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Amel Zergane

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OVERALGEBRAS AND SEPARATION OF GENERIC COADJOINT ORBITS OF $SL(n, \mathbb{R})$

AMEL ZERGANE *

ABSTRACT. For the semi simple and deployed Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, we give an explicit construction of an overalgebra $\mathfrak{g}^+ = \mathfrak{g} \rtimes V$ of \mathfrak{g} , where V is a finite dimensional vector space. In such a setup, we prove the existence of a map Φ from the dual \mathfrak{g}^* of \mathfrak{g} into the dual $(\mathfrak{g}^+)^*$ of \mathfrak{g}^+ such that the coadjoint orbits of $\Phi(\xi)$, for generic ξ in \mathfrak{g}^* , have a distinct closed convex hulls. Therefore, these closed convex hulls separate 'almost' the generic coadjoint orbits of G .

1. INTRODUCTION

In this paper, we prove that, for $n > 2$, the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ admits an overalgebra almost separating of degree n , but \mathfrak{g} does not admit an overalgebra of degree 2. More precisely:

There exist a Lie overalgebra $\mathfrak{g}^+ = \mathfrak{sl}(n, \mathbb{R}) \rtimes V$ and an application Φ of degree n , $\Phi : \mathfrak{g}^* \longrightarrow \mathfrak{g}^{+*}$ such that:

1. $p \circ \Phi = id_{\mathfrak{g}^*}$, where p is the canonical projection $p : \mathfrak{g}^{+*} \longrightarrow \mathfrak{g}^*$,
2. $\Phi(\overline{Coad}(SL(n, \mathbb{R}))\xi) = \overline{Coad}(G^+)\Phi(\xi)$,
3. if ξ is generic, then $\overline{Conv}(\Phi(\overline{Coad}(SL(n, \mathbb{R}))\xi)) = \overline{Conv}(\Phi(\overline{Coad}(SL(n, \mathbb{R}))\xi'))$ if and only if $\overline{Coad}(SL(n, \mathbb{R}))\xi'$ belongs to a finite set of coadjoint orbits of $\mathfrak{sl}(n, \mathbb{R})$ (here: a singleton if n is odd, a singleton or a set of two elements if n is even).

We identify $(\mathfrak{g}^+)^*$ the dual of \mathfrak{g}^+ with the space $\mathfrak{g}^* \oplus V^*$. The condition 1. means $\Phi(\xi) = \xi + \phi(\xi)$, where ϕ is a polynomial of degree n from \mathfrak{g}^* to V^* . We say that (\mathfrak{g}^+, ϕ) is an overalgebra almost separating of \mathfrak{g} (of degree n).

But there is no separating overalgebra of degree 2, (\mathfrak{g}_2^+, ϕ) , i.e there is neither a Lie overalgebra $\mathfrak{g}_2^+ = \mathfrak{sl}(n, \mathbb{R}) \rtimes V_2$ nor $\phi : \mathfrak{g}^* \longrightarrow V_2^*$ of degree 2 such that :

1. $p \circ \Phi = id_{\mathfrak{g}^*}$, if p is the canonical projection $p : \mathfrak{g}_2^{+*} \longrightarrow \mathfrak{g}^*$,
2. $\Phi(\overline{Coad}(SL(n, \mathbb{R}))\xi) = \overline{Coad}(G^+)\Phi(\xi)$,
3. if ξ is generic then, $\overline{Conv}(\Phi(\overline{Coad}(SL(n, \mathbb{R}))\xi)) = \overline{Conv}(\Phi(\overline{Coad}(SL(n, \mathbb{R}))\xi'))$ if and only if $\overline{Coad}(SL(n, \mathbb{R}))\xi'$ belongs to a finite family of coadjoint orbits.

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- c. If $\det P < 0$ and $r = 0$, then we replace E_1 by $E'_1 = E_2$ and E_2 by $E'_2 = E_1$. The matrix P becomes $P' = D(-1, 1, \dots, 1)P$, $P\xi P^{-1} = P'\xi P'^{-1}$ and $\det(P') > 0$. The adjoint orbit $G \cdot \xi$ of the matrix ξ contains the point $\xi' = P'\xi P'^{-1} = D^-(a_1 + ib_1, \dots, a_s + ib_s)$ of Σ .

□

Lemma 2.2.

Σ is a section for the action of $SL(n, \mathbb{R})$ in Ω , i.e each orbit of Ω contains only a single point of Σ .

Proof.

Let ξ_0 be an element in Σ and $G \cdot \xi_0$ its orbit. If $\xi \in G \cdot \xi_0 \cap \Sigma$, then, since the spectrum of ξ is the same as ξ_0 and the order of eigenvalues is fixed, we obtain:

If $r > 0$ then $\xi = \xi_0$. If $r = 0$, we have either $\xi = \xi_0$, or

$$G \cdot \xi_0 \cap \Sigma = \{\xi, \xi_0\} = \{D^+(a_k + ib_k), D^-(a_k + ib_k) = gD^+(a_k + ib_k)g^{-1}\}.$$

with $\det(g) = 1$.

In the latter case, the sub eigenspaces $Vec_{\mathbb{C}}(E_{2k-1} + iE_{2k})$ and $Vec_{\mathbb{C}}(E_{2k-1} - iE_{2k})$ are one dimensional. Thus, there exist a nonzero complex numbers z_1, \dots, z_s such that :

$$g(E_1 + iE_2) = z_1(E_1 - iE_2), \quad \text{and} \quad g(E_{2k-1} + iE_{2k}) = z_k(E_{2k-1} + iE_{2k}), \quad k > 1.$$

The matrix of g is written in the basis of eigenvectors of the first matrix as follows :

$$QgQ^{-1} = \begin{pmatrix} 0 & \overline{z_1} & & & & \\ z_1 & 0 & & & & \\ & & z_2 & & & \\ & & & \overline{z_2} & & \\ & & & & \dots & \\ & & & & & z_s \\ & & & & & & \overline{z_s} \end{pmatrix}.$$

The determinant of the matrix QgQ^{-1} is negative or zero, which is impossible. Therefore, $\xi = \xi_0$ and Σ is a section for the action of $SL(n, \mathbb{R})$ on Ω . □

Lemma 2.3.

Denote $\Omega_{r,s} = G \cdot \Sigma_{r,s}$. Let $\xi_0 \in \Omega_{r,s}$.

1. If $r > 0$, $\{\xi, \text{ such that } C_\xi = C_{\xi_0}\}$ is exactly the adjoint orbit $G \cdot \xi_0$ of ξ_0 .
2. If $r = 0$, $\{\xi, \text{ such that } C_\xi = C_{\xi_0}\}$ is the union of two adjoint orbits $G \cdot \xi_0 \sqcup G \cdot \xi_1$.

Now, show that Ω is dense. Let ξ be an arbitrary matrix of $\mathfrak{sl}(n, \mathbb{R})$. On \mathbb{C} , we can transform this matrix in Jordan form:

1. for each Jordan block $J_j(c)$ associated to the real eigenvalue c of ξ , there exist vectors E_j^1, \dots, E_j^t in \mathbb{C}^n such that $\xi E_j^1 = cE_j^1$ and, if $t > 1$, $\xi E_j^t = cE_j^t + E_j^{t-1}$. We can choose E_j^t real.

Show that the set Ω is an open :

We use the implicit function theorem. Let ξ_0 a matrix in Ω . The characteristic polynomial C_{ξ_0} of ξ_0 has n simple roots. If $C_{\xi_0}(\alpha) = 0$, then $C'_{\xi_0}(\alpha) \neq 0$.

If c_j is a real eigenvalue of ξ_0 , then we consider the map $F_j : \mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $F_j(\xi, x) = C_\xi(x)$.

If $a_k + ib_k$ is a non real eigenvalue of ξ_0 , we note $C_\xi(z) = C_\xi(x + iy) = A_\xi(x, y) + iB_\xi(x, y)$, where A_ξ and B_ξ are real. C_ξ is a polynomial in z , then

$$\frac{\partial}{\partial \bar{z}} C_\xi(z) = \frac{\partial}{\partial x} C_\xi(z) + i \frac{\partial}{\partial y} C_\xi(z) = 0,$$

for all z . Since $a_k + ib_k$ is a simple root of C_{ξ_0} , then

$$\frac{\partial}{\partial z} C_{\xi_0}(a_k + ib_k) = \frac{\partial}{\partial x} C_{\xi_0}(a_k + ib_k) - i \frac{\partial}{\partial y} C_{\xi_0}(a_k + ib_k) \neq 0.$$

Therefore, we have either $\frac{\partial}{\partial x} A_{\xi_0}(a_k + ib_k) \neq 0$ and $\frac{\partial}{\partial y} B_{\xi_0}(a_k + ib_k) \neq 0$ or $\frac{\partial}{\partial y} A_{\xi_0}(a_k + ib_k) \neq 0$ and $\frac{\partial}{\partial x} B_{\xi_0}(a_k + ib_k) \neq 0$. In all cases,

$$\frac{D(A_{\xi_0}, B_{\xi_0})}{D(x, y)}(a_k, b_k) = \begin{vmatrix} \frac{\partial}{\partial x} A_{\xi_0} & \frac{\partial}{\partial y} A_{\xi_0} \\ \frac{\partial}{\partial x} B_{\xi_0} & \frac{\partial}{\partial y} B_{\xi_0} \end{vmatrix} (a_k, b_k) = (\frac{\partial}{\partial x} A_{\xi_0}(a_k, b_k))^2 + (\frac{\partial}{\partial x} B_{\xi_0}(a_k, b_k))^2 \neq 0.$$

We define $F_j : \mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $F_k(\xi, x, y) = (A_\xi(x + iy), B_\xi(x + iy))$, then $Jac(F_k)(\xi_0, a_k, b_k) \neq 0$.

The functions F_j, F_k are differentiable, then $F_j(\xi_0, c_j) = 0$ and $\frac{\partial F_j}{\partial x}(\xi_0, c_j) = C'_{\xi_0}(c_j) \neq 0$. Similarly, $F_k(\xi_0, a_k, b_k) = 0$ and $\frac{DF_k}{D(x, y)}(\xi_0, a_k, b_k) \neq 0$.

So, there exists an open U_j (resp. U_k) of $\mathfrak{sl}(n, \mathbb{R})$, containing ξ_0 , and there is an open V_j of \mathbb{R} containing c_j (resp. V_k of \mathbb{R}^2 , containing (a_k, b_k)) and there are maps $f_j : U_j \longrightarrow V_j$ (resp. $f_k : U_k \longrightarrow V_k$) such that

$$\left. \begin{array}{l} (\xi, x) \in U_j \times V_j \\ F_j(\xi, x) = 0 \end{array} \right\} \iff \left\{ \begin{array}{l} (\xi, x) \in U_j \times V_j \\ x = f_j(\xi) \end{array} \right.$$

$$\left(\text{resp. } \begin{array}{l} (\xi, x, y) \in U_k \times V_k \\ F_k(\xi, x, y) = 0 \end{array} \right) \iff \left\{ \begin{array}{l} (\xi, x, y) \in U_k \times V_k \\ (x, y) = f_k(\xi) \end{array} \right.$$

We replace as needed the open V_r by another open small enough such that

$$V_j \cap \left(\bigcup_{j' \neq j} V_{j'} \right) = \emptyset, \quad V_k \cap \left((\mathbb{R} \times \{0\}) \cup \bigcup_{k' \neq k} V_{k'} \right) = \emptyset.$$

And we put $U = \bigcap_r U_r$. U is an open containing ξ_0 and, for all ξ in U , C_ξ vanishes at n distinct points (real or complex), then, $U \subset \Omega$.

Lemma 2.5.

The set Ω is an open in $\mathfrak{sl}(n, \mathbb{R})$.

 3. $\mathfrak{sl}(n, \mathbb{R})$ ADMITS AN OVERALGEBRA ALMOST SEPARATING OF DEGREE n
3.1. Separation of orbits of Ω by invariant functions.

This is also well known. Let ξ a $n \times n$ real matrix and C_ξ its characteristic polynomial. On \mathbb{C} , we can put ξ in Jordan form. We note z_1, \dots, z_n the diagonal terms of this Jordan form. Then :

$$\begin{aligned} C_\xi(X) &= (-1)^n \det(\xi - XI) = (X - z_1) \cdots (X - z_n) \\ &= X^n - \left(\sum_i z_i \right) X^{n-1} + \left(\sum_{i < j} z_i z_j \right) X^{n-2} + \cdots + (-1)^n z_1 \cdots z_n \\ &= X^n - \alpha_{n-1} X^{n-1} + \alpha_{n-2} X^{n-2} + \cdots + (-1)^n \alpha_0. \end{aligned}$$

Therefore, using a formula due to Newton (cf. [W]), we have, for all k ,

$$\begin{aligned} (-1)^{k+1} \sum_{i_1 < \cdots < i_k} z_{i_1} z_{i_2} \cdots z_{i_k} &= \sum_j z_j^k - \left(\sum_{i_1} z_{i_1} \right) \left(\sum_j z_j^{k-1} \right) + \left(\sum_{i_1 < i_2} z_{i_1} z_{i_2} \right) \left(\sum_j z_j^{k-2} \right) + \cdots \\ &\quad + \cdots + (-1)^{k-1} \left(\sum_{i_1 < \cdots < i_{k-1}} z_{i_1} \cdots z_{i_{k-1}} \right) \left(\sum_j z_j \right) \end{aligned}$$

or

$$(-1)^{k+1} \alpha_{n-k} = Tr(\xi^k) - \alpha_{n-1} Tr(\xi) + \alpha_{n-2} Tr(\xi^{k-1}) + \cdots + (-1)^{k-1} \alpha_{k-1} Tr(\xi).$$

This formula allows to express all α_k as functions of the numbers $Tr(\xi^j)$, and conversely, to express all $Tr(\xi^k)$ as functions of the numbers α_j .

Finally, we deduce that:

two matrices ξ and ξ' satisfying $C_\xi = C_{\xi'}$ if and only if $Tr(\xi^k) = Tr(\xi'^k)$ for all $k = 1, \dots, n$.

Proposition 3.1.

We keep all previous notations, in particular, $\Omega = \cup_{r+2s=n} \Omega_{r,s}$ is an open, dense and invariant subset of $\mathfrak{sl}(n, \mathbb{R})$. The orbits of Ω will be called generic orbits. Let $\xi_0 \in \Omega_{r,s}$.

1. If $r > 0$, $\{ \xi, \text{ such that } T_k(\xi) = T_k(\xi_0) \text{ for all } k = 2, \dots, n \}$ is exactly the adjoint orbit $G \cdot \xi_0$ of ξ_0 ,
2. If $r = 0$, $\{ \xi, \text{ such that } T_k(\xi) = T_k(\xi_0) \text{ for all } k = 2, \dots, n \}$ is the union of two adjoint orbits $G \cdot \xi_0 \sqcup G \cdot \xi_1$.

We say that the invariant functions T_k separate almost the (co)adjoint generic orbits of $\mathfrak{sl}(n, \mathbb{R})$.

3.2. Convex hull of orbits of Ω .

For $n = 2$, the convex hull of the orbits of Ω are well known (cf.[ASZ]). We deduce that, for $n = 2$:

$$\text{Conv}(G \cdot D(-c, c)) = \mathfrak{sl}(2, \mathbb{R}) \quad (c > 0), \quad \text{and}$$

$$\text{Conv}(G \cdot D^+(ib) \cup G \cdot D^-(ib)) = \mathfrak{sl}(2, \mathbb{R}) \quad (b > 0).$$

For $n = 3$, we deduce that $\Omega \subset \text{Conv}(G \cdot D(c_1, c_2, c_3))$. Indeed, if $c'_1 < c'_2 < c'_3$ such that $c'_1 + c'_2 + c'_3 = 0$, then either $c'_1 \neq -2c_3$, or $c'_2 \neq -2c_3$. Suppose $c'_1 \neq -2c_3$, the other case is trained the same by exchanging the induces 1 and 2. Let $c''_1 = c'_1 - \frac{c_1 + c_2}{2}$ and $c''_2 = -c'_1 - c_3$. We write:

$$\begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(c_1 - c_2) & \\ & \frac{1}{2}(c_2 - c_1) \end{pmatrix} + \frac{c_1 + c_2}{2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

then, there exist t in $[0, 1]$ and $g \in SL(2, \mathbb{R})$ such that :

$$\begin{pmatrix} c'_1 & \\ & c'_2 \end{pmatrix} = \begin{pmatrix} c''_1 & \\ & -c''_1 \end{pmatrix} + \frac{c_1 + c_2}{2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = t \begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix} + (1-t)g \begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix} g^{-1}.$$

We deduce that the convex hull of $G \cdot D(c_1, c_2, c_3)$ contains $\begin{pmatrix} c'_1 & \\ & c''_2 \\ & & c_3 \end{pmatrix}$ with $c''_2 \neq c_3$.

By the same argument, but with induces 2 and 3, we show that this convex hull contains $D(c'_1, c'_2, c'_3)$. Let now $a' = -\frac{1}{2}c'_1$, and $b' > 0$, then :

$$\begin{pmatrix} c'_2 & \\ & c'_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(c'_2 - c'_3) & \\ & \frac{1}{2}(c'_3 - c'_2) \end{pmatrix} + a' \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

the convex hull of $G \cdot D(c_1, c_2, c_3)$ contains also

$$D(c'_1, a' + ib') = \begin{pmatrix} c'_1 & & \\ & 0 & b' \\ & -b' & 0 \end{pmatrix} + a' \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

On the other hand, we saw that $\begin{pmatrix} c & & \\ & a & -b \\ & b & a \end{pmatrix}$ belongs to $G \cdot D(c, a + ib)$. Therefore,

if $a \neq 0$ then $\text{Conv}(G \cdot D(c, a + ib))$ contains the matrix $\begin{pmatrix} c & & \\ & a & \\ & & a \end{pmatrix}$, with $a \neq c$. So,

by our first argument, $\text{Conv}(G \cdot D(c, a + ib))$ contains the matrix $\begin{pmatrix} c'_1 & & \\ & c'_2 & \\ & & a \end{pmatrix}$ with $c'_1 \neq c'_2 \neq a \neq c'_1$. Therefore, by the above, $\text{Conv}(G \cdot D(c, a + ib))$ contains all Ω .

If $a = 0$, $\text{Conv}(G \cdot D(0, ib))$ contains the matrices $\begin{pmatrix} 0 & \\ & D^+(ib) \end{pmatrix}$ and $\begin{pmatrix} 0 & \\ & D^-(ib) \end{pmatrix}$.

So, $\text{Conv}(G \cdot D(0, ib))$ contains the matrix $\begin{pmatrix} 0 & & \\ & -1 & \\ & & 1 \end{pmatrix}$. Finally, $\text{Conv}(G \cdot D(0, ib))$ contains all Ω .

We have proved:

Lemma 3.2.

If $n = 3$ and $\xi \in \Omega$, then $\Omega \subset \text{Conv}(G \cdot \xi)$.

If $n = 4$, as above, $\Omega \subset \text{Conv}(G \cdot D(c_1, \dots, c_4))$. We deduce by using the lemma that for all c'_2, \dots, c'_4 ,

$$D(c_1, c'_2, c'_3, c'_4) \in \text{Conv}(G \cdot D(c_1, c_2, a + ib))$$

and also $\Omega \subset \text{Conv}(G \cdot D(c_1, c_2, a + ib))$. It remains the cases $D(a_1 + ib_1, a_2 + ib_2)$, $a_1 \neq 0$ and $D(ib_1, ib_2)$. In the first case, we saw that

$$\begin{pmatrix} a_1 & -b_1 & & \\ b_1 & a_1 & & \\ & & a_2 & -b_2 \\ & & b_2 & a_2 \end{pmatrix} \in G \cdot D(a_1 + ib_1, a_2 + ib_2),$$

then

$$\begin{pmatrix} a_1 & & & \\ & a_1 & & \\ & & a_2 & \\ & & & a_2 \end{pmatrix} \in \text{Conv}(G \cdot D(a_1 + ib_1, a_2 + ib_2)) \quad \text{and } a_1 \neq a_2.$$

By applying the calculation for $n = 2$, we deduce that $D(a_1, x, y, a_4)$ belongs to $\text{Conv}(G \cdot D(a_1 + ib_1, a_2 + ib_2))$, for all x and y such that $a_1 + x + y + a_4 = 0$. Therefore, $\Omega \subset \text{Conv}(G \cdot D(a_1 + ib_1, a_2 + ib_2))$.

For the latter case, we saw that, $\text{insl}(2, \mathbb{R})$, the adjoint orbit of $D(ib)$ is the set of matrices $\begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}$ with $z^2 - x^2 - y^2 = b^2$ and $z > 0$. Then, $G \cdot D(ib_1, ib_2)$ contains a matrix as follows:

$$\begin{pmatrix} x & z & & \\ -z & -x & & \\ & & 0 & b_2 \\ & & -b_2 & 0 \end{pmatrix}, \quad \text{with } 0 < b_1 < z.$$

Combining this matrix with $\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$, we obtain :

$$\xi = \begin{pmatrix} -x & -z & & \\ z & x & & \\ & & 0 & -b_2 \\ & & b_2 & 0 \end{pmatrix} \in G \cdot D(ib_1, ib_2).$$

If $t = \frac{b_1}{z + b_1}$, the matrix $t\xi + (1 - t)D(ib_1, ib_2)$ is $D(-tx, tx, i(1 - 2t)b_2) \in \text{Conv}(G \cdot D(ib_1, ib_2))$. Or

Lemma 3.3.

If $n = 4$ and $\xi \in \Omega$, then $\Omega \subset \text{Conv}(G \cdot \xi)$.

Proposition 3.4.

For all $n > 2$, the convex hull of the adjoint orbit of a point ξ in Ω contains Ω :

$$\Omega \subset \text{Conv}(G \cdot \xi).$$

This convex hull is dense in $\mathfrak{sl}(n, \mathbb{R})$.

Proof.

By induction on $n > 4$, suppose that, for all $2 < p < n$, this property is true. We consider $D(c_j, a_k + ib_k) \in \Sigma_{r,s}$, with $r + 2s = n$.

If $r \geq 2$, then $n - 2 > 2$, and we write:

$$D(c_j, a_k + ib_k) = \left(D(c'_1, c'_2) \quad D(c'_3, \dots, c'_r, a'_k + ib'_k) \right) + \begin{pmatrix} \frac{c_1 + c_2}{2} I_2 & \\ & -\frac{c_1 + c_2}{n - 2} I_{n-2} \end{pmatrix},$$

Using the fact that $\mathfrak{sl}(2, \mathbb{R}) = \text{Conv}(G \cdot D(c'_1, c'_2))$ and by the induction hypothesis for $n - 2$, we get $\Omega \subset \text{Conv}(G \cdot D(c_j, a_k + ib_k))$.

If $r = 1$, we decompose $D(c_1, a_k + ib_k)$ as : $D(c_1, a_k + ib_k) =$

$$\left(D(c'_1, a'_1 + ib_1) \quad D(a'_2 + ib'_2, \dots, a'_n + ib'_n) \right) + \begin{pmatrix} \frac{c_1 + 2a_1}{3} I_3 & \\ & -\frac{c_1 + 2a_1}{n - 3} I_{n-3} \end{pmatrix}.$$

Then, a matrix $D(c''_1, c''_2, c''_3, a_2 + ib_2, \dots, a_n + ib_n)$ belongs to $\text{Conv}(G \cdot D(c_1, a_k + ib_k))$. Therefore, the first case applies, and we still get the result.

If $r = 0$, then $s > 2$, we decompose $D(a_k + ib_k)$ as : $D(a_k + ib_k) =$

$$\left(\begin{array}{c} D(a'_1 + ib_1, a'_2 + ib'_2) \\ D(a'_3 + ib'_3, \dots, a'_n + ib'_n) \end{array} \right) + \left(\begin{array}{c} \frac{2a_1 + 2a_2}{4} I_4 \\ -\frac{2a_1 + 2a_2}{n-4} I_{n-4} \end{array} \right).$$

Then a matrix $D(c''_1, \dots, c''_4, a_3 + ib_3, \dots, a_n + ib_n)$ belongs to $\text{Conv}(G \cdot D(a_k + ib_k))$. So, the first case applies, and this completes the proof of our proposition. □

Corollary 3.5. $\mathfrak{sl}(n, \mathbb{R})$ admits an overalgebra almost separating of degree n .

Proof.

$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ admits an overalgebra of degree n , given by :

$$\mathfrak{g}^+ = \mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}^{n-1}$$

$$\Phi : \mathfrak{g}^* \longrightarrow \mathfrak{g}^{+*}, \quad \Phi(X) = (X, \phi(X)) = (X, T_2(X), T_3(X), \dots, T_n(X)).$$

Indeed, ϕ is polynomial, with degree n .

Moreover, we have for all ξ in Ω ,

$$\begin{aligned} \overline{\text{Conv}}(\Phi(\text{Coad}(SL(n, \mathbb{R}))\xi)) &= \overline{\text{Conv}}(\text{Coad}(SL(n, \mathbb{R}))\xi \times (T_2(\xi), \dots, T_n(\xi))) \\ &= \mathfrak{sl}(n, \mathbb{R})^* \times (T_2(\xi), \dots, T_n(\xi)). \end{aligned}$$

Then $\Phi(\xi')$ belongs to this set if and only if $T_k(\xi') = T_k(\xi)$ for all k , if and only if $C_{\xi'} = C_\xi$.

We saw that if n is odd, $\{\xi', \text{ such that } C_{\xi'} = C_\xi\}$ is exactly the orbit $\text{Coad}(SL(n, \mathbb{R}))\xi$ and, if n is even, $\{\xi', \text{ such that } C_{\xi'} = C_\xi\}$ is either the orbit $\text{Coad}(SL(n, \mathbb{R}))\xi$, or, if C_ξ has only non real roots, the set $\{\xi', \text{ such that } C_{\xi'} = C_\xi\}$ is the union of two disjoint orbits. This proves that (\mathfrak{g}^+, ϕ) is an overalgebra almost separating of degree n of $\mathfrak{sl}(n, \mathbb{R})$. □

4. OVERALGEBRA ALMOST SEPARATING OF DEGREE p OF A LIE ALGEBRA \mathfrak{g}

Definition 4.1. (*Semi direct product*)

Let G be a real Lie group, V a finite dimensional vector space and (π, V) a linear representation of G . Denote by $G^+ = G' \rtimes V$ the Lie group whose set $G \times V$ and law:

$$(g, v) \cdot (g', v') = (gg', v + \pi(g)v').$$

Its Lie algebra is $\mathfrak{g}^+ = \mathfrak{g}' \rtimes V$, whose space $\mathfrak{g} \oplus V$ and bracket :

$$[(X, u), (X', u')] = ([X, X'], \pi'(X)u' - \pi'(X')u).$$

(π' is the derivative of π , π' is the representation of \mathfrak{g} in V).

The exponential map is

$$\exp(X, u) = \left(\exp X, \frac{e^{\pi'(X)} - I}{\pi'(X)} u \right).$$

We also define the linear map $\psi_u : \mathfrak{g} \longrightarrow V$, by $\psi_u(X) = \pi'(X)u$, for all $u \in V$. Then, the coadjoint action is realized in $\mathfrak{g}^{+\star} = \mathfrak{g}^{\star} \times V^{\star}$ and defined by:

$$Coad'(X, u)(\xi, f) = (Coad'(X)\xi + {}^t\psi_u(f), -{}^t\pi'(X)f).$$

The group action is

$$Coad(g, v)(\xi, f) = (Coad(g)\xi + {}^t\psi_v({}^t\pi(g^{-1})f), {}^t\pi(g^{-1})f).$$

Denote by $\pi^*(g) = {}^t\pi(g^{-1})$.

Let $\Phi : \mathfrak{g}^{\star} \longrightarrow \mathfrak{g}^{+\star}$ be a map non necessarily linear. We assume that $p \circ \Phi = id$, then Φ is written $\Phi(\xi) = (\xi, \phi(\xi))$. ϕ is not necessarily linear.

Assume that $\Phi(Coad(G)\xi) = Coad(G^+)\Phi(\xi)$, then : for all g in G and all v in V , there exists $g' \in G$ ($g' = g'_{g,v,\xi}$) such that

$$\begin{cases} \pi^*(g)\phi(\xi) = \phi(Coad(g')\xi), \\ Coad(g)(\xi) + ({}^t\psi_v \circ \pi^*(g))\phi(\xi) = Coad(g')\xi = Coad(g)(\xi) + {}^t\psi_v \circ \phi(Coad(g')\xi). \end{cases}$$

In particular, if X is in \mathfrak{g} , then $\pi^*(\exp(tX))(\phi(\xi)) = \phi(Coad(g'_t)\xi)$. The continuous curve $t \mapsto \pi^*(\exp(tX))(\phi(\xi))$ is drawn on the surface $\mathcal{C} = \phi(Coad(G)\xi)$, its derivative at 0 is the vector $\pi^{*\prime}(X)\phi(\xi)$. This vector belongs to the tangent space

$$T_{\phi(\xi)}(\mathcal{C}) = \phi'(\phi(\xi))(T_{\xi}(Coad(G)\xi)) = \phi'(\phi(\xi))(Coad(\mathfrak{g})\xi).$$

We have also, for the same g in G , v in V , and $g' = g'_{g,v,\xi} \in G$,

$$Coad(g)(\xi) = (I - {}^t\psi_v \circ \phi)(Coad(g')\xi).$$

We deduce that, if $v = 0$, then $Coad(g)(\xi) = Coad(g'_{g,0,\xi})(\xi)$ and therefore $\pi^*(g)\phi(\xi) = \phi(Coad(g)\xi)$. So:

Lemma 4.2.

ϕ is an intertwining (non linear) between the coadjoint representation and the representation π^ .*

If ϕ is polynomial of degree p , then ϕ is written :

$$\phi(\xi) = \phi_1(\xi) + \phi_2(\xi) + \dots + \phi_p(\xi),$$

with ϕ_k homogeneous of degree k .

Since ϕ is an intertwining, then $\phi \circ Ad_g = \pi^*(g) \circ \phi$, and for all k , $\phi_k \circ Ad_g = \pi^*(g) \circ \phi_k$, i.e each ϕ_k is an intertwining.

On the other hand, for each k , $\phi_k(\xi)$ can be written

$$\phi_k(\xi) = b_k(\underbrace{\xi \dots \xi}_k),$$

where b_k is a linear map from $S^k(\mathfrak{g}^*)$ in V^* . The map b_k is also an intertwining, since the action $Coad^k$ of G on $S^k(\mathfrak{g}^*)$ is such that :

$$\phi_k(Coad(g)\xi) = b_k(Coad(g)\xi \cdot \dots \cdot Coad(g)\xi) = (b_k \circ Coad^k(g))(\xi \cdot \dots \cdot \xi).$$

Put then:

$$S_p(\mathfrak{g}^*) = \mathfrak{g}^* \oplus S^2(\mathfrak{g}^*) \oplus \dots \oplus S^p(\mathfrak{g}^*),$$

and

$$b: S_p(\mathfrak{g}^*) \longrightarrow V^*, \quad b(v_1 + v_2 + \dots + v_p) = b_1(v_1) + \dots + b_p(v_p).$$

Let $U = b(S_p(\mathfrak{g}^*))$. U is a submodule of V^* , isomorphic to the quotient module $S_p(\mathfrak{g}^*)/\ker(b)$. Put then $W = V/U^\perp$. W is a quotient module of the module V such that $W^* \simeq U$ (and then $W \simeq U^*$).

Lemma 4.3.

If $(\mathfrak{g} \rtimes V, \phi)$ is an overalgebra almost separating of \mathfrak{g} , then $(\mathfrak{g} \rtimes W, \tilde{\phi})$, where

$$\tilde{\phi}(\xi) = b(\xi + \xi \cdot \xi + \dots + \xi \cdot \dots \cdot \xi)$$

is also an overalgebra almost separating of \mathfrak{g} .

Proof.

In the statement of this lemma, we identify W^* with the submodule U of V^* . With this identification, if ι is the canonical injection of U in V^* , then $\phi(\xi) = \iota \circ \tilde{\phi}(\xi)$. The application Φ becomes $\tilde{\Phi}(\xi) = (\xi, \iota \circ \phi(\xi)) = (j \circ \Phi)(\xi)$ if $j(\xi, v) = (\xi, \iota(v))$. Therefore

$$\overline{Conv}(\tilde{\Phi}(CoadG \xi)) = j(\overline{Conv}(\Phi(CoadG \xi))),$$

and $\overline{Conv}(\tilde{\Phi}(CoadG \xi)) = \overline{Conv}(\tilde{\Phi}(CoadG \xi'))$ if and only if $\overline{Conv}(\Phi(CoadG \xi)) = \overline{Conv}(\Phi(CoadG \xi'))$. We deduce that $(\mathfrak{g} \rtimes W, \tilde{\phi})$ or, if we prefer, $(\mathfrak{g} \rtimes (S_p(\mathfrak{g}^*)/\ker b)^*, \tilde{\phi})$ is an overalgebra almost separating of \mathfrak{g} .

□

If \mathfrak{g} is semi-simple and deployed, then all its representations are completely reducible. Therefore $W^* = S_p(\mathfrak{g}^*)/\ker b$ is isomorphic to a submodule of $S_p(\mathfrak{g}^*) = S_p(\mathfrak{g})$. In this case, W is isomorphic to a submodule of $(S_p(\mathfrak{g}))^*$. So, we consider the application ϕ with values in $S_p(\mathfrak{g})$, and ϕ becomes :

$$\phi(\xi) = b_1(\xi) + \dots + b_p(\xi \cdot \dots \cdot \xi).$$

The application b becomes an intertwining of modules $S_p(\mathfrak{g})$.

Corollary 4.4.

If \mathfrak{g} is a deployed and semi-simple Lie algebra, admitting an overalgebra almost separating of degree p , and τ a natural application from $\mathfrak{g} = \mathfrak{g}^*$ to $S_p(\mathfrak{g})$ defined by : $\tau(\xi) = \xi + \xi \cdot \xi + \dots + \xi \cdot \dots \cdot \xi$, then there exists an intertwining b of $S_p(\mathfrak{g})$ such that $(\mathfrak{g} \rtimes (S_p(\mathfrak{g}))^*, b \circ \tau)$ is an overalgebra almost separating of \mathfrak{g} .

5. THE CASE $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ AND $p = 2$ 5.1. The case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$.

We suppose now $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. Recall the usual notations (cf. [FH]).

$\mathfrak{sl}(n, \mathbb{R})$ is a real simple algebra. A Cartan subalgebra \mathfrak{h} of $\mathfrak{sl}(n, \mathbb{R})$, of dimension $n - 1$, is given by the set of diagonal matrices $\xi = D(c_1, \dots, c_n)$. With this Cartan algebra, $\mathfrak{sl}(n, \mathbb{R})$ is deployed. For $H \in \mathfrak{h}$, we note $L_i(H) = c_i$, with $L_1 + \dots + L_n = 0$. We choose the usual system of simple roots, i.e the forms $\alpha_i = L_i - L_{i+1}$ ($1 \leq i \leq n - 1$). The system of positive roots is the set of forms $L_i - L_j$, with $i < j$. If $e_{ij} = (x_{rs})$ is the $n \times n$ matrix such that $x_{rs} = \delta_{ri}\delta_{sj}$, then for all H in \mathfrak{h} ,

$$\text{ad}(H)e_{ij} = \begin{cases} (\alpha_i + \dots + \alpha_{j-1})(H)e_{ij}, & \text{if } i < j \\ -(\alpha_j + \dots + \alpha_{i-1})(H)e_{ij}, & \text{if } i > j. \end{cases}$$

The fundamental weights are $\omega_k = L_1 + \dots + L_k$ ($1 \leq k \leq n - 1$) and the simple modules are exactly the modules noted $\Gamma_{a_1 \dots a_{n-1}}$ of highest weight $a_1\omega_1 + \dots + a_{n-1}\omega_{n-1}$, with a_k integer. Moreover, the dual of $\Gamma_{a_1, \dots, a_{n-1}}$ is the module $\Gamma_{a_{n-1}, \dots, a_1}$. (cf. [FH]).

Let $X \mapsto X^s$ be the symmetry operation relative to the second diagonal given by: if X is the matrix (x_{ij}) , then X^s is the matrix (x_{ij}^s) with :

$$x_{ij}^s = x_{(n+1-j)(n+1-i)},$$

The operation s leaves the Cartan subalgebra invariant. For all weight ω , put $\omega^s(H) = \omega(H^s)$. In particular, $L_i^s = L_{n+1-i}$ and $\omega_j^s = -\omega_{n-j}$.

Moreover, s permutes the radicial spaces, since $e_{ij}^s = e_{(n+1-j)(n+1-i)}$, or, if H belongs to \mathfrak{h} and $i < j$,

$$[H, e_{ij}^s] = -(L_i^s - L_j^s)(H)e_{ij}^s.$$

We consider now the module $S^k(\mathfrak{sl}(n, \mathbb{R}))$ i.e the space of the sums $\sum_{i_1 < \dots < i_k} \lambda_{i_1, \dots, i_k} X_{i_1} \cdot \dots \cdot X_{i_k}$ on which $\mathfrak{sl}(n, \mathbb{R})$ acts by the adjoint action ad defined by:

$$ad_X(X_{i_1} \cdot \dots \cdot X_{i_k}) = \sum_{r=1}^k X_{i_1} \cdot \dots \cdot [X, X_{i_r}] \cdot \dots \cdot X_{i_k}.$$

Lemma 5.1.

The space $S^k(\mathfrak{sl}(n, \mathbb{R}))$ is self-dual i.e $(S^k(\mathfrak{sl}(n, \mathbb{R})))^* = S^k(\mathfrak{sl}(n, \mathbb{R}))$.

Proof.

Suppose that the module $\Gamma_{a_1, \dots, a_{n-1}}$ appears in $S^k(\mathfrak{sl}(n, \mathbb{R}))$. Then, there is a non zero vector $v_{a_1, \dots, a_{n-1}}$ such that , for all H in \mathfrak{h} , and for all $i < j$,

$$ad_H(v_{a_1 \dots a_{n-1}}) = (a_1\omega_1 + \dots + a_{n-1}\omega_{n-1})(H)v_{a_1 \dots a_{n-1}}, \quad ad_{e_{ij}}(v_{a_1 \dots a_{n-1}}) = 0.$$

If $v = \sum_{i_1 < \dots < i_k} \lambda_{i_1, \dots, i_k} X_{i_1} \cdot \dots \cdot X_{i_k}$ is a vector of $S^k(\mathfrak{sl}(n, \mathbb{R}))$, then $v^s = \sum_{i_1 < \dots < i_k} \lambda_{i_1, \dots, i_k} X_{i_1}^s \cdot \dots \cdot X_{i_k}^s$. Moreover, the map $v \mapsto v^s$ is an involutive bijection : $(v^s)^s = v$.

The vector $v_{a_1 \dots a_{n-1}}^s$ is not zero and, for all H in \mathfrak{h} , and all $i < j$,

$$\begin{aligned} ad_H(v_{a_1 \dots a_{n-1}}^s) &= -(a_1 \omega_1 + \dots + a_{n-1} \omega_{n-1})^s(H) v_{a_1 \dots a_{n-1}}^s \\ &= (a_{n-1} \omega_1 + \dots + a_1 \omega_{n-1})(H) v_{a_1 \dots a_{n-1}}^s, \\ ad_{e_{ij}}(v_{a_1 \dots a_{n-1}}^s) &= 0. \end{aligned}$$

In other words,

$$(\Gamma_{a_1, \dots, a_{n-1}})^s = \Gamma_{a_{n-1}, \dots, a_1} \simeq (\Gamma_{a_1, \dots, a_{n-1}})^*.$$

□

Corollary 5.2.

If $\mathfrak{sl}(n, \mathbb{R})$ admits an overalgebra almost separating of degree p , if τ is the natural application of $\mathfrak{sl}(n, \mathbb{R})^*$ in $S_p(\mathfrak{sl}(n, \mathbb{R})) = (S_p(\mathfrak{sl}(n, \mathbb{R})))^*$ given by $\tau(\xi) = \xi + \xi \cdot \xi + \dots + \xi \cdot \dots \cdot \xi$, then there exist an intertwining b_k of $S^k(\mathfrak{sl}(n, \mathbb{R}))$ ($k = 1, \dots, p$) such that $(\mathfrak{sl}(n, \mathbb{R}) \rtimes S_p(\mathfrak{sl}(n, \mathbb{R})), \sum_k b_k \circ \tau)$ is an overalgebra almost separating of the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$.

5.2. The case $p = 2$.

For $k = 1$ and $k = 2$, looking for the intertwining between $S^k(\mathfrak{sl}(n, \mathbb{R})^*)$ and $(S^k(\mathfrak{sl}(n, \mathbb{R})))^*$.

For $k = 1$, the space of these intertwining is one dimensional and generated by P_0 defined by :

$$\langle P_0(\xi), X \rangle = Tr(\xi X).$$

Therefore, any intertwining b_1 is written $b_1 = a_0 P_0$, with a_0 real. For $k=2$:

5.2.1. Decomposition of $S^2(\mathfrak{g})$.

The module $S^2(\mathfrak{g})$ is the sum of three or four irreducible modules, all of different types. Recall the usual notations (cf. [FH]).

- The highest weight vector in $S^2(\mathfrak{sl}(n, \mathbb{R}))$ is

$$v_{20 \dots 02} = e_{1n} \cdot e_{1n}.$$

The weight of this vector is $2\omega_1 + 2\omega_n = 2L_1 - 2L_n$. Then, we deduce the existence of a simple module $\Gamma_{20 \dots 02}$ of dimension (cf. [FH]):

$$\dim \Gamma_{20 \dots 02} = \prod_{i=2}^{n-1} \frac{2+n-i}{n-i} \cdot \prod_{j=2}^{n-1} \frac{2+j-1}{j-1} \cdot \frac{4+n-1}{n-1} = \frac{n^2(n-1)(n+3)}{4}.$$

- Among the weight vectors of weight $\omega_2 + \omega_{n-2} = L_1 - L_n + L_2 - L_{n-1}$, and if $n > 3$, there is one that is annulled by the action of $e_{i(i+1)}$, $1 \leq i \leq (n-1)$. This weight vector is :

$$v_{010 \dots 010} = e_{2n} \cdot e_{1(n-1)} - e_{2(n-1)} \cdot e_{1n}.$$

We deduce then the existence of a simple module $\Gamma_{010\dots010}$ of dimension:
 $\dim \Gamma_{010\dots010} =$

$$\prod_{j=3}^{n-2} \frac{1+(j-1)}{j-1} \cdot \prod_{j=3}^{n-2} \frac{1+(j-2)}{j-2} \cdot \prod_{i=3}^{n-2} \frac{1+(n-1)-i}{n-1-i} \cdot \prod_{i=3}^{n-2} \frac{1+n-i}{n-i} \cdot \frac{n^2(n+1)}{4(n-2)^2(n-3)}$$

$$= \frac{n^2(n+1)(n-3)}{4}$$

If $n = 3$, e_{22} is not in $\mathfrak{sl}(3)$, and this dimension is 0. This sub module does not appear.

- Among the weight vectors of weight $\omega_1 + \omega_{n-1} = L_1 - L_n$, there is one that is annulled by the action of $e_{i(i+1)}$, $1 \leq i \leq (n-1)$. This weight vector is:

$$v_{10\dots01} = \sum_{i=1}^n e_{1i} \cdot e_{in} - \frac{2}{n} \sum_{j=1}^n e_{jj} \cdot e_{1n}.$$

Then, we deduce the existence of a simple module $\Gamma_{10\dots01}$ of dimension:

$$\dim \Gamma_{10\dots01} = \prod_{i=2}^{n-1} \frac{1+n-i}{n-i} \cdot \prod_{j=2}^{n-1} \frac{1+j-1}{j-1} \cdot \frac{2+n-1}{n-1} = n^2 - 1.$$

- Among the weight vectors of weight 0, there is one that is annulled by the action of $e_{i(i+1)}$, $1 \leq i \leq (n-1)$. This weight vector is:

$$v_{00\dots00} = 2n \sum_{1 \leq i < j \leq n} e_{ij} \cdot e_{ji} + \sum_{1 \leq i < j \leq n} (e_{ii} - e_{jj}) \cdot (e_{ii} - e_{jj})$$

We deduce the existence of a trivial simple module $\Gamma_{00\dots00}$ of dimension 1.

Therefore:

$$S^2(\mathfrak{sl}(n, \mathbb{R})) \cong \begin{cases} \Gamma_{20\dots02} \oplus \Gamma_{10\dots01} \oplus \Gamma_{010\dots010} \oplus \Gamma_{0\dots0}, & \text{if } n > 3 \\ \Gamma_{22} \oplus \Gamma_{11} \oplus \Gamma_{00}, & \text{if } n = 3. \end{cases}$$

since the dimensions, $\frac{n^2(n^2-1)}{2} = \frac{n^2(n-1)(n+3)}{4} + n^2 - 1 + \frac{n^2(n+1)(n-3)}{4} + 1.$

5.2.2. Intertwining of $S^2(\mathfrak{sl}(n, \mathbb{R}))$.

Let P_1, P_2, P_3 and P_4 the intertwining defined from $S^2(\mathfrak{sl}(n, \mathbb{R}))$ in $(S^2(\mathfrak{sl}(n, \mathbb{R})))^*$, such that, for all $\xi, \eta \in \mathfrak{sl}(n, \mathbb{R})$ and $X, Y \in \mathfrak{sl}(n, \mathbb{R})$:

- $\langle P_1(\xi, \eta), X \cdot Y \rangle = Tr(\xi X \eta Y) + Tr(\xi Y \eta X),$
- $\langle P_2(\xi, \eta), X \cdot Y \rangle = Tr(\xi X) Tr(\eta Y) + Tr(\xi Y) Tr(\eta X),$
- $\langle P_3(\xi, \eta), X \cdot Y \rangle = Tr(\xi \eta X Y) + Tr(\xi \eta Y X) + Tr(\eta \xi X Y) + Tr(\eta \xi Y X),$
- $\langle P_4(\xi, \eta), X \cdot Y \rangle = Tr(\xi \eta) Tr(X Y).$

In particular, we have :

- $P_2(v_{20\dots02}) = P_3(v_{20\dots02}) = P_4(v_{20\dots02}) = 0$, and $\langle P_1(v_{20\dots02}), e_{n1} \cdot e_{n1} \rangle = 1 \neq 0$,
- $P_3(v_{010\dots010}) = P_4(v_{010\dots010}) = 0$, and $\langle P_2(v_{010\dots010}), e_{n2} \cdot e_{(n-1)1} \rangle = 4 \neq 0$,
- $P_4(v_{10\dots01}) = 0$, and $\langle P_3(v_{10\dots01}), e_{n2} \cdot e_{21} \rangle \neq 0$,
- $P_4(v_{00\dots00}) \neq 0$.

Thus, if $n > 3$, P_1, P_2, P_3 and P_4 are independent and, since the dimension of the space of intertwining of $S^2(\mathfrak{sl}(n, \mathbb{R}))$ is 4, then any intertwining b_2 is written :

$$b_2 = a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4, \quad \text{where } a_i \text{ are real constants.}$$

If $n = 3$, P_1, P_3 and P_4 are independent and, since the dimension of the space of intertwining is three, then we can write :

$$b_2 = a_1P_1 + a_3P_3 + a_4P_4, \quad \text{where } a_i \text{ are real constants.}$$

Remark 5.1. *First, recall that for $\mathfrak{gl}(n, \mathbb{R})$, the forms $(A_1, \dots, A_m) \mapsto \text{Tr}(A_{i_1} \dots A_{i_k})$ are the only invariant functions which generate $K[\text{End}(V)^m]^{GL(V)}$ (see [H-C]).*

Remark that there are 24 possible products of 4 matrices, depending on the position of the matrix in the product. If we take the trace of these products, then there are only 6 distinct forms, since, for all $A_1, A_2, A_3, A_4 \in \mathfrak{sl}(n, \mathbb{R})$:

$$\text{Tr}(A_1A_2A_3A_4) = \text{Tr}(A_2A_3A_4A_1) = \text{Tr}(A_3A_4A_1A_2) = \text{Tr}(A_4A_1A_2A_3).$$

Since we are looking here to build symmetric forms in ξ, η and X, Y , there are only 4 symmetric forms obtained as product of traces of product matrices. These forms are the 4 forms described above.

5.3. $\mathfrak{sl}(n, \mathbb{R})$ does not admit an overalgebra almost separating of degree 2.

We have seen if $\mathfrak{sl}(n, \mathbb{R})$ admits an overalgebra almost separating of degree 2, then $\mathfrak{sl}(n, \mathbb{R})$ admits an overalgebra of the form

$$\mathfrak{G} = (\mathfrak{sl}(n, \mathbb{R}) \rtimes S_2(\mathfrak{sl}(n, \mathbb{R})), (\phi : \xi \mapsto b_1(\xi) + b_2(\xi \cdot \xi))),$$

with $b_1 = a_0P_0$ and $b_2 = a_1P_1 + \dots + a_4P_4$.

We assume that a such overalgebra almost separating \mathfrak{G} exists.

The generic orbits $SL(n, \mathbb{R}) \cdot \xi$ are the orbits of the points ξ of Ω . Recall that :

$$SL(n, \mathbb{R}) \cdot \xi \subset \{\xi' \in \mathfrak{sl}(n, \mathbb{R}), C_{\xi'} = C_{\xi}\}.$$

Thus, for all ξ in Ω and all $v \in S_2(\mathfrak{sl}(n, \mathbb{R}))$, we put $\zeta = \xi + {}^t\psi_v(\phi(\xi))$ such that $C_{\zeta} = C_{\xi}$.

Lemma 5.3.

If \mathfrak{G} is an overalgebra almost separating for $\mathfrak{sl}(n, \mathbb{R})$, then for all ξ of $\mathfrak{sl}(n, \mathbb{R})$ and all v of $S_2(\mathfrak{sl}(n, \mathbb{R}))$, $\zeta = \xi + {}^t\psi_v(\phi(\xi))$ has the same characteristic polynomial as ξ and the same eigenvalues.

Proof. For any matrix ξ of $\mathfrak{sl}(n, \mathbb{R})$, and all $\varepsilon > 0$, there exists ξ_ε in Ω such that :

$$\|\xi - \xi_\varepsilon\| < \varepsilon.$$

Since ξ_ε is in Ω ,

$$\det(\xi_\varepsilon + {}^t\psi_v(\phi(\xi_\varepsilon)) - \lambda I) = \det(\xi_\varepsilon - \lambda I), \quad \forall \lambda, \forall v$$

If ε tends to 0, then, for all λ ,

$$\det(\xi + {}^t\psi_v(\phi(\xi)) - \lambda I) = \det(\xi - \lambda I).$$

□

Theorem 5.4.

For $n > 2$, $\mathfrak{sl}(n, \mathbb{R})$ does not admit an overalgebra almost separating of degree 2.

Proof.

We have seen if $\mathfrak{sl}(n, \mathbb{R})$ has an overalgebra almost separating of degree 2, there exists an overalgebra \mathfrak{G} , with

$$\phi(\xi) = a_0 P_0(\xi) + (a_1 P_1 + \cdots + a_4 P_4)(\xi \cdot \xi).$$

We will show that, if for all ξ in Ω , and all v in $S_2(\mathfrak{sl}(n, \mathbb{R}))$, ξ and $\zeta = \xi + {}^t\psi_v(\phi(\xi))$ have the same eigenvalues, then $a_0 = a_1 = \cdots = a_3 = 0$, and that the function $\phi(\xi) = a_4 P_4(\xi \cdot \xi)$ does not separate the coadjoint orbits of $\mathfrak{sl}(n, \mathbb{R})$, if $n > 2$.

Taking first $v = U \in \mathfrak{sl}(n, \mathbb{R})$. Then $\langle {}^t\psi_U(\phi(\xi)), X \rangle = a_0 \text{Tr}(\xi[X, U])$, and $\zeta = \xi + a_0[U, \xi]$.

Let $U = e_{n1} + e_{(n-1)2}$ and $\xi = e_{1n} + e_{2(n-1)}$, thus

$$\zeta = a_0(-e_{11} - e_{22} + e_{(n-1)(n-1)} + e_{nn}) + e_{1n} + e_{2(n-1)}$$

and

$$\det(\zeta - \lambda I) = (-\lambda)^{n-4}(\lambda^2 - a_0^2)^2.$$

Therefore, ζ has the same spectrum as ξ implies $a_0 = 0$.

Put now $v = X.X$, then a direct calculation gives :

$${}^t\psi_{X.X}(\phi(\xi)) = \begin{cases} 4a_1[X, \xi X \xi] + 4a_2 \text{Tr}(\xi X)[X, \xi] + 4a_3[X^2, \xi^2], & \text{if } n > 3, \\ 4a_1[X, \xi X \xi] + 4a_3[X^2, \xi^2], & \text{if } n = 3. \end{cases}$$

Choose $\xi = e_{1n}$ and $X = e_{n1}$. ξ and X are nilpotent matrices : $X^2 = \xi^2 = 0$ and we obtain :

$$\xi X = e_{11}, \quad [X, \xi X \xi] = e_{nn} - e_{11}, \quad \text{Tr}(\xi X) = 1, \quad [X, \xi] = e_{nn} - e_{11},$$

thus

$${}^t\psi_{X.X}(\phi(\xi)) = \begin{cases} (4a_1 + 4a_2)(-e_{11} - e_{22} + e_{(n-1)(n-1)} + e_{nn}), & \text{if } n > 3, \\ 4a_1(e_{33} - e_{22}), & \text{if } n = 3, \end{cases}$$

and, if we note $\zeta = \xi + {}^t\psi_{X.X}(\phi(\xi))$, then

$$\det(\zeta - \lambda I) = \begin{cases} (-\lambda)^{n-2}(\lambda^2 - (4a_1 + 4a_2)^2), & \text{if } n > 3, \\ -\lambda(\lambda^2 - (4a_1)^2), & \text{if } n = 3. \end{cases}$$

Since $\det(\xi - \lambda I) = (-\lambda)^n$, then we deduce that $4a_1 + 4a_2 = 0$ if $n > 3$ and $a_1 = 0$ if $n = 3$.

Suppose now $n > 3$. We choose $\xi = e_{1n} + e_{2(n-1)}$ and $X = {}^t \xi = e_{n1} + e_{(n-1)2}$. These matrices are nilpotent and we obtain

$$\begin{aligned} \xi^2 &= 0, & \xi X &= e_{11} + e_{22}, \\ \xi X \xi &= e_{1n} + e_{2(n-1)}, & [X, \xi X \xi] &= -e_{11} - e_{22} + e_{(n-1)(n-1)} + e_{nn}, \\ Tr(\xi X) &= 2, & [X, \xi] &= -e_{11} - e_{22} + e_{(n-1)(n-1)} + e_{nn}. \end{aligned}$$

Therefore

$${}^t \psi_{X.X}(\phi(\xi)) = (4a_1 + 8a_2)(-e_{11} - e_{22} + e_{(n-1)(n-1)} + e_{nn})$$

and

$$\det(\zeta - \lambda I) = (-\lambda)^{(n-4)}(\lambda^2 - (4a_1 + 8a_2)^2).$$

Since $\det(\xi - \lambda I) = (-\lambda)^n$, we deduce that $4a_1 + 8a_2 = 0$. Thus, in all cases, $a_1 = a_2 = 0$.

Choose now $\xi = e_{1(n-1)} + e_{(n-1)n}$ and $X = {}^t \xi = e_{(n-1)1} + e_{n(n-1)}$, then

$$X^2 = e_{n1}, \quad \xi^2 = e_{1n}, \quad [X^2, \xi^2] = -e_{11} + e_{nn}.$$

Therefore

$${}^t \psi_{X^2}(\phi(\xi)) = 4a_3(-e_{11} + e_{nn})$$

and

$$\det(\zeta - \lambda I) = (-\lambda)^{(n-2)}(\lambda^2 - (4a_3)^2).$$

Hence, since $\det(\xi - \lambda I) = (-\lambda)^n$, then $a_3 = 0$.

Thus, we deduce that :

$$\phi(\xi) = b_2(\xi.\xi) = a_4 P_4(\xi.\xi) \quad \text{or} \quad \langle \phi(\xi), U + X.Y \rangle = a_4 Tr(\xi^2) Tr(XY).$$

But the overalgebra \mathfrak{G} is not separating, since, for all t in $[0, 1]$, and all matrix $M = D(c_4, \dots, c_n) \in \mathfrak{sl}(n-3, \mathbb{R})$, with $|c_k| > 2$, we define the matrix

$$\xi_t = \begin{pmatrix} \frac{1}{2}(t + \sqrt{4 - 3t^2}) & & & \\ & \frac{1}{2}(t - \sqrt{4 - 3t^2}) & & \\ & & -t & \\ & & & M \end{pmatrix}.$$

For all t ,

$$\xi_t \in \Omega, \quad \det(\xi_t) = t(1 - t^2) \prod_k c_k \quad \text{and} \quad Tr(\xi_t^2) = 2 + \sum_k c_k^2.$$

i.e, for all t ,

$$\overline{Conv}(\Phi(CoadSL(n, \mathbb{R})\xi_t)) = \mathfrak{sl}(n, \mathbb{R})^* \times \{U + X \cdot Y \mapsto a_4(2 + \sum_k c_k^2) Tr(XY)\},$$

therefore if $t \neq \frac{1}{\sqrt{3}}$,

$$Coad SL(n, \mathbb{R})(\xi_t) \neq Coad SL(n, \mathbb{R})(\xi_{\frac{1}{\sqrt{3}}}).$$

Since $t(1 - t^2) < \frac{1}{\sqrt{3}}(1 - \frac{1}{3}) = \frac{2}{3\sqrt{3}}$, $\det(\xi_t) \neq \det(\xi_{\frac{1}{\sqrt{3}}})$, thus ξ_t is not in the orbit $\text{Coad } SL(n, \mathbb{R})\xi_{\frac{1}{\sqrt{3}}}$. □

Remark 5.2.

Recall that, if $n = 2$, the overalgebra $(\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}, [\xi \mapsto \text{Tr}(\xi^2)])$ is an overalgebra almost separating of degree 2 of $\mathfrak{sl}(2, \mathbb{R})$ (cf. [ASZ] where we use the function $\det(\xi) = -\frac{1}{2}\text{Tr}(\xi^2)$).

Similarly, $\mathfrak{sl}(3, \mathbb{R})$ does not admit an overalgebra almost separating of degree 2 but $\mathfrak{sl}(3, \mathbb{R})$ admits an overalgebra almost separating of degree 3.

In this following section, we will show that $\mathfrak{sl}(4, \mathbb{R})$ does not admit an overalgebra almost separating of degree 2 or 3 but it admits one overalgebra almost separating of degree 4.

6. THE CASE $n = 4$ AND $p = 3$

As above, we shall first find the explicit decomposition of $S^3(\mathfrak{sl}(4, \mathbb{R}))$.

6.1. Decomposition of $S^3(\mathfrak{sl}(4, \mathbb{R}))$.

We have seen that the module $S^3(\mathfrak{sl}(4, \mathbb{R}))$ is self dual. Then, if the submodule $\Gamma_{a_1 a_2 a_3}$ appears in the decomposition of $S^3(\mathfrak{sl}(4, \mathbb{R}))$, the submodule $\Gamma_{a_3 a_2 a_1} \simeq (\Gamma_{a_1 a_2 a_3})^s$ appears also.

The module $S^3(\mathfrak{sl}(4, \mathbb{R}))$ is a submodule of $S^2(\mathfrak{sl}(4, \mathbb{R})) \otimes \mathfrak{sl}(4, \mathbb{R})$. The decomposition of $S^2(\mathfrak{sl}(4, \mathbb{R})) \otimes \mathfrak{sl}(4, \mathbb{R})$ is given by Littlewood-Richardson's rule (cf. [FH]), as follows :

$$\begin{aligned} S^2(\mathfrak{sl}(4, \mathbb{R})) \otimes \mathfrak{sl}(4, \mathbb{R}) &= (\Gamma_{303} + \Gamma_{212} + \Gamma_{202} + \Gamma_{101}) + (\Gamma_{121} + \Gamma_{202} + \Gamma_{101} + \Gamma_{311} + \Gamma_{113}) \\ &\quad + 3(\Gamma_{210} + \Gamma_{012}) + 2\Gamma_{020} + 3\Gamma_{101} + \Gamma_{000}. \end{aligned}$$

The highest weight vectors which appear in $S^2(\mathfrak{sl}(4, \mathbb{R}))$ are v_{202} , v_{020} , v_{101} and v_{000} . We deduce that there are 4 highest weight vectors in $S^3(\mathfrak{sl}(4, \mathbb{R}))$ which are $w_{303} = v_{202} \cdot e_{14}$, $w_{121} = v_{020} \cdot e_{14}$, $w_{202} = v_{101} \cdot e_{14}$ and $w_{101} = v_{000} \cdot e_{14}$. These vectors are the highest weight vectors for the simple modules Γ_{303} , Γ_{121} , Γ_{202} and Γ_{101} .

In $S^2(\mathfrak{sl}(4, \mathbb{R})) \otimes \mathfrak{sl}(4, \mathbb{R})$, the highest weight vectors $v_{020} \otimes e_{14} - v_{020} \cdot e_{14}$, $v_{101} \otimes e_{14} - v_{101} \cdot e_{14}$ and $v_{000} \otimes e_{14} - v_{000} \cdot e_{14}$ appear also. The corresponding simple modules of these vectors are, respectively, Γ_{121} , Γ_{202} and Γ_{101} . Since these vectors are not symmetric, then their corresponding modules are not submodules of $S^3(\mathfrak{sl}(4, \mathbb{R}))$.

The highest weight vector of Γ_{311} is $e_{14} \otimes e_{13} \otimes e_{14} - e_{14} \cdot e_{13} \cdot e_{14}$ which is not symmetric, then Γ_{311} does not appear in $S^3(\mathfrak{sl}(4, \mathbb{R}))$, and Γ_{113} does not appear also.

We conclude:

$$\Gamma_{303} + \Gamma_{121} + \Gamma_{202} + \Gamma_{101} \subset S^3(\mathfrak{sl}(4, \mathbb{R})).$$

The additional invariant space of $(\Gamma_{303} + \Gamma_{212} + \Gamma_{202} + \Gamma_{101})$ in $S^3(\mathfrak{sl}(4, \mathbb{R}))$ has the following decomposition, by using the dimensions :

$$S^3(\mathfrak{sl}(4, \mathbb{R})) / (\Gamma_{303} + \Gamma_{212} + \Gamma_{202} + \Gamma_{101}) = (\Gamma_{210} + \Gamma_{012}) + \Gamma_{101} + \Gamma_{000}.$$

Therefore :

$$S^3(\mathfrak{sl}(4, \mathbb{R})) = (\Gamma_{303} + \Gamma_{212} + \Gamma_{202} + \Gamma_{101}) + (\Gamma_{210} + \Gamma_{012} + \Gamma_{101} + \Gamma_{000}).$$

The highest weight vectors w_{303} , w_{121} , w_{202} and w_{101} are:

$$w_{303} = e_{14} \cdot e_{14} \cdot e_{14},$$

$$w_{121} = e_{24} \cdot e_{13} \cdot e_{14} - e_{23} \cdot e_{14} \cdot e_{14},$$

$$w_{202} = e_{12} \cdot e_{24} \cdot e_{14} + e_{13} \cdot e_{34} \cdot e_{14} + \frac{1}{2}((e_{11} - e_{22}) - (e_{33} - e_{44})) \cdot e_{14} \cdot e_{14},$$

$$\begin{aligned} w_{101} = & 8(e_{12} \cdot e_{21} \cdot e_{14} + e_{13} \cdot e_{31} \cdot e_{14} + e_{14} \cdot e_{41} \cdot e_{14} + e_{23} \cdot e_{32} \cdot e_{14} + e_{34} \cdot e_{43} \cdot e_{14}) + \\ & + 3(e_{11} \cdot e_{11} \cdot e_{14} + e_{22} \cdot e_{22} \cdot e_{14} + e_{33} \cdot e_{33} \cdot e_{14} + e_{44} \cdot e_{44} \cdot e_{14}) - \\ & - 2(e_{11} \cdot e_{22} \cdot e_{14} + e_{11} \cdot e_{33} \cdot e_{14} + e_{11} \cdot e_{44} \cdot e_{14} + e_{22} \cdot e_{33} \cdot e_{14} + e_{22} \cdot e_{44} \cdot e_{14} + e_{33} \cdot e_{44} \cdot e_{14}). \end{aligned}$$

Now looking for the highest weight vectors of the four remaining simple modules.

By Littlewood-Richardson's rule, $(\Gamma_{210} + \Gamma_{012})$ appears in the tensorial product $\Gamma_{020} \otimes \Gamma_{101}$ where Γ_{020} is in $S^2(\mathfrak{sl}(4, \mathbb{R}))$, and Γ_{101} is in $\mathfrak{sl}(4, \mathbb{R})$.

The highest weight vector of the module Γ_{020} is :

$$v_{020} = e_{24} \cdot e_{13} - e_{23} \cdot e_{14}.$$

We deduce also two other vectors of Γ_{020} given by :

$$ad_{e_{42}} v_{020} = ((e_{44} - e_{22}) \cdot e_{13} - e_{43} \cdot e_{14} + e_{23} \cdot e_{12}),$$

$$ad_{e_{32}} v_{020} = (e_{34} \cdot e_{13} - e_{24} \cdot e_{12} - (e_{33} - e_{22}) \cdot e_{14}).$$

Thus, there is a highest weight vector of Γ_{210} , defined by :

$$w_{210} = e_{12} \cdot e_{24} \cdot e_{13} - e_{12} \cdot e_{23} \cdot e_{14} - e_{14} \cdot e_{43} \cdot e_{14} + e_{13} \cdot e_{34} \cdot e_{13} + (e_{44} - e_{33}) \cdot e_{13} \cdot e_{14}.$$

w_{210} is a non zero vector and its weight is $4L_1 + 2L_2 + 2L_3 = 2\omega_1 + 2\omega_3$. Indeed :

$$ad_{e_{12}} w_{210} = 0, \quad ad_{e_{23}} w_{210} = 0 \quad \text{and} \quad ad_{e_{34}} w_{210} = 0.$$

Using the application s , the highest weight vector of the module Γ_{012} is v_{210}^s or :

$$w_{012} = e_{34} \cdot e_{13} \cdot e_{24} - e_{34} \cdot e_{23} \cdot e_{14} - e_{14} \cdot e_{21} \cdot e_{14} + e_{24} \cdot e_{12} \cdot e_{24} + (e_{11} - e_{22}) \cdot e_{24} \cdot e_{14}.$$

It remains the modules Γ_{101} and Γ_{000} which appear in the tensorial product $\Gamma_{101} \otimes \Gamma_{101}$. The first factor is in $S^2(\mathfrak{sl}(4, \mathbb{R}))$, the second is in $\mathfrak{sl}(4, \mathbb{R})$.

There is a basis for the first factor defined by the following vectors :

$$e'_{ij} = e_{i1} \cdot e_{1j} + e_{i2} \cdot e_{2j} + e_{i3} \cdot e_{3j} + e_{i4} \cdot e_{4j} - \frac{1}{2}(e_{11} + e_{22} + e_{33} + e_{44}) \cdot e_{ij}.$$

In $S^2(\mathfrak{sl}(4, \mathbb{R})) \subset \Gamma_{101} \otimes \Gamma_{101}$, we have seen that the corresponding highest weight vectors are:

$$\begin{aligned} v_{101} &= e_{12} \cdot e_{24} + e_{13} \cdot e_{34} + \frac{1}{2}((e_{11} - e_{22}) - (e_{33} - e_{44})) \cdot e_{14}, \\ w_{000} &= 8 \sum_{1 \leq i < j \leq 4} e_{ij} \cdot e_{ji} + \sum_{1 \leq i < j \leq 4} (e_{ii} - e_{jj}) \cdot (e_{ii} - e_{jj}). \end{aligned}$$

By replacing the first factor e_{ij} by the factor e'_{ij} , we obtain the highest weight vectors w'_{101} and w_{000} (w_{000} is not developed) :

$$\begin{aligned} w'_{101} &= 2e_{12}(2e_{23} \cdot e_{34} + 2e_{21} \cdot e_{14} + (e_{22} - e_{11}) \cdot e_{24} - (e_{33} - e_{44}) \cdot e_{24}) + \\ &\quad + 2e_{13}(2e_{32} \cdot e_{24} + 2e_{31} \cdot e_{14} + (e_{33} - e_{11}) \cdot e_{34} + (e_{44} - e_{22}) \cdot e_{34}) + \\ &\quad + (e_{11} - e_{22})(2e_{12} \cdot e_{24} + 2e_{13} \cdot e_{34} + (e_{11} - e_{22}) \cdot e_{14} - \\ &\quad - (e_{33} - e_{44}) \cdot e_{14}) - (e_{33} - e_{44})(2e_{12} \cdot e_{24} + 2e_{13} \cdot e_{34} + (e_{11} - e_{22}) \cdot e_{14} - (e_{33} - e_{44}) \cdot e_{14}), \\ w_{000} &= 4(e_{12} \cdot e'_{21} + e'_{12} \cdot e_{21} + e_{13} \cdot e'_{31} + e'_{13} \cdot e_{31} + e_{14} \cdot e'_{41} + e'_{14} \cdot e_{41} + \\ &\quad + e_{23} \cdot e'_{32} + e'_{23} \cdot e_{32} + e_{24} \cdot e'_{42} + e'_{24} \cdot e_{42} + e_{34} \cdot e'_{43} + e'_{34} \cdot e_{43}) + \\ &\quad + (e_{11} - e_{22})(e'_{11} - e'_{22}) + (e_{11} - e_{33})(e'_{11} - e'_{33}) + (e_{11} - e_{44})(e'_{11} - e'_{44}) + \\ &\quad + (e_{22} - e_{33})(e'_{22} - e'_{33}) + (e_{22} - e_{44})(e'_{22} - e'_{44}) + (e_{33} - e_{44})(e'_{33} - e'_{44}). \end{aligned}$$

6.2. Trace forms and intertwining of $S^3(\mathfrak{sl}(4, \mathbb{R}))$.

As for $S^2(\mathfrak{sl}(n, \mathbb{R}))$, we know 12 trace forms. Denote by ξ , η and ζ elements in $(\mathfrak{sl}(4, \mathbb{R}))^* = \mathfrak{sl}(4, \mathbb{R})$, and X , Y , Z elements in $\mathfrak{sl}(4, \mathbb{R})$. The trace forms are the following :

$$\begin{aligned} T_1 &= Tr(\xi\eta\zeta XYZ), & T_2 &= Tr(\xi\eta X\zeta YZ), & T_3 &= Tr(\xi\eta XY\zeta Z), \\ T_4 &= Tr(\xi X\eta Y\zeta Z), & T_5 &= Tr(\xi\eta\zeta X)Tr(YZ), & T_6 &= Tr(\xi\eta XY)Tr(\zeta Z), \\ T_7 &= Tr(\xi XYZ)Tr(\eta\zeta), & T_8 &= Tr(\xi X\eta Y)Tr(\zeta Z), & T_9 &= Tr(\xi\eta\zeta)Tr(XYZ), \\ T_{10} &= Tr(\xi\eta X)Tr(\zeta YZ), & T_{11} &= Tr(\xi\eta)Tr(\zeta X)Tr(YZ), & T_{12} &= Tr(\xi X)Tr(\eta Y)Tr(\zeta Z). \end{aligned}$$

Recall that, in the previous section, we calculated the 8 highest weight vectors of the decomposition of $S^3(\mathfrak{sl}(4, \mathbb{R}))$, i.e the free system

$$(w^1, \dots, w^8) = (w_{303}, w_{121}, w_{202}, w_{210}, w_{012}, w_{101}, w'_{101}, w_{000}).$$

Let M the matrix with 8 rows and 12 columns whose entries are the numbers $\langle T_i(w^k), (w^k)^t \rangle$ ($i = 1, \dots, 12, k = 1, \dots, 8$) where the vector $e_{j_1 i_1} \cdot e_{j_2 i_2} \cdot e_{j_3 i_3}$ of $S^3(\mathfrak{sl}(4, \mathbb{R}))$ is noted $(e_{i_1 j_1} \cdot e_{i_2 j_2} \cdot e_{i_3 j_3})^t$.

We obtain, by using a symbolic computation program, the following matrix:

$$M = \begin{pmatrix} 0 & 0 & 0 & 36 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 12 \\ 0 & 1 & 1 & 3 & 0 & 1 & 0 & 4 & 0 & 1 & 0 & 6 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 6 \\ 1 & 1 & 2 & 0 & 2 & 3 & 2 & 2 & 0 & 0 & 4 & 6 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 6 \\ 3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 9 & 0 & 0 & 6 \end{pmatrix}$$

The rank of this matrix is 8.

We extract the columns 1, 2, 3, 4, 5, 8, 10, 9, so we obtain the following intertwining. Explicitly :

$$\begin{aligned} \langle P_1(\xi\eta\zeta), XYZ \rangle &= \text{Sym}(\text{Tr}(\xi\eta\zeta XYZ)), \\ \langle P_2(\xi\eta\zeta), XYZ \rangle &= \text{Sym}(\text{Tr}(\xi\eta X\zeta YZ)), \\ \langle P_3(\xi\eta\zeta), XYZ \rangle &= \text{Sym}(\text{Tr}(\xi\eta XY\zeta Z)), \\ \langle P_4(\xi\eta\zeta), XYZ \rangle &= \text{Sym}(\text{Tr}(\xi X\eta Y\zeta Z)), \\ \langle P_5(\xi\eta\zeta), XYZ \rangle &= \text{Sym}(\text{Tr}(\xi\eta\zeta X)\text{Tr}(YZ)), \\ \langle P_6(\xi\eta\zeta), XYZ \rangle &= \text{Sym}(\text{Tr}(\xi X\eta Y)\text{Tr}(\zeta Z)), \\ \langle P_7(\xi\eta\zeta), XYZ \rangle &= \text{Sym}(\text{Tr}(\xi\eta X)\text{Tr}(\zeta YZ)), \\ \langle P_8(\xi\eta\zeta), XYZ \rangle &= \text{Sym}(\text{Tr}(\xi\eta\zeta)\text{Tr}(XYZ)). \end{aligned}$$

The notation 'Sym' means that the expression is symmetrical in ξ, η, ζ .

If N is the sub-matrix of M , with 8 rows and 8 columns whose entries are $\langle P_i(w^k), (w^k)^t \rangle$, $i = 1, \dots, 8$, then

$$N = \begin{pmatrix} 0 & 0 & 0 & 36 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}$$

The rank of this matrix is also 8. Thus $(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8)$ are independent. Therefore:

Lemma 6.1.

The applications $P_i : S^3(\mathfrak{sl}(4, \mathbb{R})^) \longrightarrow (S^3(\mathfrak{sl}(4, \mathbb{R})))^*$ defined above, form a basis of the space of intertwining of the module $S^3(\mathfrak{sl}(4, \mathbb{R}))$.*

6.3. $\mathfrak{sl}(4, \mathbb{R})$ does not admit an overalgebra almost separating of degree 3.

Theorem 6.2.

The algebra $\mathfrak{sl}(4, \mathbb{R})$ does not admit an overalgebra almost separating of degree 3.

Proof.

We have seen that if $\mathfrak{sl}(4, \mathbb{R})$ admits an overalgebra of degree 3, then $\mathfrak{sl}(4, \mathbb{R})$ admits an overalgebra of the form

$$(\mathfrak{sl}(4, \mathbb{R}) \rtimes S_3(\mathfrak{sl}(4, \mathbb{R})), (b_1 + b_2 + b_3) \circ \tau).$$

In this case, b_i are an intertwining, $b_1 = 0$, $\langle b_2(\xi.\eta), X.Y \rangle = aTr(\xi\eta)Tr(XY)$ and b_3 is written :

$$b_3(\xi.\eta.\zeta) = \sum_{j=1}^8 c_j P_j(\xi.\eta.\zeta).$$

Then, we choose $v = X.X.X$ in $S_3(\mathfrak{sl}(4, \mathbb{R}))$ and we calculate ${}^t\psi_v(P_j(\xi.\xi.\xi))$. Explicitly:

$$\begin{aligned} {}^t\psi_v(P_1(\xi.\xi.\xi)) &= [X^3, \xi^3], \\ {}^t\psi_v(P_2(\xi.\xi.\xi)) &= [X^2\xi^2X, \xi] + [X\xi X^2, \xi^2], \\ {}^t\psi_v(P_3(\xi.\xi.\xi)) &= [X^2\xi X, \xi^2] + [X\xi^2 X^2, \xi], \\ {}^t\psi_v(P_4(\xi.\xi.\xi)) &= 3[X\xi X\xi X, \xi], \\ {}^t\psi_v(P_5(\xi.\xi.\xi)) &= Tr(X^2)[X, \xi^3], \\ {}^t\psi_v(P_6(\xi.\xi.\xi)) &= 2Tr(\xi X)[X\xi X, \xi] + Tr(\xi X\xi X)[X, \xi], \\ {}^t\psi_v(P_7(\xi.\xi.\xi)) &= Tr(\xi X^2)[X, \xi^2] + Tr(\xi^2 X)[X^2, \xi], \\ {}^t\psi_v(P_8(\xi.\xi.\xi)) &= 0. \end{aligned}$$

Let $\xi = e_{14}$, then $\xi^2 = 0$ and ${}^t\psi_v(P_j(\xi.\xi.\xi)) = 0$, for $j = 1, 2, 3, 5, 7$.

Let now $X = e_{14} + e_{41}$, then $\xi X = e_{11}$ and $X\xi X\xi X = X - \xi = e_{41}$.

So, we obtain ${}^t\psi_v(P_4(\xi.\xi.\xi)) = -3(e_{11} - e_{44})$ and ${}^t\psi_v(P_6(\xi.\xi.\xi)) = 3(e_{11} - e_{44})$. Thus, with same notations as above,

$$\zeta = \xi + {}^t\psi_v(\phi(\xi)) = 3(c_6 - c_4)(e_{11} - e_{44}) + e_{14}.$$

Therefore,

$$\det(\zeta - \lambda I) = \lambda^2(\lambda^2 - 9(c_6 - c_4)^2).$$

We deduce the relation $c_6 - c_4 = 0$.

On the other hand, let $X = e_{14} - e_{41}$, then :

$$\begin{array}{lll} \xi X &= -e_{11}, & \xi X \xi X &= e_{11}, & X \xi X \xi X &= -e_{41}, \\ X \xi X &= e_{41}, & [X \xi X \xi X, \xi] &= e_{11} - e_{44}, & X \xi X &= e_{41}, \\ [X \xi X, \xi] &= -e_{11} + e_{44}, & [X, \xi] &= e_{11} - e_{44}, & \xi X \xi X &= e_{11}. \end{array}$$

Thus, we get ${}^t\psi_v(P_6(\xi.\xi.\xi)) = 3(e_{11} - e_{44})$ and ${}^t\psi_v(P_4(\xi.\xi.\xi)) = 3(e_{11} - e_{44})$. Therefore, if $\zeta = \xi + {}^t\psi_v(\phi(\xi))$,

$$\det(\zeta - \lambda I) = \lambda^2(\lambda^2 - 9(c_6 + c_4)^2)$$

then $c_6 + c_4 = 0$. This shows that $c_6 = c_4 = 0$.

We choose now $\xi = e_{13} + e_{34}$, and $X = {}^t\xi = e_{31} + e_{43}$. Then:

$$\begin{array}{llll} X^2 & = e_{41}, & X^2\xi^2X & = e_{43}, & [\xi, X^2\xi^2X] & = e_{33} - e_{44}, & X\xi X^2 & = e_{41}, \\ [\xi^2, X\xi X^2] & = e_{11} - e_{44}, & X^2\xi X & = e_{41}, & [\xi^2, X^2\xi X] & = e_{11} - e_{44}, & X\xi^2 X^2 & = e_{31}, \\ [\xi, X\xi^2 X^2] & = e_{11} - e_{33}, & X^2\xi^2 X & = e_{43}, & [X^2\xi^2 X, \xi] & = e_{44} - e_{33}, & X\xi X^2 & = e_{41}, \\ [X\xi X^2, \xi^2] & = e_{44} - e_{11}. \end{array}$$

We deduce that ${}^t\psi_v(P_2(\xi.\xi.\xi)) = e_{11} + e_{33} - 2e_{44}$ and ${}^t\psi_v(P_3(\xi.\xi.\xi)) = 2e_{11} - e_{33} - e_{44}$. Therefore,

$$\zeta = \xi + {}^t\psi_v(\phi(\xi)) = e_{13} + e_{34} + (c_2 + 2c_3)e_{11} + (c_2 - c_3)e_{33} - (2c_2 + c_3)e_{44}$$

and

$$\det(\zeta - \lambda I) = -\lambda(c_2 + 2c_3 - \lambda)(c_2 - c_3 - \lambda)(-2c_2 - c_3 - \lambda)$$

Hence, the spectrum of ζ is the same as ξ , i.e $\{0\}$ implies $c_2 + 2c_3 = 0$, $c_2 - c_3 = 0$, and $2c_2 + c_3 = 0$, so $c_2 = c_3 = 0$.

Now, let $\xi = e_{13} + e_{14} + e_{34}$ and $X = {}^t\xi = e_{31} + e_{41} + e_{43}$, then

$$\begin{array}{llll} \xi^2 & = e_{14}, & X^2 & = e_{41}, & \xi X^2 & = e_{11} + e_{31}, \\ \xi^2 X & = e_{11} + e_{13}, & [\xi^2, X] & = e_{11} + e_{13} - e_{34} - e_{44}, & [\xi, X^2] & = e_{11} + e_{31} - e_{43} - e_{44}. \end{array}$$

Thus,

$${}^t\psi_v(P_7(\xi.\xi.\xi)) = 2e_{11} + e_{13} + e_{31} - e_{34} - e_{43} - 2e_{44}$$

and, if $\zeta = \xi + {}^t\psi_v(\phi(\xi))$,

$$\det(\zeta - \lambda I) = -\lambda(-\lambda^3 + \lambda(5c_7^2 + c_7) + 2c_7^3 + c_7^2 + 2c_7).$$

Therefore, the spectrum of ζ is the same as ξ , i.e $\{0\}$ implies $c_7 = 0$.

Later, we choose $\xi = e_{12} + e_{23} + e_{34}$ and $X = {}^t\xi$. Then, $\xi^2 = e_{13} + e_{24}$, $\xi^3 = e_{14}$, $X^3 = e_{41}$. Thus ${}^t\psi_v(P_1(\xi.\xi.\xi)) = e_{11} - e_{44}$ and the spectrum of $\zeta = \xi + {}^t\psi_v(\phi(\xi))$ is $\{0\}$ implies $c_1 = 0$.

Finally, we choose another $X = e_{14} + e_{41}$ and we allowed $\xi = e_{12} + e_{23} + e_{34}$. Then $X^2 = e_{11} + e_{44}$ and $\xi^3 = e_{11} - e_{44}$. Therefore, ${}^t\psi_v(P_5(\xi.\xi.\xi)) = 2(e_{11} - e_{44})$ and $\det(\zeta - \lambda I) = \lambda^4$ implies $c_5 = 0$.

We finally get:

$$\langle \phi(\xi), U + X.Y + X'.Y'.Z' \rangle = a_4 Tr(\xi^2) Tr(XY) + c_8 Tr(\xi^3) Tr(X'Y'Z').$$

But we consider, for $0 < t < 1$, the matrices

$$\xi_t = \begin{pmatrix} \sqrt{1+t} & & & \\ & -\sqrt{1+t} & & \\ & & \sqrt{1-t} & \\ & & & -\sqrt{1-t} \end{pmatrix}.$$

ξ_t is an element of Ω for all t , $Tr(\xi_t^2) = 4$ and $Tr(\xi_t^3) = 0$ for all t . Although, $\det(\xi_t) = (1 - t^2)^2$. Therefore, with the same argument as in a previous section, we have, for all t ,

$$\overline{Conv}(\Phi(Coad SL(4, \mathbb{R})\xi_t)) = (\mathfrak{sl}(4, \mathbb{R}))^* \times \{U + X.Y + X'.Y'.Z' \mapsto 4a_4 Tr(XY)\}.$$

But, if $t \neq \frac{1}{2}$, ξ_t is not in the orbit $Coad SL(4, \mathbb{R})\xi_{\frac{1}{2}}$.

Thus, $\mathfrak{sl}(4, \mathbb{R})$ does not admit an overalgebra almost separating of degree 3.

□

In fact, we think that the following conjecture is always true :

Conjecture 6.3. *For all n , $\mathfrak{sl}(n, \mathbb{R})$ does not admit an overalgebra almost separating of degree $n - 1$, but it admits an overalgebra almost separating of degree n .*

More generally, if \mathfrak{g} is a real and deployed semi simple Lie algebra and if k is the greatest degree of the generators of the algebra of invariant functions on \mathfrak{g} , then \mathfrak{g} admits an overalgebra almost separating of degree k . But \mathfrak{g} does not admit an overalgebra almost separating of degree $k - 1$.

The hypothesis ‘ \mathfrak{g} deployed’ is necessary. Indeed, we remark that $\mathfrak{sl}(2, \mathbb{R})$ does not admit an overalgebra almost separating of degree 1, but the Lie algebra $\mathfrak{su}(2)$ admits an overalgebra almost separating of degree 1 since its adjoint orbits are spheres which are characterized by the closure of their convex hull.

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* INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR CNRS 5584, UNIVERSITÉ DE BOURGOGNE, U.F.R. SCIENCES ET TECHNIQUES B.P. 47870, F-21078 DIJON CEDEX, FRANCE

* LABORATOIRE DE MATHÉMATIQUE PHYSIQUE, FONCTIONS SPÉCIALES ET APPLICATIONS,
UNIVERSITÉ DE SOUSSE, ECOLE SUPÉRIEURE DES SCIENCES ET DE TECHNOLOGIE DE HAMMAM-
SOUSSE, RUE LAMINE ABASSI 4011 H. SOUSSE, TUNISIE

E-mail address: `amel.zergane@u-bourgogne.fr`

E-mail address: `amel.zergan@yahoo.fr`