A generalized lens equation for light deflection in weak gravitational fields

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A generalized lens equation for small light-deflection angles

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Abstract.

A generalized lens equation in Schwarzschild metric and valid for small light-deflection angles is suggested. The magnitude of neglected terms in the generalized lens equation is estimated to be smaller or equal than $\frac{15\,\pi}{4}\,\frac{m^2}{d'^2}$, where m is the Schwarzschild radius of masive body and d' is Chandrasekhar's impact vector. The main applications of this generalized lens equation are extreme astrometrical configurations, where *Standard post-Newtonian approach* as well as *Classical lens equation* cannot be applied. It is shown that in the appropriate limits the proposed lens equation yields the known post-Newtonian terms, 'enhanced' post-post-Newtonian terms and the Classical lens equation, thus provides a link between these both essential approaches for determining the light-deflection.

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1. Introduction

Todays astrometry necessitates theoretical predictions of light-deflection by massive bodies on microarcsecond (μas) level, e.g. NASA mission SIM or ESA mission GAIA. In principle, an astrometric precision on microarcsecond level can be achieved by numerical integration of geodesic equations of light-propagation. On the other side, modern astrometric missions like GAIA determine the positions and proper motions of approximately one billion objects, each of which is observed about one hundred times. The data reduction of such huge amount of observations implies the need of analytical solutions, because the numerical investigation of geodesic equations is by far too time-consuming. The metric of a massive body is usually expanded in terms of multipoles, i.e. monopole term, quadrupole term and higher multipoles. Numerically, the largest contributions of light-deflection originates from the spherically symmetric part (Schwarzschild) of the massive body under consideration. The exact analytical solution of light-propagation in Schwarzschild metric [5] inherits elliptic integrals, those evaluation becomes comparable with the time effort needed for a numerical integration

of geodesic equation. Thus, approximative analytical solutions valid on microarcsecond level of accuracy are indispensible for a highly time-efficient data reduction.

Basically, two essential approximative approaches are known. The first one is the *Standard parameterized post-Newtonian approach* (PPN) which is of the order $\mathcal{O}(m)$. During the last decades, it has been the common understanding that the higher order terms $\mathcal{O}(m^2)$ are negligible even on microarcsecond level, except for observations in the vicinity of the Sun. However, recent investigations [17, 1, 12, 18] have revealed that the post-post-Newtonian approximation [3, 4], which is of the order $\mathcal{O}(m^2)$, is needed for such high accuracy. Both approximations are applicable for $d \gg m$, where d being the impact vector.

The second one is the *Classical lens equation*. Soon after the light-deflection by the Sun has been measured in 1919 [7], the gravitational lens effect at stars was discussed [15, 6, 8]. In 1937 Fritz Zwicky hypothesized not only could stars act as lenses but galaxy clusters could as well. Later in 1963, by detailed calculations it has been recognized that quasars as pointlike sources are ideal in order to determine the gravitational lensing effect [14, 16, 10]. One decisive advantage of classical lens equation is it's validity for arbitrarily small values of impact vector d. The classical lens equation is valid for astrometrical configurations where source and observer are far ernough from the lens, especially in case of $a \gg d$ and $b \gg d$, where $a = \boldsymbol{k} \cdot \boldsymbol{x}_1$ and $b = -\boldsymbol{k} \cdot \boldsymbol{x}_0$, where $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are the three-vectors from the center of the massive body to the source and observer, respectively, and $\boldsymbol{k}$ is the unit vector from the source to the observer.

However, there are astrometric configurations where neither the post-Newtonian approach nor the classical lens equation are applicable. In order to investigate the light-deflection in such systems a link between these both approaches is needed. Such a link can be provided by a *Generalized lens equation* which, in the appropriate limits, coincides with *Standard post-Newtonian approach* and *Classical lens equation*. Accordingly, the aim of our investigation is an analytical expression for the generalized lens equation having a form very similar to the classical lens equation. We formulate the following conditions under which our generlized lens equation should be applicable (see Figure 1):

1. valid for $d = 0\ ,\ a = x_1 \gg m\ ,\ b = x_0 \gg m$,
2. valid for $a = 0\ ,\ d \gg m\ ,\ b \neq 0$,
3. valid for $b = 0\ ,\ d \gg m\ ,\ a \neq 0$.

These conditions imply that the light-path is always far enough from the lens, thus inherit small light deflection angles. Our paper is organized as follows: In Section 2 we present the standard post-Newtonian approach. In Section 3 we show the steps of post-post-Newtonian approach relevant for our investigation, and we will briefly summarize the main results of our paper [12]. The classical lens equation is derived in Section 4. The generalized lens equation is obtained in Section 5 and discussed in Section 6. A summary is given in Section 7.

We will use fairly standard notations:

- G is the Newtonian constant of gravitation.
- c is the velocity of light.
- β and γ are the parameters of the Parametrized Post-Newtonian (PPN) formalism which characterize possible deviation of the physical reality from general relativity theory ($\beta = \gamma = 1$ in general relativity).
- Lower case Latin indices i, j, ... take values $1, 2, 3$.
- Lower case Greek indices μ, ν, ... take values $0, 1, 2, 3$.
- A dot over any quantity designates the total derivative with respect to the coordinate time of the corresponding reference system: e.g. $\dot{a} = \dfrac{da}{dt}$.
- The 3-dimensional coordinate quantities ("3-vectors") referred to the spatial axes of the corresponding reference system are set in boldface: $\boldsymbol{a} = a^i$.
- The absolute value (Euclidean norm) of a "3-vector" $\boldsymbol{a}$ is denoted as $|\boldsymbol{a}|$ or, simply, a and can be computed as $a = |\boldsymbol{a}| = (a^1\, a^1 + a^2\, a^2 + a^3\, a^3)^{1/2}$.
- The scalar product of any two "3-vectors" $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to the Euclidean metric δ_{ij} is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$ and can be computed as $\boldsymbol{a} \cdot \boldsymbol{b} = \delta_{ij}\, a^i\, b^j = a^i\, b^i$.
- The vector product of any two "3-vectors" $\boldsymbol{a}$ and $\boldsymbol{b}$ is designated by $\boldsymbol{a} \times \boldsymbol{b}$ and can be computed as $(\boldsymbol{a} \times \boldsymbol{b})^i = \varepsilon_{ijk}\, a^j\, b^k$, where $\varepsilon_{ijk} = (i-j)(j-k)(k-i)/2$ is the fully antisymmetric Levi-Civita symbol.
- For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the angle between them is designated as $\delta(\boldsymbol{a}, \boldsymbol{b})$. Clearly, for an angle between two vectors one has $0 \leq \delta(\boldsymbol{a}, \boldsymbol{b}) \leq \pi$. Angle $\delta(\boldsymbol{a}, \boldsymbol{b})$ can be computed in many ways, for example, as $\delta(\boldsymbol{a}, \boldsymbol{b}) = \arccos \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{a\, b}$.

Throughout the paper, we work in harmonic gauge, where the components of metric tensor of the Schwarzschild solution is given by $g_{00} = -\dfrac{1-a}{1+a}$, $g_{i0} = 0$, $g_{ij} = (1+a)^2\, \delta_{ij} + \dfrac{a^2}{x^2}\, \dfrac{1+a}{1-a}\, x^i\, x^j$, where $a = \dfrac{m}{x}$, $m = \dfrac{G\, M}{c^2}$, and M is the mass of massive body.

2. Standard post-Newtonian solution

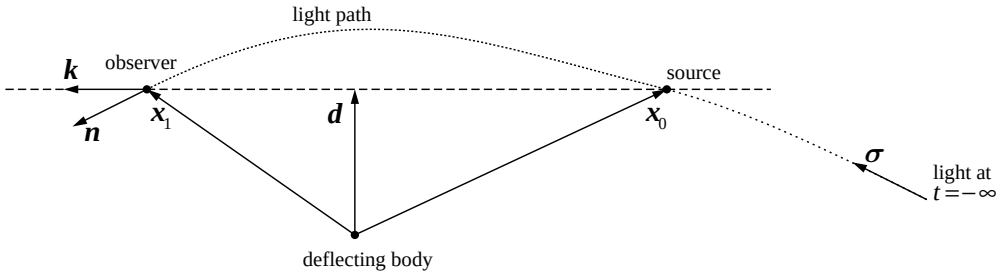
Let us consider the trajectory of a light-signal in post-Newtonian Schwarzschild metric: $g_{00} = -1 + 2\, a + \mathcal{O}\,(c^{-4})$, $g_{i0} = 0$, $g_{ij} = \delta_{ij} + 2\, \gamma\, a\, \delta_{ij} + \mathcal{O}\,(c^{-4})$. The light-ray is being emitted at a position $\boldsymbol{x}_0$ at time moment t_0 and received at position $\boldsymbol{x}_1$ at a time moment t_1, see Figure 1.
Light propagation is governed by geodesic equation, in post-Newtonian order given by

$$\ddot{\boldsymbol{x}} = -\,(1+\gamma)\, c^2\, \frac{a\, \boldsymbol{x}}{x^2} + 2\,(1+\gamma)\, \frac{a\, \dot{\boldsymbol{x}}\, (\dot{\boldsymbol{x}} \cdot \boldsymbol{x})}{x^2} + \mathcal{O}(c^{-2})\,. \tag{1}$$

The unit tangent vector $\boldsymbol{n}$ at the point of observation is defined by

$$\boldsymbol{n} = \frac{\dot{\boldsymbol{x}}(t_1)}{|\dot{\boldsymbol{x}}(t_1)|}\,, \tag{2}$$

Figure 1. A geometrical representation of the boundary problem under consideration for a light-propagation from the source to the observer.



and the unit tangent vector $\boldsymbol{\sigma} = \lim\limits_{t\to-\infty} \dfrac{\dot{\boldsymbol{x}}(t)}{c}$. Furthermore, we define the unit vector $\boldsymbol{k} = \dfrac{\boldsymbol{R}}{R}$, where $\boldsymbol{R} = \boldsymbol{x}_1 - \boldsymbol{x}_0$ and the absolute value is $R = |\boldsymbol{R}|$. In the post-Newtonian standard approach [11], the transformation between $\boldsymbol{n}$ and $\boldsymbol{k}$ reads:

$$\boldsymbol{n} = \boldsymbol{k} - (1+\gamma)\,\frac{m}{d}\,\frac{\boldsymbol{d}}{d}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} + \mathcal{O}\left(c^{-4}\right), \tag{3}$$

where $\boldsymbol{d}$ is the impact parameter

$$\boldsymbol{d} = \boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{k}) = \boldsymbol{k}\times(\boldsymbol{x}_1\times\boldsymbol{k})\,. \tag{4}$$

Eq. (3) is the standard post-Newtonian formula of light-deflection. By means of $\sin\varphi = |\boldsymbol{n}\times\boldsymbol{k}|$, we find the light-deflection angle $\varphi = \delta(\boldsymbol{n},\boldsymbol{k})$ in post-Newtonian approximation:

$$\varphi = (1+\gamma)\,\frac{m}{d}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} + \mathcal{O}\left(c^{-4}\right). \tag{5}$$

A post-post-Newtonian approximation shows that the neglected terms in Eq. (3) are of the order $\mathcal{O}\left(\frac{m^2}{d^2}\,\frac{x_1}{d}\right)$; see Eq. (94) in [12]. Due to the fact that $\frac{x_1}{d}$ can be arbitrarily large, the contribution of the neglected terms in Eq. (3) can also be arbitrarily large in general, depending on the concrete astrometrical configuration under consideration.

3. Post-post Newtonian solution

Now we will consider the trajectory of a light-signal in post-post-Newtonian Schwarzschild metric: $g_{00} = -1 + 2\,a - 2\,\beta\,a^2 + \mathcal{O}\,(c^{-6})$, $g_{i0} = 0$, $g_{ij} = \delta_{ij} + 2\,\gamma\,a\,\delta_{ij} + \epsilon\left(\delta_{ij} + \dfrac{x^i\,x^j}{x^2}\right)a^2 + \mathcal{O}\left(c^{-6}\right)$. The geodesic equation of light-propagation in post-post-Newtonian approximation is given by [12]

$$\begin{aligned}\ddot{\boldsymbol{x}} = &-(1+\gamma)\,c^2\,\frac{a\,\boldsymbol{x}}{x^2} + 2\,(1+\gamma)\,\frac{a\,\dot{\boldsymbol{x}}\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2}\\ &+ 2\,c^2\,\alpha\,\left(\beta - \epsilon + 2\,\gamma\,(1+\gamma)\right)\,\frac{a^2\,\boldsymbol{x}}{x^2} + 2\,\alpha\,\epsilon\,\frac{a^2\,\boldsymbol{x}\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})^2}{x^4}\\ &+ 2\,\alpha\,(2(1-\beta) + \epsilon - 2\,\gamma^2)\,\frac{a^2\,\dot{\boldsymbol{x}}\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} + \mathcal{O}\left(c^{-4}\right).\end{aligned} \tag{6}$$

The solution of (6) and the transformation between the unit vectors $\boldsymbol{n}$ and $\boldsymbol{k}$ in post-post-Newtonian order has been given in [12]. Here, we will write this transformation in the following form:

$$\boldsymbol{n} = \boldsymbol{k} - (1+\gamma)\,\frac{m}{d}\,\frac{\boldsymbol{d}}{d}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} + (1+\gamma)^2\,\frac{m^2}{d^2}\,\frac{\boldsymbol{d}}{d}\,\frac{\left(x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1\right)^2}{R^2\,d\,x_1}$$

$$+\,\boldsymbol{n}_1^{\mathrm{scaling}} + \boldsymbol{n}_1^{\mathrm{ppN}} + \mathcal{O}\left(c^{-6}\right), \tag{7}$$

where the terms $\boldsymbol{n}_1^{\mathrm{scaling}}$ and $\boldsymbol{n}_1^{\mathrm{ppN}}$ are given in Appendix A. While $\boldsymbol{n}_1^{\mathrm{scaling}} \times \boldsymbol{k} = 0$, we note the esimate (for a proof see [19, 20]):

$$\omega_1^{\mathrm{ppN}} = \left|\boldsymbol{n}_1^{\mathrm{ppN}} \times \boldsymbol{k}\right| \leq \frac{15\,\pi}{4}\,\frac{m^2}{d^2}\,. \tag{8}$$

From (7) we obtain

$$\varphi = (1+\gamma)\,\frac{m}{d}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} - (1+\gamma)^2\,\frac{m^2}{d^2}\,\frac{\left(x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1\right)^2}{R^2\;d\;x_1} - \omega_1^{\mathrm{ppN}} + \mathcal{O}\left(c^{-6}\right). \tag{9}$$

Accordingly, the post-post-Newtonian solution contains 'regular' terms of order m^2 which have been estimated in (8) and so-called 'enhanced' terms of order m^2 (i.e. the last term in the first line of Eq. (7) or second term in (9)) which can be much larger than the 'regular' ones. Note, that the 'enhanced' terms of order m^2 contain only the parameter γ, thus these terms come from the post-Newtonian terms in the metric and geodesic equation. Therefore, their origin is the second-order (post-post-Newtonian) solution of the first-order (post-Newtonian) geodesic equation given in (1). This fact has been clarified in [12]. Furthermore, in [12] it has been demonstrated that the 'enhanced' terms result from the use of impact vector (4). This impact vector is convinient for determining the light-propagation when positions of source and observer are given in a concrete reference systems, e.g. the BCRS [11].

The light-deflection formula in (7) can be transformed in a form which resembles the post-Newtonian formula (3) by means of the coordinate-independent impact vector $\boldsymbol{d}'$, cf. Eq. (57) of [12]:

$$\boldsymbol{d}' = \lim_{t\to-\infty} \boldsymbol{\sigma} \times \left(\boldsymbol{x}(t) \times \boldsymbol{\sigma}\right). \tag{10}$$

This impact parameter has the property $d' \geq d$ and is identical to Chandrasekhar's impact parameter (see Eq.(215) in chapter 20 of [5]), that means in vectorial form $\boldsymbol{d}' = \frac{\boldsymbol{L}}{E}$, where $\boldsymbol{L}$ is the orbital three-momentum and E is the energy of the photon on the light-trajectory. While the impact vector $\boldsymbol{d}$ is a simple expression in terms of $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$, the impact parameter $\boldsymbol{d}'$ is uniquely determined by $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ and by the mass of the light deflecting body, but a complicated expression in terms of elliptic integrals. An approximative relation between the impact vectors (4) and (10) has been given in Eq. (62) in [12] and reads:

$$\boldsymbol{d}' = \frac{\boldsymbol{d}}{d}\left(d + (1+\gamma)\,\frac{m}{d}\,\frac{x_1+x_0}{R}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R}\right) - (1+\gamma)m\,\boldsymbol{k}\,\frac{x_1 - x_0 + R}{R} + \mathcal{O}\left(c^{-4}\right), \tag{11}$$

$$d' = d + (1+\gamma)\frac{m}{d}\,\frac{x_0 + x_1}{R}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R} + \mathcal{O}\left(c^{-4}\right). \tag{12}$$

With the aid of the coordinate-independent impact vector Eq. (10), the transformation between the vectors $\boldsymbol{n}$ and $\boldsymbol{k}$ in post-post-Newtonian order can be written as follows, cf. Eqs. (108) and (109) of [12]:

$$\boldsymbol{n} = \boldsymbol{k} - (1+\gamma)\,m\,\frac{\boldsymbol{d}'}{d'^2}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} + \boldsymbol{n}_2^{\mathrm{scaling}} + \boldsymbol{n}_2^{\mathrm{ppN}} + \mathcal{O}\left(c^{-6}\right). \tag{13}$$

The scaling term $\boldsymbol{n}_2^{\mathrm{scaling}}$ and $\boldsymbol{n}_2^{\mathrm{ppN}}$ are given in Appendix A. The expressions (7) and (13) are identical. While $\boldsymbol{n}_2^{\mathrm{scaling}} \times \boldsymbol{k} = 0$, we have the inequality (for a proof see [21])

$$\omega_2^{\mathrm{ppN}} = \left|\boldsymbol{n}_2^{\mathrm{ppN}} \times \boldsymbol{k}\right| \le \frac{15\,\pi}{4}\,\frac{m^2}{d^2}\,. \tag{14}$$

From Eq. (13) we obtain the expression

$$\varphi = (1+\gamma)\,\frac{m}{d'}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} - \omega_2^{\mathrm{ppN}} + \mathcal{O}\left(c^{-6}\right). \tag{15}$$

The form of Eq. (15) resembles the post-Newtonian formula (3).

4. Classical lens equation

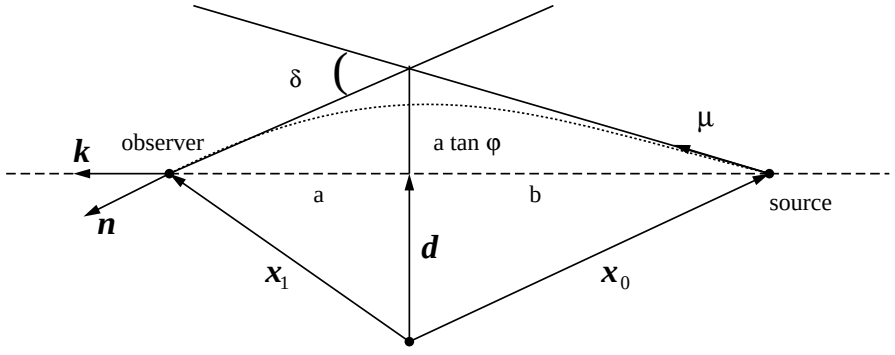
In this Section we consider some basic steps leading to the Classical lens equation. According to the scheme in Figure 2, we obtain the following geometrical relations,

$$\varphi + \psi \;\; = \delta\,, \tag{16}$$

$$a\,\tan\varphi = b\,\tan\psi\,, \tag{17}$$

where the angles $\varphi = \delta(\boldsymbol{n}, \boldsymbol{k}), \psi = \delta(\boldsymbol{k}, \boldsymbol{\mu})$, and the unit tangent vector $\boldsymbol{\mu} = \frac{\dot{\boldsymbol{x}}(t_0)}{|\,\dot{\boldsymbol{x}}(t_0)\,|}$.

Figure 2. A geometrical representation of gravitational lens.



If source and observer are infinitely far from the massive body, then the total light-deflection angle $\delta = \delta\,(\boldsymbol{n}, \boldsymbol{\mu})$ in Schwarzschild metric reads [2]

$$\delta = 2\,(1+\gamma)\,\frac{m}{d'} + \mathcal{O}\left(\frac{m^2}{d'^2}\right) + \mathcal{O}\left(c^{-6}\right), \tag{18}$$

which is a coordinate independent result. The terms of order $\mathcal{O}\left(\frac{m^2}{d'^2}\right)$ can be estimated to be smaller or equal than $\frac{15\,\pi}{4}\,\frac{m^2}{d'^2}$, see [2]. According to (16) in the limit $x_0 = x_1 \to \infty$ we should have $\varphi = \frac{\delta}{2}$, a strict condition which is satisfied by the expression (15) up to the given order.

In classical lens approach, the approximation $d' \simeq d + a\,\tan\varphi$ is used, see Figure 2. Inserting this relation into (18), by means of geometrical relations (16) and (17), and using $\tan\varphi = \varphi + \mathcal{O}(\varphi^3)$ and $\tan\psi = \psi + \mathcal{O}(\varphi^3)$, we obtain the quadratic equation

$$\varphi^2 + \frac{d}{a}\,\varphi - 2\,(1+\gamma)\,\frac{m}{a}\,\frac{b}{a+b} = 0\,. \tag{19}$$

The solution of Eq. (19) is the classical lens equation,

$$\varphi_{1,2}^{\text{class}} = \frac{1}{2}\left(\sqrt{\frac{d^2}{a^2} + 8(1+\gamma)\,\frac{m}{a}\,\frac{b}{a+b}} \mp \frac{d}{a}\right), \tag{20}$$

where the solution with the upper (lower) sign is denoted by φ_1 (φ_2). The classical lens equation is valid for arbitrary small values of d. Below we will see that (20) is restricted for configurations where $a \gg d$ and $b \gg d$. This is because relation (18) and the used approximation for d' is only valid when source and observer are at infinity.

5. Generalized lens equation

There are extreme astrometrical configurations, where $b \sim d$ or even $b \ll d$ are possible. That means, in general astrometric configurations, neither $\frac{m}{d}$ nor $\frac{d}{b}$ are small parameters. For instance, in binary systems the situations $d = 0$ or $b = 0$ are possible. However, a small parameter it given by $\frac{m}{d'} \ll 1$, because for $b \geq 0$ the impact parameter d' is always larger than the radius of massive body. Accordingly, the aim of our investigation is to derive a generalized lens equation valid up to terms of the order $\mathcal{O}\left(\frac{m^2}{d'^2}\right)$. Such a generalized lens equation will provide a link between *Standard post-Newtonian approach* and *Classical lens equation*.

In the indroductionary Section we have formulated the three conditions under which the generalized lens equation should be valid. We first notice that according to $\boldsymbol{d}' = \frac{\boldsymbol{L}}{E}$, the impact vector $\boldsymbol{d}'$ is finite in the limit $d \to 0$. On the other side, the approximative relation (12) is not finite in this limit. However, up to the given order we can rewrite (12) as follows:

$$d' = d + (1+\gamma)\,\frac{m}{d'}\,\frac{x_0 + x_1}{R}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R} + \mathcal{O}\left(c^{-4}\right), \tag{21}$$

which is a quadratic equation for Chandrasekhar's impact parameter, having the solution

$$d' = \frac{d}{2} + \sqrt{\frac{d^2}{4} + (1+\gamma)\, m\, \frac{x_0 + x_1}{R}\, \frac{x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R}} + \mathcal{O}\left(c^{-4}\right). \tag{22}$$

We note, that expression (22) is finite for $d \to 0$ as well as in the limit $R \to 0$. A comparison of (21) with (15) yields the relation

$$d' = d + x_1\, \varphi + \frac{x_0 + x_1 - R}{R}\, x_1\, \varphi + \mathcal{O}\left(c^{-4}\right), \tag{23}$$

where φ is given by (15). By inserting (23) into (15) we obtain the quadratic equation

$$\varphi^2 + \frac{d}{x_1}\, \varphi - (1+\gamma)\, \frac{m}{x_1}\, \frac{x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R\, x_1} + \delta_1 + \delta_2 = \mathcal{O}(c^{-6}), \tag{24}$$

where $\delta_1 = \dfrac{d}{x_1}\, \omega_{\mathrm{ppN}}$ and $\delta_2 = \dfrac{x_0 + x_1 - R}{R}\, \varphi^2$, and we note $\delta_1 \ll \varphi^2$ and $\delta_2 \ll \varphi^2$. The solution of quadratic equation (24) reads

$$\varphi_{1,2} = \frac{1}{2}\left(\sqrt{\frac{d^2}{x_1^2} + 4\,(1+\gamma)\, \frac{m}{x_1}\, \frac{x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R\, x_1}} \mp \frac{d}{x_1}\right) + \Omega_{\mathrm{ppN}} + \mathcal{O}\left(c^{-6}\right), \tag{25}$$

where φ_1 (φ_2) is the solution with the upper (lower) sign. Eq. (25) represents the generalized lens equation, which is valid in all those extreme astrometrical configurations defined in 1. - 3. in the introductionary Section. One can show, that expression Ω_{ppN} can be estimated as follows:

$$\Omega_{\mathrm{ppN}} = \frac{1}{4}\,(\delta_1 + \delta_2)\left(\frac{d^2}{x_1^2} + 4\,(1+\gamma)\, \frac{m}{x_1}\, \frac{x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R\, x_1}\right)^{-1/2} \leq \frac{15\,\pi}{4}\, \frac{m^2}{d'^2}. \tag{26}$$

In the following Section we will show that the formula (25) represents a link between *Standard post-Newtonian approach* and *Classical lens equation*.

6. Discussion of generalized lens equation

6.1. Comparison with post-Newtonian and post-post-Newtonian solution

In this Section we compare the generalized lens equation (25) with the standard post-Newtonian and post-post-Newtonian approach of light-deflection. A series expansion of the solution φ_1 in Eq. (25) for $d \gg m$ yields

$$\varphi_1 = \varphi_{\mathrm{pN}} + \varphi_{\mathrm{ppN}} + \Omega_{\mathrm{ppN}} + \mathcal{O}\left(c^{-6}\right), \tag{27}$$

with

$$\varphi_{\mathrm{pN}} = (1+\gamma)\, \frac{m}{d}\, \frac{x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R\, x_1}, \tag{28}$$

$$\varphi_{\mathrm{ppN}} = -\,(1+\gamma)^2\, \frac{m^2}{d^2}\, \frac{\left(x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1\right)^2}{R^2\, d\, x_1}. \tag{29}$$

This expression coincides with (9) up to the given order. That means, a series expansion of the generalized lens equation (25) yields the standard post-Newtonian solution (3) and the 'enhanced' post-post-Newtonian term in (7); cf. Eqs. (3) and (4) in [21]. A further series expansion of Eq. (25) would yield the 'enhanced' terms beyond post-post-Newtonian terms.

6.2. Derivation of classical lens equation from generalized lens equation

In order to consider the relation of generalized lens equation (25) and classical lens equation (20), we rewrite (25) in terms of a and b and obtain

$$\varphi_{1,2} = \frac{1}{2}\left(\sqrt{\frac{d^2}{a^2} + 2(1+\gamma)\frac{m}{a}\frac{(a+b)^2 - \left(a\sqrt{1+\epsilon_1^2} - b\sqrt{1+\epsilon_2^2}\right)^2}{(a+b)\ a}} \mp \frac{d}{a}\right)\frac{1}{\sqrt{1+\epsilon_1^2}}$$
$$+\ \Omega_{\mathrm{ppN}} + \mathcal{O}\left(c^{-6}\right), \tag{30}$$

where $\epsilon_1 = \dfrac{d}{a}$ and $\epsilon_2 = \dfrac{d}{b}$. The classical lens equation (20) is valid in case of $a, b \gg d$. A corresponding series expansion of generalized lens equation (30) in terms of ϵ_1 and ϵ_2 leads to (see Appendix B)

$$\varphi_{1,2} = \varphi_{1,2}^{\mathrm{class}} + \mathcal{O}\left(\frac{m}{a}\,\frac{d}{a}\right) + \mathcal{O}\left(\frac{m}{a}\,\frac{d}{b}\right) + \Omega_{\mathrm{ppN}} + \mathcal{O}\left(c^{-6}\right), \tag{31}$$

where $\varphi_{1,2}^{\mathrm{class}}$ is defined by Eq. (20). For astrometry the relevant configurations are such that $d \ll a$, while in extreme but nevertheless real astrometrical configurations the situation $b \ll d$ is possible, e.g. in binary systems. Therefore, the classical lens equation is not applicable for such configurations.

Finally, we also note that in the limit $d \to 0$, known as *Einstein ring solution*, we obtain

$$\lim_{d\to 0}\ \varphi_{1,2} = \sqrt{2\,(1+\gamma)\ \frac{m}{x_1}\,\frac{x_0}{x_0+x_1}}\,. \tag{32}$$

In the limit $d \to 0$ the classical lens equation coincides with (32); note, that in this limit $a = x_1$ and $b = x_0$. In the extreme configuration $b = 0$ (in this limit φ_2 does not exist) we obtain the finite result

$$\lim_{b\to 0}\varphi_1 = \frac{1}{2}\left(\sqrt{\frac{d^2}{x_1^2} + 4\,(1+\gamma)\,\frac{m}{x_1}\,\frac{d\,a}{(x_1+d)\ x_1}} - \frac{d}{x_1}\right), \tag{33}$$

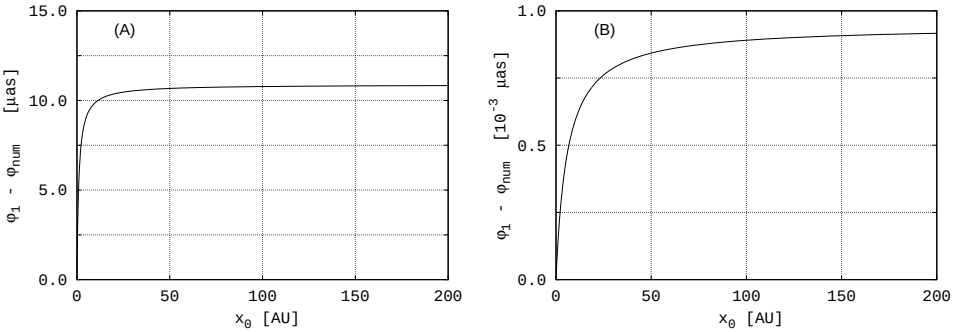
while the classical lens equation yields simply $\varphi_1^{\mathrm{class}} = 0$. Obviously, in the limit $a \to 0$ the expression (33) yields zero as it has to be.

6.3. Comparison with exact solution

The accuracy of (25) and the estimate (26) has been confirmed by a comparison with the exact numerical solution. For that, we have solved the geodesic equations in Schwarzschild metric by numerical integrator ODEX [9] for several extreme astrometrical configurations. Using forth and back integration a numerical accuracy of at least 10^{-24} in the components of position and velocity of the photon is guaranteed. Thus, the numerical integration can be considered as an exact solution of geodesic equation. This numerical approach has been described in some detail in [12]. In all considered extreme

configurations the validity of (25) and the estimate (26) have been confirmed. As a typical example, in Figure 3 we present the results for light-deflection of a grazing ray at Sun and Jupiter. The accuracy of generalized lens equation (25) is considerably better than the post-post-Newtonian solution investigated in detail in [19], cf. FIG. 3 (B) with FIG. 2 in [19].

Figure 3. Comparison of generalized lens equation (25) with exact numerical solution. (A) for the case of a grazing ray at Sun ($m_\odot = 1476.6\,\mathrm{m}$, $d' = 696.0 \times 10^6\,\mathrm{m}$). (B) for the case of a grazing ray at Jupiter ($m = 1.40987\,\mathrm{m}$, $d' = 71.492 \times 10^6\,\mathrm{m}$).



7. Summary

Modern astrometry of microarcsecond level, e.g. ESA mission GAIA, needs highly precise approximative solutions for light-deflection. In order to determine the light deflection analytically, there are two essential approaches: *Standard post-Newtonian approach* valid for $m \ll d$ and *Classical lens equation* valid for source and observer far enough from the lens, especially when $a \gg d$ and $b \gg d$. However, there are astrometrical configurations, where none of them are applicable, e.g. in binary systems. In order to investigate the light-deflection in such systems a link between these both approaches is needed. Such a link can be provided by a *Generalized lens equation* which, in the appropriate limits, coincides with *Standard post-Newtonian approach* and *Classical lens equation*. In our study we have suggested a generalized lens equation (25) for Schwarzschild metric which is valid for small light-deflection angles. The derivation is based on the solution of geodesic equation in post-Newtonian metric and Chandrasekhar's coordinate independent impact parameter d' (10). The neglected terms in (25) can be estimated by $\Omega_{\mathrm{ppN}} \leq \frac{15\,\pi}{4}\,\frac{m^2}{d'^2}$. The accuracy of generalized lens equation (25) is considerably better than the post-post-Newtonian solution investigated in some detail in [19].

The generalized lens equation satisfies three conditions formulated in the introductionary Section. Especially, we have shown that in the appropriate limits we obtain the post-Newtonian terms, 'enhanced' post-post-Newtonian terms and the Classical lens equation. Thus, the generalized lens equation (25) provides a link

between these both essential approaches to determine the light-deflection. Numerical investigations have confirmed the analytical results obtained. The generalized lens equation (25) will allow an analytical understanding and investigation of light-deflection in extreme astrometric configurations.

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Appendix A. The terms $\boldsymbol{n}_1^{\rm scaling}$, $\boldsymbol{n}_1^{\rm ppN}$ and $\boldsymbol{n}_2^{\rm scaling}$, $\boldsymbol{n}_2^{\rm ppN}$

According to Eq. (87) in [12], the transformation between $\boldsymbol{n}$ and $\boldsymbol{k}$ in post-post-Newtonain order is given by Eq. (13), where the scaling term is given by

$$\boldsymbol{n}_1^{\rm scaling} = -\frac{1}{8}\,(1+\gamma)^2\,\frac{m^2}{d^2}\,\boldsymbol{k}\,\frac{\left((x_1-x_0)^2-R^2\right)^2}{R^2\,x_1^2}\,. \tag{A.1}$$

Furthermore,

$$\begin{aligned}
\boldsymbol{n}_1^{\rm ppN} = {}& m^2\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)\left[\frac{1}{2}\,(1+\gamma)^2\,\frac{R^2-(x_1-x_0)^2}{x_1^2\,|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2}\right.\\
&+\frac{1}{4}\,\alpha\,\epsilon\,\frac{1}{R}\left(\frac{1}{R\,x_0^2}-\frac{1}{R\,x_1^2}-2\,\frac{\boldsymbol{k}\cdot\boldsymbol{x_1}}{x_1^4}\right)\\
&-\frac{1}{4}\,\left(\,8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\,\alpha\,\epsilon\,\right)\,R\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1^2\,|\,\boldsymbol{x}_1\times\boldsymbol{x}_0\,|^2}\\
&\left.+\frac{1}{8}\,\left(8(1+\gamma-\alpha\,\gamma)(1+\gamma)-4\,\alpha\,\beta+3\alpha\,\epsilon\right)\,\frac{x_1^2-x_0^2-R^2}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^3}\,\delta(\boldsymbol{x}_1,\boldsymbol{x}_0)\right]\\
&+(1+\gamma)^2\,\frac{m^2}{d^2}\,\frac{\boldsymbol{d}}{d}\,\frac{\left(x_0\,x_1-\boldsymbol{x}_0\cdot\boldsymbol{x}_1\right)^2}{R^2\,d\,x_1}\,\frac{x_0+x_1-R}{R}\,.
\end{aligned} \tag{A.2}$$

In Eq. (13) we have introduced the scaling term

$$\boldsymbol{n}_2^{\rm scaling} = \boldsymbol{n}_1^{\rm scaling} - (1+\gamma)^2\,\frac{m^2}{d^2}\,\boldsymbol{k}\,\frac{x_0\,x_1-\boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1}\frac{x_1-x_0+R}{R}\,, \tag{A.3}$$

and

$$\boldsymbol{n}_2^{\rm ppN} = \boldsymbol{n}_1^{\rm ppN} - (1+\gamma)^2\,\frac{m^2}{d^2}\,\frac{\boldsymbol{d}}{d}\,\frac{\left(x_0\,x_1-\boldsymbol{x}_0\cdot\boldsymbol{x}_1\right)^2}{R^2\,d\,x_1}\,\frac{x_0+x_1-R}{R}\,. \tag{A.4}$$

In case of general theory of relativity we have $\beta=\gamma=\epsilon=1$. The choice $\alpha=0$ means that we obtain the solution of post-Newtonian geodesic equation (1).

Appendix B. Series expansion of generaized lens equation

The series expansion of generalized lens equation (25) in the form (30) yields

$$\varphi_{1,2} = \varphi_{1,2}^{\mathrm{class}} - \frac{1}{4}\varphi_{1,2}^{\mathrm{class}}\,\epsilon_1^2 + (1+\gamma)\,\frac{m}{a}\,\frac{b-a}{a+b}\,\frac{1}{\sqrt{\dfrac{d^2}{a^2} + 8\,(1+\gamma)\,\dfrac{m}{a}\,\dfrac{b}{a+b}}}\,\epsilon_1^2$$
$$+ (1+\gamma)\,\frac{m}{a}\,\frac{b}{a}\,\frac{a-b}{a+b}\,\frac{1}{\sqrt{\dfrac{d^2}{a^2} + 8\,(1+\gamma)\,\dfrac{m}{a}\,\dfrac{b}{a+b}}}\,\epsilon_2^2 + \mathcal{O}\left(\epsilon_1^4\right) + \mathcal{O}\left(\epsilon_2^4\right) + \mathcal{O}\left(\epsilon_1^2\,\epsilon_2^2\right)\,. \tag{B.1}$$

By means of $\dfrac{|\,b-a\,|}{a+b} \leq 1$ and $\left(\dfrac{d^2}{a^2} + 8\,(1+\gamma)\,\dfrac{m}{a}\,\dfrac{b}{a+b}\right)^{-1/2} \leq \dfrac{a}{d}$ we obtain (31).

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