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Oracle inequalities for the Lasso in the high-dimensional multiplicative Aalen intensity model

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Abstract

We consider the problem of obtaining a prognostic on the survival time adjusted on covariates in a high-dimensional setting. Towards this end, we construct an estimator of the conditional intensity function that does not rely on an underlying model. We estimate it by the best Cox proportional hazards model given two dictionaries of functions. The first dictionary is used to construct an approximation of the logarithm of the baseline hazard function and the second to approximate the relative risk. As we are in high-dimension, we consider the Lasso procedure to estimate the unknown parameters of the best Cox model approximating the conditional hazard rate function. We provide non-asymptotic oracle inequalities for the Lasso estimator of the conditional hazard risk function. Our results rely on an empirical Bernstein's inequality for martingales with jumps.

Keywords: Survival analysis; Right-censored data; Conditional hazard rate function; Cox proportional hazards model; Semiparametric model; Nonparametric model; High-dimensional covariates; Lasso; Non-asymptotic oracle inequalities; Empirical Bernstein's inequality

Introduction

We consider one of the statistical challenges brought by the recent advances in biomedical technology to clinical applications. We aim at estimating the complete conditional intensity among a very large number of covariates to predict the survival. For example, in Dave et al. [14], the considered data relate 191 patients with follicular lymphoma. The observed variables are the survival time, that can be right-censored, clinical variables, as the age or the disease stage, and 44 929 levels of gene expression. In this high-dimensional right-censored setting, there are two clinical questions. One is to determine prognostic biomarkers, the second is to predict the survival from follicular lymphoma adjusted on covariates. We focus our interest on the second to obtain a prognostic on the survival time adjusted on the covariates (see Gourlay [17] and Steyerberg [26]). As a consequence, we consider

the statistical question of estimating the whole conditional intensity. To adjust on covariates, the most popular semi-parametric regression model is the Cox proportional hazards model (see Cox [13]) : the conditional hazard rate function of the survival time T given the vector of covariates $\mathbf{Z} = (Z_1, \dots, Z_p)^T$ is defined by

$$\lambda_0(t, \mathbf{Z}) = \alpha_0(t) \exp(\boldsymbol{\beta}_0^T \mathbf{Z}), \quad (1)$$

where $\boldsymbol{\beta}_0 = (\beta_{0_1}, \dots, \beta_{0_p})^T$ is the vector of regression coefficients and α_0 is the baseline hazard function. The unknown parameters of the model are $\boldsymbol{\beta}_0 \in \mathbb{R}^p$ and the functional parameter α_0 . To construct an estimator of λ_0 , one usually considers the partial likelihood introduced by Cox [13] to derive an estimator of $\boldsymbol{\beta}_0$ and then plug this estimator to obtain the well-known Breslow estimator of α_0 . We propose in this paper an alternative one-step strategy.

Framework

We consider the setting of counting processes. For $i = 1, \dots, n$, let N_i be a marked counting process and Y_i a predictable random process with values in $[0, 1]$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t \geq 0}$ be the filtration defined by

$$\mathcal{F}_t = \sigma\{N_i(s), Y_i(s), 0 \leq s \leq t, \mathbf{Z}_i, i = 1, \dots, n\},$$

where $\mathbf{Z}_i = (Z_{i,1}, \dots, Z_{i,p})^T \in \mathbb{R}^p$ is the \mathcal{F}_0 -measurable random vector of covariates of individual i . Let $\Lambda_i(t)$ be the compensator of the process $N_i(t)$ with respect to $(\mathcal{F}_t)_{t \geq 0}$, so that $M_i(t) = N_i(t) - \Lambda_i(t)$ is a $(\mathcal{F}_t)_{t \geq 0}$ -martingale.

The process N_i satisfies the Aalen multiplicative intensity model : for all $t \geq 0$,

$$\Lambda_i(t) = \int_0^t \lambda_0(s, \mathbf{Z}_i) Y_i(s) ds, \quad (\mathbf{A}_1)$$

where λ_0 is an unknown nonnegative function called intensity.

This general setting, introduced by Aalen [1], embeds several particular examples as censored data, marked Poisson processes and Markov processes (see Andersen et al. [2] for further details).

Remark 0.1. *In the specific case of right censoring, let $(T_i)_{i=1, \dots, n}$ be i.i.d. survival times of n individuals and $(C_i)_{i=1, \dots, n}$ their i.i.d. censoring times. We observe $\{(X_i, \mathbf{Z}_i, \delta_i)\}_{i=1, \dots, n}$ where $X_i = \min(T_i, C_i)$ is the event time, $\mathbf{Z}_i = (Z_{i,1}, \dots, Z_{i,p})^T$ is the vector of covariates and $\delta_i = \mathbb{1}_{\{T_i \leq C_i\}}$ is the censoring indicator. The survival times T_i are supposed to be conditionally independent of the censoring times C_i given some vector of covariates $\mathbf{Z}_i = (Z_{i,1}, \dots, Z_{i,p})^T \in \mathbb{R}^p$ for $i = 1, \dots, n$. With these notations, the \mathcal{F}_t -adapted processes $Y_i(t)$ and $N_i(t)$ are respectively defined as the at-risk process $Y_i(t) = \mathbb{1}_{\{X_i \geq t\}}$ and the counting process $N_i(t) = \mathbb{1}_{\{X_i \leq t, \delta_i = 1\}}$ which jumps when the i th individual dies.*

We observe the independent and identically distributed (i.i.d.) data $(\mathbf{Z}_i, N_i(t), Y_i(t), i = 1, \dots, n, 0 \leq t \leq \tau)$, where $[0, \tau]$ is the time interval between the beginning and the end of the study.

$$\text{On } [0, \tau], \text{ we assume that } A_0 = \sup_{1 \leq i \leq n} \left\{ \int_0^\tau \lambda_0(s, \mathbf{Z}_i) ds \right\} < \infty. \quad (\mathbf{A}_2)$$

This assumption is classic (see Andersen et al. [2] for instance). We also precise that in the following, we work conditionally to the covariates and from now, all probabilities \mathbb{P} and expectations \mathbb{E} are conditional to the covariates. Our goal is to estimate λ_0 non-parametrically in a high-dimensional setting, i.e. when the number of covariates p is larger than the sample size n ($p \gg n$).

Previous results

In high-dimensional regression, the benchmarks for results are the ones obtained in the additive regression model. In this setting, Tibshirani [28] has introduced the Lasso procedure, which consists in minimizing an ℓ_1 -penalized criterion. The Lasso estimator has been widely studied for this model, with consistency results (see Meinshausen and Bühlmann [23]) and variable selection results (see Zhao and Yu [37], Zhang and Huang [33]). Recently, attention has been directed on establishing non-asymptotic oracle inequalities for the Lasso (see Bunea et al. [10, 11], Bickel et al. [6] and Massart and Meynet [22] among others).

In the setting of survival analysis, the Lasso procedure has been first considered by Tibshirani [29] and applied to the partial log-likelihood. More generally, several adaptive works exist for the Cox model, as for example the adaptive Lasso, the smooth clipped absolute deviation penalizations and the Dantzig selector that are considered, respectively in Zou [39], Zhang and Lu [34], Fan and Li [15] and Antoniadis et al. [3]. However few results exist on the Lasso estimator in the Cox model and all results are based on the partial log-likelihood. Antoniadis et al. [3] have established asymptotic estimation inequalities in the Cox proportional hazard model for the Dantzig estimator, which is similar to the Lasso estimator (see Bickel et al. [6] for a comparison between these two estimators for an additive regression model). In Bradic et al. [7], asymptotic estimation inequalities for the Lasso estimator have been also obtained in the Cox model. More recently, Kong and Nan [19] and Bradic and Song [8] have established non-asymptotic oracle inequalities for the Lasso in the generalized Cox model

$$\lambda_0(t, \mathbf{Z}) = \alpha_0(t) \exp(f_0(\mathbf{Z})), \quad (2)$$

where α_0 is the baseline hazard function and f_0 a function of the covariates. However, since they have used the Cox partial log-likelihood to focus on the function of the covariates f_0 , the obtained results are either on $f_{\hat{\beta}_L} - f_0$ or on $\hat{\beta}_L - \beta_0$ for $f_0(\mathbf{Z}) = \beta_0^T \mathbf{Z}$ and the problem of estimating the whole conditional hazard rate function λ_0 is not considered, as needed for the prevision of the survival time. Non-asymptotic results for the Lasso in the alternative Aalen additive model may be found in Gaïffas and Guilloux [16].

Our contribution

As in practice, one can not consider that the asymptotic regime has been reached, cf. in Dave et al. [14] for example, we state in this paper only non-asymptotic results. In addition, to the best of our knowledge, no non-asymptotic results for the estimation of the whole intensity in high dimension exist in the literature. As a consequence, we focus our attention on the problem of obtaining non-asymptotic oracle inequalities for Lasso estimators of the complete intensity λ_0 . We will consider the total log-likelihood rather than the partial one. The second motivation of the present paper is to address the problem of estimating λ_0 defined in **(A₁)** regardless of an underlying model. We opt for an agnostic approach, see Rigollet [24], to construct an estimator that mimics the performance of the best Cox model, whether this model is true or not. More precisely, we will consider candidates for the estimation of λ_0 of the form

$$\lambda_{\beta,\gamma}(t, \mathbf{Z}) = \alpha_\gamma(t)e^{f_\beta(\mathbf{Z})} \text{ for } (\beta, \gamma) \in \mathbb{R}^M \times \mathbb{R}^N,$$

where f_β and α_γ are respectively linear combinations of functions of two dictionaries \mathbb{F}_M and \mathbb{G}_N . The estimator of λ_0 is defined as the candidate which minimizes a weighted ℓ_1 -penalized full log-likelihood.

Towards this end, we will proceed in two steps. First we start with assuming that λ_0 verifies model **(2)**, where α_0 is assumed to be known. In this particular case, the only nonparametric function to estimate is f_0 and we estimate it by a linear combination of functions of the dictionary \mathbb{F}_M . In this setting, we obtain non-asymptotic oracle inequalities for the Cox model when α_0 is supposed to be known. In a second time, we consider the general problem of estimating the whole conditional hazard rate function λ_0 . We state non-asymptotic oracle inequalities in terms of both empirical Kullback divergence and weighted empirical quadratic norm for our Lasso estimators.

These results are obtained via three ingredients. First, we establish empirical versions of Bernstein's inequality involving the optional variation for martingales with jumps (similar to Gaïffas and Guillaux [16] and Hansen et al. [18]). This allows us to define a fully data-driven weighted ℓ_1 -penalization. For the resulting estimation, we obtain the first fast non-asymptotic oracle inequalities for the intensity under a classical Restricted Eigenvalue condition (see Bickel et al. [6]), by adapting the notion of self-concordance of Bach [4].

The paper is organized as follows. In Section 2, we describe the framework and the Lasso procedure for estimating the conditional hazard rate function. The estimation risk that we consider and its associated loss function are presented. In Section 3, prediction and estimation oracle inequalities in the particular Cox model with known baseline hazard function are stated. In Section 4, non-asymptotic oracle inequalities with different convergence rates are given for a general conditional hazard rate function. Section 5 is devoted to statement of empirical Bernstein's inequalities associated to our processes. Proofs are gathered in section 6.

1 Estimation procedure

1.1 The estimation criterion and the loss function

Since we are aiming at estimating the complete conditional hazard rate function λ_0 , we choose to consider the total empirical log-likelihood. By Jacod's Formula (see Andersen et al. [2]), the log-likelihood based on the data $(\mathbf{Z}_i, N_i(t), Y_i(t), i = 1, \dots, n, 0 \leq t \leq \tau)$ is given by

$$C_n(\lambda) = -\frac{1}{n} \sum_{i=1}^n \left\{ \int_0^\tau \log \lambda(t, \mathbf{Z}_i) dN_i(t) - \int_0^\tau \lambda(t, \mathbf{Z}_i) Y_i(t) dt \right\}.$$

Our estimation procedure is based on the minimization of this empirical risk. To this empirical risk, we associate the empirical Kullback divergence defined by

$$\begin{aligned} \widetilde{K}_n(\lambda_0, \lambda) &= \frac{1}{n} \sum_{i=1}^n \int_0^\tau (\log \lambda_0(t, \mathbf{Z}_i) - \log \lambda(t, \mathbf{Z}_i)) \lambda_0(t, \mathbf{Z}_i) Y_i(t) dt \\ &\quad - \frac{1}{n} \sum_{i=1}^n \int_0^\tau (\lambda_0(t, \mathbf{Z}_i) - \lambda(t, \mathbf{Z}_i)) Y_i(t) dt. \end{aligned} \quad (3)$$

We refer to van de Geer [31] and Senoussi [25] for close definitions. Notice in addition, that this loss function is closed to the Kullback-Leibler information considered in the density framework (see Stone [27] and Cohen and Le Pennec [20]). The following proposition justify the choice of this criterion.

Proposition 1.1. *The empirical Kullback divergence $\widetilde{K}_n(\lambda_0, \lambda)$ is nonnegative and equals zero if and only if $\lambda = \lambda_0$ almost surely.*

Let introduce the weighted empirical quadratic norm defined for all function h on $[0, \tau] \times \mathbb{R}^p$ by

$$\|h\|_{n,\Lambda} = \sqrt{\frac{1}{n} \sum_{i=1}^n \int_0^\tau (h(t, \mathbf{Z}_i))^2 d\Lambda_i(t)}, \quad (4)$$

where Λ_i is defined in (A₁). Notice that, in this definition, the higher the intensity of the process N_i is, the higher the contribution of individual i to the empirical norm is. This norm is connected to the empirical Kullback divergence, as it will be shown in Proposition 5.3. Finally, for a vector \mathbf{b} in \mathbb{R}^M , we define, $\|\mathbf{b}\|_1 = \sum_{j=1}^M |b_j|$ and $\|\mathbf{b}\|_2^2 = \sum_{j=1}^M b_j^2$.

1.2 Weighted Lasso estimation procedure

The estimation procedure is based on the choice of two finite sets of functions, called dictionaries. Let $\mathbb{F}_M = \{f_1, \dots, f_M\}$ where $f_j : \mathbb{R}^p \rightarrow \mathbb{R}$ for $j = 1, \dots, M$, and $\mathbb{G}_N = \{\theta_1, \dots, \theta_N\}$ where $\theta_k : \mathbb{R}_+ \rightarrow \mathbb{R}$ for $k = 1, \dots, N$, be two dictionaries. Typically the size of the dictionary \mathbb{F}_M used to estimate the function of the covariates in a high-dimensional setting is large, i.e. $M \gg n$, whereas to estimate a function on \mathbb{R}_+ , we consider a dictionary \mathbb{G}_N with size N of the order of n . The sets \mathbb{F}_M and

\mathbb{G}_N can be collections of basis functions such as wavelets, splines, step functions, etc. They can also be collections of several estimators computed using different tuning parameters. To make sure that no identification problems appear by using two dictionaries, it is assumed that only the dictionary $\mathbb{G}_N = \{\theta_1, \dots, \theta_N\}$ can contain the constant function, not $\mathbb{F}_M = \{f_1, \dots, f_M\}$. The candidates for the estimator of λ_0 are of the form

$$\lambda_{\beta, \gamma}(t, \mathbf{Z}_i) = \alpha_\gamma(t) e^{f_\beta(\mathbf{Z}_i)} \text{ with } \log \alpha_\gamma = \sum_{k=1}^N \gamma_k \theta_k \text{ and } f_\beta = \sum_{j=1}^M \beta_j f_j.$$

The dictionaries \mathbb{F}_M and \mathbb{G}_N are chosen such that the two following assumptions are fulfilled.

$$\text{For all } j \text{ in } \{1, \dots, M\}, \|f_j\|_{n, \infty} = \max_{1 \leq i \leq n} |f_j(\mathbf{Z}_i)| < \infty. \quad (\mathbf{A}_3)$$

$$\text{For all } k \text{ in } \{1, \dots, N\}, \|\theta_k\|_\infty = \max_{t \in [0, \tau]} |\theta_k(t)| < \infty. \quad (\mathbf{A}_4)$$

We consider a weighted Lasso procedure for estimating λ_0 .

Estimation procedure 1.2. *The Lasso estimator of λ_0 is defined by $\lambda_{\hat{\beta}_L, \hat{\gamma}_L}$, where*

$$(\hat{\beta}_L, \hat{\gamma}_L) = \arg \min_{(\beta, \gamma) \in \mathbb{R}^M \times \mathbb{R}^N} \{C_n(\lambda_{\beta, \gamma}) + \text{pen}(\beta) + \text{pen}(\gamma)\},$$

with

$$\text{pen}(\beta) = \sum_{j=1}^M \omega_j |\beta_j| \text{ and } \text{pen}(\gamma) = \sum_{k=1}^N \delta_k |\gamma_k|.$$

The weights ω_j , $j = 1, \dots, M$ and δ_k , $k = 1, \dots, N$, are positive data-driven weights defined, for $x > 0$, $y > 0$, $\varepsilon > 0$ and $\varepsilon' > 0$ fixed, by

$$\omega_j = c_{1, \varepsilon} \sqrt{\frac{x + \log M + \hat{\ell}_{n, x}(f_j)}{n}} \hat{V}_n(f_j) + c_{2, \varepsilon} \frac{x + 1 + \log M + \hat{\ell}_{n, x}(f_j)}{n} \|f_j\|_{n, \infty} \approx \sqrt{\frac{\log M}{n}} \hat{V}_n(f_j) \quad (5)$$

and

$$\delta_k = c'_{1, \varepsilon'} \sqrt{\frac{y + \log N + \hat{\ell}'_{n, y}(\theta_k)}{n}} \hat{R}_n(\theta_k) + c'_{2, \varepsilon'} \frac{y + 1 + \log N + \hat{\ell}'_{n, y}(\theta_k)}{n} \|\theta_k\|_\infty \approx \sqrt{\frac{\log N}{n}} \hat{R}_n(\theta_k), \quad (6)$$

where $\hat{V}_n(f_j)$ and $\hat{R}_n(\theta_k)$ are the "observable" empirical variance of f_j and θ_k respectively, given by

$$\hat{V}_n(f_j) = \frac{1}{n} \sum_{i=1}^n \int_0^\tau (f_j(\mathbf{Z}_i))^2 dN_i(s) \text{ and } \hat{R}_n(\theta_k) = \frac{1}{n} \sum_{i=1}^n \int_0^\tau (\theta_k(s))^2 dN_i(s).$$

The additional terms are defined in Theorem 4.1.

Remark 1.3. *The general Lasso estimator for β is classically defined by*

$$\hat{\beta}_L = \arg \min_{\beta \in \mathbb{R}^M} \{C_n(\lambda_\beta) + \Gamma \sum_{j=1}^M |\beta_j|\},$$

with $\Gamma > 0$ a smoothing parameter. Usually, Γ is of order $\sqrt{\log M/n}$ (see Massart and Meynet [22] for the usual additive regression model and Antoniadis et al. [3] for the Cox model among other). The Lasso penalization for β corresponds to the simple choice $\omega_j = \Gamma$ where $\Gamma > 0$ is a smoothing parameter. Our weights could be compared with those of Bickel and al. [6] in the case of an additive regression model with a gaussian noise. They have considered a weighted Lasso with a penalty term of the form $\Gamma \sum_{j=1}^M \|f_j\|_n |\beta_j|$, with Γ of order $\sqrt{\log M/n}$ and $\|\cdot\|_n$ the usual empirical norm. We can deduce from the weights ω_j defined by (5) higher suitable weights that can be written $\Gamma_{n,M}^1 \tilde{\omega}_j$ with $\tilde{\omega}_j = \sqrt{\hat{V}_n(f_j)}$ and

$$\Gamma_{n,M}^1 = c_{1,\varepsilon} \sqrt{\frac{x + \log M + \max_{1 \leq j \leq M} \hat{\ell}_{n,x}(f_j)}{n}} + c_{2,\varepsilon} \frac{x + 1 + \log M + \max_{1 \leq j \leq M} \hat{\ell}_{n,x}(f_j)}{n} \max_{1 \leq j \leq M} \frac{\|f_j\|_{n,\infty}}{\sqrt{\hat{V}_n(f_j)}}.$$

The regularization parameter $\Gamma_{n,M}^1$ is still of order $\sqrt{\log M/n}$. The weights $\tilde{\omega}_j$ correspond to the estimation of the weighted empirical norm $\|\cdot\|_{n,\Lambda}$ that is not observable and play the same role than the empirical norm $\|f_j\|_n$ in Bickel et al. [6]. These weights are also of the same form as those of van de Geer [32] for the logistic model.

The idea of adding some weights in the penalization comes from the adaptive Lasso, although it is not the same procedure. Indeed, in the adaptive Lasso (see Zou [38]) one chooses $\omega_j = |\tilde{\beta}_j|^{-a}$ where $\tilde{\beta}_j$ is a preliminary estimator and $a > 0$ a constant. The idea behind this is to correct the bias of the Lasso in terms of variables selection accuracy (see Zou [38] and Zhang [36] for regression analysis and Zhang and Lu [35] for the Cox model). The weights ω_j can also be used to scale each variable at the same level, which is suitable when some variables have a large variance compared to the others.

2 Oracle inequalities for the Cox model when the baseline hazard function is known

As a first step, we suppose that the conditional hazard rate function satisfies the generalization of the Cox model (2) with a known baseline function α_0 . In this context, only f_0 has to be estimated and λ_0 is estimated by

$$\lambda_{\hat{\beta}_L}(t, \mathbf{Z}_i) = \alpha_0(t) e^{f_{\hat{\beta}_L}(\mathbf{Z}_i)} \quad \text{and} \quad \hat{\beta}_L = \arg \min_{\beta \in \mathbb{R}^M} \{C_n(\lambda_\beta) + \text{pen}(\beta)\}. \quad (7)$$

In this section, we state non-asymptotic oracle inequalities for the prediction loss of the Lasso in terms of the Kullback divergence. These inequalities allow us to compare the prediction error of the estimator and the best approximation of the regression function by a linear combination of the functions of the dictionary in a non-asymptotic way.

2.1 A slow oracle inequality

In the following theorem, we state an oracle inequality in the Cox model with slow rate of convergence, i.e. with a rate of convergence of order $\sqrt{\log M/n}$. This inequality is obtained under a very light assumption on the dictionary \mathbb{F}_M .

Proposition 2.1. *Consider Model (2) with known α_0 . Let $x > 0$ be fixed, ω_j be defined by (5) and for $\beta \in \mathbb{R}^M$,*

$$\text{pen}(\beta) = \sum_{j=1}^M \omega_j |\beta_j|.$$

Let A_{ε, c_ℓ} be some numerical positive constant depending only on ε and c_ℓ , and $x > 0$ be fixed. Under Assumption **A₃**, with a probability larger than $1 - A_{\varepsilon, c_\ell} e^{-x}$, then

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L}) \leq \inf_{\beta \in \mathbb{R}^M} \left(\widetilde{K}_n(\lambda_0, \lambda_\beta) + 2 \text{pen}(\beta) \right). \quad (8)$$

This theorem states a non-asymptotic oracle inequality in prediction on the conditional hazard rate function in the Cox model. The ω_j are the order of $\sqrt{\log M/n}$ and the penalty term is of order $\|\beta\|_1 \sqrt{\log M/n}$. This variance order is usually referred as a slow rate of convergence in high dimension (see Bickel et al. [6] for the additive regression model, Bertin et al. [5] and Bunea et al. [12] for density estimation).

2.2 A fast oracle inequality

Now, we are interested in obtaining a non-asymptotic oracle inequality with a fast rate of convergence of order $\log M/n$ and we need further assumptions in order to prove such result. In this subsection, we shall work locally, for $\mu > 0$, on the set $\Gamma_M(\mu) = \{\beta \in \mathbb{R}^M : \|\log \lambda_\beta - \log \lambda_0\|_{n, \infty} \leq \mu\}$, simply denoted $\Gamma(\mu)$ to simplify the notations and we consider the following assumption :

$$\text{There exists } \mu > 0, \text{ such that } \Gamma(\mu) \text{ contains a non-empty open set of } \mathbb{R}^M. \quad (\mathbf{A}_5)$$

This assumption has already been considered by van de Geer [32] or Kong and Nan [19]. Roughly speaking, it means that one can find a set where we can restrict our attention for finding good estimator of f_0 . The weighted Lasso estimator becomes

$$\hat{\beta}_L^\mu = \arg \min_{\beta \in \Gamma(\mu)} \{C_n(\lambda_\beta) + \text{pen}(\beta)\}. \quad (9)$$

To achieve a fast rate of convergence, one needs an additional assumption on the Gram matrix. We choose to work under the Restricted Eigenvalue condition, introduced in Bickel et al. [6] for the additive regression model. This condition is one of the weakest assumption on the design matrix. See Bühlmann and van de Geer [9] and Bickel et al. [6] for further details on assumptions required for oracle inequalities.

Let us first introduce further notations :

$$\Delta = \mathbf{D}(\hat{\beta}_L^\mu - \beta) \text{ with } \beta \in \Gamma(\mu) \text{ and } \mathbf{D} = (\text{diag}(\omega_j))_{1 \leq j \leq M},$$

$$\mathbf{X} = (f_j(\mathbf{Z}_i))_{i,j}, \text{ with } i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, M\},$$

$$\mathbf{G}_n = \frac{1}{n} \mathbf{X}^T \mathbf{C} \mathbf{X} \text{ with } \mathbf{C} = (\text{diag}(\Lambda_i(\tau)))_{1 \leq i \leq n}. \quad (10)$$

In the matrix \mathbf{G}_n , the covariates of individual i is re-weighted by its cumulative risk $\Lambda_i(\tau)$, which is consistent with the definition of the empirical norm in (4). Let also $J(\beta)$ be the sparsity set of vector $\beta \in \Gamma(\mu)$ defined by $J(\beta) = \{j \in \{1, \dots, M\} : \beta_j \neq 0\}$, and the sparsity index is then given by $|J(\beta)| = \text{Card}\{J(\beta)\}$. For $J \subset \{1, \dots, M\}$, we denote by β_J the vector β restricted to the set J : $(\beta_J)_j = \beta_j$ if $j \in J$ and $(\beta_J)_j = 0$ if $j \in J^c$ where $J^c = \{1, \dots, M\} \setminus J$.

In order to obtain a fast oracle inequality, we need to assume a Restricted Eigenvalue condition on the Gram matrix \mathbf{G}_n . However, since \mathbf{G}_n is random, we impose the Restricted Eigenvalue condition to $\mathbb{E}(\mathbf{G}_n)$, where the expectation is taken conditionally to the covariates.

For some integer $s \in \{1, \dots, M\}$ and a constant $a_0 > 0$, the following condition holds :

$$0 < \kappa_0(s, a_0) = \min_{\substack{J \subset \{1, \dots, M\}, \\ |J| \leq s}} \min_{\substack{\mathbf{b} \in \mathbb{R}^M \setminus \{0\}, \\ \|\mathbf{b}_{J^c}\|_1 \leq a_0 \|\mathbf{b}_J\|_1}} \frac{(b^T \mathbb{E}(\mathbf{G}_n) b)^{1/2}}{\|\mathbf{b}_J\|_2}. \quad (\mathbf{RE}(s, \mathbf{a}_0))$$

The integer s here plays the role of an upper bound on the sparsity $|J(\beta)|$ of a vector of coefficients β .

The following lemma implies that if the Restricted Eigenvalue condition is verified for $\mathbb{E}(\mathbf{G}_n)$, then the empirical version of the Restricted Eigenvalue condition applied to \mathbf{G}_n holds true with large probability.

Lemma 2.2. *Let $L > 0$ such that $\max_{1 \leq j \leq M} \max_{1 \leq i \leq n} |f_j(\mathbf{Z}_i)| \leq L$. Under Assumptions **A₂** and **RE(s, a₀)**, we have*

$$0 < \kappa = \min_{\substack{J \subset \{1, \dots, M\}, \\ |J| \leq s}} \min_{\substack{\mathbf{b} \in \mathbb{R}^M \setminus \{0\}, \\ \|\mathbf{b}_{J^c}\|_1 \leq a_0 \|\mathbf{b}_J\|_1}} \frac{(\mathbf{b}^T \mathbf{G}_n \mathbf{b})^{1/2}}{\|\mathbf{b}_J\|_2} \text{ and } \kappa = (1/\sqrt{2A_0}) \kappa_0(s, a_0), \quad (11)$$

with probability larger than $1 - \pi_n$, where

$$\pi_n = \exp\left(-\frac{n\kappa^4}{2L^2(1+a_0)^2 s(L^2(1+a_0)^2 s + \kappa^2)}\right).$$

Thanks to Lemma 2.2, the empirical Restricted Eigenvalue condition will be fulfilled on an event of large probability, on which we establish a fast non-asymptotic oracle inequality.

Theorem 2.3. Consider Model (2) with known α_0 and for $x > 0$, let ω_j be defined by (5) and $\hat{\beta}_L^\mu$ be defined by (9). Let $A_{\varepsilon, c_\ell} > 0$ be a numerical positive constant only depending on ε and c_ℓ , $\zeta > 0$ and $s \in \{1, \dots, M\}$ be fixed. Let Assumptions **A₃**, **A₅** and **RE(s, a₀)** be satisfied with $a_0 = (3 + 4/\zeta)$ and let $\kappa = (1/\sqrt{2A_0})\kappa_0(s, a_0)$. Then, with a probability larger than $1 - A_{\varepsilon, c_\ell}e^{-x} - \pi_n$, the following inequality holds

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) \leq (1 + \zeta) \inf_{\substack{\beta \in \Gamma(\mu) \\ |J(\beta)| \leq s}} \left\{ \widetilde{K}_n(\lambda_0, \lambda_\beta) + C(\zeta, \mu) \frac{|J(\beta)|}{\kappa^2} \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \right\}, \quad (12)$$

where $C(\zeta, \mu) > 0$ is a constant depending on ζ and μ .

This result permits to compare the prediction error of the estimator and the best sparse approximation of the regression function by an oracle that knows the truth, but is constrained by sparsity. The Lasso estimator approaches the best approximation in the dictionary with a fast error term of order $\log M/n$.

Thanks to Proposition 5.1, which states a connection between the empirical Kullback divergence (3) and the weighted empirical quadratic norm (4), we deduce from Theorem 2.3 a non-asymptotic oracle inequality in weighted empirical quadratic norm.

Corollary 2.4. Under the assumptions of Theorem 2.3, with a probability larger than $1 - A_{\varepsilon, c_\ell}e^{-x} - \pi_n$,

$$\|\log \lambda_{\hat{\beta}_L^\mu} - \log \lambda_0\|_{n, \Lambda}^2 \leq (1 + \zeta) \inf_{\substack{\beta \in \Gamma(\mu) \\ |J(\beta)| \leq s}} \left\{ \|\log \lambda_\beta - \log \lambda_0\|_{n, \Lambda}^2 + C'(\zeta, \mu) \frac{|J(\beta)|}{\kappa^2} \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \right\},$$

where $C'(\zeta, \mu)$ is a positive constant depending on ζ and μ .

Note that for α_0 supposed to be known, this oracle inequality is also equivalent to

$$\|f_{\hat{\beta}_L^\mu} - f_0\|_{n, \Lambda}^2 \leq (1 + \zeta) \inf_{\substack{\beta \in \Gamma(\mu) \\ |J(\beta)| \leq s}} \left\{ \|f_\beta - f_0\|_{n, \Lambda}^2 + C'(\zeta, \mu) \frac{|J(\beta)|}{\kappa^2} \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \right\}.$$

2.3 Particular case : variable selection in the Cox model

We now consider the case of variable selection in the Cox model (2) with $f_0(Z_i) = \beta_0^T Z_i$. In this case, $M = p$ and the functions of the dictionary are such that for $i = 1, \dots, n$ and $j = 1, \dots, p$

$$f_j(\mathbf{Z}_i) = Z_{i,j} \text{ and } \hat{E}f_\beta(\mathbf{Z}_i) = \sum_{j=1}^p \beta_j Z_{i,j} = \beta^T \mathbf{Z}_i.$$

Let $\mathbf{X} = (Z_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$ be the design matrix and for $\hat{\beta}_L$ defined by (7), let

$$\Delta_0 = \mathbf{D}(\hat{\beta}_L - \beta_0), \mathbf{D} = (\text{diag}(\omega_j))_{1 \leq j \leq M}, J_0 = J(\beta_0) \text{ and } |J_0| = \text{Card}\{J_0\}.$$

We now state non-asymptotic inequalities for prediction on $\mathbf{X}\beta_0$ and for estimation on β_0 . In this subsection, we don't need to work locally on the set $\Gamma(\mu)$ to obtain Proposition 5.2 and instead of considering Assumption (A₅), we only have to introduce the following assumption to connect the empirical Kullback divergence and the weighted empirical quadratic norm :

$$\text{Let } R \text{ be a positive constant, such that } \max_{i \in \{1, \dots, n\}} \|Z_i\|_2 \leq R. \quad (\mathbf{A}_6)$$

We consider the Lasso estimator defined with the regularization parameter $\Gamma_1 > 0$:

$$\hat{\beta}_L = \arg \min_{\beta \in \mathbb{R}^p} \{C_n(\lambda_\beta) + \Gamma_1 \sum_{j=1}^p \omega_j |\beta_j|\},$$

Theorem 2.5. *Consider Model (1) with known α_0 . For $x > 0$, let ω_j be defined by (5) and denote $\kappa' = (1/\sqrt{2A_0})\kappa_0(s, 3)$. Let A_{ε, c_ℓ} be some numerical positive constant depending on ε and c_ℓ . Under Assumptions A₃, A₆ and RE(s, a₀) with $a_0 = 3$, for all Γ_1 such that*

$$\Gamma_1 \leq \frac{1}{48Rs} \frac{\min_{1 \leq j \leq M} \omega_j^2}{\max_{1 \leq j \leq M} \omega_j^2} \frac{\kappa'^2}{\max_{1 \leq j \leq M} \omega_j},$$

with a probability larger than $1 - A_{\varepsilon, c_\ell} e^{-\Gamma_1 x} - \pi_n$, then

$$\|\mathbf{X}(\hat{\beta}_L - \beta_0)\|_{n, \Lambda}^2 \leq \frac{4}{\xi^2} \frac{|J_0|}{\kappa'^2} \Gamma_1^2 (\max_{1 \leq j \leq p} \omega_j)^2 \quad (13)$$

and

$$\|\hat{\beta}_L - \beta_0\|_1 \leq 8 \frac{\max_{1 \leq j \leq p} \omega_j}{\min_{1 \leq j \leq p} \omega_j} \frac{|J_0|}{\xi \kappa'^2} \Gamma_1 \max_{1 \leq j \leq p} \omega_j. \quad (14)$$

This theorem gives non-asymptotic upper bounds for two types of loss functions. Inequality (13) gives a non-asymptotic bound on prediction loss with a rate of convergence in $\log M/n$, while Inequality (14) states a bound on $\hat{\beta}_L - \beta_0$.

3 Oracle inequalities for general conditional hazard rate

In the previous section, we have assumed α_0 known and have obtained results on the relative risk. Now, we consider a general conditional hazard rate function λ_0 that does not rely on an underlying model. Oracle inequalities are established under different assumptions with slow and fast rates of convergence.

3.1 A slow oracle inequality

The slow oracle inequality for a general conditional hazard rate is obtained under light assumptions that concern only the construction of the two dictionaries \mathbb{F}_M and \mathbb{G}_N .

Theorem 3.1. *For $x > 0$ and $y > 0$, let ω_j and δ_k be defined by (5) and (6) respectively and $(\hat{\beta}_L, \hat{\gamma}_L)$ be defined in Estimation procedure 1.2. Let A_{ε, c_ℓ} and $B_{\varepsilon', c'_\ell} > 0$ be two positive numerical constants depending on ε, c_ℓ and ε', c'_ℓ respectively and Assumptions **A₃**, **A₄** be satisfied. Then, with probability larger than $1 - A_{\varepsilon, c_\ell} e^{-x} - B_{\varepsilon', c'_\ell} e^{-y}$*

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L, \hat{\gamma}_L}) \leq \inf_{(\beta, \gamma) \in \mathbb{R}^M \times \mathbb{R}^N} \{ \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + 2 \text{pen}(\beta) + 2 \text{pen}(\gamma) \}. \quad (15)$$

We have chosen to estimate the complete conditional hazard rate function, which involves two different parts : the first part is the baseline function $\alpha_\gamma : \mathbb{R} \rightarrow \mathbb{R}$ and the second part is the function of the covariates $f_\beta : \mathbb{R}^M \rightarrow \mathbb{R}$. The double ℓ_1 -penalization considered here is tuned to concurrently estimate the function f_0 depending on high-dimensional covariates and the non-parametric function α_0 . Examples of Lasso algorithms for the estimation of non-parametric density or intensity may be found in Bertin et al. [5] and Hansen et al. [18] respectively. As f_0 and α_0 are estimated at once, the resulting rate of convergence is the sum of the two expected rates in both situations considered separately ($\sim \sqrt{\log M/n} + \sqrt{\log N/n}$). Nevertheless, from Bertin et al. [5], we expect that a choice of N of order n would suitably estimate α_0 . As a consequence, in a very high-dimensional setting the leading error term in (15) would be of order $\sqrt{\log M/n}$, which again is the classical slow rate of convergence in a regression setting.

3.2 A fast oracle inequality

We are now interested in obtaining the fast non-asymptotic oracle inequality and as usual, we need to introduce further notations and assumptions. In this subsection, we shall again work locally for $\rho > 0$ on the set $\widetilde{\Gamma}_{M,N}(\rho) = \{(\beta, \gamma) \in \mathbb{R}^M \times \mathbb{R}^N : \|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty} \leq \rho\}$, simply denoted $\widetilde{\Gamma}(\rho)$ and we consider the following assumption :

$$\text{There exists } \rho > 0, \text{ such that } \widetilde{\Gamma}(\rho) \text{ contains a non-empty open set of } \mathbb{R}^M \times \mathbb{R}^N. \quad (\mathbf{A}_7)$$

On $\widetilde{\Gamma}(\rho)$, we define the weighted Lasso estimator as

$$(\hat{\beta}_L^\rho, \hat{\gamma}_L^\rho) = \arg \min_{(\beta, \gamma) \in \widetilde{\Gamma}(\rho)} \{ C_n(\lambda_{\beta, \gamma}) + \text{pen}(\beta) + \text{pen}(\gamma) \}.$$

Let us give the additional notations. Set $\tilde{\Delta}$ be

$$\tilde{\Delta} = \tilde{D} \begin{pmatrix} \hat{\beta}_L - \beta \\ \hat{\gamma}_L - \gamma \end{pmatrix} \in \mathbb{R}^{M+N} \text{ with } (\beta, \gamma) \in \widetilde{\Gamma}(\rho) \text{ and } \tilde{D} = \text{diag}(\omega_1, \dots, \omega_M, \delta_1, \dots, \delta_N).$$

Let $\mathbf{1}_{n \times N}$ be the matrix $n \times N$ with all coefficients equal to one,

$$\tilde{\mathbf{X}}(t) = \begin{bmatrix} (f_j(\mathbf{Z}_i))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq M}} & \mathbf{1}_{n \times N}(\text{diag}(\theta_k(t)))_{1 \leq k \leq N} \end{bmatrix} = \begin{bmatrix} \mathbf{X} & \begin{bmatrix} \theta_1(t) & \dots & \theta_N(t) \\ \vdots & & \vdots \\ \theta_1(t) & \dots & \theta_N(t) \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{n \times (M+N)}$$

and

$$\tilde{\mathbf{G}}_n = \frac{1}{n} \int_0^\tau \tilde{\mathbf{X}}(t)^T \tilde{\mathbf{C}}(t) \tilde{\mathbf{X}}(t) dt \text{ with } \tilde{\mathbf{C}}(t) = (\text{diag}(\lambda_0(t, \mathbf{Z}_i) Y_i(t)))_{1 \leq i \leq n}, \forall t \geq 0.$$

Let also $J(\boldsymbol{\beta})$ and $J(\boldsymbol{\gamma})$ be the sparsity sets of vectors $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \tilde{\Gamma}(\rho)$ respectively defined by

$$J(\boldsymbol{\beta}) = \{j \in \{1, \dots, M\} : \beta_j \neq 0\} \text{ and } J(\boldsymbol{\gamma}) = \{k \in \{1, \dots, N\} : \gamma_k \neq 0\},$$

and the sparsity indexes are then given by

$$|J(\boldsymbol{\beta})| = \sum_{j=1}^M \mathbf{1}_{\{\beta_j \neq 0\}} = \text{Card}\{J(\boldsymbol{\beta})\} \text{ and } |J(\boldsymbol{\gamma})| = \sum_{k=1}^N \mathbf{1}_{\{\gamma_k \neq 0\}} = \text{Card}\{J(\boldsymbol{\gamma})\}.$$

To obtain the fast non-asymptotic oracle inequality, we consider the Restricted Eigenvalue condition applied to the matrix $\mathbb{E}(\tilde{\mathbf{G}}_n)$.

For some integer $s \in \{1, \dots, M\}$ and a constant $r_0 > 0$, we assume that $\tilde{\mathbf{G}}_n$ satisfies :

$$0 < \tilde{\boldsymbol{\kappa}}_0(s, r_0) = \min_{\substack{J \subset \{1, \dots, M+N\}, \\ |J| \leq s}} \min_{\substack{\mathbf{b} \in \mathbb{R}^{M+N} \setminus \{0\}, \\ \|\mathbf{b}_{J^c}\|_1 \leq r_0 \|\mathbf{b}_J\|_1}} \frac{(\mathbf{b}^T \mathbb{E}(\tilde{\mathbf{G}}_n) \mathbf{b})^{1/2}}{\|\mathbf{b}_J\|_2}. \quad (\widetilde{\mathbf{RE}}(\mathbf{s}, \mathbf{r}_0))$$

The condition on the matrix $\mathbb{E}(\tilde{\mathbf{G}}_n)$ is rather strong because the block matrix involves both functions of the covariates of \mathbb{F}_M and functions of time which belong to \mathbb{G}_N . This is the price to pay for an oracle inequality on the full conditional hazard rate function. If we had instead considered two restricted eigenvalue assumptions on each block, we would have established an oracle inequality on the sum of the two unknown parameters α_0 and f_0 and not on λ_0 . As in Lemma 2.2, we can show that under Assumption $\widetilde{\mathbf{RE}}(\mathbf{s}, \mathbf{r}_0)$, we have an empirical Restricted Eigenvalue condition on the matrix $\tilde{\mathbf{G}}_n$.

Lemma 3.2. *Let L defined as in Lemma 2.2. Under Assumptions \mathbf{A}_2 and $\widetilde{\mathbf{RE}}(\mathbf{s}, \mathbf{r}_0)$, we have*

$$0 < \tilde{\boldsymbol{\kappa}} = \min_{\substack{J \subset \{1, \dots, M\}, \\ |J| \leq s}} \min_{\substack{\mathbf{b} \in \mathbb{R}^M \setminus \{0\}, \\ \|\mathbf{b}_{J^c}\|_1 \leq r_0 \|\mathbf{b}_J\|_1}} \frac{(\mathbf{b}^T \tilde{\mathbf{G}}_n \mathbf{b})^{1/2}}{\|\mathbf{b}_J\|_2} \text{ and } \tilde{\boldsymbol{\kappa}} = (1/\sqrt{2A_0}) \tilde{\boldsymbol{\kappa}}_0(s, r_0), \quad (16)$$

with probability larger than $1 - \tilde{\pi}_n$, where

$$\tilde{\pi}_n = \exp\left(-\frac{n\tilde{\boldsymbol{\kappa}}^4}{2L^2(1+r_0)^2s(L^2(1+r_0)^2s + \tilde{\boldsymbol{\kappa}}^2)}\right).$$

Theorem 3.3. For $x > 0$ and $y > 0$, let ω_j and δ_k be defined by (5) and (6) respectively. Let $A_{\varepsilon, c_\ell} > 0$ and $B_{\varepsilon', c'_\ell} > 0$ be two numerical positive constants depending on ε, c_ℓ and ε', c'_ℓ respectively, $\zeta > 0$ and $s \in \{1, \dots, M + N\}$ be fixed. Let Assumptions **A₃**, **A₄**, **A₇** and **RE**(**s**, **r₀**) be satisfied with

$$r_0 = \left(3 + 8 \max \left(\sqrt{|J(\boldsymbol{\beta})|}, \sqrt{|J(\boldsymbol{\gamma})|} \right) / \zeta \right),$$

and let $\tilde{\boldsymbol{\kappa}} = (1/\sqrt{2A_0})\tilde{\boldsymbol{\kappa}}_0(s, r_0)$. Then, with probability larger than $1 - A_{\varepsilon, c_\ell}e^{-x} - B_{\varepsilon', c'_\ell}e^{-y} - \tilde{\pi}_n$

$$\tilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^p, \hat{\gamma}_L^p}) \leq (1 + \zeta) \inf_{\substack{(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \tilde{\Gamma}(\rho) \\ \max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|) \leq s}} \left\{ \tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}) + \tilde{C}(\zeta, \rho) \frac{\max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|)}{\tilde{\boldsymbol{\kappa}}^2} \max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j^2, \delta_k^2\} \right\}, \quad (17)$$

and

$$\begin{aligned} & \|\log \lambda_0 - \log \lambda_{\hat{\beta}_L^p, \hat{\gamma}_L^p}\|_{n, \Lambda}^2 \\ & \leq (1 + \zeta) \inf_{\substack{(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \tilde{\Gamma}(\rho) \\ \max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|) \leq s}} \left\{ \|\log \lambda_0 - \log \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}\|_{n, \Lambda}^2 + \tilde{C}'(\zeta, \rho) \frac{\max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|)}{\tilde{\boldsymbol{\kappa}}^2} \max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j^2, \delta_k^2\} \right\}, \end{aligned} \quad (18)$$

where $\tilde{C}(\zeta, \rho) > 0$ and $\tilde{C}'(\zeta, \rho) > 0$ are constants depending only on ζ and ρ .

We obtain a non-asymptotic fast oracle inequality in prediction. Indeed, the rate of convergence of this oracle inequality is of order

$$\left(\max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \right)^2 \approx \max \left\{ \frac{\log M}{n}, \frac{\log N}{n} \right\},$$

namely, if we choose \mathbb{G}_N of size n , the rate of convergence of this oracle inequality is then of order $\log M/n$ (see Subsection 3.1 for more details). While Algorithm 1.2 allows to derive a prediction for the survival time through the conditional intensity, Theorem 3.3 measures the accuracy of this prediction. In that sense, the clinical problem of establishing a prognosis has been addressed at this point. To our best knowledge, this oracle inequality is the first non-asymptotic oracle inequality in prediction for the whole conditional hazard rate function with a fast rate of convergence of order $\log M/n$. This result should be compared to the non-asymptotic oracle inequality for the Lasso estimator of f_0 established recently by Kong and Nan [19]. The first difference is that we don't have the same goal : their result concerns f_0 whereas we are interested in estimating the whole hazard function, and so we don't use the same criteria. Indeed, Kong and Nan [19] consider the log partial likelihood function, whereas our criterion is the total log-likelihood to obtain a result on the whole hazard function. The second difference is that their rate of convergence is of order $K_M^2 \log(2M)/n$ with $K_M := \max\{\|f_j\|_{n, \infty} / [\mathbb{E}f_j^2(\mathbf{Z})]^{1/2}, j = 1, \dots, M\}$, that is to say lower than our. Finally, the confidence (probability that the result holds true) of Inequality (12) does not depend on M or n , whereas in Kong and Nan [19] the confidence of their result depends on n and M .

The remaining part of the paper is devoted to the technical results and proofs

4 An empirical Bernstein's inequality

The main ingredient of Theorems 2.1, 2.3, 3.1 and 3.3 are Bernstein's concentration inequalities that we present in this section. To clarify the relation between the stated oracle inequalities and the Bernstein's inequality, we sketch here the proof of Theorem 3.1. Using the Doob-Meier decomposition $N_i = M_i + \Lambda_i$, we can easily show that for all $\beta \in \mathbb{R}^M$ and for all $\gamma \in \mathbb{R}^N$

$$C_n(\lambda_{\hat{\beta}_L, \hat{\gamma}_L}) - C_n(\lambda_{\beta, \gamma}) = \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L, \hat{\gamma}_L}) - \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + (\hat{\gamma}_L - \gamma)^T \nu_{n, \tau} + (\hat{\beta}_L - \beta)^T \eta_{n, \tau}, \quad (19)$$

where

$$\eta_{n, \tau} = \frac{1}{n} \sum_{i=1}^n \int_0^\tau \vec{f}(\mathbf{Z}_i) dM_i(t) \text{ and } \nu_{n, \tau} = \frac{1}{n} \sum_{i=1}^n \int_0^\tau \vec{\theta}(t) dM_i(t), \quad (20)$$

with $\vec{f} = (f_1, \dots, f_M)^T$ and $\vec{\theta} = (\theta_1, \dots, \theta_N)^T$. By definition of the Lasso estimator, we have for all (β, γ) in $\mathbb{R}^M \times \mathbb{R}^N$

$$C_n(\lambda_{\hat{\beta}_L, \hat{\gamma}_L}) + \text{pen}(\hat{\beta}_L) + \text{pen}(\hat{\gamma}_L) \leq C_n(\lambda_{\beta, \gamma}) + \text{pen}(\beta) + \text{pen}(\gamma),$$

and we finally obtain

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L, \hat{\gamma}_L}) \leq \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + (\hat{\gamma}_L - \gamma)^T \nu_{n, \tau} + (\hat{\beta}_L - \beta)^T \eta_{n, \tau} + \text{pen}(\beta) - \text{pen}(\hat{\beta}_L) + \text{pen}(\gamma) - \text{pen}(\hat{\gamma}_L).$$

Consequently, $\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L, \hat{\gamma}_L})$ is bounded by

$$\widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + \sum_{j=1}^M (\hat{\beta}_{L,j} - \beta_j) \eta_{n, \tau}(f_j) + \sum_{j=1}^M \omega_j (|\beta_j| - |\hat{\beta}_{L,j}|) + \sum_{k=1}^N (\hat{\gamma}_{L,k} - \gamma_k)^T \nu_{n, \tau}(\theta_k) + \sum_{k=1}^N \delta_k (|\gamma_k| - |\hat{\gamma}_{L,k}|),$$

with

$$\eta_{n, t}(f_j) = \frac{1}{n} \sum_{i=1}^n \int_0^t f_j(\mathbf{Z}_i) dM_i(s) \text{ and } \nu_{n, t}(\theta_k) = \frac{1}{n} \sum_{i=1}^n \int_0^t \theta_k(s) dM_i(s).$$

We will control $\eta_{n, t}(f_j)$ and $\nu_{n, t}(\theta_k)$ respectively by ω_j and δ_k . More precisely, the weights ω_j (respectively δ_k) will be chosen such that $|\eta_{n, t}(f_j)| \leq \omega_j$ (respectively $|\nu_{n, t}(\theta_k)| \leq \delta_k$) and $\mathbb{P}(|\eta_{n, t}(f_j)| > \omega_j)$ (respectively $\mathbb{P}(|\nu_{n, t}(\theta_k)| > \delta_k)$) large. As $\eta_{n, t}(f_j)$ and $\nu_{n, t}(\theta_k)$ involve martingales, we could directly apply classical Bernstein's inequalities for martingales with $x > 0$ and $y > 0$

$$\mathbb{P}\left[\eta_{n, t}(f_j) \geq \sqrt{\frac{2V_{n, t}(f_j)x}{n}} + \frac{x}{3n}\right] \leq e^{-x} \text{ and } \mathbb{P}\left[\nu_{n, t}(\theta_k) \geq \sqrt{\frac{2R_{n, t}(\theta_k)y}{n}} + \frac{y}{3n}\right] \leq e^{-y},$$

where the predictable variations $V_{n,t}(f_j)$ and $R_{n,t}(\theta_k)$ of $\eta_{n,t}(f_j)$ and $\nu_{n,t}(\theta_k)$ are respectively defined by

$$V_{n,t}(f_j) = n \langle \eta_n(f_j) \rangle_t = \frac{1}{n} \sum_{i=1}^n \int_0^t (f_j(\mathbf{Z}_i))^2 \lambda_0(t, \mathbf{Z}_i) Y_i(s) ds,$$

$$R_{n,t}(\theta_k) = n \langle \nu_n(\theta_k) \rangle_t = \frac{1}{n} \sum_{i=1}^n \int_0^t (\theta_k(t))^2 \lambda_0(t, \mathbf{Z}_i) Y_i(s) ds,$$

see e.g. van de Geer [31]. Applying these inequalities, the weights of Algorithm 1.2 would have the forms $\omega_j = \sqrt{2V_{n,t}(f_j)x/n + x/3n}$ and $\delta_k = \sqrt{2R_{n,t}(\theta_k)y/n + y/3n}$. As $V_{n,t}(f_j)$ and $R_{n,t}(\theta_k)$ both depend on λ_0 , this would not result a statistical procedure. We propose to replace in the Bernstein's inequality the predictable variations by the optional variations of the processes $\eta_{n,t}(f_j)$ and $\nu_{n,t}(\theta_k)$ defined by

$$\hat{V}_{n,t}(f_j) = n[\eta_n(f_j)]_t = \frac{1}{n} \sum_{i=1}^n \int_0^t (f_j(\mathbf{Z}_i))^2 dN_i(s) \text{ and } \hat{R}_{n,t}(\theta_k) = n[\nu_n(\theta_k)]_t = \frac{1}{n} \sum_{i=1}^n \int_0^t (\theta_k(t))^2 dN_i(s).$$

This ensures that the weights ω_j and δ_k will depends on $\hat{V}_{n,t}(f_j)$ and $\hat{R}_{n,t}(\theta_k)$ respectively. Equivalent strategies in different models have been considered in Gaïffas and Guillaou [16] or Hansen et al. [18]. The following theorem states the resulting Bernstein's inequalities.

Theorem 4.1. *For any numerical constant $c_\ell > 1$, $c'_\ell > 1$, $\varepsilon > 0$, $\varepsilon' > 0$ and $c_0 > 0$, $c'_0 > 0$ such that $ec_0 > 2(4/3 + \varepsilon)c_\ell$ and $ec'_0 > 2(4/3 + \varepsilon')c'_\ell$, the following holds for any $x > 0$, $y > 0$:*

$$\mathbb{P} \left[|\eta_{n,t}(f_j)| \geq c_{1,\varepsilon} \sqrt{\frac{x + \hat{\ell}_{n,x}(f_j)}{n} \hat{V}_{n,t}(f_j)} + c_{2,\varepsilon} \frac{x + 1 + \hat{\ell}_{n,x}(f_j)}{n} \|f_j\|_{n,\infty} \right] \leq c_{3,\varepsilon,c_\ell} e^{-x}, \quad (21)$$

$$\mathbb{P} \left[|\nu_{n,t}(\theta_k)| \geq c'_{1,\varepsilon'} \sqrt{\frac{y + \hat{\ell}'_{n,y}(\theta_k)}{n} \hat{R}_{n,t}(\theta_k)} + c'_{2,\varepsilon'} \frac{y + 1 + \hat{\ell}'_{n,y}(\theta_k)}{n} \|\theta_k\|_{n,\infty} \right] \leq c'_{3,\varepsilon',c'_\ell} e^{-y}, \quad (22)$$

where

$$\hat{\ell}_{n,x}(f_j) = c_\ell \log \log \left(\frac{2en\hat{V}_{n,t}(f_j) + 8e(4/3 + \varepsilon)x\|f_j\|_{n,\infty}^2}{4(ec_0 - 2(4/3 + \varepsilon)c_\ell)\|f_j\|_{n,\infty}^2} \vee e \right), \quad \|f_j\|_{n,\infty} = \max_{i=1,\dots,n} |f_j(\mathbf{Z}_i)|,$$

$$\hat{\ell}'_{n,y}(\theta_k) = c'_\ell \log \log \left(\frac{2en\hat{R}_{n,t}(\theta_k) + 8e(4/3 + \varepsilon')y\|\theta_k\|_{n,\infty}^2}{4(ec'_0 - 2(4/3 + \varepsilon')c'_\ell)\|\theta_k\|_{n,\infty}^2} \vee e \right), \quad \|\theta_k\|_{n,\infty} = \max_{t \in [0,\tau]} |\theta_k(t)|,$$

and where

$$c_{1,\varepsilon} = 2\sqrt{1 + \varepsilon}, c_{2,\varepsilon} = 2\sqrt{2 \max(c_0, 2(1 + \varepsilon)(4/3 + \varepsilon))} + 2/3 \text{ and } c_{3,\varepsilon,c_\ell} = 8 + 6(\log(1 + \varepsilon))^{-c_\ell} \sum_{k \geq 1} k^{-c_\ell},$$

$$c'_{1,\varepsilon'} = 2\sqrt{1 + \varepsilon'}, c'_{2,\varepsilon'} = 2\sqrt{2 \max(c'_0, 2(1 + \varepsilon')(4/3 + \varepsilon'))} + 2/3 \text{ and } c'_{3,\varepsilon',c'_\ell} = 8 + 6(\log(1 + \varepsilon'))^{-c'_\ell} \sum_{k \geq 1} k^{-c'_\ell}.$$

We deduce the weights ω_j and δ_k defined in (5) and (6) respectively, from Theorem 4.1. These empirical Bernstein's inequalities are new in the Cox model and then are interesting per se.

Theorem 4.1 is closer to Theorem 3 in Gaïffas and Guillaux [16] than Theorem 3 in Hansen et al. [18], in the sense that we do not seek simpler weights by adding an assumption such that $\max_{1 \leq i \leq n} |N_i(\tau)| < \sigma < \infty$. Nevertheless in precise situations, such as censored data, our weights can be simplified. Indeed, in this case, for real numbers $(u, v) \in (0, 3)^2$ such that $u > \phi(u)$ and $v > \phi(v)$, where $\phi(z) = \exp(z) - z - 1$, and $c_{1,\varepsilon} = \sqrt{2(1 + \varepsilon)}$, we would get

$$\mathbb{P}\left[\eta_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{W}_n^u(f_j)x}{n}} + \frac{x}{3n} \|f_j\|_{n,\infty}\right] \leq 4 \left(\frac{\log(1 + u/x)}{\log(1 + \varepsilon)} + 1 \right) e^{-x}. \quad (23)$$

with

$$\hat{W}_n^u(f_j) = \frac{u/n}{u/n - \phi(u/n)} \hat{V}_{n,t}(f_j) + \frac{(x/n) \|f_j\|_{n,\infty}^2}{u/n - \phi(u/n)},$$

We deduce from (23) the following weights

$$\tilde{\omega}_j = c_{1,\varepsilon} \sqrt{\frac{\hat{W}_n^u(f_j)(x + \log M)}{n}} + 2 \frac{x + \log M}{3n} \|f_j\|_{n,\infty}.$$

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5 Proofs

5.1 Proof of Proposition 1.1

Following the proof of Theorem 1 in Senoussi [25], we rewrite the empirical Kullback divergence (3) as

$$\begin{aligned} \tilde{K}_n(\lambda_0, \lambda) &= \frac{1}{n} \sum_{i=1}^n \int_0^\tau \left[\log \lambda_0(t, \mathbf{Z}_i) - \log \lambda(t, \mathbf{Z}_i) - \left(1 - \frac{\lambda(t, \mathbf{Z}_i)}{\lambda_0(t, \mathbf{Z}_i)} \right) \right] \lambda_0(t, \mathbf{Z}_i) Y_i(t) dt \\ &= \frac{1}{n} \sum_{i=1}^n \int_0^\tau \left[\exp \left(\log \frac{\lambda(t, \mathbf{Z}_i)}{\lambda_0(t, \mathbf{Z}_i)} \right) - \log \frac{\lambda(t, \mathbf{Z}_i)}{\lambda_0(t, \mathbf{Z}_i)} - 1 \right] \lambda_0(t, \mathbf{Z}_i) Y_i(t) dt. \end{aligned}$$

Since the map $t \rightarrow e^t - t - 1$ is a positive function on \mathbb{R} , we deduce that except for $\lambda = \lambda_0$,

$$\exp \left(\log \frac{\lambda(t, \mathbf{Z}_i)}{\lambda_0(t, \mathbf{Z}_i)} \right) - \log \frac{\lambda(t, \mathbf{Z}_i)}{\lambda_0(t, \mathbf{Z}_i)} - 1 > 0.$$

Thus $\tilde{K}_n(\lambda_0, \lambda)$ is positive and vanishes only if $(\log \lambda_0 - \log \lambda)(t, \mathbf{Z}_i) = 0$ almost surely, namely if $\lambda_0 = \lambda$ almost surely. \square

5.2 Proof of Proposition 2.1

According to the definition (9) of $\hat{\beta}_L$, for all β in \mathbb{R}^M , we have

$$C_n(\lambda_{\hat{\beta}_L}) + \text{pen}(\hat{\beta}_L) \leq C_n(\lambda_\beta) + \text{pen}(\beta).$$

Here α_0 is assumed to be known. Hence applying (19), we obtain

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L}) \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + (\hat{\beta}_L - \beta)^T \boldsymbol{\eta}_{n,\tau} + \text{pen}(\beta) - \text{pen}(\hat{\beta}_L). \quad (24)$$

It remains to control the term $(\hat{\beta}_L - \beta)^T \boldsymbol{\eta}_{n,\tau}$. For ω_j defined in (5), set

$$\mathcal{A} = \bigcap_{j=1}^M \left\{ |\eta_{n,\tau}(f_j)| \leq \frac{\omega_j}{2} \right\}.$$

On \mathcal{A} , we have

$$|(\hat{\beta}_L - \beta)^T \boldsymbol{\eta}_{n,\tau}| \leq \sum_{j=1}^M \frac{\omega_j}{2} |(\hat{\beta}_L - \beta)_j| \leq \sum_{j=1}^M \omega_j |(\hat{\beta}_L - \beta)_j|.$$

The result (8) follows since $\text{pen}(\beta) = \sum_{j=1}^M \omega_j |\beta_j|$. It remains to bound up $\mathbb{P}(\mathcal{A}^c)$. By applying Theorem 4.1

$$\mathbb{P}(\mathcal{A}^c) \leq \sum_{j=1}^M \mathbb{P}\left(|\eta_{n,\tau}(f_j)| > \frac{\omega_j}{2}\right) \leq c_{3,\varepsilon,c\ell} e^{-x}.$$

Consequently, by taking $A_{\varepsilon,c\ell} = c_{3,\varepsilon,c\ell}$, we conclude $\mathbb{P}(\mathcal{A}) \geq 1 - A_{\varepsilon,c\ell} e^{-x}$, which ends up the proof of Theorem 2.1. \square

5.3 Proof of Lemma 2.2

Let $J \subset \{1, \dots, M\}$ and \mathbf{b} in $\mathbb{R}^M \setminus \{0\}$ such that $|J| \leq s$ and $\|\mathbf{b}_{J^c}\|_1 \leq a_0 \|\mathbf{b}_J\|_1$. We show with high probability, that under $\mathbf{RE}(\mathbf{s}, \mathbf{a}_0)$,

$$\frac{\mathbf{b}^T \mathbf{G}_n \mathbf{b}}{\|\mathbf{b}_J\|_2^2} > \kappa^2, \quad \text{with } \kappa = (1/\sqrt{2A_0}) \kappa_0(s, a_0) \quad \text{and } A_0 \text{ defined in Assumption } \mathbf{A}_2.$$

Under Assumption $\mathbf{RE}(\mathbf{s}, \mathbf{a}_0)$, we can write

$$\frac{\mathbf{b}^T \mathbf{G}_n \mathbf{b}}{\|\mathbf{b}_J\|_2^2} = \frac{\mathbf{b}^T (\mathbf{G}_n - \mathbb{E}(\mathbf{G}_n)) \mathbf{b}}{\|\mathbf{b}_J\|_2^2} + \frac{\mathbf{b}^T \mathbb{E}(\mathbf{G}_n) \mathbf{b}}{\|\mathbf{b}_J\|_2^2} > \frac{\mathbf{b}^T (\mathbf{G}_n - \mathbb{E}(\mathbf{G}_n)) \mathbf{b}}{\|\mathbf{b}_J\|_2^2} + \kappa_0^2(s, a_0).$$

By definition of \mathbf{G}_n (see (10)) and recalling that $C = (\text{diag}(\Lambda_i(\tau)))_{i=1,\dots,n}$, we have

$$\frac{\mathbf{b}^T (\mathbf{G}_n - \mathbb{E}(\mathbf{G}_n)) \mathbf{b}}{\|\mathbf{b}_J\|_2^2} = \frac{1}{n} \frac{\mathbf{b}^T \mathbf{X}^T (C - \mathbb{E}(C)) \mathbf{X} \mathbf{b}}{\|\mathbf{b}_J\|_2^2} = \frac{1}{n} \frac{\sum_{i=1}^n \left(\sum_{j=1}^M b_j f_j(\mathbf{Z}_i) \right)^2 (\Lambda_i(\tau) - \mathbb{E}(\Lambda_i(\tau)))}{\|\mathbf{b}_J\|_2^2}.$$

For the sake of simplicity, we put $\zeta_i = -(\sum_{j=1}^M b_j f_j(\mathbf{Z}_i))^2 \Lambda_i(\tau)$ for $i = 1, \dots, n$. We have

$$\mathbb{P}\left(\frac{\mathbf{b}^T \mathbf{G}_n \mathbf{b}}{\|\mathbf{b}_J\|_2^2} > \kappa^2\right) \geq \mathbb{P}\left(\frac{\mathbf{b}^T (\mathbf{G}_n - \mathbb{E}(\mathbf{G}_n)) \mathbf{b}}{\|\mathbf{b}_J\|_2^2} > -A_0 \kappa^2\right) \geq 1 - \mathbb{P}\left(\sum_{i=1}^n \zeta_i - \mathbb{E}[\zeta_i] > n A_0 \kappa^2 \|\mathbf{b}_J\|_2^2\right).$$

So we want to show that $\mathbb{P}(\sum_{i=1}^n \zeta_i - \mathbb{E}[\zeta_i] > n A_0 \kappa^2 \|\mathbf{b}_J\|_2^2)$ is small. To apply a standard Bernstein's inequality to the independent random variables ζ_1, \dots, ζ_n , we have to verify that $(1/n) \sum_{i=1}^n \mathbb{E}|\zeta_i|^m \leq m! w^2 d^{m-2}$ for some positive constants w and d and for all integers $m \geq 2$ (see Proposition 2.9 in Massart [21]).

We consider here two cases.

- If $A_0 > 1$, under Assumption **A₂**,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \mathbb{E}|\zeta_i|^m &= \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^M b_j f_j(\mathbf{Z}_i) \right)^{2m} \mathbb{E}[|\Lambda_i(\tau)|^m] \\ &\leq \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^M b_j f_j(\mathbf{Z}_i) \right)^{2m} m! A_0^m \\ &\leq m! \left[A_0 L^2 \|\mathbf{b}\|_1^2 \right]^{m-2} \left[A_0 L^2 \|\mathbf{b}\|_1^2 \right]^2 \end{aligned}$$

By taking $w = d$ equal to $A_0 L^2 \|\mathbf{b}\|_1^2$, we apply the standard Bernstein's inequality to get

$$\mathbb{P}\left(\sum_{i=1}^n \zeta_i - \mathbb{E}[\zeta_i] > n A_0 \kappa^2 \|\mathbf{b}_J\|_2^2\right) \leq \exp\left(-\frac{n A_0^2 \kappa^4 \|\mathbf{b}_J\|_2^4}{2(A_0^2 L^4 \|\mathbf{b}\|_1^4 + A_0 L^2 \|\mathbf{b}\|_1^2 A_0 \kappa^2 \|\mathbf{b}_J\|_2^2)}\right).$$

Since $\|\mathbf{b}\|_2 \leq \|\mathbf{b}\|_1 \leq (1 + a_0) \|\mathbf{b}_J\|_1 \leq (1 + a_0) \sqrt{s} \|\mathbf{b}_J\|_2$, we finally obtain

$$\mathbb{P}\left(\sum_{i=1}^n \zeta_i - \mathbb{E}[\zeta_i] > n \kappa^2 \|\mathbf{b}_J\|_2^2\right) \leq \exp\left(-\frac{n \kappa^4}{2L^2(1 + a_0)^2 s (L^2(1 + a_0)^2 s + \kappa^2)}\right).$$

We finally obtain that

$$\frac{\mathbf{b}^T \mathbf{G}_n \mathbf{b}}{\|\mathbf{b}_J\|_2^2} > -A_0 \kappa^2 + \kappa_0^2(s, a_0) = \frac{\kappa_0^2(s, a_0)}{2} > \kappa^2,$$

with probability larger than

$$1 - \exp\left(-\frac{n \kappa^4}{2L^2(1 + a_0)^2 s (L^2(1 + a_0)^2 s + \kappa^2)}\right).$$

- If $A_0 < 1$, we repeat the same proof with $A_0 = 1$ and $\kappa = (1/\sqrt{2}) \kappa_0(s, a_0)$. \square

In the particular case of the right censoring (see Remark 0.1), we obtain a better version of Lemma 2.2. Indeed, in this case, $\Lambda_i([0, \tau])$ is exponentially distributed with rate parameter 1 and since $\Lambda_i([0, \tau]) \leq \Lambda_i([0, \infty])$ almost surely, its expectation is then just less than 1 and then

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}|\zeta_i|^m = \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^M b_j f_j(\mathbf{Z}_i) \right)^{2m} \mathbb{E}[|\Lambda_i(\tau)|^m] \leq m! \left[L^2 \|\mathbf{b}\|_1^2 \right]^{m-2} \left[L^2 \|\mathbf{b}\|_1^2 \right]^2.$$

We obtain (11) with probability larger than

$$1 - \exp\left(-\frac{n\kappa^4}{2L^2(1+a_0)^2s(L^2(1+a_0)^2s+\kappa^2)}\right) \text{ with } \kappa = (1/\sqrt{2})\kappa_0(s, a_0).$$

5.4 Proof of Theorem 2.3

Let introduce the event $\Omega_{\mathbf{RE}_n(s, a_0)}(\kappa) = \left\{0 < \kappa = \min_{\substack{J \subset \{1, \dots, M\}, \\ |J| \leq s}} \min_{\substack{\mathbf{b} \in \mathbb{R}^M \setminus \{0\}, \\ \|\mathbf{b}_{J^c}\|_1 \leq a_0 \|\mathbf{b}_J\|_1}} \frac{(\mathbf{b}^T \mathbf{G}_n \mathbf{b})^{1/2}}{\|\mathbf{b}_J\|_2}\right\}$. We start from Inequality (24) and the fact that on \mathcal{A} , for $\beta \in \Gamma(\mu)$,

$$|(\hat{\beta}_L^\mu - \beta)^T \boldsymbol{\eta}_{n, \tau}| \leq \sum_{j=1}^M \frac{\omega_j}{2} |(\hat{\beta}_L^\mu - \beta)_j|.$$

It follows that

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) + \sum_{j=1}^M \frac{\omega_j}{2} |(\hat{\beta}_L^\mu - \beta)_j| \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + \sum_{j=1}^M \omega_j (|(\hat{\beta}_L^\mu - \beta)_j| + |\beta_j| - |(\hat{\beta}_L^\mu)_j|).$$

On $J(\beta)^c$, $|(\hat{\beta}_L^\mu - \beta)_j| + |\beta_j| - |(\hat{\beta}_L^\mu)_j| = 0$, so on \mathcal{A} we obtain

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) + \sum_{j=1}^M \frac{\omega_j}{2} |(\hat{\beta}_L^\mu - \beta)_j| \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2 \sum_{j \in J(\beta)} \omega_j |(\hat{\beta}_L^\mu - \beta)_j|. \quad (25)$$

We apply Cauchy-Schwarz Inequality to the second right hand side of (25) to get

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) + \sum_{j=1}^M \frac{\omega_j}{2} |(\hat{\beta}_L^\mu - \beta)_j| \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2\sqrt{|J(\beta)|} \sqrt{\sum_{j \in J(\beta)} \omega_j^2 |\hat{\beta}_L^\mu - \beta|_j^2}. \quad (26)$$

With the notations $\Delta = \mathbf{D}(\hat{\beta}_L^\mu - \beta)$ and $\mathbf{D} = (\text{diag}(\omega_j))_{1 \leq j \leq M}$ introduced in Subsection 2.2, Inequalities (25) and (26) become

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) + \frac{1}{2} \|\Delta\|_1 \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2\|\Delta_{J(\beta)}\|_1, \quad (27)$$

and

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2\sqrt{|J(\beta)|} \|\Delta_{J(\beta)}\|_2. \quad (28)$$

Consider,

$$\mathcal{A}_1 = \{\zeta \widetilde{K}_n(\lambda_0, \lambda_\beta) \leq 2\|\Delta_{J(\beta)}\|_1\}. \quad (29)$$

On $\mathcal{A} \cap \mathcal{A}_1^c$, the result of the theorem follows immediately from (27). As soon as, $\|\Delta_{J(\beta)^c}\|_1 \leq (3 + 4/\zeta) \|\Delta_{J(\beta)}\|_1$, on $\Omega_{\mathbf{RE}_n(s, a_0)}(\kappa)$, with $a_0 = (3 + 4/\zeta)$ and $\kappa = (1/\sqrt{2A_0})\kappa_0(s, a_0)$ we get

$$\kappa^2 \|\Delta_{J(\beta)}\|_2^2 \leq \Delta^T \mathbf{G}_n \Delta.$$

So, initially we will assume that $\|\Delta_{J(\beta)^c}\|_1 \leq (3 + 4/\zeta) \|\Delta_{J(\beta)}\|_1$, and we will verify later that this inequality holds. Since,

$$\begin{aligned} \Delta^T \mathbf{G}_n \Delta &= \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^M \omega_j (\hat{\beta}_{L,j}^\mu - \beta_j) f_j(\mathbf{Z}_i) \right)^2 \Lambda_i(\tau) \\ &\leq \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \frac{1}{n} \sum_{i=1}^n \int_0^\tau \left(\log(\alpha_0(t) e^{f_{\hat{\beta}_L^\mu}(\mathbf{Z}_i)}) - \log(\alpha_0(t) e^{f_\beta(\mathbf{Z}_i)}) \right)^2 d\Lambda_i(t) \\ &\leq \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \|\log \lambda_{\hat{\beta}_L^\mu} - \log \lambda_\beta\|_{n,\Lambda}^2, \end{aligned}$$

on $\Omega_{\mathbf{RE}_n(s,a_0)}(\boldsymbol{\kappa})$, Inequality (28) becomes on $\mathcal{A} \cap \Omega_{\mathbf{RE}_n(s,a_0)}(\boldsymbol{\kappa})$

$$\begin{aligned} \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) &\leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2\sqrt{|J(\boldsymbol{\beta})|} \left(\max_{1 \leq j \leq M} \omega_j \right) \boldsymbol{\kappa}^{-1} \|\log \lambda_{\hat{\beta}_L^\mu} - \log \lambda_\beta\|_{n,\Lambda} \\ &\leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2\sqrt{|J(\boldsymbol{\beta})|} \left(\max_{1 \leq j \leq M} \omega_j \right) \boldsymbol{\kappa}^{-1} (\|\log \lambda_{\hat{\beta}_L^\mu} - \log \lambda_0\|_{n,\Lambda} + \|\log \lambda_0 - \log \lambda_\beta\|_{n,\Lambda}). \end{aligned}$$

The following proposition (proof in Annexe A) connects the weighted empirical norm and the empirical Kullback divergence.

Proposition 5.1. *Under Assumption A₅, for all $\boldsymbol{\beta} \in \Gamma(\mu)$,*

$$\mu' \|\log \lambda_\beta - \log \lambda_0\|_{n,\Lambda}^2 \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) \leq \mu'' \|\log \lambda_\beta - \log \lambda_0\|_{n,\Lambda}^2,$$

where $\mu' = \phi(\mu)/\mu^2$, $\mu'' = \phi(-\mu)/\mu^2$ and $\phi(t) = e^{-t} + t - 1$.

Now, applying Proposition 5.1, it follows that

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2\sqrt{|J(\boldsymbol{\beta})|} \left(\max_{1 \leq j \leq M} \omega_j \right) \frac{\boldsymbol{\kappa}^{-1}}{\sqrt{\mu'}} \left(\sqrt{\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu})} + \sqrt{\widetilde{K}_n(\lambda_0, \lambda_\beta)} \right).$$

We now use the elementary inequality $2uv \leq bu^2 + \frac{v^2}{b}$ with $b > 1$, $u = \sqrt{|J(\boldsymbol{\beta})|} \left(\max_{1 \leq j \leq M} \omega_j \right) \boldsymbol{\kappa}^{-1}$ and v

being either $\sqrt{\frac{1}{\mu'} \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu})}$ or $\sqrt{\frac{1}{\mu'} \widetilde{K}_n(\lambda_0, \lambda_\beta)}$. Consequently

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) \leq \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2b|J(\boldsymbol{\beta})| \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \boldsymbol{\kappa}^{-2} + \frac{1}{b\mu'} \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) + \frac{1}{b\mu'} \widetilde{K}_n(\lambda_0, \lambda_\beta).$$

Hence,

$$\left(1 - \frac{1}{\mu'b} \right) \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) \leq \left(1 + \frac{1}{b\mu'} \right) \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2b|J(\boldsymbol{\beta})| \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \boldsymbol{\kappa}^{-2},$$

and

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) \leq \frac{b\mu' + 1}{b\mu' - 1} \widetilde{K}_n(\lambda_0, \lambda_\beta) + 2\frac{b^2\mu'}{b\mu' - 1} |J(\boldsymbol{\beta})| \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \boldsymbol{\kappa}^{-2}.$$

We take $\frac{b\mu' + 1}{b\mu' - 1} = 1 + \zeta$ and $C(\zeta, \mu) = 2\frac{b^2\mu'}{b\mu' + 1}$ a constant depending on ζ and μ . It follows that for any $\beta \in \Gamma(\mu)$:

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\mu}) \leq (1 + \zeta) \left\{ \widetilde{K}_n(\lambda_0, \lambda_\beta) + C(\zeta, \mu) |J(\beta)| \left(\max_{1 \leq j \leq M} \omega_j \right)^2 \kappa^{-2} \right\}.$$

Finally, taking the infimum over all $\beta \in \Gamma(\mu)$ such that $|J(\beta)| \leq s$, we obtain (12).

We have now to verify that $\|\Delta_{J(\beta)^c}\|_1 \leq (3 + 4/\zeta) \|\Delta_{J(\beta)}\|_1$. On $\mathcal{A} \cap \mathcal{A}_1$, applying (27) we get that

$$\|\Delta\|_1 \leq 4 \left(1 + \frac{1}{\zeta} \right) \|\Delta_{J(\beta)}\|_1,$$

so by splitting $\Delta = \Delta_{J(\beta)} + \Delta_{J(\beta)^c}$, we finally obtain

$$\|\Delta_{J(\beta)^c}\|_1 \leq \left(3 + \frac{4}{\zeta} \right) \|\Delta_{J(\beta)}\|_1.$$

Finally, Lemma 2.2 ensures that $\mathbb{P}(\mathcal{A}^c \cup \Omega_{\mathbf{RE}_n(s, a_0)}^c(\kappa)) \leq A_{\varepsilon, c_\ell} e^{-x} + \pi_n$, which achieves the proof of Theorem 2.3. \square

5.5 Proof of Corollary 2.4

Corollary 2.4 follows from Proposition 5.1 and same arguments as in the proof of Theorem 2.3 with

$$b = \frac{\mu'(1 + \zeta) + \mu''}{\mu'(1 + \zeta) - \mu''}.$$

\square

5.6 Proof of Theorem 2.5

To prove Inequality (13) of Theorem 2.5, we start from (25) with $\beta = \beta_0$ and $\hat{\beta}_L$ defined by (7). Consequently $\widetilde{K}_n(\lambda_0, \lambda_\beta) = 0$. Here we give the proposition that gives the relation between the empirical Kullback divergence and the empirical norm, in the case of variable selection.

Proposition 5.2. *Under Assumption (A₆), there exist two positive numerical constants ξ and ξ' such that*

$$\xi \|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n, \Lambda}^2 \leq \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L}) \leq \xi' \|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n, \Lambda}^2.$$

The proof of Proposition (5.2) is given in Annexe B. Applying Proposition 5.2 with $\lambda_0(t, \mathbf{Z}_i) = \alpha_0(t) e^{\beta_0^T \mathbf{Z}_i}$ and $\lambda_{\hat{\beta}_L}(t, \mathbf{Z}_i) = \alpha_0(t) e^{\hat{\beta}_L^T \mathbf{Z}_i}$, we obtain that, on $\mathcal{A} = \bigcap_{j=1}^p \left\{ |\eta_{n, \tau}(f_j)| \leq \Gamma_1 \frac{\omega_j}{2} \right\}$

$$\xi \|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n, \Lambda}^2 + \Gamma_1 \sum_{j=1}^p \frac{\omega_j}{2} |\hat{\beta}_L - \beta_0|_j \leq 2\Gamma_1 \sum_{j \in J_0} \omega_j |\hat{\beta}_L - \beta_0|_j. \quad (30)$$

From this inequality, we deduce

$$\xi \|\mathbf{X}(\hat{\beta}_L - \beta_0)\|_{n,\Lambda}^2 \leq 2\Gamma_1 \sum_{j \in J_0} \omega_j |\hat{\beta}_L - \beta_0|_j \leq 2\sqrt{|J_0|}\Gamma_1 \|\Delta_{0,J_0}\|_2. \quad (31)$$

From (30), we also have

$$\sum_{j=1}^p \omega_j |\hat{\beta}_L - \beta_0|_j \leq 4 \sum_{j \in J_0} \omega_j |\hat{\beta}_L - \beta_0|_j$$

and we obtain $\|\Delta_0\|_1 \leq 4\|\Delta_{0,J_0}\|_1$. We then split $\|\Delta_0\|_1 = \|\Delta_{0,J_0}\|_1 + \|\Delta_{0,J_0^c}\|_1$ to get

$$\|\Delta_{0,J_0^c}\|_1 \leq 3\|\Delta_{0,J_0}\|_1. \quad (32)$$

On $\Omega_{\text{RE}_n(s,a_0)}(\kappa')$, with $a_0 = 3$ and $\kappa' = (1/\sqrt{2A_0})\kappa_0(s, 3)$ we get

$$\|\mathbf{X}\Delta_0\|_{n,\Lambda}^2 \geq \kappa'^2 \|\Delta_{0,J_0}\|_2^2. \quad (33)$$

According to (31), we conclude that on $\mathcal{A} \cap \Omega_{\text{RE}_n(s,a_0)}(\kappa')$

$$\xi \|\mathbf{X}(\hat{\beta}_L - \beta_0)\|_{n,\Lambda}^2 \leq 2\sqrt{|J_0|}\Gamma_1 \max_{1 \leq j \leq p} \omega_j \frac{\|\mathbf{X}(\hat{\beta}_L - \beta_0)\|_{n,\Lambda}}{\kappa'},$$

which entails that

$$\|\mathbf{X}(\hat{\beta}_L - \beta_0)\|_{n,\Lambda}^2 \leq \frac{4|J_0|}{\xi^2 \kappa'^2} \Gamma_1^2 (\max_{1 \leq j \leq p} \omega_j)^2,$$

with $\mathbb{P}(\mathcal{A} \cap \Omega_{\text{RE}_n(s,a_0)}(\kappa')) \geq 1 - A_{\varepsilon,c_\ell} e^{-\Gamma_1 x} - \pi_n$.

Let us come to the proof of Inequality (14) in Theorem 2.5. On $\mathcal{A} \cap \Omega_{\text{RE}_n(s,a_0)}(\kappa')$, with $a_0 = 3$, Inequality (31) becomes

$$\xi \frac{\kappa'^2}{\max_{1 \leq j \leq M} \omega_j^2} \|\Delta_{0,J_0}\|_2^2 \leq 2\sqrt{|J_0|}\Gamma_1 \|\Delta_{0,J_0}\|_2,$$

and hence

$$\|\Delta_{0,J_0}\|_2 \leq \frac{2\sqrt{|J_0|}}{\xi \kappa'^2} \Gamma_1 \max_{1 \leq j \leq p} \omega_j^2. \quad (34)$$

According to (32) and thanks to Cauchy-Schwarz Inequality, we have

$$\|\Delta_0\|_1 = \|\Delta_{0,J_0}\|_1 + \|\Delta_{0,J_0^c}\|_1 \leq 4\|\Delta_{0,J_0}\|_1 \leq 4\sqrt{|J_0|} \|\Delta_{0,J_0}\|_2.$$

From (34), we get

$$\frac{\|\Delta_0\|_1}{4\sqrt{|J_0|}} \leq \frac{2\sqrt{|J_0|}}{\xi \kappa'^2} \Gamma_1 \max_{1 \leq j \leq p} \omega_j^2,$$

and finally

$$\|\hat{\beta}_L - \beta_0\|_1 \leq 8 \frac{|J_0|}{\xi \kappa'^2} \Gamma_1 \frac{\max_{1 \leq j \leq p} \omega_j^2}{\min_{1 \leq j \leq p} \omega_j},$$

with $\mathbb{P}(\mathcal{A} \cap \Omega_{\text{RE}_n(s,a_0)}(\kappa')) \geq 1 - A_{\varepsilon,c_\ell} e^{-\Gamma_1 x} - \pi_n$. □

5.7 Proof of Theorem 3.1

The proof is very similar to the one of Theorem 2.1. We start from (19) and (20), and write

$$\begin{aligned} \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L, \hat{\gamma}_L}) &\leq \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + (\hat{\gamma}_L - \gamma)^T \boldsymbol{\nu}_{n, \tau} + \text{pen}(\gamma) - \text{pen}(\hat{\gamma}_L) \\ &\quad + (\hat{\beta}_L - \beta)^T \boldsymbol{\eta}_{n, \tau} + \text{pen}(\beta) - \text{pen}(\hat{\beta}_L). \end{aligned} \quad (35)$$

Set \mathcal{A} and \mathcal{B} such that

$$\mathcal{A} = \bigcap_{j=1}^M \left\{ |\eta_{n, \tau}(f_j)| \leq \frac{\omega_j}{2} \right\} \text{ and } \mathcal{B} = \bigcap_{k=1}^N \left\{ |\nu_{n, \tau}(\theta_k)| \leq \frac{\delta_k}{2} \right\}. \quad (36)$$

We apply Theorem 4.1 to bound up $\mathbb{P}(\mathcal{A}^c)$ and $\mathbb{P}(\mathcal{B}^c)$ and obtain that

$$\mathbb{P}(\mathcal{A}^c) \leq c_{3, \varepsilon, c_\ell} e^{-x} \text{ and } \mathbb{P}(\mathcal{B}^c) \leq c'_{3, \varepsilon', c'_\ell} e^{-y}.$$

Hence for $A_{\varepsilon, c_\ell} = c_{3, \varepsilon, c_\ell}$ and $B_{\varepsilon', c'_\ell} = c'_{3, \varepsilon', c'_\ell}$, we have

$$\mathbb{P}[(\mathcal{A} \cap \mathcal{B})^c] = \mathbb{P}(\mathcal{A}^c \cup \mathcal{B}^c) \leq \mathbb{P}(\mathcal{A}^c) + \mathbb{P}(\mathcal{B}^c) \leq c_{3, \varepsilon, c_\ell} e^{-x} + c'_{3, \varepsilon', c'_\ell} e^{-y} \leq A_{\varepsilon, c_\ell} e^{-x} + B_{\varepsilon', c'_\ell} e^{-y} \quad (37)$$

On $\mathcal{A} \cap \mathcal{B}$ arguing as in the proof of Theorem 2.1, with probability larger than $1 - A_{\varepsilon, c_\ell} e^{-x} - B_{\varepsilon', c'_\ell} e^{-y}$, we finish the proof by writing (15). \square

5.8 Proof of Theorem 3.3

Let introduce the event $\Omega_{\widetilde{\text{RE}}_n(s, r_0)}(\tilde{\boldsymbol{\kappa}}) = \left\{ 0 < \tilde{\boldsymbol{\kappa}} = \min_{\substack{J \subset \{1, \dots, M\}, \\ |J| \leq s}} \min_{\substack{\mathbf{b} \in \mathbb{R}^M \setminus \{0\}, \\ \|\mathbf{b}_{J^c}\|_1 \leq r_0}} \frac{(\mathbf{b}^T \tilde{\mathbf{G}}_n \mathbf{b})^{1/2}}{\|\mathbf{b}_J\|_2} \right\}$. We start

from Inequality (35). On $\mathcal{A} \cap \mathcal{B}$ defined in (36), for $(\beta, \gamma) \in \tilde{\Gamma}(\rho)$,

$$|(\hat{\beta}_L - \beta)^T \boldsymbol{\eta}_{n, \tau}| \leq \sum_{j=1}^M \frac{\omega_j}{2} |(\hat{\beta}_L - \beta)_j| \text{ and } |(\hat{\gamma}_L - \gamma)^T \boldsymbol{\nu}_{n, \tau}| \leq \sum_{k=1}^N \frac{\delta_k}{2} |(\hat{\gamma}_L - \gamma)_k|,$$

and therefore

$$\begin{aligned} \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\rho, \hat{\gamma}_L^\rho}) &+ \sum_{j=1}^M \frac{\omega_j}{2} |(\hat{\beta}_L^\rho - \beta)_j| + \sum_{k=1}^N \frac{\delta_k}{2} |(\hat{\gamma}_L^\rho - \gamma)_k| \\ &\leq \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + 2 \sum_{j \in J(\beta)} \omega_j |(\hat{\beta}_L^\rho - \beta)_j| + 2 \sum_{k \in J(\gamma)} \delta_k |(\hat{\gamma}_L^\rho - \gamma)_k|. \end{aligned} \quad (38)$$

We then apply Cauchy-Schwarz inequality to the second right-term of (38) and obtain

$$\begin{aligned} \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^\rho, \hat{\gamma}_L^\rho}) &+ \sum_{j=1}^M \frac{\omega_j}{2} |(\hat{\beta}_L^\rho - \beta)_j| + \sum_{k=1}^N \frac{\delta_k}{2} |(\hat{\gamma}_L^\rho - \gamma)_k| \\ &\leq \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + 2\sqrt{|J(\beta)|} \sqrt{\sum_{j \in J(\beta)} \omega_j^2 |\hat{\beta}_L^\rho - \beta|_j^2} + 2\sqrt{|J(\gamma)|} \sqrt{\sum_{k \in J(\gamma)} \delta_k^2 |\hat{\gamma}_L^\rho - \gamma|_k^2}. \end{aligned} \quad (39)$$

With the notation of Subsection 3.2, Inequality (38) is rewritten as :

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}}) + \frac{1}{2} \|\widetilde{\Delta}\|_1 \leq \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + 2 \|\widetilde{\Delta}_{J(\beta), J(\gamma)}\|_1, \quad (40)$$

where $\widetilde{\Delta}_{J(\beta), J(\gamma)} = \widetilde{D} \begin{pmatrix} (\hat{\beta}_L^{\rho} - \beta)_{J(\beta)} \\ (\hat{\gamma}_L^{\rho} - \gamma)_{J(\gamma)} \end{pmatrix}$. In the same way, Inequality (39) becomes :

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}}) \leq \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + 4 \max \left(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|} \right) \|\widetilde{\Delta}_{J(\beta), J(\gamma)}\|_2. \quad (41)$$

Consider

$$\mathcal{A}_1 = \zeta \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) \leq 2 \|\widetilde{\Delta}_{J(\beta), J(\gamma)}\|_1. \quad (42)$$

On $\mathcal{A} \cap \mathcal{B} \cap \mathcal{A}_1$, Inequality (17) in Theorem 3.3 follows immediately from (40). As soon as, $\|\widetilde{\Delta}_{J(\beta)^c, J(\gamma)^c}\|_1 \leq (3 + 8 \max(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|}) / \zeta) \|\widetilde{\Delta}_{J(\beta), J(\gamma)}\|_1$, on $\Omega_{\widetilde{\mathbf{RE}}_n(s, r_0)}(\tilde{\kappa})$, with

$$\tilde{\kappa} = (1/\sqrt{2}) \tilde{\kappa}_0(s, r_0) \quad \text{and} \quad r_0 = \left(3 + 8 \max \left(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|} \right) / \zeta \right),$$

we get that

$$\tilde{\kappa}^2 \|\widetilde{\Delta}_{J(\beta), J(\gamma)}\|_2^2 \leq \widetilde{\Delta}^T \widetilde{\mathbf{G}}_n \widetilde{\Delta} \quad \text{with} \quad \widetilde{\Delta}^T \widetilde{\mathbf{G}}_n \widetilde{\Delta} \leq \max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \|\log \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}} - \log \lambda_{\beta, \gamma}\|_{n, \Lambda}^2.$$

On $\mathcal{A} \cap \mathcal{B} \cap \Omega_{\widetilde{\mathbf{RE}}_n(s, r_0)}(\tilde{\kappa})$, Equation (41) becomes

$$\begin{aligned} \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}}) &\leq \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + 4 \max \left(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|} \right) \|\widetilde{\Delta}_{J(\beta), J(\gamma)}\|_2 \\ &\leq \widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + 4 \max \left(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|} \right) \max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \tilde{\kappa}^{-1} \|\log \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}} - \log \lambda_{\beta, \gamma}\|_{n, \Lambda}. \end{aligned}$$

Using that $\|\log \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}} - \log \lambda_{\beta, \gamma}\|_{n, \Lambda} \leq \|\log \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}} - \log \lambda_0\|_{n, \Lambda} + \|\log \lambda_0 - \log \lambda_{\beta, \gamma}\|_{n, \Lambda}$, we obtain that $\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}})$ is less than

$$\widetilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) + 4 \max \left(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|} \right) \max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \tilde{\kappa}^{-1} (\|\log \lambda_{\hat{\beta}_L^{\rho}, \hat{\gamma}_L^{\rho}} - \log \lambda_0\|_{n, \Lambda} + \|\log \lambda_0 - \log \lambda_{\beta, \gamma}\|_{n, \Lambda}).$$

This inequality involves both oracle inequalities in empirical Kullback divergence and in weighted empirical norm.

In the same way that Proposition 5.1, we obtain a Proposition that connect the empirical Kullback divergence and the weighted empirical norm.

Proposition 5.3. Under Assumption **A7**, for all $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \tilde{\Gamma}(\rho)$,

$$\rho' \|\log \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}} - \log \lambda_0\|_{n, \Lambda}^2 \leq \tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}) \leq \rho'' \|\log \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}} - \log \lambda_0\|_{n, \Lambda}^2,$$

where $\rho' = \phi(\rho)/\rho^2$, $\rho'' = \phi(-\rho)/\rho^2$ and $\phi(t) = e^{-t} + t - 1$.

Applying Proposition 5.3, we obtain that $\tilde{K}_n(\lambda_0, \lambda_{\hat{\boldsymbol{\beta}}_L^\rho, \hat{\boldsymbol{\gamma}}_L^\rho})$ is less than

$$\tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}) + 4 \max \left(\sqrt{|J(\boldsymbol{\beta})|}, \sqrt{|J(\boldsymbol{\gamma})|} \right) \max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \frac{\tilde{\boldsymbol{\kappa}}^{-1}}{\sqrt{\rho'}} \left(\sqrt{\tilde{K}_n(\lambda_0, \lambda_{\hat{\boldsymbol{\beta}}_L^\rho, \hat{\boldsymbol{\gamma}}_L^\rho})} + \sqrt{\tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}})} \right).$$

Using again $2uv \leq bu^2 + \frac{v^2}{b}$ with $b > 1$, $u = 2 \max \left(\sqrt{|J(\boldsymbol{\beta})|}, \sqrt{|J(\boldsymbol{\gamma})|} \right) \max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \tilde{\boldsymbol{\kappa}}^{-1}$ and v being

either $\sqrt{\frac{1}{\rho'} \tilde{K}_n(\lambda_0, \lambda_{\hat{\boldsymbol{\beta}}_L^\rho, \hat{\boldsymbol{\gamma}}_L^\rho})}$ or $\sqrt{\frac{1}{\rho'} \tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}})}$, we obtain

$$\begin{aligned} \tilde{K}_n(\lambda_0, \lambda_{\hat{\boldsymbol{\beta}}_L^\rho, \hat{\boldsymbol{\gamma}}_L^\rho}) &\leq \tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}) + 8b \max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|) \left(\max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \right)^2 \tilde{\boldsymbol{\kappa}}^{-2} \\ &\quad + \frac{1}{b\rho'} \tilde{K}_n(\lambda_0, \lambda_{\hat{\boldsymbol{\beta}}_L^\rho, \hat{\boldsymbol{\gamma}}_L^\rho}) + \frac{1}{b\rho'} \tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}). \end{aligned}$$

Hence,

$$\left(1 - \frac{1}{b\rho'}\right) \tilde{K}_n(\lambda_0, \lambda_{\hat{\boldsymbol{\beta}}_L^\rho, \hat{\boldsymbol{\gamma}}_L^\rho}) \leq \left(1 + \frac{1}{b\rho'}\right) \tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}) + 8b \max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|) \left(\max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \right)^2 \tilde{\boldsymbol{\kappa}}^{-2},$$

and

$$\tilde{K}_n(\lambda_0, \lambda_{\hat{\boldsymbol{\beta}}_L^\rho, \hat{\boldsymbol{\gamma}}_L^\rho}) \leq \frac{b\rho' + 1}{b\rho' - 1} \tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}) + 8 \frac{b^2 \rho'}{b\rho' - 1} \max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|) \left(\max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \right)^2 \frac{\tilde{\boldsymbol{\kappa}}^{-2}}{\rho'}. \quad (43)$$

We take $\frac{b\rho' + 1}{b\rho' - 1} = 1 + \zeta$ and we introduce $\tilde{C}(\zeta, \rho) = 8 \frac{b^2 \rho'}{b\rho' + 1}$ a constant depending on ζ and ρ . For all $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ in $\tilde{\Gamma}(\rho)$, we obtain

$$\tilde{K}_n(\lambda_0, \lambda_{\hat{\boldsymbol{\beta}}_L^\rho, \hat{\boldsymbol{\gamma}}_L^\rho}) \leq (1 + \zeta) \left\{ \tilde{K}_n(\lambda_0, \lambda_{\boldsymbol{\beta}, \boldsymbol{\gamma}}) + \tilde{C}(\zeta, \rho) \max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|) \left(\max_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N}} \{\omega_j, \delta_k\} \right)^2 \tilde{\boldsymbol{\kappa}}^{-2} \right\}.$$

Finally, taking the infimum over all $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \tilde{\Gamma}(\rho)$ such that $\max(|J(\boldsymbol{\beta})|, |J(\boldsymbol{\gamma})|) \leq s$, we obtain Inequality (17). Inequality (18) follows by applying Proposition 5.1 with $b = \frac{(1 + \zeta)\rho' + \rho''}{(1 + \zeta)\rho' - \rho''}$ in (43).

We have now to verify that $\|\tilde{\Delta}_{J(\beta)^c, J(\gamma)^c}\|_1 \leq (3 + 8 \max(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|}) / \zeta) \|\tilde{\Delta}_{J(\beta), J(\gamma)}\|_1$. We deduce from (40) that, on $\mathcal{A} \cap \mathcal{B} \cap \mathcal{A}_1$,

$$\|\tilde{\Delta}\|_1 \leq 4 \left(1 + \frac{2}{\zeta} \max(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|}) \right) \|\tilde{\Delta}_{J(\beta), J(\gamma)}\|_1.$$

By splitting $\tilde{\Delta} = \tilde{\Delta}_{J(\beta), J(\gamma)} + \tilde{\Delta}_{J(\beta)^c, J(\gamma)^c}$, we infer that

$$\|\tilde{\Delta}_{J(\beta)^c, J(\gamma)^c}\|_1 \leq \left(3 + \frac{8}{\zeta} \max(\sqrt{|J(\beta)|}, \sqrt{|J(\gamma)|}) \right) \|\tilde{\Delta}_{J(\beta), J(\gamma)}\|_1.$$

To achieve the proof of Theorem 3.3, we combine Equation (37) with Lemma 3.2 to conclude

$$\mathbb{P} \left[\left(\mathcal{A} \cap \mathcal{B} \cap \Omega_{\widetilde{\mathbf{RE}}_n(s, r_0)}(\tilde{\kappa}) \right)^c \right] \leq A_{\varepsilon, c_\ell} e^{-x} + B_{\varepsilon', c'_\ell} e^{-y} + \mathbb{P}(\widetilde{\mathbf{RE}}_n(s, r_0)).$$

□

5.9 Proof of Theorem 4.1

The proofs of (22) and (21) are quite similar, so we only present the one of (21). To prove (22), it suffices to replace $\eta_{n,t}(f_j)$ by the process $\nu_{n,t}(\theta_k)$ throughout the following. Denote by $U_{n,t}$ and $H_i(f_j)$ the quantities

$$U_{n,t}(f_j) = \frac{1}{n} \sum_{i=1}^n \int_0^t H_i(f_j) dM_i(s) \text{ and } H_i(f_j) := \frac{f_j(\mathbf{Z}_i)}{\max_{1 \leq i \leq n} |f_j(\mathbf{Z}_i)|}.$$

Since $H_i(f_j)$ is a bounded predictable process with respect to \mathcal{F}_t , $U_{n,t}(f_j)$ is a square integrable martingale. Its predictable variation is given by

$$\vartheta_{n,t}(f_j) = n \langle U_n(f_j) \rangle_t = \frac{1}{n} \sum_{i=1}^n \int_0^t (H_i(f_j))^2 d\Lambda_i(s)$$

and the optional variation of $U_{n,t}(f_j)$ is

$$\hat{\vartheta}_{n,t}(f_j) = n[U_n(f_j)]_t = \frac{1}{n} \sum_{i=1}^n \int_0^t (H_i(f_j))^2 dN_i(s).$$

The standard Bernstein's inequality (see Uspensky [30] or Massart [21] for the classical Bernstein's inequality and van de Geer [31] for the Bernstein's inequality for martingales) states that

$$\mathbb{P} \left[U_{n,t}(f_j) \geq \sqrt{\frac{2\omega x}{n}} + \frac{x}{3n}, \vartheta_{n,t}(f_j) \leq \omega \right] \leq e^{-x}. \quad (44)$$

For any $0 < v < \omega < +\infty$, we have

$$\begin{aligned} & \left\{ U_{n,t}(f_j) \geq \sqrt{\frac{2\omega\vartheta_{n,t}(f_j)x}{vn}} + \frac{x}{3n} \right\} \cap \{v < \vartheta_{n,t}(f_j) \leq \omega\} \\ & \subset \left\{ U_{n,t}(f_j) \geq \sqrt{\frac{2\omega x}{n}} + \frac{x}{3n} \right\} \cap \{v < \vartheta_{n,t}(f_j) \leq \omega\}. \end{aligned}$$

Consequently,

$$\mathbb{P}\left[U_{n,t}(f_j) \geq \sqrt{\frac{2\omega\vartheta_{n,t}(f_j)x}{vn}} + \frac{x}{3n}, v < \vartheta_{n,t}(f_j) \leq \omega\right] \leq e^{-x}. \quad (45)$$

Now, we aim at replacing $\vartheta_{n,t}(f_j)$ which is non observable by the observable $\hat{\vartheta}_{n,t}(f_j)$ in Bound (45). Let us consider the martingale given \mathcal{F}_t

$$\hat{\vartheta}_{n,t}(f_j) - \vartheta_{n,t}(f_j) = \frac{1}{n} \sum_{i=1}^n \int_0^t (H_i(f_j))^2 \left(dN_i(s) - d\Lambda_i(s) \right) = \frac{1}{n} \sum_{i=1}^n \int_0^t (H_i(f_j))^2 dM_i(s).$$

From (45), we deduce that

$$\mathbb{P}\left[|\hat{\vartheta}_{n,t}(f_j) - \vartheta_{n,t}(f_j)| \geq \sqrt{\frac{2\omega\vartheta_{n,t}(f_j)x}{vn}} + \frac{x}{3n}, v < \vartheta_{n,t}(f_j) \leq \omega\right] \leq 2e^{-x}. \quad (46)$$

If $\vartheta_{n,t}(f_j)$ satisfies

$$|\hat{\vartheta}_{n,t}(f_j) - \vartheta_{n,t}(f_j)| \leq \sqrt{\frac{2\omega\vartheta_{n,t}(f_j)x}{vn}} + \frac{x}{3n}, \quad (47)$$

thanks to the fact that $A \leq b + \sqrt{aA}$ entails $A \leq a + 2b$ for any $a, A, b > 0$, taking $A = \vartheta_{n,t}(f_j)$, $a = \frac{2\omega x}{vn}$ and $b = \hat{\vartheta}_{n,t}(f_j) + \frac{x}{3n}$, we obtain

$$\vartheta_{n,t}(f_j) \leq 2\hat{\vartheta}_{n,t}(f_j) + 2\left(\frac{\omega}{v} + \frac{1}{3}\right)\frac{x}{n}. \quad (48)$$

Combining Inequality (48) and Inequality (47), we get

$$\hat{\vartheta}_{n,t}(f_j) \leq \vartheta_{n,t}(f_j) + \sqrt{\frac{2\omega x}{vn} \left(2\hat{\vartheta}_{n,t}(f_j) + 2\left(\frac{\omega}{v} + \frac{1}{3}\right)\frac{x}{n}\right)} + \frac{x}{3n}.$$

Once again using that $A \leq b + \sqrt{aA}$ entails $A \leq a + 2b$ with

$$A = \hat{\vartheta}_{n,t}(f_j), a = \frac{4\omega x}{vn} \text{ and } b = \vartheta_{n,t}(f_j) + \sqrt{\frac{4\omega x}{vn} \left(\frac{\omega}{v} + \frac{1}{3}\right)\frac{x}{n}} + \frac{x}{3n},$$

we obtain $\hat{\vartheta}_{n,t}(f_j) \leq 2\vartheta_{n,t}(f_j) + 2\left(\frac{1}{3} + 2\sqrt{\frac{\omega}{v}\left(\frac{\omega}{v} + \frac{1}{3}\right)} + \frac{2\omega}{v}\right)\frac{x}{n}$. Now, we deduce from Inequality (48) that

$$\begin{aligned} \left\{U_{n,t}(f_j) \leq \sqrt{\frac{2\omega\vartheta_{n,t}(f_j)x}{vn}} + \frac{x}{3n}\right\} &\cap \left\{|\hat{\vartheta}_{n,t}(f_j) - \vartheta_{n,t}(f_j)| \leq \sqrt{\frac{2\omega\vartheta_{n,t}(f_j)x}{vn}} + \frac{x}{3n}\right\} \\ &\subset \left\{U_{n,t}(f_j) \leq 2\sqrt{\frac{\omega x}{vn}\hat{\vartheta}_{n,t}(f_j)} + \left(2\sqrt{\frac{\omega}{v}\left(\frac{\omega}{v} + \frac{1}{3}\right)} + \frac{1}{3}\right)\frac{x}{n}\right\}. \end{aligned} \quad (49)$$

Using (45) and (46), we finally obtain

$$\mathbb{P}\left[U_{n,t}(f_j) \geq 2\sqrt{\frac{\omega x}{vn}\hat{\vartheta}_{n,t}(f_j)} + \left(2\sqrt{\frac{\omega}{v}\left(\frac{\omega}{v} + \frac{1}{3}\right)} + \frac{1}{3}\right)\frac{x}{n}, v < \vartheta_{n,t}(f_j) \leq \omega\right] \leq 3e^{-x}. \quad (50)$$

It remains to remove the event $\{v \leq \vartheta_{n,t}(f_j) < \omega\}$ in (50). For $k \geq 0$, set $v_k = c_0\frac{x+1}{n}(1+\varepsilon)^k$, and use the following decomposition into disjoint sets :

$$\{\vartheta_{n,t}(f_j) > c_0x/n\} = \bigcup_{k \geq 0} \{v_k < \vartheta_{n,t}(f_j) \leq v_{k+1}\}. \quad (51)$$

Instead of considering the event $\{v < \vartheta_{n,t}(f_j) \leq \omega\}$, we calculate the probabilities on $\{\vartheta_{n,t}(f_j) > v_0\}$ and on its complementary to finally get the expected probability. According to (50)

$$\mathbb{P}\left[U_{n,t}(f_j) \geq c_{1,\varepsilon}\sqrt{\frac{x}{n}\hat{\vartheta}_{n,t}(f_j)} + c_{2,\varepsilon}\frac{x}{n}, v_k < \vartheta_{n,t}(f_j) \leq v_{k+1}\right] \leq 3e^{-x},$$

with

$$c_{1,\varepsilon} = 2\sqrt{1+\varepsilon} \text{ and } c_{2,\varepsilon} = 2\sqrt{(1+\varepsilon)(4/3+\varepsilon)} + 1/3.$$

Set for some constant $c_\ell > 1$,

$$\ell = c_\ell \log \log \left(\frac{\vartheta_{n,t}(f_j)}{v_0} \vee e\right).$$

On the event

$$D_{n,\ell,\varepsilon} = \left\{|\hat{\vartheta}_{n,t}(f_j) - \vartheta_{n,t}(f_j)| \leq \sqrt{\frac{2(1+\varepsilon)\vartheta_{n,t}(f_j)(x+\ell)}{n}} + \frac{x+\ell}{3n}\right\}$$

applying (48) with $\frac{\omega}{v} = 1 + \varepsilon$ and replacing x by $x + \ell$, we have

$$\vartheta_{n,t}(f_j) \leq 2\hat{\vartheta}_{n,t}(f_j) + 2(4/3 + \varepsilon)\frac{x}{n} + \frac{2(4/3 + \varepsilon)c_\ell}{n} \log \log \left(\frac{\vartheta_{n,t}(f_j)}{v_0} \vee e\right).$$

We now use the fact that $\log \log(x) \leq x/e - 1$ for any $x \geq e$, and since $ec_0 > 2(4/3 + \varepsilon)c_\ell$ we get

$$\vartheta_{n,t}(f_j) \leq \frac{ec_0}{ec_0 - 2(4/3 + \varepsilon)c_\ell} \left(2\hat{\vartheta}_{n,t}(f_j) + 2(4/3 + \varepsilon)\frac{x}{n} \right).$$

Combining the last inequality with (49), we obtain the following embeddings :

$$\begin{aligned} & \left\{ U_{n,t}(f_j) \leq \sqrt{\frac{2(1 + \varepsilon)\vartheta_{n,t}(f_j)(x + \ell)}{n}} + \frac{x + \ell}{3n} \right\} \cap D_{n,\ell,\varepsilon} \\ & \subset \left\{ U_{n,t}(f_j) \leq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{2,\varepsilon} \frac{x + \hat{\ell}}{n} \right\}, \end{aligned} \quad (52)$$

where $\hat{\ell} = c_\ell \log \log \left(\frac{2en\hat{\vartheta}_{n,t}(f_j) + 2e(4/3 + \varepsilon)x}{ec_0 - 2(4/3 + \varepsilon)c_\ell} \vee e \right)$. From (51), we have

$$\begin{aligned} & \mathbb{P} \left[U_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{2,\varepsilon} \frac{x + \hat{\ell}}{n}, \vartheta_{n,t}(f_j) > v_0 \right] \\ & \leq \sum_{k \geq 0} \mathbb{P} \left[U_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{2,\varepsilon} \frac{x + \hat{\ell}}{n}, v_k < \vartheta_{n,t}(f_j) \leq v_{k+1} \right]. \end{aligned}$$

Then, we write

$$\begin{aligned} & \sum_{k \geq 0} \mathbb{P} \left[U_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{2,\varepsilon} \frac{x + \hat{\ell}}{n}, v_k < \vartheta_{n,t}(f_j) \leq v_{k+1} \right] \\ & = \sum_{k \geq 0} \mathbb{P} \left[U_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{2,\varepsilon} \frac{x + \hat{\ell}}{n}, D_{n,\ell,\varepsilon}, v_k < \vartheta_{n,t}(f_j) \leq v_{k+1} \right] \\ & + \sum_{k \geq 0} \mathbb{P} \left[U_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{2,\varepsilon} \frac{x + \hat{\ell}}{n}, D_{n,\ell,\varepsilon}^c, v_k < \vartheta_{n,t}(f_j) \leq v_{k+1} \right], \end{aligned}$$

where $D_{n,\ell,\varepsilon}^c$ is the complementary of $D_{n,\ell,\varepsilon}$. Applying (52), we get

$$\begin{aligned} & \sum_{k \geq 0} \mathbb{P} \left[U_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{2,\varepsilon} \frac{x + \hat{\ell}}{n}, v_k < \vartheta_{n,t}(f_j) \leq v_{k+1} \right] \\ & \leq \sum_{k \geq 0} \mathbb{P} \left[U_{n,t}(f_j) \geq \sqrt{\frac{2(1 + \varepsilon)\vartheta_{n,t}(f_j)(x + \ell)}{n}} + \frac{x + \ell}{3n}, v_k < \vartheta_{n,t}(f_j) \leq v_{k+1} \right] \\ & + \sum_{k \geq 0} \mathbb{P} \left[D_{n,\ell,\varepsilon}^c, v_k < \vartheta_{n,t}(f_j) \leq v_{k+1} \right]. \end{aligned}$$

Gathering (45) and (46), we finally obtain

$$\begin{aligned}
& \mathbb{P}\left[U_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{2,\varepsilon} \frac{x + \hat{\ell}}{n}, \vartheta_{n,t}(f_j) > v_0\right] \\
& \leq 3 \left(e^{-x} + \sum_{j \geq 1} e^{-(x + c_\ell \log \log(v_j/v_0))} \right) \\
& = 3(1 + (\log(1 + \varepsilon))^{-c_\ell} \sum_{j \geq 1} j^{-c_\ell}) e^{-x}.
\end{aligned}$$

We apply Inequality (44) with $\omega = v_0 = c_0(x + 1)/n$ to get

$$\mathbb{P}\left[U_{n,t}(f_j) \geq \left(\sqrt{2c_0} + \frac{1}{3}\right) \frac{x + 1}{n}, \vartheta_{n,t}(f_j) \leq v_0\right] \leq e^{-x}. \quad (53)$$

According to (53), for $c_{3,\varepsilon} = \sqrt{2 \max(c_0, 2(1 + \varepsilon)(4/3 + \varepsilon))} + 1/3$, we get

$$\mathbb{P}\left[U_{n,t}(f_j) \geq c_{1,\varepsilon} \sqrt{\frac{\hat{\vartheta}_{n,t}(f_j)(x + \hat{\ell})}{n}} + c_{3,\varepsilon} \frac{x + 1 + \hat{\ell}}{n}\right] \leq \left(4 + 3(\log(1 + \varepsilon))^{-c_\ell} \sum_{j \geq 1} j^{-c_\ell}\right) e^{-x}.$$

Now it suffices to multiply both sides of the inequality inside the probability by $\|f_j\|_{n,\infty} = \max_{i=1,\dots,n} |f_j(\mathbf{Z}_i)|$ to end up the proof of Theorem 4.1. \square

A Proof of Proposition 5.3

The proof of Proposition 5.1 and Proposition 5.3 are similar. So we only dprove Proposition 5.3 which corresponds to the general case. To compare the empirical Kullback divergence (3) and the weighted empirical norm (4), we use Lemma 1 in Bach [4], that we recall here :

Lemma A.1. *Let g be a convex three times differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $t \in \mathbb{R}$, $|g'''(t)| \leq Sg''(t)$, for some $S \geq 0$. Then, for all $t \geq 0$:*

$$\frac{g''(0)}{S^2} \phi(St) \leq g(t) - g(0) - g'(0)t \leq \frac{g''(0)}{S^2} \phi(-St) \text{ with } \phi(u) = e^{-u} + u - 1$$

This Lemma gives upper and lower Taylor expansions for some convex and three times differentiable function. It has been introduced to extend tools from self-concordant functions (i.e. which verify $|g'''(t)| \leq 2g''(t)^{3/2}$) and provide simple extensions of theoretical results for the square loss for logistic regression.

Let h be a function on $[0, \tau] \times \mathbb{R}^p$ and define

$$G(h) = -\frac{1}{n} \sum_{i=1}^n \int_0^\tau h(s, \mathbf{Z}_i) d\Lambda_i(s) + \frac{1}{n} \sum_{i=1}^n \int_0^\tau e^{h(s, \mathbf{Z}_i)} Y_i(s) ds.$$

Consider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(t) = G(h + tk)$, where h and k are two functions defined on \mathbb{R}^p . By differentiating G with respect to t we get :

$$\begin{aligned} g'(t) &= -\frac{1}{n} \sum_{i=1}^n \int_0^\tau k(s, \mathbf{Z}_i) d\Lambda_i(s) + \frac{1}{n} \sum_{i=1}^n \int_0^\tau k(s, \mathbf{Z}_i) e^{h(s, \mathbf{Z}_i) + tk(s, \mathbf{Z}_i)} Y_i(s) ds, \\ g''(t) &= \frac{1}{n} \sum_{i=1}^n \int_0^\tau (k(s, \mathbf{Z}_i))^2 e^{h(s, \mathbf{Z}_i) + tk(s, \mathbf{Z}_i)} Y_i(s) ds, \\ g'''(t) &= \frac{1}{n} \sum_{i=1}^n \int_0^\tau (k(s, \mathbf{Z}_i))^3 e^{h(s, \mathbf{Z}_i) + tk(s, \mathbf{Z}_i)} Y_i(s) ds. \end{aligned}$$

It follows that

$$|g'''(t)| \leq \|k\|_{n, \infty} g''(t).$$

Applying Lemma A.1 with $S = \|k\|_{n, \infty}$, we obtain for all $t \geq 0$,

$$\frac{g''(0)}{\|k\|_{n, \infty}^2} \phi(t\|k\|_{n, \infty}) \leq g(t) - g(0) - g'(0)t \leq \frac{g''(0)}{\|k\|_{n, \infty}^2} \phi(-t\|k\|_{n, \infty}).$$

Take $t = 1$, $h(s, \mathbf{Z}_i) = \log \lambda_0(s, \mathbf{Z}_i)$ and for $(\beta, \gamma) \in \tilde{\Gamma}(\rho)$, $k(s, \mathbf{Z}_i) = \log \lambda_{\beta, \gamma}(s, \mathbf{Z}_i) - \log \lambda_0(s, \mathbf{Z}_i)$. We obtain

$$g''(0) \frac{\phi(\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty})}{\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty}^2} \leq G(\log \lambda_{\beta, \gamma}) - G(\log \lambda_0) - g'(0) \leq g''(0) \frac{\phi(-\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty})}{\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty}^2}. \quad (54)$$

Now straightforward calculations show that $g'(0) = 0$ and

$$\begin{aligned} g''(0) &= \frac{1}{n} \sum_{i=1}^n \int_0^\tau ((\log \lambda_{\beta, \gamma} - \log \lambda_0)(s, \mathbf{Z}_i))^2 d\Lambda_i(s) \\ &= \|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \Lambda}^2. \end{aligned}$$

Replacing $g'(0)$ and $g''(0)$ by their expressions in (54) and noting that

$$G(\log \lambda_{\beta, \gamma}) - G(\log \lambda_0) = \tilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}),$$

we get

$$\frac{\phi(\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty})}{\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty}^2} \|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \Lambda}^2 \leq \tilde{K}_n(\lambda_0, \lambda_{\beta, \gamma}) \leq \frac{\phi(-\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty})}{\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty}^2} \|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty}^2$$

According to Assumption A₅ for $(\beta, \gamma) \in \tilde{\Gamma}(\rho)$,

$$\|\log \lambda_{\beta, \gamma} - \log \lambda_0\|_{n, \infty} \leq \rho.$$

Since $\phi(t)/t^2$ is decreasing and bounded below by 0, we can deduce that

$$\frac{\phi(\|\log \lambda_{\beta,\gamma} - \log \lambda_0\|_{n,\infty})}{\|\log \lambda_{\beta,\gamma} - \log \lambda_0\|_{n,\infty}^2} \geq \frac{\phi(\rho)}{\rho^2}$$

and

$$\frac{\phi(-\|\log \lambda_{\beta,\gamma} - \log \lambda_0\|_{n,\infty})}{\|\log \lambda_{\beta,\gamma} - \log \lambda_0\|_{n,\infty}^2} \leq \frac{\phi(-\rho)}{\rho^2}.$$

Take $\rho' := \phi(\rho)/\rho^2 > 0$ and $\rho'' := \phi(-\rho)/\rho^2 > 0$ to finish the proof. \square

B Proof of Proposition 5.2

The beginning of this proof is similar to the proof of Proposition 5.3.

- For β and η in \mathbb{R}^M , let $G : \mathbb{R}^M \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ define by

$$G(\beta) = -\frac{1}{n} \sum_{i=1}^n \int_0^\tau \log(\alpha_0(s) e^{\beta^T \mathbf{Z}_i}) d\Lambda_i(s) + \frac{1}{n} \sum_{i=1}^n \int_0^\tau \alpha_0(s) e^{\beta^T \mathbf{Z}_i} Y_i(s) ds \text{ and } g(t) = G(\beta + t\eta).$$

By differentiating G with respect to t , we get

$$\begin{aligned} g'(t) &= -\frac{1}{n} \sum_{i=1}^n \int_0^\tau \eta^T \mathbf{Z}_i d\Lambda_i(s) + \frac{1}{n} \sum_{i=1}^n \int_0^\tau \alpha_0(s) \eta^T \mathbf{Z}_i e^{(\beta+t\eta)^T \mathbf{Z}_i} Y_i(s) ds \\ g''(t) &= \frac{1}{n} \sum_{i=1}^n \int_0^\tau \alpha_0(s) (\eta^T \mathbf{Z}_i)^2 e^{(\beta+t\eta)^T \mathbf{Z}_i} Y_i(s) ds \\ g'''(t) &= \frac{1}{n} \sum_{i=1}^n \int_0^\tau \alpha_0(s) (\eta^T \mathbf{Z}_i)^3 e^{(\beta+t\eta)^T \mathbf{Z}_i} Y_i(s) ds \end{aligned}$$

It follows that

$$|g'''(t)| \leq \frac{1}{n} \sum_{i=1}^n \int_0^\tau \alpha_0(s) \|\eta\|_2 \|\mathbf{Z}_i\|_2 (\eta^T \mathbf{Z}_i)^2 e^{(\beta+t\eta)^T \mathbf{Z}_i} Y_i(s) ds,$$

Under Assumption **A₆**, we can deduce that $|g'''(t)| \leq R \|\eta\|_2 g''(t)$. Now applying Lemma **A.1** with $S = R \|\eta\|_2$, we obtain for all $t \geq 0$,

$$\frac{g''(0)}{R^2 \|\eta\|_2^2} \phi(R \|\eta\|_2 t) \leq g(t) - g(0) - g'(0)t \leq \frac{g''(0)}{R^2 \|\eta\|_2^2} \phi(-R \|\eta\|_2 t)$$

Take $t = 1$, $\beta = \beta_0$ and $\eta = \hat{\beta}_L - \beta_0$, to write

$$g''(0) \frac{\phi(R \|\hat{\beta}_L - \beta_0\|_2)}{R^2 \|\hat{\beta}_L - \beta_0\|_2^2} \leq G(\hat{\beta}_L) - G(\beta_0) - g'(0) \leq g''(0) \frac{\phi(-R \|\hat{\beta}_L - \beta_0\|_2)}{R^2 \|\hat{\beta}_L - \beta_0\|_2^2} \quad (55)$$

Now straightforward calculations show that $g'(0) = 0$ and

$$g''(0) = \frac{1}{n} \sum_{i=1}^n \int_0^\tau ((\hat{\beta}_L - \beta_0)^T \mathbf{Z}_i)^2 \alpha_0(s) e^{\beta_0^T \mathbf{Z}_i} Y_i(t) dt = \frac{1}{n} \sum_{i=1}^n \int_0^\tau ((\hat{\beta}_L - \beta_0)^T \mathbf{Z}_i)^2 d\Lambda_i(s) = \|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n,\Lambda}^2,$$

Replacing $g'(0)$ and $g''(0)$ by their expressions in (55) and noting that

$$G(\hat{\beta}_L) - G(\beta_0) = \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L}),$$

we get

$$\|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n,\Lambda}^2 \frac{\phi(R\|\hat{\beta}_L - \beta_0\|_2)}{R^2\|\hat{\beta}_L - \beta_0\|_2^2} \leq \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L}) \leq \|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n,\Lambda}^2 \frac{\phi(-R\|\hat{\beta}_L - \beta_0\|_2)}{R^2\|\hat{\beta}_L - \beta_0\|_2^2}. \quad (56)$$

• Now, we will show that $R\|\hat{\beta}_L - \beta_0\|_2$ is bounded. From Equation (27) with $\hat{\beta}_L^\mu = \hat{\beta}_L$ and $\beta = \beta_0$, we can deduce that

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L}) \leq \frac{3}{2} \Gamma_1 \|\Delta_0\|_1,$$

where $\Delta_0 = \mathbf{D}(\hat{\beta}_L - \beta_0)$ and $\mathbf{D} = (\text{diag}(\omega_j))_{1 \leq j \leq M}$. From (56), we have

$$\widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L}) \geq \frac{\|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n,\Lambda}^2}{R^2\|\hat{\beta}_L - \beta_0\|_2^2} \phi(R\|\hat{\beta}_L - \beta_0\|_2)$$

We apply Assumption **RE**(\mathbf{s}, \mathbf{a}_0) with $a_0 = 3$ and $\kappa' = \kappa'(s, 3)$ and we infer that

$$\kappa'^2 \|\Delta_{0,J_0}\|_2^2 \leq \|\Delta_0^T \mathbf{X}\|_{n,\Lambda}^2.$$

So we have,

$$\frac{\kappa'^2 \|\Delta_{0,J_0}\|_2^2}{\max_{1 \leq j \leq M} \omega_j^2} \frac{\phi(R\|\hat{\beta}_L - \beta_0\|_2)}{R^2\|\hat{\beta}_L - \beta_0\|_2^2} \leq \frac{3}{2} \Gamma_1 \|\Delta_0\|_1.$$

We can now use, with $s = |J_0|$, $\|\Delta_0\|_2 \leq \|\Delta_0\|_1 \leq 4\|\Delta_{0,J_0}\|_1 \leq 4\sqrt{s}\|\Delta_{0,J_0}\|_2$ to get

$$\begin{aligned} \kappa'^2 \phi(R\|\hat{\beta}_L - \beta_0\|_2) &\leq \frac{3}{2} \Gamma_1 \frac{\max_{1 \leq j \leq M} \omega_j^2}{\min_{1 \leq j \leq M} \omega_j^2} \max_{1 \leq j \leq M} \omega_j \frac{(4\sqrt{s}\|\hat{\beta}_L - \beta_0\|_{J_0})^2 R^2 \|\hat{\beta}_L - \beta_0\|_2}{\|(\hat{\beta}_L - \beta_0)_{J_0}\|_2^2} \\ &\leq 24 \Gamma_1 \frac{\max_{1 \leq j \leq M} \omega_j^2}{\min_{1 \leq j \leq M} \omega_j^2} \max_{1 \leq j \leq M} \omega_j s R^2 \|\Delta_0\|_2. \end{aligned}$$

A short calculation shows that for all $k \in (0, 1]$:

$$e^{-2k(1-k)^{-1}} + (1-k)2k(1-k)^{-1} - 1 \geq 0.$$

(see Bach [4] for more details) So by taking $2k(1-k)^{-1} = R\|\hat{\beta}_L - \beta_0\|_2$, we have

$$e^{-R\|\hat{\beta}_L - \beta_0\|_2} + R\|\hat{\beta}_L - \beta_0\|_2 - 1 \geq \frac{R^2\|\hat{\beta}_L - \beta_0\|_2^2}{2 + R\|\hat{\beta}_L - \beta_0\|_2}$$

and we deduce that

$$\frac{\kappa'^2 R^2\|\hat{\beta}_L - \beta_0\|_2^2}{2 + R\|\hat{\beta}_L - \beta_0\|_2} \leq 24\Gamma_1 \frac{\max_{1 \leq j \leq M} \omega_j^2}{\min_{1 \leq j \leq M} \omega_j^2} \max_{1 \leq j \leq M} \omega_j s R^2\|\hat{\beta}_L - \beta_0\|_2.$$

This implies that $R\|\hat{\beta}_L - \beta_0\|_2 \leq \frac{\frac{48\Gamma_1 R s \max_{1 \leq j \leq M} \omega_j^2}{\kappa'^2 \min_{1 \leq j \leq M} \omega_j^2} \max_{1 \leq j \leq M} \omega_j^2}{1 - \frac{24\Gamma_1 R s \max_{1 \leq j \leq M} \omega_j^2}{\kappa'^2 \min_{1 \leq j \leq M} \omega_j^2} \max_{1 \leq j \leq M} \omega_j^2} \leq 2$ as soon as $\Gamma_1 \leq \frac{1}{48 R s} \frac{\min_{1 \leq j \leq M} \omega_j^2}{\max_{1 \leq j \leq M} \omega_j^2} \frac{\kappa'^2}{\max_{1 \leq j \leq M} \omega_j}$.

- Since $\phi(t)/t^2$ is decreasing and bounded below by 0, we can deduce that

$$\frac{\phi(R\|\hat{\beta}_L - \beta_0\|_2)}{R^2\|\hat{\beta}_L - \beta_0\|_2^2} \geq \frac{\phi(2)}{4}$$

and

$$\frac{\phi(-R\|\hat{\beta}_L - \beta_0\|_2)}{R^2\|\hat{\beta}_L - \beta_0\|_2^2} \leq \frac{\phi(-2)}{4}$$

Take $\xi := \phi(2)/4 > 0$ and $\xi' := \phi(-2)/4 > 0$ and conclude that

$$\xi\|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n,\Lambda}^2 \leq \widetilde{K}_n(\lambda_0, \lambda_{\hat{\beta}_L}) \leq \xi'\|(\hat{\beta}_L - \beta_0)^T \mathbf{X}\|_{n,\Lambda}^2.$$

□

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