  $(\mathcal{M}, \|\cdot\|_K)$  is  $< 1$ .

Proof. It will be a direct consequence of the continuity of the map  $s \mapsto \mathcal{R}_s$  on  $K_\delta(R_o)$  and the inequality  $\|\mathcal{R}_{R_o}\|_K < 1$ . Indeed, one gets, using the definition of  $\mathcal{R}$  and setting  $\phi := 1_{[-x, +\infty[}$

$$\begin{aligned} \|\mathcal{R}_{R_o}\|_K &\leq \sup_{x \in \mathbb{N}_0} \sum_{y \geq 1} \left( \sum_{n \geq 0} R_o^n \mathbb{P} [\phi(S_1) \cdots \phi(S_{n-1}) 1_{\{-x-y\}}(S_n)] \right) K(y) \\ &= \sup_{x \in \mathbb{N}_0} \sum_{y \geq 1} \left( \sum_{n \geq 0} R_o^n r_0^{x+y} K(y) \mathbb{P}^\circ [\phi(S_1) \cdots \phi(S_{n-1}) 1_{\{-x-y\}}(S_n)] \right) \\ &\leq r_0 \sup_{x \in \mathbb{N}_0} \left( \sum_{n \geq 0} R_o^n \mathbb{P}^\circ [\phi(S_1) \cdots \phi(S_{n-1}) (1 - \phi)(S_n)] \right) \quad \text{if } r_0^{y-1} K(y) \leq 1 \quad \text{for all } y \geq 1 \\ &\leq r_0 \end{aligned}$$

which achieves the proof, assuming that  $y \mapsto r_0^{y-1} K(y)$  is  $\leq 1$  on  $\mathbb{N}_0$ . □

As a direct consequence, one may write

$$(I - \mathcal{R}_s)^{-1} = \sum_{n \geq 0} \mathcal{R}_s^n.$$

Furthermore, the map  $s \mapsto (I - \mathcal{R}_s)^{-1}$  is analytic in the variable  $s$  on  $\Omega$  and analytic in the variable  $\sqrt{R_o - s}$  on  $\mathcal{O}_\delta(R_o)$  and the local expansion near  $R_o$  is

$$(I - \mathcal{R}_s)^{-1} = (I - \mathcal{R})^{-1} + \sqrt{R_o - s} (I - \mathcal{R})^{-1} \tilde{\mathcal{R}} (I - \mathcal{R})^{-1} + (R_o - s) \cdots \quad (40)$$

## 5.2 The return probabilities in the non centered case: proof of the main theorem

We use here the identity  $\mathcal{G}_s = \left( I - \mathcal{R}_s \right)^{-1} \mathcal{E}_s$  given in the introduction. By Proposition 5.1.2 and Fact 5.1.3, for any fixed  $y \in \mathbb{N}_0$ , the function  $s \mapsto \mathcal{G}_s(\cdot, y)$  is analytic on a neighborhood of

$\overline{B(0, R_o)} \setminus \{R_o\}$ . Furthermore, for  $\delta > 0$  small enough and  $s \in \mathcal{O}_\delta(R_o)$ , one may write, using Proposition 5.1.2 and the local expansion (40)

$$\mathcal{G}_s(\cdot, y) = (I - \mathcal{R})^{-1}(\cdot, y) + \sqrt{R_o - s} \left( (I - \mathcal{R})^{-1} \tilde{\mathcal{R}} (I - \mathcal{R})^{-1} \mathcal{E}(\cdot, y) + (I - \mathcal{R})^{-1} \tilde{\mathcal{E}}(\cdot, y) \right) + (R_o - s) \mathbf{O}_s$$

with  $s \mapsto \mathbf{O}_s$  analytic on  $\mathcal{O}_\delta(R_o)$  in the variable  $\sqrt{R_o - s}$  and uniformly bounded in  $(\mathcal{M}, \|\cdot\|_K)$ .

We may thus apply Darboux's theorem 1.0.1 with  $R = R_o, \alpha = \frac{1}{2}$  (and so  $\Gamma(-\alpha) = -2\sqrt{\pi}$ ) and  $\mathfrak{A}(R_o) = (I - \mathcal{R})^{-1} \tilde{\mathcal{R}} (I - \mathcal{R})^{-1} \mathcal{E}(\cdot, y) + (I - \mathcal{R})^{-1} \tilde{\mathcal{E}}(\cdot, y) < 0$ .

One gets, for all  $x, y \in \mathbb{N}_0$

$$\mathbb{P}_x[X_n = y] \sim C_{x,y} \frac{\rho_o^n}{n^{3/2}}$$

with  $\rho_o = \frac{1}{R_o} = \hat{\mu}(r_0) \in ]0, 1[$  and

$$C_{x,y} = -\frac{1}{2\rho_o\sqrt{\pi}} \times \left( (I - \mathcal{R})^{-1} \tilde{\mathcal{R}} (I - \mathcal{R})^{-1} \mathcal{E}(x, y) + (I - \mathcal{R})^{-1} \tilde{\mathcal{E}}(x, y) \right) > 0 \quad (41)$$

where the matrices  $\mathcal{R}, \tilde{\mathcal{R}}, \mathcal{E}$  and  $\tilde{\mathcal{E}}$  are given explicitly in Proposition 5.1.2. □

## References

- [1] BAXENDALE P.H. *Renewal theory and computable convergence rates for geometrically ergodic Markov chains*, Annals of Appl. Probab. 15 (2005) 700–738.
- [2] CHAE S.B. *Holomorphy and calculus in normed spaces*, Pure and Applied Mathematics
- [3] ESSIFI R. & PEIGN M. *Some aspects of fluctuations of random walks on  $\mathbb{R}$* , in preparation.
- [4] FELLER W. *An Introduction to Probability Theory and Its Applications, Vol. 2*, Wiley series.
- [5] LALLEY S. *Return probabilities for random walk on a half-line*, J. Theoret. Probab. 8 (1995) 571–599.
- [6] PRESMAN, . L. *A Boundary Problem for Sums of Lattice Random Variables Defined on a Regular Finite Markov Chain*, Teor. Veroyatnost. i Primenen., 1967, Volume 12, Issue 2, Pages 373380.
- [7] PEIGNÉ M. & WOESS W. *On recurrence of a reflected random walk on the half line*, <http://arxiv.org/abs/math/0612306>.