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Edmund Stoner and white dwarf stars

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Abstract

The discovery of a limiting mass for white dwarf stars is today usually attributed to Subramanian Chandrasekhar. However it appears that an article by Edmund Stoner, which appeared in the *Philosophical Magazine* in 1930, was the first publication to give a convincing demonstration of the existence of a limiting mass for white dwarfs. We examine here why it is that the contributions of Stoner and others towards this discovery have been largely forgotten.

1. Introduction

The Biographical Memoires of Fellows of the Royal Society [1] tell us that Edmund Clifton Stoner was born in Esher in the county of Surrey in 1899. When he was in his teens his parents moved to Bolton in Lancashire where he won a scholarship that enabled him to attend Bolton Grammar School. From there he entered Cambridge University in 1918 graduating in 1921 with a first-class degree in the Natural Sciences Tripos. Surprisingly, although the physics lectures given to undergraduates were quite mathematical in content there was no mathematics teaching in the syllabus, students being expected to pick up what they needed on their own. Stoner was conscious of this lack of formal mathematics training throughout his career.

Between 1921 and 1924 Stoner worked for a PhD in the Cavendish laboratory with Ernest Rutherford as his official supervisor. His research topic was the measurement of the absorption of X-rays by a selected group of elements. He obtained his PhD in 1924, despite a prolonged period in hospital after being diagnosed with diabetes. Towards the end of his time at the Cavendish he wrote a paper entitled *The distribution of electrons among atomic energy levels*. This paper, published in the *Philosophical Magazine* [2] preceded Pauli's enunciation of his exclusion principle and played a significant role in its creation: in effect it contained an explicit statement of the exclusion principle. It should be noted that the exclusion principle was to play a crucial role in the theory of white dwarfs.

After deciding that he did not wish to remain in Cambridge, Stoner successfully applied for a lectureship in the Physics Department of the University of Leeds. There, where he remained until his retirement in 1963, he switched to research in theoretical physics and

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embarked on pioneering work to understand the magnetic properties of matter: this is the work for which he is best known. His interest in white dwarfs was sparked by Fowler's paper *On Dense Matter* [3]. Over the period 1929 to 1932 he published the pioneering papers on white dwarfs, which are the subject of this article. In particular he deduced the existence of a limiting mass for white dwarfs now known as the Chandrasekhar mass.

2. White dwarfs

White dwarfs are hot, very compact, stars that have ceased nuclear burning. Their high temperature and low luminosity places them in the bottom left-hand corner of the Hertzprung- Russell diagram [4]. The white dwarf, 40 Eridani B, recorded by William Herschel in 1783 was the first to be detected [5]. By the late 1920s, when Stoner started his investigations, the masses, luminosities and surface temperatures of a small number of white dwarfs had been measured. The best-known example, Sirius B, is the faint companion of Sirius, the brightest star in the night sky. The observational data at this time for Sirius B gave its mass to be 0.85 solar masses, its surface temperature 8000 K and its radius 21 000 km. These figures imply a mean density of about 5 x 10^7 kg m^{-3}. At the time this density exceeded that of any other known object by several orders of magnitude. For example the mean density of the Sun is about 1 400 kg m^{-3} and the density of osmium metal, the densest element, is 2.2 x 10^4 kg m^{-3}. It is interesting to note that, with these values for the mass and radius of Sirius B, the General Theory of Relativity predicts that light coming from its surface would have a red shift of 20 km s^{-1}. A measurement of this red shift published by W S Adams [6] produced a value of 19 km s^{-1}. However, we now know that this apparent agreement between The General Theory of Relativity and the data on Sirius B, which so delighted Eddington, is coincidental. Adam's measurement of redshift is too low by a factor of 4 and the radius is too large by a factor of almost 4. The modern values for the mass, temperature, radius and red shift of Sirius B are $0.98M_{solar}$, 22 000 K and 6 000 km respectively and the red shift, measured using the Hubble space telescope, is 80 km s^{-1} [7]. With these measurements the density of Sirius B becomes 2 x 10^9 kg m^{-3} or 2 tonnes cm^{-3}.

Such high densities seemed at the time to raise a paradox: matter compressed to this degree must be completely ionised so what happens to the star as its reaches the end of its lifetime and starts to cool? In the words of Sir Arthur Eddington [8]

'The star could not stop losing heat, but it would not have enough energy to cool down'.

Eddington supposed that for the star to reach a stable cold state, its nuclei and electrons needed to recombine into atoms. But to do this the star needed to regain normal density by expanding against the force of gravity, which it could not do. According to classical physics, a

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cold ionised gas cannot exert any pressure so the star was doomed to continue contracting indefinitely, a situation that Eddington regarded as absurd.

The resolution of this paradox was provided by the newly discovered quantum mechanics and the Pauli Exclusion Principle. The Cambridge theoretician R H Fowler realised that even at the absolute zero of temperature a very dense electron gas will exert a large pressure and that this pressure could support a white dwarf star against collapse [3]. Fowler assumed that the electrons occupied the lowest energy states allowed by the Pauli Principle, which states that only two electrons, one in each of two spin states, can occupy a state of volume h^3 in six-dimensional phase space, where h is the Planck constant. Such a gas is said to be *degenerate*; the uppermost filled state has a momentum p_0 which, as a function of number density of electrons n, is given, at absolute zero, by

$$p_0 = h\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3}. \qquad (1)$$

So a high density of electrons in physical space implies that they have high momenta and hence that they exert a large pressure which, for non-relativistic electrons, is given as a function of the mass density $\rho = \mu_e m_H n$ by the expression

$$P = \frac{h^2}{5m_e}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{\rho}{\mu_e m_H}\right)^{5/3}, \qquad (2)$$

where m_e is the mass of an electron, m_H the mass of hydrogen and μ_e the average atomic weight per electron. At the time, white dwarfs, being highly evolved stars, were thought to be composed of heavy elements so the value of μ_e was taken to be about 2.5. We now know that white dwarfs are composed mainly of carbon and oxygen so the correct value of μ_e is very close to 2. The nuclei being much more massive than the electrons do not contribute significantly to the pressure.

3. Stoner's papers on white dwarfs

In the first of three papers that appeared in the *Philosophical Magazine* [9], Stoner took Fowler's idea that white dwarfs are supported by electron degeneracy pressure and asked the question: is there a limit to the density of such stars? He modelled a star as a sphere of ionised gas having *uniform* density at zero temperature. Then, from the kinetic energy E_K of the electrons and the gravitational binding energy E_G of the star, the density is obtained by minimising the total energy with respect to the number density n, that is he imposed the condition

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$$\frac{d}{dn}\left(E_K + E_G\right) = 0. \qquad (3)$$

From this condition he obtained the following expression for the mean mass density of a star of mass *M*:

$$\rho = 3.85 \times 10^9 \left(\frac{M}{M_{solar}}\right)^2 \text{ kg m}^{-3}. \qquad (4)$$

Note that this expression gives a value for the mean density of Sirius B in fairly good agreement with the modern value but not in good agreement with the 1930's value. Note also that there is no suggestion of a limiting mass with Fowler's non-relativistic equation (equation 2).

Following the appearance of Stoner's paper, Wilhelm Anderson, a German physicist working at Tartu University in Estonia, noticed that for masses beyond about a solar mass the electron energies become relativistic [10]. So Fowler's equation of state, which is derived using Newtonian Mechanics, will not be valid for the more massive stars. Anderson derived a relativistic equation of state and, despite his derivation being not strictly correct, deduced from it that there is a limiting mass for white dwarfs of 0.68 M_{solar}; with the modern value $\mu_e = 2$ this becomes 1.1 M_{solar}.

Stoner responded to Anderson in a second paper [11] (reproduced as a facsimile following this commentary) in which he pointed out the error in Anderson's derivation of the relativistic equation of state and derived the correct equation – Stoner's equation is now known as the Stoner-Anderson equation. In the extreme relativistic limit this equation of state tends to the simple form

$$P = \frac{ch}{4}\left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{\rho}{\mu_e m_H}\right)^{4/3}. \qquad (5)$$

Stoner used his energy minimisation method with the constant density approximation and his relativistic equation of state to obtain the mass of a star for any value of the electron density *n*, or, equivalently, ρ. He found that as the density increases, the mass of the star increases more and more slowly, eventually reaching a limiting value given by the expression

$$M_L = \frac{3}{16\pi}\left(\frac{5hc}{2G}\right)^{3/2}\frac{1}{\left(\mu_e m_H\right)^2}. \qquad (6)$$

With $\mu_e = 2$ this gives M_L = 1.56 M_{solar}, which should be compared with the more accurate value of 1.44 M_{solar} obtained by numerical integration of the equations of stellar structure. It is interesting to note that Stoner did not write out the expression above for M_L explicitly, so

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missed the significant observation that M_L is proportional to the Planck mass $\left(\frac{hc}{G}\right)^{1/2}$ cubed. The Planck mass occurs whenever gravity, quantum mechanics and relativity play an essential role; for example see [12].

We can understand the reason for the existence of a limiting mass using the constant energy approximation as follows. With the non-relativistic equation of state [equation (2)], as the star shrinks the kinetic energy E_K increases faster than the potential energy E_G so there is always an equilibrium radius. But with the limiting form of the equation of state [equation (5)], E_K becomes proportional to $N^{4/3}/R$, where R is the radius of the star and N the number of electrons. As the gravitational energy E_G is proportional to N^2/R, both E_K and E_G increase at the same rate as the star contracts. So if $|E_G| > E_K$ the star will collapse and if $|E_G| < E_K$ the star will expand until the dependence of E_K on R alters allowing equilibrium to be reached. The limiting mass will occur when the two terms are equal. This condition determines the value of N and hence the limiting mass of the star.

Stoner also looked for and failed to find a relationship between the measured masses and luminosities of the few known white dwarfs. We now know that there is no relationship because white dwarfs have ceased nuclear burning and are cooling down, so the luminosity of a white dwarf depends on its age.

Stoner's third paper [13] written in collaboration with his research student Frank Tyler is a sequel to his two previous papers. In it he attempts to obtain the actual density distribution in order to determine to what extent his previous results based on the constant density approximation would need to be modified. In his own words:

'The problem has proved more intractable than was anticipated, and a completely satisfactory solution has not been reached'.

He also addressed the fate of a star whose mass lay above the limit. He concluded that the star would contract, heat up and gradually lose mass through radiation until its mass dropped below the limiting value, allowing it to stabilise. It should be born in mind that, in the early 1930s, nuclear physics was in its infancy so his conclusions were reasonable given the state of knowledge at the time. We now know that, rather than a slow contraction, a catastrophic collapse would occur initiated by the onset of the inverse beta decay reaction $e^- + p \rightarrow n + \nu_e$, which undermines the support of the star by removing electrons. The collapse gives rise to very rapid heating, causing thermonuclear burning of the carbon and oxygen. The resultant release of energy is rapid enough to blow the star apart – an event now identified as a supernova of type 1a.

4. Chandrasekhar and Landau

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The name that is attached to the limiting mass of white dwarfs is that of the Indian astrophysicist Subramanian Chandrasekhar, who was also working on the structure of white dwarfs in the early 1930s. He read Stoner's paper-*The limiting density of white dwarf stars*[9] and realized that, with Fowler's equation of state (2) , the density as a function of radius is given by the solution of the Lane-Emden equation with polytropic index n = 3/2. This solution is given in tabulated form in Eddington's book *The Internal Constitution of the Stars* [8] so he was able to rederive Stoner's result for the dependence of density on the mass, i.e. $\rho \propto M^2$, without making Stoner's constant-density approximation [14]. The next year, following the publication of Stoner's second paper *The Equilibrium of Dense Stars* [11], Chandrasekhar applied the theory of polytropic gas spheres once again but this time with the equation of state (5) which applies only in the extreme relativistic limit. The solution of the stellar structure equations with this equation of state is a polytrope of index n = 3. This solution is also tabulated in Eddington's book, and yields the unique mass 0.91 solar masses (1.4 M_{solar} with $\mu_e = 2$) which Chandrasekhar took, without proof, to be the maximum mass [15] of a white dwarf. By comparison, Stoner's maximum mass, arrived at with the constant-density approximation, was about 20% higher than this. It seems that at this stage Chandrasekhar did not understand the full significance of his result [16]. Finally four years later Chandrasekhar succeeded in solving the stellar structure equations with the Stoner-Anderson equation of state for the mass-radius relation for all stellar masses up to the limiting mass [17]. The solutions show that, as the limit is approached, the radius of the star shrinks to zero and the density goes to infinity. In practice the equation of state ceases to be valid when the electron energies exceed the threshold for inverse beta decay to occur, so the limiting mass is never reached.

It is interesting to note that L Landau also obtained a value of $M \sim 1.5 M_{solar}$ for the limiting mass of stars supported by relativistic electron degeneracy pressure [18]. But then, surprisingly, he went on to interpret this limit as arising from a failure of quantum mechanics:

'As in reality such masses exist quietly as stars and do not show any such ridiculous tendencies we must conclude that all stars heavier than 1.5M_{solar} certainly possess regions in which the laws of quantum mechanics (and therefore of quantum statistics) are violated'.

5. Discussion

As has often been the case in the history of science an important discovery is not solely due to one person. Such is the case with the discovery of a limiting mass for white dwarf stars. Stoner's paper *The equilibrium of Dense Stars* [11] was the first paper to give a convincing demonstration of the existence of a limiting mass. So why is it that Chandrasekhar's name is the one that has subsequently come to be associated with this discovery and why has the important role played by Stoner has been largely forgotten? There seem to be several

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reasons for this neglect of Stoner's work. Chandrasekhar pursued the study of white dwarfs further than Stoner and was the first person to carry out the accurate numerical integration of the stellar structure equations and to obtain the value 1.4 solar masses [16]. Also he tended to give rather sparing acknowledgment of Stoner's pioneering work in his papers and in the numerous interviews he gave later in his life. For a more detailed account of this matter see the article by M Nauenberg [19]. A conspicuous example of this neglect occurred In 1983 when Chandrasekhar was awarded a Nobel Prize for his work on stars. His Nobel lecture entitled *On Stars and their Stability* recounted in some detail the steps leading to the discovery of a limiting mass for white dwarfs. Surprisingly, in this account, acknowledgement of Stoner's work is completely absent.

Finally there is the often recounted story of the dramatic meeting of the Royal Astronomical Society that took place on 11th January 1935 [20] in which Chandrasekhar presented his results, only for Eddington to get up and ridicule them with the words

'The star has to go on radiating and radiating and contracting and contracting until I suppose it gets down to a few kilometres radius when gravity becomes strong enough to hold in the radiation and the star can at last find peace…. I think there should be a law of nature to prevent a star behaving in this absurd way'.

Stoner's name does not appear in this famous story and he does not appear to have taken part in the subsequent debate with Eddington over the existence of a limiting mass for white dwarfs.

Such was Eddington's preeminent position as the world's leading astrophysicist that, although many of the leading physicists of the day, such as Fowler, Rosenfeld, Bohr, Dirac and Pauli were convinced that Eddington was wrong, none of them were prepared to challenge him openly [20] [21]. Most probably it was Eddington's unyielding opposition to the idea of gravitational collapse that delayed the exploration of its full implications until the 1960s when research into black holes began.

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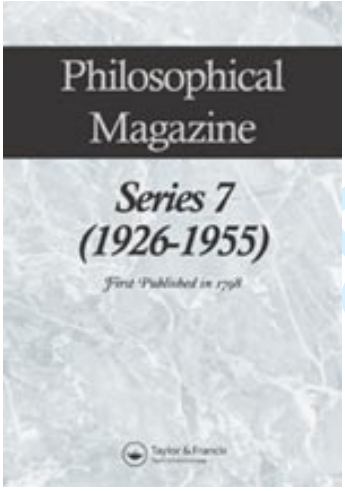

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LXXXVII. *The equilibrium of dense stars*

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LXXXVII. *The Equilibrium of Dense Stars.* By EDMUND C. STONER, *Ph.D.* (*Camb.*), *Reader in Physics, University of Leeds* *.

Introduction.

IN white dwarf stars the last stages of the ionization process are reached, so that at least in the central parts the molecules consist almost entirely of atomic nuclei and free electrons. Under these conditions very high densities are possible. The mean densities deduced from observations, though large (of the order 50 to 100 thousand), fall very far short of those corresponding to a "close-packed" arrangement, if a naive view of the "sizes" of electrons and nuclei is taken. It was suggested that some limitation might be introduced by the "jamming" of a few remaining atoms with K-ring electrons. In a previous paper †, however, it was shown that a limitation is imposed by the exclusion principle applying to free electrons, as embodied in the Fermi statistics. The number of electrons with momenta within a definite range cannot exceed a certain maximum. Any increase in density involves an increase of energy. In the limiting case, at absolute zero, the star can contract until the decrease in gravitational energy becomes insufficient to balance the increase of kinetic energy of the electrons. With a few simplifying assumptions, an expression was derived which indicated that the maximum density varied as the square of the mass of the star. It has been pointed out by Anderson ‡ that in this calculation the effect of the relativity change of mass was neglected. He has taken this effect into account, but in a manner which seems open to criticism. His general conclusions, which seem to be correct, are that the simple expression holds provided the electron densities are not too large, but that the mass corresponding to large electron densities is smaller than that previously calculated, and that it reaches a limit. The correction becomes important for stars of mass about half that of the Sun, and so for the white dwarfs which were actually considered. The main purpose of this paper is to calculate the effect of the relativity change of mass, using a method which seems more rigorous than

* Communicated by Prof. R. Whiddington, F.R.S.

† E. C. Stoner, Phil. Mag. vii. p. 63 (1929). (This paper will be referred to as I.)

‡ W. Anderson, *Zeits. für Phys.* lvi. p. 851 (1929).

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that of Anderson. The conclusion drawn is that the range in which the simple expression holds approximately is wider than that indicated by Anderson's results, the "limiting mass" being somewhat greater, so that there is still an equilibrium state, under the particular conditions, for the stars under consideration. The significance of the results for very dense stars is discussed, and the effect of the electron gas following the Fermi statistics at the smaller densities found in normal stars is briefly considered.

Effect of the Relativity Change of Mass on the Zero-point Energy of Electron Gas.

Let E_G be the gravitational energy of the idealized star, E_K the total kinetic energy of the electrons, n the number of electrons per unit volume. Then, as previously shown, the equilibrium condition is given by

$$\frac{d}{dn}(E_G+E_K)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $\bar{\epsilon}$ be the mean kinetic energy of the electrons at absolute zero, V the volume of the star, supposed to be of uniform density. (The distribution of density will be considered later.) When the relativity change of mass is neglected, the total kinetic energy at absolute zero is given by

$$E_K=nV\bar{\epsilon}=nV\frac{3}{40}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2n^{2/3}}{m_0}. \quad . \quad . \quad . \quad (2)$$

It is this expression which has to be modified.

In Anderson's treatment m_0 is replaced by m, where m is derived from the equation

$$c^2(m-m_0)=\frac{3}{40}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2n^{2/3}}{m}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is thus assumed that the total kinetic energy can be expressed in the alternative forms

$$nVc^2(m-m_0) \quad \text{and} \quad nV\frac{3}{40}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2n^{2/3}}{m},$$

where m has the same value in both expressions, being a "mean" mass; further, that this mean mass can be employed in subsequent calculations in place of m_0. The value of a mean mass, however, depends on the manner in which the mass enters into the particular expressions for which the appropriate mean value is being derived, and it is not legitimate to use one particular mean value indiscriminately

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in more general calculations. (Perhaps the simplest illustration is that the arithmetic mean is not in general equal to the root mean square. The objection to Anderson's treatment has a similar basis.) It seems more satisfactory to attack the problem of determining the zero-point energy without introducing the idea of mean mass at all.

Let ϵ be the kinetic energy of an electron, p the momentum, and let $\beta = v/c$,

$$\epsilon = (m - m_0)c^2 = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right), \quad . \quad . \quad (4)$$

$$p^2 = (m\beta c)^2 = m_0{}^2c^2\left(\frac{1}{1-\beta^2} - 1\right). \quad . \quad . \quad . \quad . \quad (5)$$

Substituting for $1/\sqrt{1-\beta^2}$ in (4),

$$\epsilon = m_0c^2\left[\left(1 + \frac{p^2}{m_0{}^2c^2}\right)^{1/2} - 1\right]. \quad . \quad . \quad . \quad . \quad (6)$$

In the completely condensed state there are two electrons in each cell of the 6-dimensional phase-space (of 6-volume h^3), so that the number of electrons in a volume V with momenta between p and $p + dp$ is equal to

$$2 \times \frac{4\pi p^2 dp}{h^3} \times \mathrm{V}.$$

Thus for the total energy $\mathrm{E_K}$ of the electrons in a condensed state with maximum momentum p_0

$$\mathrm{E_K} = \int_0^{p_0} 2m_0c^2\left[\left(1 + \frac{p^2}{m_0{}^2c^2}\right)^{1/2} - 1\right]\frac{4\pi p^2 dp\,\mathrm{V}}{h^3}. \quad . \quad (7)$$

Since $n\mathrm{V}$ is the total number of electrons (double the number of cells), the maximum momentum p_0 is related to n by

$$(\mathrm{V}/h^3)\int_0^{p_0} 4\pi p^2\,dp = \tfrac{1}{2}\,n\mathrm{V}.$$

Let

$$\left.\begin{aligned} n &= \frac{8\pi p_0{}^3}{h^3}, & p_0 &= \left(\frac{3h^3n}{8\pi}\right)^{1/3}. \\ y &= \frac{p}{m_0c}, & x &= \frac{p_0}{m_0c}. \end{aligned}\right\} \quad . \quad . \quad (8)$$

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Then

$$E_K = \frac{8\pi V m_0 c^2}{h^3} \times (m_0 c)^3 \int_0^x \{(1+y^2)^{1/2} - 1\} y^2\, dy,$$

$$= \frac{8\pi V m_0^4 c^5}{h^3} \left[-\frac{x^3}{3} + \int_0^x (1+y^2)^{1/2} y^2\, dy \right] \quad . \quad (9)$$

$$= \frac{8\pi V m_0^4 c^5}{h^3} \left[-\frac{x^3}{3} + f(x) \right] . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

As long as x is small (i. e., $p_0 << m_0 c$), $f(x)$ may be readily evaluated by integration in series,

$$[f(x)]_{x<} = \frac{1}{3} x^3 + \frac{1}{10} x^5 - \frac{1}{56} x^7 \ldots,$$

$$E_K = \frac{8\pi V m_0^4 c^5}{10 h^3} x^5 \left(1 - \frac{5}{28} x^2 \ldots \right) . \quad . \quad . \quad (10)$$

Substituting for x

$$E_K = \frac{8\pi V}{3 h^3 m_0} p_0^5 \left(1 - \frac{5}{28} \frac{p_0^2}{m_0^2 c^2} \ldots \right), \quad . \quad . \quad . \quad . \quad (10a)$$

$$= n V \frac{3}{40} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2 n^{2/3}}{m_0} \left[1 - \frac{5 h^2}{112} \left(\frac{3}{\pi} \right)^{2/3} n^{2/3} \right], \quad . \quad (10b)$$

agreeing in the limit with (2) for n small.

The above expression may be used for $p_0 << m_0 c$, corresponding to $n << \frac{8\pi}{3} \left(\frac{m_0 c}{h} \right)^3$, or, substituting numerical values, $n << 5{\cdot}88 \times 10^{29}$. For stars of mass ·44 and ·85 that of the Sun, the value of n came out on the simple theory as about $1{\cdot}86 \times 10^{29}$ and $6{\cdot}9 \times 10^{29}$. The approximation given by series integration is thus inadequate, and it is necessary to use the complete expression for the integral

$$f(x) = \int_0^x (1+y^2)^{1/2} y^2\, dy,$$

$$= \int_0^x (1+y^2)^{3/2}\, dy - \int_0^x (1+y^2)^{1/2}\, dy.$$

The second integral is known, and the first may be reduced by noting that

$$\frac{d}{dy} \{ y(1+y^2)^{1/2} \} = 4(1+y^2)^{3/2} - 3(1+y^2)^{1/2}.$$

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The final result is

$$f(x)=\tfrac{1}{8}[x(1+x^2)^{1/2}(1+2x^2)-\log\{x+(1+x^2)^{1/2}\}]. \quad (11)$$

For x small this may be expanded (noting that the log is equal to

$$\int(1+x^2)^{-1/2}\,dx),$$

giving the result (10) on substitution in (9 a).

The complete expression for the total kinetic energy of the electrons thus becomes

$$\mathrm{E_K}=\frac{8\pi \mathrm{V} m_0{}^4c^5}{h^3}[\tfrac{1}{8}\,x(1+x^2)^{1/2}(1+2x^2)-\tfrac{1}{3}\,x^3 -\log\{x+(1+x^2)^{1/2}\}], \quad . \quad (12)$$

with

$$x=\frac{p_0}{m_0c}=\frac{1}{m_0c}\left(\frac{3h^3n}{8\pi}\right)^{1/3}.$$

The energy is less than that given by (2). In the limit when x is large ($x>>1$, $n>>5{\cdot}88\times10^{29}$),

$$(\mathrm{E_K})_{x>>1}=\frac{8\pi \mathrm{V} m_0{}^4c^5}{h^3}\,\frac{x^4}{4},$$

$$=\frac{2\pi \mathrm{V}c}{h^3}p_0{}^4=n\mathrm{V}\frac{3}{8}\left(\frac{3}{\pi}\right)^{1/3}hcn^{1/3}. \quad . \quad (12\,a)$$

These expressions may be compared with (10a) and (10b). They show that when n is large the mean kinetic energy increases as $n^{1/3}$ (instead of n^2). For a constant total number of electrons (nV constant), $\mathrm{E_K}$ then varies as $n^{1/3}$, as does also $-\mathrm{E_G}$. When this is so there will obviously be no equilibrium under the conditions imposed.

The Equilibrium Condition.

If the conditions are such that an equilibrium state is possible, there will be a limiting density—"limiting" because the calculation refers to a sphere of uniform density at zero temperature. The limiting density is that corresponding to the value of n when

$$\frac{d}{dn}(\mathrm{E_K}+\mathrm{E_G})=0. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It is more convenient, for the present application, to treat x, defined by (8), as the variable, so that the equilibrium condition becomes

$$\frac{d}{dx}(\mathrm{E_K}+\mathrm{E_G})=0. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

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For the gravitational potential energy (as shown in I.) with $2{\cdot}5\ m_{\mathrm{H}}$ as the mean molecular weight of the material of the star,

$$\mathrm{E_G} = -\frac{3}{5}\frac{\mathrm{GM}^2}{\mathrm{M}^{1/3}}\left(\frac{4}{3}\pi n\right)^{1/3}(2{\cdot}5\ m_{\mathrm{H}})^{1/3}. \quad . \quad . \quad (15a)$$

Substituting $\left(\frac{8\pi}{3}\right)^{1/3}\frac{m_0 c x}{h}$ for $n^{1/3}$ from (8),

$$\mathrm{E_G} = 3^{1/3}\left(\frac{4\pi}{5}\right)^{2/3}\frac{1}{h}\mathrm{GM}^{5/3}m_{\mathrm{H}}{}^{1/3}m_0 c x. \quad . \quad . \quad (15b)$$

From (12) $\mathrm{E_K}$ may be written as

$$\mathrm{E_K} = \frac{8\pi \mathrm{V} m_0{}^4 c^5}{h^3} f_1(x).$$

Substituting $\mathrm{M}/(2{\cdot}5\ m_{\mathrm{H}} n)$ for V, and further substituting for n as above,

$$\mathrm{E_K} = \frac{3\mathrm{M}m_0 c^2}{2{\cdot}5\ m_{\mathrm{H}}}\frac{f_1(x)}{x^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Using the values (15) and (16) in (14), the result is

$$\frac{d}{dx}\left[\frac{1}{x^3}f_1(x)\right] = 10^{1/3}\left(\frac{\pi}{3}\right)^{2/3}\frac{\mathrm{G}m_{\mathrm{H}}{}^{4/3}\mathrm{M}^{2/3}}{hc}. \quad . \quad (17)$$

Inserting numerical values,

$$\frac{d}{dx}\left[\frac{1}{x^3}f_1(x)\right] = \mathrm{F}(x) = 1{\cdot}483 \times 10^{-23}\mathrm{M}^{2/3}, \quad . \quad (18)$$

$$\mathrm{M} = 1{\cdot}751 \times 10^{34}[\mathrm{F}(x)]^{3/2}. \quad . \quad . \quad . \quad . \quad (18a)$$

$\mathrm{F}(x)$ is obtained by straightforward differentiation of $\frac{1}{x^3}f_1(x)$, where $f_1(x)$ is the bracketed quantity in (12). The final result is

$$\mathrm{F}(x) = \frac{1}{8x^3}\left[\frac{3}{x}\log\{x + (1+x^2)^{1/2}\} + (1+x^2)^{1/2}(2x^2-3)\right]. \quad . \quad . \quad . \quad (19)$$

By means of equations (18 *a*) and (19) the mass may be found corresponding to any value of x, and so to any limiting electron concentration. Since the mean molecular weight is about $2{\cdot}5\ m_{\mathrm{H}}$, the limiting density is given by

$$\rho_0 = 2{\cdot}5\ m_{\mathrm{H}} n = 4{\cdot}15 \times 10^{-24} n. \quad . \quad . \quad . \quad (20)$$

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Numerical Results.

The method of procedure adopted is to calculate $F(x)$ for different values of x, from which a $F(x)$, x curve may be plotted. From this, using (18 *a*), an M, x curve may be derived. If desired, this may be converted into an M, n curve, using the relation

$$n=\left(\frac{8\pi}{3}\right)\left(\frac{m_0c}{h}\right)^3 x^3,$$

giving

$$n=5{\cdot}876\times10^{29}x^3 \quad \text{or} \quad x=1{\cdot}194\times10^{-10}n^{1/3}. \quad . \quad (21)$$

The full expression for $F(x)$ (19) is inconvenient to use when x is small. The expression may then be expanded, with the result

$$[F(x)]_{x<<1}=\frac{1}{5}x-\frac{1}{14}x^3+\frac{1}{24}x^5 \ldots . \quad . \quad . \quad (19\,a)$$

When x is small enough for the first term in the expansion to be sufficient, the same result is obtained as before, namely,

$$n=10^4\left(\frac{\pi}{3}\right)^3\frac{1}{h^6}G^3M^2m_H{}^4m^3, \quad . \quad . \quad . \quad (19\,a')$$

or, substituting numerical values from (18 *a*), (20), and (21),

$$\left.\begin{aligned} n&=2{\cdot}396\times10^{-37}M^2,\\ \rho&=9{\cdot}95\times10^{-61}M^2.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (22)^*$$

The first two terms in (19 *a*) give $F(x)$ correct to less than 1 per cent. up to $x={\cdot}5$, and form the most convenient expression for this range.

When x is large, as may be seen from (19),

$$\begin{aligned}[F(x)]_{x>>1}&=\frac{1}{8x^3}(2x^3-3x),\\ &={\cdot}2500-\frac{3}{8x^2}. \quad . \quad . \quad . \quad . \quad (19\,b)\end{aligned}$$

This expression gives $F(x)$ correct to 1 per cent. for $x>10$. It indicates, moreover, the limiting mass for which an

* This is in agreement with Anderson's result. In I., as pointed out by Anderson, a slight numerical error was made, and the result was given as $n=2{\cdot}31\times10^{-37}M^2$. Correspondingly, the result for the maximum density, given as $\rho=3{\cdot}85\times10^6\,(M/M_s)^2$, should be $\rho=3{\cdot}977\times10^6\,(M/M_s)^2$, with the Sun's mass, M_s, taken as $2{\cdot}0\times10^{33}$.

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equilibrium state can be attained under the conditions, namely, from (18 *a*),

$$M_0 = 1{\cdot}751 \times 10^{34} ({\cdot}25)^{3/2},$$
$$= 2{\cdot}19 \times 10^{33}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

TABLE I.

Corresponding values of x, $F(x)$, $\log_{10} n$, and $\log_{10} M$.

$$n = 5{\cdot}876 \times 10^{29} x^3,$$
$$M = 1{\cdot}751 \times 10^{34} [F(x)]^{3/2}.$$

x.	$F(x)$.	$\log_{10} n$.	$\log_{10} M$.
·1	·0200	26·7690	31·6947
·2	·0394	27·6720	32·1364
·3	·0581	28·2003	·3895
·4	·0756	·5753	·5609
·6	·1071	29·1036	·7879
·8	·1330	·4783	·9291
1·0	·1537	·7690	33·0233
1·5	·1887	30·2973	·1569
2	·2085	·6720	·2220
3	·2280	31·2003	·2801
4	·2366	·5753	·3042
5	·2410	·8660	·3162
6	·2436	32·1036	·3233
8	·2463	·4783	·3303
10	·2476	·7690	·3339
20	·2494	33·6720	·3385
40	·2498	34·5753	·3396
60	·2499	35·1036	·3399
100	·2500	·7690	·3401

The value obtained by Anderson for M_0 is $1{\cdot}37 \times 10^{33}$, so that the range of mass in which this particular type of equilibrium can occur is shown by this method to be considerably greater than that indicated by Anderson's approximate treatment.

A series of corresponding values of x and $F(x)$ calculated from (19), supplemented by (19 *a*) and (19 *b*), is shown in Table I. The relation required is that between n and M, related to x and $F(x)$ by (21) and (18 *a*). It is convenient

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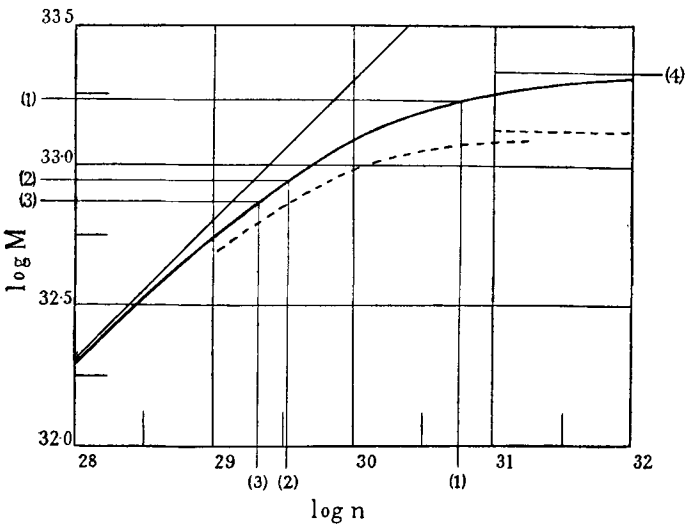
to use logarithms (base 10) when the following equations are obtained:

$$\log_{10} n = 29{\cdot}7690 + 3 \log x, \quad . \quad . \quad . \quad . \quad (24)$$

$$\log_{10} \mathrm{M} = 34{\cdot}2432 + \frac{3}{2} \log \mathrm{F}(x). \quad . \quad . \quad . \quad (25)$$

The conversion is then readily carried out. Corresponding values of n and M are shown in the last two columns of the table. It is unnecessary to give values for log M less than 32, as no stars are known of mass less than a tenth that of the Sun ($\log \mathrm{M}_s = 33{\cdot}3010$), and in any case, for stars of small mass, the approximate expression (22) will be sufficiently accurate. For any value of M the maximum value of $\log n$ can be found by interpolation, and from that



Variation of limiting electron concentration (n) with mass (M) in a sphere of uniform density.

(1)	Sirus B	$\log \mathrm{M} = 33{\cdot}230$	$\log n = 30{\cdot}748$
(2)	o_2 Eridani B	$\log \mathrm{M} = 32{\cdot}944$	$\log n = 29{\cdot}524$
(3)	Procyon B	$\log \mathrm{M} = 32{\cdot}869$	$\log n = 29{\cdot}313$
(4)	Limiting M	$\log \mathrm{M}_0 = 33{\cdot}340$	$(\mathrm{M}_0 = 2{\cdot}19 \times 10^{33})$

For the limiting density $\log \rho_0 = \log n - 24 + 0{\cdot}618$.
The straight line corresponds to the formula in which the relativity effect is neglected. The dotted curve gives Anderson's results.

the value of n. If the density is required it can be found from (20), while $\log \rho_0$ is given by

$$\log \rho_0 = \log n - 24 + {\cdot}6180. \quad . \quad . \quad . \quad . \quad (26)$$

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The relation between log M and log n is shown by the curve in the figure. The "positions" of a number of stars are indicated. The straight line is the result found by the original approximation, in which the relativity effect was neglected (giving n varying as M^2). The main conclusion is that for stars for which this equilibrium can occur the limiting density is somewhat less than that calculated by Anderson, and that the "limiting mass" is larger; his results are indicated by the dotted line.

White Dwarfs.

The number of stars known to be of the white dwarf type is small, but this does not necessarily indicate that stars of very high density are uncommon. Dense stars of ordinary mass will have a small radius, and so will be faint objects unless they are near the Sun (on the stellar scale of nearness) or have a high-surface temperature. "Black dwarfs" (to use Fowler's term) would not be observed. According to Jeans *, four white dwarfs are known with certainty, but for one of these—the companion of the red giant o Ceti—he does not give any quantitative data. Procyon B is a possible white dwarf, but its spectral type is unknown. In the following table are collected the relevant observational data derived from Jeans †. Procyon B is included because its mass is known. The four columns give the spectral type, the absolute visual and bolometric magnitudes, and the mass in terms of the Sun's mass (M_s).

TABLE II.

Observational Data.

	Spectral Type.	$M_{vis.}$	$M_{bol.}$	M/M_s.
Sirius B	A 7	11·3	11·2	·85
o_2 Eridani B	A 0	11·2	10·8	·44
Procyon B	—	16	—	·37
Van Maanen's star	F	14·3	14·3	—

* J. H. Jeans, 'Astronomy and Cosmogony,' p. 63 (Camb. Univ. Press, 1928).

† *Loc. cit.* p. 59. I am indebted to Prof. Eddington for informing me that the value ·21 given for the mass of o_2 Eridani B in his 'Internal Constitution of the Stars' is based on an earlier determination of the orbit, and that the value ·44 is presumably the more trustworthy. Also to Sir James Jeans for confirming the value ·44.

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From these observational data deductions may be made as to other characteristics of the stars. From the spectral type the surface temperature (T_e) may be estimated, and from the effective temperature and the absolute magnitudes the radius of the star. The following formulæ are given by Jeans for the radius in terms of that of the sun (r/r_s denoted by R):—

$$\log R = -0{\cdot}2\, M_{bol.} - 2 \log T_e + 8{\cdot}53, \quad . \quad . \quad (27a)$$

$$\log R = -0{\cdot}2\, M_{vis.} + \frac{5880}{T_e} - 0{\cdot}01. \quad . \quad . \quad . \quad (27b)$$

It is further possible to calculate the total radiation of energy in ergs per second (E),

$$\log E = -0{\cdot}4\, M_{bol.} + 35{\cdot}52. \quad . \quad . \quad . \quad . \quad (28)$$

TABLE III.

Deduced Characteristics of Sirius B and o_2 Eridani B.

	M/M_s.	T_e.	R.	E′.	ρ.	ρ_0 calc.
Sirius B	·85	8,000	·0312	·0065	$3{\cdot}97\times10^4$	$2{\cdot}3\times10^7$
o_2 Eridani B...	·44	11,200	·0187	·0185	$9{\cdot}55\times10^4$	$1{\cdot}4\times10^6$

The amount of radiation per gram (E′) is a quantity of great interest:

$$\begin{aligned} \log E' &= \log E - \log M, \\ &= \log E + \log (M_s/M) - \log M_s, \\ &= -0{\cdot}4\, M_{bol.} + 2{\cdot}22 + \log (M_s/M). \quad . \quad (29) \end{aligned}$$

(The sun's mass is taken as $2{\cdot}0 \times 10^{33}$ grams.)

The mean density is given by

$$\begin{aligned} \rho &= \frac{3M}{4\pi r^3} = \left(\frac{M}{M_s}\right)\left(\frac{r_s}{r}\right)^3 \frac{3M_s}{4\pi r_s^3}, \\ &= (M/M_s)(1/R)^3 \rho_s = 1{\cdot}42 (M/M_s)(1/R)^3. \quad . \quad . \quad (30) \end{aligned}$$

The values given in Table III. have been calculated from these equations. It may be noted that for Sirius B the radius deduced from the relativity shift is ·03, in remarkable agreement with the value ·0312 calculated in the way indicated. (In general the values are not significant to more than two figures.) The available data for Procyon B and Van Maanen's Star are insufficient for the calculation

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of ρ or E′; these stars will be discussed later. The value of M/M_s is included in the table, and also the value of the limiting density, ρ_0, calculated as already described.

The calculated limiting densities are considerably greater than those obtained by the simple formula (19 *a*) (used in I.), which gives for stars of the mass of Sirius B and o_2 Eridani B the values $\rho = 2{\cdot}8 \times 10^6$ and $7{\cdot}5 \times 10^5$. The order of magnitude of the difference for stars of different mass will be apparent from the figure.

The calculated radiation per gram (E′) of o_2 Eridani B is about 10 times greater than the value given by Jeans. This will be discussed later. The mean density for Sirius B is somewhat smaller than that usually given (about 5×10^4). This results from the use of a slightly larger value for the radius. Taking the relativity shift value (R = ·03), the mean density comes out as $4{\cdot}5 \times 10^5$.

Mean Molecular Weight.

The mean molecular weight has been taken as $2{\cdot}5\, m_H$. This is the value for completely ionized atoms of lead $\left(\frac{207{\cdot}2}{82+1}\right)$. For atoms of lower atomic number the value is slightly smaller, and for those of high atomic number, larger. Jeans derives a value $2{\cdot}6\, m_H$ on the assumption, for which there is strong evidence, that in many stars atoms of the atomic number equal to that of uranium, and even higher, are present. In condensed stars of the type considered, however, most of these atoms will have been transformed into atoms of an inert type, and the estimate $2{\cdot}5\, m_H$ is probably sufficiently accurate. In any case, a small change will not affect the numerical results appreciably.

It might be thought that the change of mass of the electron would produce a not negligible effect, but it may readily be shown that the effect will be very small. Let m be the mass of an electron with momentum p. The molecular weight becomes $2{\cdot}5\, m_H + \mu$, where

$$2{\cdot}5\, m_H + \mu = m_H \left(2{\cdot}5 + \frac{m - m_0}{m_H}\right).$$

Now

$$m^2(1-\beta) = m_0^2,$$

$$\beta = \frac{p}{mc} = \left(\frac{p}{m_0 c}\right)\frac{m_0}{m}$$

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$$\frac{m-m_0}{m_0}=\left[1+\left(\frac{p}{m_0c}\right)^2\right]^{1/2},$$

$$2{\cdot}5\,m_{\mathrm{H}}+\mu=m_{\mathrm{H}}\left[2{\cdot}5+\frac{1}{1845}\left\{1+\left(\frac{p}{m_0c}\right)^2\right\}^{1/2}\right]. \quad . \quad . \quad . \quad . \quad (31)$$

From (8) the maximum momentum is given by

$$p_0=\left(\frac{3h^3n}{8\pi}\right)^{1/3},$$

so that

$$\frac{p_0}{m_0c}=\frac{h}{m_0c}\left(\frac{3}{8\pi}\right)^{1/3}n^{1/3},$$

$$=1{\cdot}194\times10^{-10}n^{1/3}. \quad . \quad . \quad . \quad (32)$$

For an electron concentration $n=10^{33}$ (corresponding to a density greater than 10^9) the maximum molecular weight is

$$2{\cdot}5\,m_{\mathrm{H}}+\mu=m_{\mathrm{H}}(2{\cdot}5+{\cdot}0065). \quad . \quad . \quad . \quad (31a)$$

The correction is thus quite negligible for all the densities which come into consideration.

Mean Density.

In the foregoing treatment the star has been idealized as having a uniform density. This seems legitimate when the aim has been to obtain an estimate of the maximum density under gravitational-kinetic equilibrium. In an actual star, however, the density will not be uniform, even in the condensed limit. When n is small the energy per unit volume is proportional to $n^{5/3}$ (eq. 10 *b*), and when n is large, to $n^{4/3}$ (eq. 12 *a*). Since the pressure is proportional to the energy per unit volume, and the density is proportional to n, the following relations hold between the pressure and density:—

$$n<<5{\cdot}9\times10^{29},\quad \rho<<2{\cdot}4\times10^{6},\quad p=\kappa\rho^{5/3}, \quad . \quad (33a)$$

$$n>>5{\cdot}9\times10^{29},\quad \rho>>2{\cdot}4\times10^{6},\quad p=\kappa\rho^{4/3}. \quad . \quad (33b)$$

In an ideal condensed star the distribution will thus be polytropic, the relation between pressure and density obeying the relation $p=\kappa\rho^{\gamma}$. The distributions in polytropic stars for a number of values of γ (or its equivalent n', defined by $\gamma=1+1/n'$) have been worked out by Emden. The results are quoted by Eddington *. Table IV. gives the ratio of the

* A. S. Eddington, 'The Internal Constitution of the Stars,' p. 89 (C. U. P. 1926).

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maximum (central) to the mean density. Uniform density corresponds to $n'=0$.

This table certainly indicates that the maximum mean density may be considerably less than the maximum central density, and it might seem that all that is necessary to convert the previously calculated results for limiting densities to mean densities is to divide by the appropriate ratio. There are a number of other factors, however, which have to be taken into consideration. The gravitational energy of a star is given by *

$$E_G = -\frac{3}{5-n'} G \frac{M^2}{r} \quad . \quad . \quad . \qquad (34)$$

For uniform density the coefficient is 3/5, the value used (eq. 15 *a*). For $\gamma=5/3$ the factor becomes 6/7, so that for a given M the calculated n would be increased. The whole

TABLE IV.

Ratio of Maximum to Mean Density in Polytropes.

γ	∞	5/3	3/2	7/5	4/3
n'	0	3/2		5/2	3
ρ_0/ρ	1	6·00	11·4	24·1	54·4

calculation, however, is based on the assumption that n is uniform, and different averaging processes would have to be carried out in deriving expressions for the gravitational and kinetic energies in terms either of n_0 (the central electron concentration) or the mean n. Some rough calculations indicate that the values derived for the limiting concentrations approximate to those for n_0; but a much more elaborate investigation would be necessary if the variation of density were to be properly taken into account. The indications are that the values obtained for the limiting density approach those for the limiting central density, and that the limiting mass may be slightly changed; that Sirius B still falls within the range for which gravitational-kinetic equilibrium can occur; and that the limiting mean densities may be smaller than those calculated approximately in the ratios suggested by the values in Table IV. These results, however, are somewhat uncertain, and for the present the calculations are left in the form which is strictly appropriate for the case of the homogeneous sphere.

* Eddington, *loc. cit.* p. 87.

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Van Maanen's Star and Procyon B.

Van Maanen's Star is of great interest, as it is the smallest star known, being about the same size as the earth. From the data in Table II. the radius may be calculated (eq. 27), the result being $R = \cdot 00955$, taking the surface temperature as 7000 corresponding to an F-type star. The mass is unknown, and since the star is not a binary component, there is no method available by which it may be estimated. It is, however, possible to calculate *compatible* values of the mass (giving the mean density) and of the radiation per gram (E') from the known data. From (29),

$$\log E' = -5\cdot 72 + 2\cdot 22 + \log (M_s/M),$$

$$E'(M/M_s) = 3\cdot 162 \times 10^{-4}. \quad . \quad . \quad . \quad . \quad (35)$$

TABLE V.

Compatible Values for Van Maanen's Star.

$T_e = 7000.$ $R = \cdot 00955.$

M/M_s.	E'.	$10^{-6}\rho$.	$10^{-6}\rho_0$ calc.
·2	·0016	·33	·19
·4	·0008	·65	1·14
·6	·0005	·98	3·7
·8	·0004	1·30	13·1
1·0	·0003	1·63	104

In Table V. are given compatible values of M/M_s, ρ, and E', together with the calculated ρ_0 for each value of the mass. Incidentally, this table gives the values of ρ_0 for a range of masses.

As will be seen from the above table, the values which Jeans suggests as a pure guess* for M/M_s and E'—·2 and ·00055—are not compatible with each other. The interesting point which emerges, however, is that unless the numerical results of the equilibrium theory are very wide of the mark owing to the essential factors being ignored, a lower limit for the mass of Van Maanen's Star can be roughly estimated. The mean density cannot be greater than the calculated limiting density, and it is probably several times smaller.

* J. H. Jeans, 'The Universe Around Us,' p. 310 (C. U. P. 1929).

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From this it is possible to draw the following conclusions for Van Maanen's Star :—

$$M/M_s > \cdot 4, \quad \rho > 650{,}000, \quad E' < \cdot 0008.$$

For Procyon B the spectral type is unknown, though it is probably M or K *. It is, however, of interest to examine how far the star approaches the condensed type if a reasonable value of the surface temperature is assumed. For assumed temperatures of 3000 (M type) and 4000 (K type) the estimated correction † to convert $M_{vis.}$ to $M_{bol.}$ may be applied, and using (27), (29), and (30), the following results are obtained :—

TABLE VI.

Compatible Values for Procyon B.

$M/M_s = \cdot 37. \qquad \rho_0$ calc. $= \cdot 84 \times 10^6$.

T_e.	$M_{bol.}$	R.	ρ.	E'.
3000	14	·056	$2{\cdot}4 \times 10^3$	·001
4000	15·5	·018	$1{\cdot}1 \times 10^5$	·0003

Of course nothing definite can be deduced from these results, as in the case of Van Maanen's Star. They simply serve to show that unless the surface temperature is considerably greater than 3000, Procyon B does not approach the limiting condensed state, the mean density being much smaller than that of other white dwarfs.

Generation of Energy.

Some of the more important observed and deduced data for the three white dwarfs about which most is known are collected together, for convenience in comparison, in Table VII. The total energy radiated per second is symbolized by E, the energy per gram by E'.

As was to be expected, the calculated limiting density ρ_0 is considerably greater than the "observed" mean density. The polytrope ratios (Table IV.) suggest that the maximum mean density may be something approaching 50 times smaller than ρ_0 for Sirius B, and 6 times smaller for

* J. H. Jeans, 'Astronomy and Cosmogony,' p. 128.
† *Ibid.* p. 46.

o_2 Eridani B. If these ratios are approximately correct, o_2 Eridani B may be said to approach the limiting condensed state fairly closely, while Sirius B is still far removed from it. If the observational data (Table II.) and the essentials of the equilibrium theory are correct, Van Maanen's Star is certainly the most dense that is known, though it is impossible to say how nearly it approaches the condensed limit.

If the observational data are correct, again, there is for stars of this type no simple relation between the mass and luminosity—as is shown by the absolute bolometric magnitude of o_2 Eridani B being smaller than that of Sirius B, although its mass is smaller. The radiation per gram, moreover, does not decrease steadily with the mass of the star, as had been suggested *.

TABLE VII.

Observed and Deduced Data for White Dwarfs.

	M/M_s.	T_e.	R.	ρ.	ρ_0 calc.	ϵ.	E'.
Sirius B	·85	8,000	0312	40,000	23×10^6	$1{\cdot}1\times10^{31}$	0065
o_2 Eridani B	·44	11,200	·0187	95,000	$1{\cdot}4\times10^6$	$1{\cdot}6\times10^{31}$	·0185
Van Maanen's Star .	>·4	7,000	·0095	>650,000	$>1{\cdot}1\times10^6$	$6{\cdot}4\times10^{29}$	<·0008

According to Jeans's hypothesis, which, in spite of its speculative nature, certainly satisfies most of the conditions, the energy generation in stars is to be traced to electron-proton annihilation occurring in hyper-uranium atoms as a result of one of the extra-nuclear electrons of the atom falling into the nucleus. In completely ionized stars of the white dwarf type this process could not occur, so that the small radiation per gram of these stars is in complete accordance with the hypothesis. The radiation would be mainly due to the atoms in the outer layers of the star, so that there is not necessarily any simple relation between either the mass or the mean density and the amount of radiation per gram. It does none the less remain peculiar that o_2 Eridani B, which approaches the condensed state, should have a much higher surface temperature than Sirius B, and that it should generate more energy per gram. The material available in this connexion, however, is too slight to permit of any inductive generalizations being made.

* J. H. Jeans, 'The Universe Around Us,' p. 310.

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The Fermi Statistics of Electron Gas at Lower Densities.

The densities of white dwarfs are very much greater than those of other stars, in which the densities are too low and the temperatures too high for the electron gas to approach the condensed state. According to Jeans's theory of stellar evolution (outlined in I.), the tenanted portions of the Russell temperature-luminosity diagram correspond to stable states in which the pressures exceed considerably the perfect gas-pressures. There will be a range of stability for each successive stage of ionization of the atoms, and this, on the liquid-star hypothesis, corresponds to the "jamming" of the incompletely-ionized atoms. The difficulty about this hypothesis is that the size which must be attributed to the atoms is very much greater than that indicated by other evidence. Now, with the Fermi statistics the pressure is always greater than that of a perfect gas, and it is interesting to note that there will be a maximum deviation corresponding to each stage of ionization. When with increasing temperature there is an increase in the number of electrons through ionization, there will be a deviation from the perfect gas relation between pressure and number of molecules. This deviation will decrease as the temperature increases, and will increase again when the next stage of ionization will increase. The deviations required, however, are much greater than those indicated by the Fermi statistics. This may be illustrated by taking a particular case and making the most favourable estimates. To ensure stability at the centre of the star, according to Jeans, the pressure must be at least 1·107 times the normal pressure *. The pressure of electron gas is given by

$$p = nk\mathrm{T}\left\{1 + \frac{1}{16 \times 2}\frac{nh^3}{(\pi mk\mathrm{T})^{3/2}}\cdots\right\}. \quad . \quad . \quad (36)$$

Inserting numerical values

$$p = nk\mathrm{T}\left\{1 + 3{\cdot}62 \times 10^{-17}\frac{n}{\mathrm{T}^{3/2}}\cdots\right\}. \quad . \quad . \quad (37)$$

Taking ρ as 2·5 nm_{H}, Table VIII. gives the corresponding values of T, n, and ρ for the pressure to exceed the normal by 10 per cent.

In stars generally, for a given central temperature the central density is of order of ten times less than that in

* J. H. Jeans, 'Astronomy and Cosmogony,' p. 141.

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the table. For the sun * the central density is about 140, and the temperature can hardly be less than 40×10^6. This gives

$$p = nk\mathrm{T}(1 + 0{\cdot}48 \times 10^{-2}). \quad . \quad . \quad . \quad . \quad (38)$$

The pressure is thus about $\frac{1}{2}$ per cent. above normal instead of the required 10 per cent. It would seem, therefore, that unless there are other parts of the theory requiring modification, a simple application of the Fermi statistics is unable to remove the difficulties, although it does lead to deviations from the perfect gas laws of the required type.

TABLE VIII.

Corresponding Values for the Pressure to exceed normal by 10 per cent.

T.	n.	ρ.
10^2	$2{\cdot}76 \times 10^{18}$	$1{\cdot}145 \times 10^{-5}$
10^4	$2{\cdot}76 \times 10^{21}$	$1{\cdot}145 \times 10^{-2}$
10^5	$2{\cdot}76 \times 10^{24}$	$1{\cdot}145 \times 10$
10^8	$2{\cdot}76 \times 10^{27}$	$1{\cdot}145 \times 10^4$

Conclusion.

It is generally agreed that the dense dwarf stars represent an advanced stage of stellar evolution. It is reasonable to suppose that the *white* dwarf stage, corresponding to high surface temperature, will be a comparatively short one, and since the stars are small, it is probable that most condensed stars are too faint to be observed. From the mass luminosity relation, moreover, even treating this as a purely empirical generalization, the masses of condensed stars will in general be small. There is therefore a strong probability that stars of the type in which the gravitational kinetic equilibrium discussed in this paper is of importance are much more numerous than is suggested by the number of known examples. For these the theory may be said to account in a general way for the order of magnitude of the densities deduced from the observational data.

For more normal stars the application of the Fermi statistics shows that there will be an appreciable deviation from the perfect gas laws for the electron gas. Although the deviations will be of the kind required in Jeans's theory

* *Ibid.* p. 104.

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of stability, they are not nearly large enough. It would seem, therefore, either that the theory as to the conditions for stability requires modification, or that there are factors involved which are at present unrecognized.

Summary.

In a previous paper the conclusion was reached that there was a limiting density for stars in which the atoms were completely ionized, varying as the square of the mass of the star. The limiting state occurs when the decrease in gravitational energy on contraction is equal to the increase in the total kinetic energy of the electron gas. In the treatment the relativity change of mass with velocity was neglected. Some approximate calculations by Anderson indicate the general effect of this change of mass, which necessitates a modification of the previous conclusions when the mass of the star becomes comparable with that of the sun.

In the present paper the effect of the relativity change of mass is worked out with more rigour for the idealized case for a sphere of uniform density. For spheres of increasing mass the limiting density varies at first as the square of the mass, and then more rapidly, there being a limiting mass ($2{\cdot}19 \times 10^{33}$) above which the gravitational kinetic equilibrium considered will not occur. Tables and curves are given showing the relation between mass and limiting density.

It is shown that the distribution of density in condensed stars will be polytropic, and rough estimates are made of the ratio of the central to the mean density.

The observational and deduced data for known white dwarfs are considered in some detail. The density of Sirius B is well below the limit calculated, while that of o_2 Eridani B approaches it. The theory enables a lower limit for the mass and density of Van Maanen's Star to be roughly estimated. The observational, deduced, and calculated data for these three stars, and also for Procyon B, are given in tables.

The deduced data indicate that the generation of energy per gram in known dense stars is not simply related to the mass of the star.

The application of the Fermi statistics to the electron gas in normal stars is briefly discussed in connexion with Jeans's theory of stability. There will be deviations from the perfect gas laws of the type required, but they are about twenty times too small to satisfy the stability conditions.

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