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Super-Poincaré and Nash-type inequalities for Subordinated Semigroups

Ivan Gentil ^{*}, Patrick Maheux [†]

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Abstract

We prove that if a super-Poincaré inequality is satisfied by an infinitesimal generator $-A$ of a symmetric contraction semigroup on L^2 and that is contracting on L^1 , then it implies a corresponding super-Poincaré inequality for $-g(A)$ for any Bernstein function g . We also study the converse of this statement. We prove similar results for Nash-type inequalities. We apply our results to Euclidean, Riemannian, hypoelliptic and Ornstein-Uhlenbeck settings.

Key words: Super-Poincaré inequality, Nash-type inequality, Symmetric semigroup, Subordination in the sense of Bochner, Bernstein function, Super-Poincaré profile.

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1 Introduction and Main Results

Let $(T_t)_{t \geq 0}$ be a symmetric strongly continuous contraction semigroup on $L^2(X, \mu)$ with (X, μ) a σ -finite measure space. The infinitesimal generator $-A$ of $(T_t)_{t \geq 0}$ on L^2 is a non-positive self-adjoint operator on L^2 . We suppose that each operator T_t with $t > 0$ maps $L^1 \cap L^2$ into itself and is a contraction in the L^1 -norm. So, for each $t > 0$ the operator T_t can be extended uniquely to a contraction on $L^p(\mu) := L^p(X, \mu)$, $1 \leq p \leq \infty$ by interpolation and duality. This extension $T_t^{(p)}$ will again be denoted by T_t . Recall that the infinitesimal generator $-A$ on $L^2(\mu)$ of $(T_t)_{t \geq 0}$ is defined by

$$-Af = \lim_{t \rightarrow 0^+} \frac{T_t f - f}{t} \in L^2$$

on the domain $\mathcal{D}(A)$ that is the set of functions f of $L^2(\mu)$ such that the above limit exists in $L^2(\mu)$. Let (\cdot, \cdot) denote the inner product on $L^2(\mu)$, and let $\|\cdot\|_p$ denote the L^p -norm with respect to the measure μ .

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We say that A satisfies a *super-Poincaré inequality* with *rate function* $\beta : (0, \infty) \rightarrow [0, \infty)$ if the following inequality

$$\|f\|_2^2 \leq r(Af, f) + \beta(r)\|f\|_1^2 \quad (1)$$

holds for all $f \in \mathcal{D}(A) \cap L^1(\mu)$ and all $r > 0$. More generally, we say that A satisfies a (r_0, r_1) -super-Poincaré inequality if (1) holds for all $r \in (r_0, r_1)$ with $0 \leq r_0 < r_1 \leq \infty$. We may always assume that the function β is non-increasing by considering the function $\tilde{\beta}(r) = \inf_{r_0 < s \leq r \wedge r_1} \beta(s)$ for all $r \in (r_0, \infty)$ in place of $\beta(r)$ in (1). Consequently, we may also assume that $r_1 = \infty$.

We also say that A satisfies a *Nash-type inequality* with *rate function* $D : [0, \infty) \rightarrow [0, \infty)$ if the following inequality

$$\|f\|_2^2 D(\|f\|_2^2) \leq (Af, f) \quad (2)$$

holds for all $f \in \mathcal{D}(A) \cap L^1(\mu)$ such that $\|f\|_1 \leq 1$. We can always consider that the function D is non-negative in (2) by replacing D by $D_+ = \sup(0, D)$ since $(Af, f) \geq 0$. Moreover we shall assume that the function D is non-decreasing. The Nash-type inequality (2) is usually written in the following form

$$\Theta(\|f\|_2^2) \leq (Af, f)$$

for all $f \in \mathcal{D}(A) \cap L^1(\mu)$ such that $\|f\|_1 \leq 1$. But the equivalent expression (2) of the Nash-type inequality is more appropriate to deal with the Bernstein functions $g(x) = x^\alpha$, $\alpha \in (0, 1)$ as shown in [B-M] and more generally for any Bernstein functions as shown in Theorem 1.2 below. In this paper, we shall use in a crucial way that the super-Poincaré inequality (1) and the Nash-type inequality (2) are in the fact equivalent for large classes of rate functions β and D , see Proposition 2.2 below.

We now briefly recall some definitions and some facts about the subordination of semigroups in the sense of Bochner. A Bernstein function g is a continuous function from $[0, \infty)$ into itself, smooth on $(0, \infty)$ and satisfying

$$(-1)^{n-1} g^{(n)}(s) \geq 0, \quad s > 0, \quad n \in \mathbb{N}.$$

Then there exists a convolution semigroup of sub-probability measures $(\nu_t^g)_{t>0}$ on $(0, \infty)$ with density $(\eta_t^g)_{t>0}$ with respect to the Lebesgue measure ds such that the Laplace transform of ν_t^g is given by

$$\int_0^\infty e^{-sx} d\nu_t^g(s) = \int_0^\infty e^{-sx} \eta_t^g(s) ds = e^{-tg(x)}, \quad x \geq 0. \quad (3)$$

There is a one-to-one correspondence between g and $(\nu_t^g)_{t>0}$, see [J, p.177]. We recall that each Bernstein function g is described by the following Lévy-Khintchine formula

$$g(x) = a + bx + \int_0^\infty (1 - e^{-\lambda x}) d\nu(\lambda) \quad (4)$$

with $a, b \geq 0$ and ν a positive measure on $(0, \infty)$ such that $\int_0^\infty \frac{\lambda}{1+\lambda} d\nu(\lambda) < \infty$. The triplet (a, b, ν) is uniquely determined by g , see [J, Theorem 3.9.4]. We have $a = g(0) = 0$ if and only if ν_t^g is a probability measure for all $t > 0$. For instance, the Lévy measures ν associated with $g(x) = x^\alpha$, $\alpha \in (0, 1)$, and $g(x) = \log(1 + x)$ are given by $d\nu(\lambda) =$

$\frac{\alpha}{\Gamma(1-\alpha)}\lambda^{-1-\alpha}d\lambda$ and $d\nu(\lambda) = \frac{e^{-\lambda}}{\lambda}d\lambda$ respectively.

Now, let A be a densely defined non-negative definite self-adjoint operator on L^2 and $(E_\lambda)_{\lambda \in [0, \infty)}$ be its spectral resolution. Let $\Psi : [0, \infty) \rightarrow \mathbb{R}$ be any Borel function. The operator $\Psi(A)$ is defined on $L^2(\mu) := L^2(X, \mu)$ by the formula

$$\Psi(A)f = \int_0^\infty \Psi(\lambda) dE(\lambda)f \quad (5)$$

with domain $\mathcal{D}(\Psi(A))$ given by

$$\mathcal{D}(\Psi(A)) = \{f \in L^2 : \int_0^\infty |\Psi(\lambda)|^2 d(E(\lambda)f, f) < \infty\}. \quad (6)$$

If the function Ψ is non-negative, then the operator $\Psi(A)$ is non-negative and self-adjoint on $L^2(\mu)$. It defines a symmetric semigroup of contractions on L^2 by the spectral formula

$$e^{-t\Psi(A)}f = \int_0^\infty e^{-t\Psi(\lambda)} dE(\lambda)f, \quad f \in L^2(\mu).$$

The operator $e^{-t\Psi(A)}$ will also be denoted by T_t^Ψ . If $\Psi = g$ is a Bernstein function, it can be easily shown that the semigroup $(T_t^g)_{t \geq 0}$ satisfies also the so-called subordination formula

$$T_t^g = \int_0^\infty T_s d\nu_t^g(s) = \int_0^\infty T_s \eta_t^g(s) ds \quad (7)$$

where ν_t^g and η_t^g are given by (3). If T_t is a contraction on L^p for all $t > 0$ (respectively positive or sub-markovian on L^2), then T_t^g is a contraction on L^p for all $t > 0$ (respectively positive or sub-markovian on L^2), see [Sc-S-V, Chapter 12]. We always have the inclusion $\mathcal{D}(A) \subset \mathcal{D}(g(A))$ for any Bernstein function g . Now using the formula (4), it can be proved that

$$g(A)f = af + bAf + \int_0^\infty (f - T_\lambda f) d\nu(\lambda) \quad (8)$$

for all $f \in \mathcal{D}(A)$, see [Sc-S-V, Example 11.6] (note that our A is their $-A$). Our results apply to general Bernstein functions, and in particular to the following classical examples.

1. *The fractional subordinator (one-sided α -stable process):* $g(x) = x^\alpha$, $x > 0$, and $g(A) = A^\alpha$ where $0 < \alpha < 1$.
2. *The Gamma subordinator:* $g(x) = \log(1+x)$ and $g(A) = \log(I+A)$ where I denotes the identity operator on $L^2(\mu)$.
3. *The generalization of the previous example:* $g(x) = [\log(1+x^\alpha)]^\gamma$, $0 < \alpha, \gamma \leq 1$ and $g(A) = [\log(I+A^\alpha)]^\gamma$. If $\gamma = 1$ then g is called the *geometric α -stable subordinator*, see [S-S-V] for a recent study.
4. *Elementary functions:* $g_\lambda(x) = 1 - e^{-\lambda x}$, $\lambda > 0$ and $g_\lambda(A) = I - T_\lambda$.

See [Sc-S-V, Chapter 15] for a long list of examples of Bernstein functions.

Throughout this paper, we shall always assume implicitly that the functions f belongs to the domain of the operator under consideration.

We now state the main results of this paper.

Theorem 1.1 *Let $-A$ be the infinitesimal generator of a semigroup as defined above satisfying the super-Poincaré inequality (1) with rate function β . Let g be any Bernstein function. Then the infinitesimal generator $-g(A)$ satisfies the (r_0, r_1) -super-Poincaré inequality (1) with rate function $\beta_g(r) = \beta\left(\frac{1}{g^{-1}(1/r)}\right)$ for all $r \in (r_0, r_1)$, where $r_0 = \frac{1}{g(\infty)}$ and $r_1 = \frac{1}{g(0)}$.*

Note that by (4), the Bernstein function g is either strictly increasing and its inverse g^{-1} is well defined from $(g(0), g(\infty))$ into $(0, \infty)$ or g is constant and (r_0, r_1) is empty. A Bernstein function g is bounded if and only if $b = 0$ and ν is a bounded measure, see [J, p.174]. If g is not bounded, i.e. $g(\infty) = \infty$, then we have $r_0 = 0$. If $g(0) = a = 0$ then one has $r_1 = \infty$ by the definition of r_1 . If $a = g(0) > 0$ then we obtain immediately a spectral gap inequality

$$\|f\|_2^2 \leq \frac{1}{a}(g(A)f, f)$$

by using (8). In other words,

$$\|f\|_2^2 \leq r(g(A)f, f) + \beta_g(r)\|f\|_1^2, \quad r > \frac{1}{a}$$

with $\beta_g(r) = 0$ for all $r \in [1/a, \infty)$. In any case, and as already mentioned above, we may assume that $r_1 = \infty$.

Theorem 1.1 applies to several important examples. Here is a non-exhaustive list of couples $(g(A), \beta_g)$.

1. *Fractional powers.* If $g(A) = A^\alpha$ with $0 < \alpha < 1$ then we get $\beta_\alpha(r) = \beta(r^{1/\alpha})$ for all $r > 0$. This improves the result obtained in [W1].
2. *Gamma subordinator.* If $g(A) = \log(I + A)$ then we get $\beta_{\log}(r) = \beta((e^{1/r} - 1)^{-1})$ for all $r > 0$ where I denotes the identity operator on $L^2(X)$.
3. *Generalized geometric stable subordinators.* If $g(A) = [\log(I + A^\alpha)]^\gamma$ with $\alpha > 0$ and $\gamma \leq 1$, then we get

$$\beta_g(r) = \beta\left(\left[e^{(\frac{1}{r})^\alpha} - 1\right]^{-\frac{1}{\alpha}}\right), \quad r > 0.$$

4. *Poisson subordination.* If $g(A) = I - T_\lambda = I - e^{-\lambda A}$ with $\lambda > 0$ then we get

$$\beta^{(\lambda)}(r) = \beta\left(\frac{\lambda}{\log(1 + \frac{1}{r-1})}\right), \quad r > 1.$$

In case (iv) the super Poincaré inequality obtained for the generator $I - T_\lambda$ can also be proved with the same rate function $\beta^{(\lambda)}$ but by a different route, see (ii) of Proposition 2.1 below. For a detailed discussion related to the case (iv) and optimality results, see Corollaries 2.3 and 2.4 below.

The next theorem is similar to Theorem 1.1. Instead of the super Poincaré inequality assumption, we suppose that the Nash-type inequality (2) holds for the operator A .

Theorem 1.2 *1. Let $-A$ be the infinitesimal generator of a semigroup as defined above satisfying the Nash-type inequality (2) with rate function D given by*

$$D(x) = \sup_{t>0} (t - tx^{-1}\beta(t^{-1})), \quad x > 0 \tag{9}$$

where $\beta : (0, \infty) \rightarrow (0, \infty)$ is a continuous decreasing function such that $\beta(0^+) = \infty$. Let g be any Bernstein function. Then the generator $-g(A)$ satisfies the following Nash-type inequality

$$\|f\|_2^2 D_{g,\beta}(\|f\|_2^2) \leq (g(A)f, f) \quad (10)$$

for all $f \in \mathcal{D}(g(A))$ with $\|f\|_1 \leq 1$ and where

$$D_{g,\beta}(x) = \sup_{\rho \in (g(0), g(\infty))} \left[\rho - \rho x^{-1} \beta \left(\frac{1}{g^{-1}(\rho)} \right) \right], \quad x > 0. \quad (11)$$

2. If in addition to the hypotheses of (i) we assume that the Bernstein function g and the rate function β are bijections. Then we have

$$\|f\|_2^2 H_{g,\beta}(\|f\|_2^2) \leq (g(A)f, f), \quad \|f\|_1 \leq 1 \quad (12)$$

where

$$H_{g,\beta}(x) = \sup_{0 < \varepsilon < 1} (1 - \varepsilon) (g \circ D)(\varepsilon x), \quad x > 0.$$

Moreover, we have the following estimates

$$\sup_{0 < \varepsilon < 1} (1 - \varepsilon) (g \circ D)(\varepsilon x) \leq D_{g,\beta}(x) \leq g \circ D(x), \quad x > 0$$

and we have $\frac{1}{4} g \circ D \leq D_{g,\beta} \leq g \circ D$ if $g \circ D$ is concave.

The Legendre transform is hidden behind the relationship between the functions D and β given by (9) and hence behind the equivalence between the Nash-type inequality (2) and the super-Poincaré inequality (1), see Proposition 2.2 below. In Section 8, we discuss this relationship (9) and its inversion. In particular, the usual conditions on the N-functions associated with β and D are somewhat relaxed to deal with the examples presented in this paper, see Lemmas 8.1 and 8.2.

Our approach simplifies and generalizes the proof of the main result of [B-M] and [W1]. In particular, the inequality (12) clarifies the constants obtained in [B-M] for the fractional powers A^α . With the same arguments of proof, we can replace the expression $\|f\|_1$ in Theorems 1.1 and 1.2 by any non-negative functional $\Phi(f)$ satisfying a contraction property of the form $\Phi(T_t f) \leq \Phi(f)$ for all $t \geq 0$. Here $(T_t)_{t \geq 0}$ stands for the semigroup generated by $-A$. Our results can be generalized to Hilbert spaces in the same way as done in Wang's paper [W1]. We shall not give the details.

The paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2. More precisely, in Section 2.1 we first recall the equivalence between the super-Poincaré inequality satisfied by the generator of the semigroup and the *integrated* version satisfied by the semigroup itself (see Proposition 2.1 (i)). In Proposition 2.1 (ii), we show that the integrated version of the super-Poincaré inequality is equivalent to a $(1, \infty)$ -super-Poincaré inequality for the bounded operator $I - T_s$ seen as the generator of a semigroup. We also describe quantitatively the relationship between the super-Poincaré inequality (1) and the Nash-type inequality (2) (see Proposition 2.2). Sections 2.2 and 2.3 are devoted to the proofs of Theorems 1.1 and 1.2 respectively. In Section 3, we briefly apply our results to the study of ultracontractivity of subordinated semigroups. In Section 4, we provide several examples of settings where our results apply. In Section 5, we study results similar to Theorems 1.1 and 1.2 but for convex functions of the infinitesimal generator. In particular, we prove the converses of Theorems 1.1 and 1.2. In Section 6, we briefly revisit

the spectral gap for $g(A)$ in $L^p(\mu)$. In Section 7, we prove super-Poincaré inequalities for a wide class of functions of the Laplacian on the Euclidean space, including the class of Bernstein functions. Finally, in Section 8 we exhibit the underlying connection with the Legendre transform between the rate functions β and D as they appear in the super-Poincaré inequality (1) and the Nash-type inequality (2) respectively.

2 Proofs of the Main Theorems

2.1 Super-Poincaré, Semigroup and Nash-type inequalities

We recall some known connections between several functional inequalities. We are mostly concerned with super-Poincaré, semigroup and Nash-type inequalities useful for the proofs of Theorems 1.1 and 1.2.

For the proof of Theorem 1.1, we use in a crucial way the following result of F-Y. Wang; namely the equivalence between the super-Poincaré inequality (13) below satisfied by the infinitesimal generator A and the integrated version (14) expressed only in terms of the associated semigroup $(T_t)_{t \geq 0}$. See [W2, p.230] or [W4, Lemma 3.3.5], see also [W3, p.3]. This result is the analogue of the well-known equivalence between the Poincaré inequality and the exponential decay of the semigroup. In addition, we show that the inequality (14) satisfied by the operator T_t turns out to be the $(1, \infty)$ -super-Poincaré inequality (15) for the generator $I - T_{2t}$ for each fixed $t > 0$. Note that the generator $I - T_{2t}$ is related to the elementary Bernstein function $g_{2t}(x) = 1 - e^{-2tx}$ by the spectral formula $I - T_{2t} = g_{2t}(A)$.

Proposition 2.1 *Let $(T_t)_{t \geq 0}$ be a semigroup as in Section 1 with infinitesimal generator $-A$ and let β be a non-negative function defined on $(0, \infty)$.*

1. *Let $r > 0$ be fixed. Then the two following statements are equivalent.*

(a) *For all $f \in \mathcal{D}(A) \cap L^1(\mu)$,*

$$\|f\|_2^2 \leq r(Af, f) + \beta(r)\|f\|_1^2. \quad (13)$$

(b) *For all $f \in L^2(\mu) \cap L^1(\mu)$ and all $t > 0$,*

$$\|T_t f\|_2^2 \leq e^{-2t/r} \|f\|_2^2 + (1 - e^{-2t/r})\beta(r) \|f\|_1^2. \quad (14)$$

2. *Assume that the inequality (14) holds true for all $r > 0$, for a fixed $t > 0$ and a fixed $f \in L^2(\mu) \cap L^1(\mu)$. Then we have*

$$\|f\|_2^2 \leq \rho((I - T_{2t})f, f) + \beta\left(\frac{2t}{\log(1 + \frac{1}{\rho-1})}\right) \|f\|_1^2 \quad (15)$$

for all $\rho > 1$. The converse holds true.

Remark 2.1 (a) *The exponential term $e^{-2t/r}$ of the inequality (14) is well adapted to deal with the Laplace transforms defined by (3). This is the key point of our paper. From this remark, the inequality (14) can easily be transferred from A to $g(A)$.*

(b) *Note that the equivalence between the inequalities (13) and (14) is valid for any fixed $r > 0$. Thus the inequality (13) holds on some interval (r_0, r_1) if and only if (14) holds on the same interval (for all $t > 0$).*

(c) *To the authors' knowledge, although very simple, the fact that the inequalities (14) and (15) are equivalent does not seem to have been noticed previously.*

Proof (i) We show that (a) implies (b). Let $r > 0$ be fixed. First, we prove (14) in the case $f \in \mathcal{D}(A) \cap L^1(\mu)$. We put $H(t) = e^{2t/r} \|T_t f\|_2^2$ with $t > 0$. We can write

$$H(t) - H(0) = \int_0^t H'(u) du = \int_0^t 2e^{2u/r} \left(\frac{1}{r} \|T_u f\|_2^2 - (AT_u f, T_u f) \right) du.$$

We apply the inequality (13) to $T_u f \in \mathcal{D}(A)$. Using the fact that T_u is a contraction on $L^1(\mu)$ and that β is non-negative, we obtain

$$H(t) - H(0) \leq \frac{2}{r} \beta(r) \|f\|_1^2 \left(\int_0^t e^{2u/r} du \right).$$

This establishes (14) for any $f \in \mathcal{D}(A) \cap L^1(\mu)$.

We now treat the general case. Let $f \in L^2(\mu) \cap L^1(\mu)$. For all $u > 0$, we put $f_u = u^{-1} \int_0^u T_s f ds$. Then it is well known that $f_u \in \mathcal{D}(A)$. Since the semigroup $(T_t)_{t \geq 0}$ is contracting on $L^1(\mu)$ and $L^2(\mu)$, we have $\|f_u\|_p \leq \|f\|_p$ for $p = 1, 2$ and for all $u > 0$. We apply (14) to f_u and conclude by taking the limit as u goes to zero using the facts that f_u converges to f in $L^2(\mu)$ and that the operators T_t are continuous on $L^2(\mu)$. This proves that (a) implies (b).

Converse. Let $r > 0$ be fixed. For all $f \in \mathcal{D}(A) \cap L^1(\mu)$ and all $t > 0$, the inequality (14) can be rewritten in the form

$$\frac{\|T_t f\|_2^2 - \|f\|_2^2}{2t} \leq \left(\frac{e^{-2t/r} - 1}{2t} \right) \|f\|_2^2 + \beta(r) \|f\|_1^2 \left(\frac{1 - e^{-2t/r}}{2t} \right).$$

We conclude that (13) holds by taking the limit as t goes to 0.

(ii) Let $t > 0$ and $f \in L^2(\mu) \cap L^1(\mu)$ be fixed. From (14), we have by symmetry of the semigroup

$$(T_{2t} f, f) = \|T_t f\|_2^2 \leq e^{-2t/r} \|f\|_2^2 + (1 - e^{-2t/r}) \beta(r) \|f\|_1^2$$

for all $r > 0$. This is equivalent to

$$(1 - e^{-2t/r}) \|f\|_2^2 \leq (f - T_{2t} f, f) + (1 - e^{-2t/r}) \beta(r) \|f\|_1^2.$$

Let $g(x) = g_{2t}(x) = 1 - e^{-2tx}$. By using spectral theory, this inequality reads as

$$\|f\|_2^2 \leq \frac{1}{g(1/r)} (g(A)f, f) + \beta(r) \|f\|_1^2.$$

Let $\rho > 1$ and choose $r > 0$ such that $\rho = \frac{1}{g(1/r)}$, i.e. $r = \frac{1}{g^{-1}(1/\rho)} = \frac{2t}{\log(1 + \frac{1}{\rho-1})}$. This yields the $(1, \infty)$ -super-Poincaré (15) for the operator $I - T_{2t}$ as expected. The converse is clear. \square

Now we recall that super-Poincaré and Nash-type inequalities are essentially equivalent under natural conditions on the rate functions β in (1) and D in (2). This equivalence is more or less known and the proof given here is in the spirit of [W4, Proposition 3.3.16] but with a different formulation.

Proposition 2.2 *Let A be a non-negative symmetric operator on $L^2(\mu)$.*

1. Assume that the operator A satisfies the super-Poincaré inequality (1) with rate function β . Then A satisfies the Nash-type inequality (2) with rate function D defined by

$$D(x) = \sup_{t>0} (t - tx^{-1}\beta(t^{-1})) \in (-\infty, \infty], \quad x > 0. \quad (16)$$

In particular, the function D is non-decreasing, finite on the subset $(0, \sup \mathcal{G})$ of \mathbb{R} where $\mathcal{G} = \{\|f\|_2^2, f \in \mathcal{D}(A) \cap L^1(\mu), \|f\|_1 \leq 1\}$ and $D(\infty) = \infty$.

2. Conversely, assume that the Nash-type inequality (2) holds true for the operator A with a rate function D . Then A satisfies the super-Poincaré inequality (1) with rate function β defined by

$$\beta(r) = \sup_{x>0} (x - rxD(x)) \in [0, \infty], \quad r > 0. \quad (17)$$

If the set \mathcal{G} of Proposition 2.2 (i) is unbounded above as a subset of \mathbb{R} , then the rate function $D(x)$ defined by (16) is finite for all $x \in (0, \infty)$. Moreover many examples of rate functions β are non-negative, non-increasing and satisfy the following properties: $\lim_{t \rightarrow 0^+} \beta(t) = \infty$ and $\lim_{t \rightarrow 0^+} t\beta(1/t) = 0$. Under these conditions, the rate function $D(x)$ is finite for all $x > 0$, non-negative, non-decreasing and continuous. Conversely, the rate function β defined by (17) is non-negative since $\lim_{x \rightarrow 0^+} (x - rxD(x)) = 0$. For the detailed proof of these facts, see Section 8.

Proof (i) We assume that the super-Poincaré inequality (1) holds true. We easily deduce from (1) that

$$\|f\|_2^2 \left(\frac{1}{r} - \frac{\beta(r)}{r\|f\|_2^2} \right) \leq (Af, f)$$

for all $f \neq 0$, $\|f\|_1 \leq 1$, $f \in \mathcal{D}(A)$ and all $r > 0$. By taking the supremum over $r > 0$, we obtain the Nash-type inequality (2) with rate function D defined by (16). Note that D is finite on the subset $\mathcal{G} \setminus \{0\}$ of \mathbb{R} since (Af, f) is finite for all $f \in \mathcal{D}(A)$. On one hand, the set \mathcal{G} is not empty since it contains 0. On the other hand, if $f \in \mathcal{G}$ and $\lambda \in (0, 1)$, then $\lambda f \in \mathcal{G}$. Hence $(0, \sup \mathcal{G}) \subset \mathcal{G}$ and D is finite on $(0, \sup \mathcal{G})$. It is easily proved that D is non-decreasing using the fact that $\beta \geq 0$. Moreover, we have

$$D(x) \geq t - tx^{-1}\beta(1/t)$$

for all $x, t > 0$. Therefore, we obtain $\liminf_{x \rightarrow +\infty} D(x) \geq t$ for all $t > 0$. This implies $\lim_{x \rightarrow \infty} D(x) = \infty$. The statement (i) is proved.

(ii) The proof is straightforward. We apply the inequality $x \leq rxD(x) + \beta(r)$ valid for all $r > 0$ to $x = \|f\|_2^2$ with $f \in \mathcal{D}(A) \cap L^1(\mu)$, $\|f\|_1 \leq 1$. The proof is completed by using the Nash-type inequality assumption (2). \square

We now describe an application of Proposition 2.1 to the relationship between the profiles of super-Poincaré inequalities defined below for the generators A and $B_\lambda = I - e^{-\lambda A}$. As a corollary, we obtain the explicit profile of the operator $B_\lambda = I - e^{-\lambda \Delta}$, $\lambda > 0$, from the profile of the Laplacian $A = \Delta$ on the Euclidean space \mathbb{R}^n .

1. *Profile of the super-Poincaré inequality*

Let A be a non-negative self-adjoint operator on $L^2(\mu)$. We define the (r_0, r_1) -super-Poincaré profile β_p of A by

$$\beta_p(r) := \sup\{\|f\|_2^2 - r(Af, f) : f \in \mathcal{D}(A) \cap L^1(\mu), \|f\|_1 \leq 1\}, \quad r_0 < r < r_1. \quad (18)$$

The optimal rate function β_p is non-increasing and satisfies $0 \leq \beta_p \leq \beta$ with β as in (1). To prove that $0 \leq \beta_p$, we evaluate the expression $\{\|f\|_2^2 - r(Af, f)\}$ at $f = 0$. If $\beta_p(r_2) = 0$ at some point $r_2 > 0$, say, then $\beta_p(r) = 0$ for all $r \geq r_2$ and the following Poincaré inequality holds true

$$\|f\|_2^2 \leq r_2(Af, f).$$

2. *Relationship between the super Poincaré profiles of A and $B_\lambda = I - e^{-\lambda A}$*

The equivalence between the inequalities (13) and (15) is particularly interesting in terms of relationship between the super-Poincaré profiles of the generators A and $B_\lambda = I - e^{-\lambda A}$ for any fixed $\lambda > 0$.

Corollary 2.3 *Let $\lambda > 0$. Under the assumptions of Proposition 2.1, if β_p and $\gamma_p^{(\lambda)}$ are the super-Poincaré profiles of A and $B_\lambda = I - e^{-\lambda A}$ respectively, then the profiles are related by the formulas*

$$\gamma_p^{(\lambda)}(r) = \beta_p\left(\frac{\lambda}{\log\left(1 + \frac{1}{r-1}\right)}\right), \quad r > 1$$

or equivalently

$$\beta_p(s) = \gamma_p^{(\lambda)}\left(1 + (e^{\lambda/s} - 1)^{-1}\right), \quad s > 0.$$

The proof is immediate from Proposition 2.1.

3. *Applications to the Laplacian on \mathbb{R}^n*

The optimal Nash inequality (29) below provides the super-Poincaré profile for the Laplacian Δ on \mathbb{R}^n , namely $\beta_p(s) = C_n s^{-n/2}$ for the optimal constant C_n given by (30).

Corollary 2.4 *Let Δ be the Laplacian on \mathbb{R}^n and $\lambda > 0$. The super-Poincaré profile of $B_\lambda = I - e^{-\lambda \Delta}$ on \mathbb{R}^n is given by*

$$\gamma_p^{(\lambda)}(r) = C_n \lambda^{-n/2} \left(\log\left[1 + \frac{1}{r-1}\right]\right)^{n/2}, \quad r > 1 \quad (19)$$

where C_n is given by (30).

4. *Interpretation in terms of continuous time random walks*

We have the following interpretation of Corollary 2.4 in the probability theory framework. More precisely, our result can be closely connected to the *continuous time random walk* with generator $B_\lambda = I - e^{-\lambda \Delta}$ where Δ is the Laplacian on \mathbb{R}^n . In the next discussion, we describe the well-known underlying Markov process associated with the generator B_λ .

Let $\lambda > 0$ be fixed. Let $\{X_k, k \geq 1\}$ be a sequence of independent \mathbb{R}^n -valued random variables of same law ν given by

$$\nu(B) = P(X_k \in B) = \int_B h_\lambda(y) dy, \quad k \geq 1$$

for all Borel sets B of \mathbb{R}^n . Here h_s denotes the Gaussian (heat) kernel defined by

$$h_s(y) = \frac{1}{(4\pi s)^{n/2}} \exp\left(-\frac{|y|^2}{4s}\right), \quad y \in \mathbb{R}^n, \quad s > 0.$$

We define the Markov chain $(Z_k^x)_{k \geq 0}$ starting from $x \in \mathbb{R}^n$ by $Z_k^x = x + S_k$ with $S_k = X_1 + \cdots + X_k$, $k \geq 1$ and $Z_0^x = x$. The law of this random walk is given by

$$P(Z_k^x \in B) = \nu^{*(k)}(B - x) = \int_B h_{k\lambda}(y - x) dy$$

with $\nu^{*(0)}(B - x) = \delta_x(B)$ and by abuse of notation $h_0(y - x) dy = \delta_x(dy)$. Now, let $\{N(t), t \geq 0\}$ be a Poisson process independent of the process $(Z_k^x)_{k \geq 0}$ and with

$$P(N(t) = k) = e^{-t} \frac{t^k}{k!}, \quad k = 0, 1, 2, \dots.$$

For $x \in \mathbb{R}^n$ and $t \geq 0$, we define a continuous time Markov process by $Y_t^x = Z_{N(t)}^x$. The transition probability kernels $Q_t(x, \cdot)$ of this process are given by

$$Q_t(x, B) = e^{-t} \sum_{k \geq 0} \frac{t^k}{k!} \nu^{*(k)}(B - x) = \int_B e^{-t} \sum_{k \geq 0} \frac{t^k}{k!} h_{k\lambda}(y - x) dy.$$

The process $\{Y(t), t \geq 0\}$ is called a *compound* Poisson process. We also say that $\{Y(t), t \geq 0\}$ is subordinated to $\{S_k, k \geq 1\}$ using the Poisson process $\{N(t), t \geq 0\}$ as a directing process.

We are now in a position to describe the connection between the process $\{Y(t), t \geq 0\}$ and the result of Corollary 2.4 obtained in terms of semigroup. The process $\{Y(t), t \geq 0\}$ is related to the semigroup $R_t = e^{-tB_\lambda}$ generated by the bounded operator $B_\lambda = I - e^{-\lambda\Delta}$ as follows. By the semigroup property of $T_s = e^{-s\Delta}$, we have

$$R_t = e^{-t(I - e^{-\lambda\Delta})} = e^{-t} \sum_{k \geq 0} \frac{t^k}{k!} T_{k\lambda} = \int_0^\infty T_s \nu_t(ds), \quad t > 0.$$

Thus the semigroup $(R_t)_{t \geq 0}$ is subordinated to the heat semigroup $(T_s)_{s \geq 0}$ on \mathbb{R}^n by the Poisson semigroup with jumps of size λ defined on $[0, \infty)$ by $\nu_t = \sum_{k \geq 0} \frac{t^k}{k!} e^{-t} \delta_{k\lambda}$. See (7) and [J, p.180]. We recall that $(T_s)_{s \geq 0}$ is the heat semigroup defined by

$$T_s f(x) = \int_{\mathbb{R}^n} h_s(y - x) f(y) dy, \quad s > 0, \quad x \in \mathbb{R}^n.$$

Finally, one easily checks that the semigroup $(R_t)_{t \geq 0}$ possesses $Q_t(x, B)$ as transition probability kernels

$$R_t f(x) = \int_{\mathbb{R}^n} Q_t(x, dy) f(y).$$

We conclude that we have obtained in Corollary 2.4 the super Poincaré profile (19) for the generator $B_\lambda = I - e^{-\lambda\Delta}$ of the *continuous time random walk* $\{Y(t), t \geq 0\}$ as described above. For a recent study of the discrete subordination of random walks see [B-SC].

2.2 Proof of Theorem 1.1

We suppose that the Bernstein function g is non-constant (if not there is nothing to prove since $(r_0, r_1) = \emptyset$). Assume that the super-Poincaré inequality (1) holds true for the operator A with rate function β . By Proposition 2.1, the above inequality (14) is satisfied and can be rewritten as

$$\|T_t f\|_2^2 = (T_{2t} f, f) \leq e^{-2t/r} \|f\|_2^2 + \beta(r) \|f\|_1^2 (1 - e^{-2t/r}), \quad t > 0$$

because $(T_t)_{t \geq 0}$ is a symmetric semigroup. Let $s > 0$ and put $t = s/2$. Finally we have

$$(T_s f, f) \leq e^{-s/r} \|f\|_2^2 + \beta(r) \|f\|_1^2 (1 - e^{-s/r})$$

for all $r > 0$ and all $s > 0$. By the subordination formula (7) and Fubini's theorem, we obtain the following inequality

$$\begin{aligned} (T_t^g f, f) &= \int_0^\infty (T_s f, f) d\nu_t^g(s) \leq \left(\int_0^\infty e^{-s/r} d\nu_t^g(s) \right) \|f\|_2^2 \\ &\quad + \beta(r) \|f\|_1^2 \left(\int_0^\infty (1 - e^{-s/r}) d\nu_t^g(s) \right) \end{aligned}$$

for all $t, r > 0$, $f \in L^1(\mu) \cap L^2(\mu)$ and any Bernstein function g . By the Laplace transform (3) of the (sub-)probability ν_t^g , we deduce that

$$(T_t^g f, f) \leq e^{-tg(1/r)} \|f\|_2^2 + \beta(r) \|f\|_1^2 (1 - e^{-tg(1/r)}), \quad t, r > 0.$$

Replacing t by $2t$ and using the fact that $(T_t^g)_{t \geq 0}$ is a symmetric semigroup, we obtain

$$\|T_t^g f\|_2^2 \leq e^{-2tg(1/r)} \|f\|_2^2 + \beta(r) \|f\|_1^2 (1 - e^{-2tg(1/r)}), \quad t, r > 0.$$

Now let $\rho \in (r_0, r_1) := (\frac{1}{g(\infty)}, \frac{1}{g(0)})$. Since $r \rightarrow \frac{1}{g(1/r)}$ is a bijection from $(0, \infty)$ to (r_0, r_1) , there exists a (unique) $r > 0$ such that $\rho = [g(1/r)]^{-1}$, i.e. $r = [g^{-1}(1/\rho)]^{-1}$ and

$$\|T_t^g f\|_2^2 \leq e^{-2t/\rho} \|f\|_2^2 + \beta([g^{-1}(1/\rho)]^{-1}) \|f\|_1^2 (1 - e^{-2t/\rho}), \quad t > 0.$$

We finish the proof of Theorem 1.1 by applying "(b) \Rightarrow (a)" of (i) in Proposition 2.1 and Remark 2.1(b) to the semigroup $(T_t^g)_{t \geq 0}$ associated with the generator $g(A)$. Note that neither the existence of the density of the measures ν_t^g nor additional properties of the function β are needed for the proof. This finishes the proof of Theorem 1.1.

2.3 Proof of Theorem 1.2

The main ingredient of the proof of Theorem 1.2 is Theorem 1.1. Let us describe briefly the steps of the proof. From the Nash-type inequality (2) satisfied by the generator A , we obtain the super Poincaré inequality (1) for A by Proposition 2.2 (ii), hence for $g(A)$ by Theorem 1.1. By reversing the steps, we conclude that the Nash-type inequality (2) holds for the generator $g(A)$ by Proposition 2.2 (i).

Proof of (i). From the definition (9) of the rate function D , we deduce that

$$\beta(r) \geq \tilde{\beta}(r) := \sup_{x>0} (x - rxD(x)), \quad r > 0.$$

Hence $\tilde{\beta}(r)$ is finite for all $r > 0$. By applying Proposition 2.2 (ii), we conclude that the super-Poincaré inequality (1) with rate function $\tilde{\beta}$ holds for the operator A . But $\tilde{\beta} \leq \beta$, so we can write

$$\|f\|_2^2 \leq r(Af, f) + \beta(r)\|f\|_1^2, \quad r > 0$$

for all $f \in \mathcal{D}(A) \cap L^1(\mu)$. We now apply Theorem 1.1 to obtain the super-Poincaré inequality (1) for the operator $g(A)$ with any Bernstein function g . It yields

$$\|f\|_2^2 \leq r(g(A)f, f) + \beta_g(r)\|f\|_1^2$$

where $\beta_g(r) = \beta\left(\frac{1}{g^{-1}(1/r)}\right)$ for all $r \in (r_0, r_1) = \left(\frac{1}{g(\infty)}, \frac{1}{g(0)}\right)$.

We finish the proof of (i) of Theorem 1.2 by applying Proposition 2.2 (i), replacing A and the rate function β by $g(A)$ and β_g respectively. This proves (10) with $D_{g,\beta}$ given by (11).

Proof of (ii). The proof consists in estimating $D_{g,\beta}$ in (10). For that purpose, the next lemma compares $D_{g,\beta}$ with $g \circ D$ whenever $D_{g,\beta}$ and D are defined by (11) and (9) respectively. Applying this lemma will finish the proof of (ii) of Theorem 1.2.

Lemma 2.5 *Let $\beta : (0, \infty) \rightarrow (0, \infty)$ be a continuous bijective decreasing function and $g : [0, \infty) \rightarrow [0, \infty)$ be a continuous bijective increasing concave function (e.g. a bijective Bernstein function). Let $D_{g,\beta}$ be the function defined as in (11) with $g(0) = 0$ and $g(\infty) = \infty$ and let $D = D_{\text{id},\beta}$. Then*

$$(1 - \varepsilon)(g \circ D)(\varepsilon x) \leq D_{g,\beta}(x) \leq g \circ D(x) \quad (20)$$

for all $\varepsilon \in (0, 1)$ and all $x > 0$. Moreover, if $g \circ D$ is concave (in particular if D is concave) then

$$\frac{1}{4}g \circ D \leq D_{g,\beta} \leq g \circ D. \quad (21)$$

For the proof, we need some preparations.

To simplify the presentation, we put $V(t) = \beta(1/t)$ for $t > 0$ and $V(0) = 0$ (because $\beta(\infty) = 0$). Then the function $V : [0, \infty) \rightarrow [0, \infty)$ is an increasing continuous bijection. We can rewrite $D_{g,\beta}$ as follows. We have

$$D_{g,\beta}(x) = \sup_{\rho \geq 0} \rho [1 - x^{-1}V(g^{-1}(\rho))], \quad x > 0.$$

By putting $u = g^{-1}(\rho)$ in the expression of $D_{g,\beta}$ above, we arrive at

$$D_{g,\beta}(x) = \sup_{u \geq 0} g(u) [1 - x^{-1}V(u)].$$

Let $H_x(u) = g(u) [1 - x^{-1}V(u)]$ for all $u \geq 0$ and all $x > 0$. We have $H_x(0) = H_x(V^{-1}(x)) = 0$, $H_x(u) > 0$ if $0 < u < V^{-1}(x)$ and $H_x(u) < 0$ if $u > V^{-1}(x)$. By continuity of the function H_x , it follows that

$$D_{g,\beta}(x) = \max_{u \in [0, V^{-1}(x)]} g(u) [1 - x^{-1}V(u)]. \quad (22)$$

This implies that $D_{g,\beta}(x)$ is positive and finite for any $x > 0$. In particular, it applies to the identity function $g = \text{id}$ and yields

$$D(x) = D_{\text{id},\beta}(x) = \max_{u \in [0, V^{-1}(x)]} u [1 - x^{-1}V(u)]. \quad (23)$$

We now prove Lemma 2.5.

Proof of Lemma 2.5.

(i) *The upper bound in (20).* Let $x > 0$ be fixed. Since g is increasing and continuous, it follows from (23) that

$$g \circ D(x) = g \circ D_{\text{id},\beta}(x) = \max_{u \in [0, V^{-1}(x)]} g(u [1 - x^{-1}V(u)]). \quad (24)$$

Let $u \in [0, V^{-1}(x)]$ and put $a = 1 - x^{-1}V(u)$. Since $a \in [0, 1]$, $g(0) = 0$ and g is concave, it follows that

$$ag(u) = ag(u) + (1 - a)g(0) \leq g(au + (1 - a)0) = g(au)$$

which yields

$$g(u) [1 - x^{-1}V(u)] \leq g(u [1 - x^{-1}V(u)])$$

for all $u \geq 0$. By taking the supremum over $u \in [0, V^{-1}(x)]$ and using the relations (22) and (24), we deduce that

$$D_{g,\beta}(x) \leq g \circ D(x).$$

This proves the upper bound in (20).

(ii) *The lower bound in (20).* Let $x > 0$ be fixed. By (22), it follows that

$$D_{g,\beta}(x) \geq g(u) [1 - x^{-1}V(u)]$$

for all $u \in [0, V^{-1}(x)]$. Let $\varepsilon \in (0, 1)$ and take $u = V^{-1}(\varepsilon x)$, we obtain

$$D_{g,\beta}(x) \geq (1 - \varepsilon) g(V^{-1}(\varepsilon x)). \quad (25)$$

On the other hand, by (23) applied to εx in place of x , there exists $r \in (0, V^{-1}(\varepsilon x))$ such that

$$D(\varepsilon x) = r [1 - (\varepsilon x)^{-1}V(r)].$$

It follows that $0 < D(\varepsilon x) \leq r$ and $r \leq V^{-1}(\varepsilon x)$. As a consequence, it implies that

$$D(\varepsilon x) \leq V^{-1}(\varepsilon x).$$

Since g is increasing, we deduce that $g \circ D(\varepsilon x) \leq g \circ V^{-1}(\varepsilon x)$. Now by (25), we conclude that

$$D_{g,\beta}(x) \geq (1 - \varepsilon)(g \circ D)(\varepsilon x). \quad (26)$$

This proves the lower bound in (20).

(iii) *The lower bound in (21).*

Here, we assume that $g \circ D$ is concave. We have already seen that $0 \leq D(x) \leq V^{-1}(x)$ for all $x > 0$. Therefore $\lim_{x \rightarrow 0^+} D(x) = 0$ by continuity of V^{-1} . We put $D(0) = 0$. Let $\varepsilon \in (0, 1)$ and $x > 0$. Since $g \circ D(0) = g(0) = 0$ and $g \circ D$ is concave, we obtain

$$\varepsilon(g \circ D)(x) \leq (g \circ D)(\varepsilon x).$$

By (26), we deduce that

$$\frac{1}{4}(g \circ D)(x) = \sup_{\varepsilon \in (0,1)} (1 - \varepsilon)\varepsilon(g \circ D)(x) \leq D_{g,\beta}(x)$$

for all $x > 0$. This completes the proof of Lemma 2.5. \square

This finishes the proof of Theorem 1.2. \square

3 Ultracontractivity of Subordinated Semigroups

Recall that a symmetric semigroup $(T_t)_{t \geq 0}$ of contractions on $L^p(\mu)$, $1 \leq p \leq \infty$, is said to be ultracontractive if

$$\|T_t f\|_2 \leq b(t)\|f\|_1 \quad (27)$$

for all $t > 0$ where $b : (0, \infty) \rightarrow (0, \infty)$ is a non-increasing function, see [D]. In practice, we often have the additional property $b(0^+) = \infty$. Ultracontractivity implies the super-Poincaré inequality (1) with rate function $\beta(r) = b^2(r/2)$. Indeed the function $s \rightarrow (AT_s f, T_s f)$ is non-increasing for all $f \in \mathcal{D}(A)$, then we obtain

$$\begin{aligned} \|f\|_2^2 - b^2(r/2)\|f\|_1^2 &\leq \|f\|_2^2 - \|T_{r/2} f\|_2^2 \\ &= (f - T_r f, f) = \int_0^r (AT_{s/2} f, T_{s/2} f) ds \leq r(Af, f) \end{aligned}$$

for all $r > 0$ which is the desired inequality.

By duality and interpolation, ultracontractivity is equivalent to

$$\|T_t f\|_\infty \leq a(t)\|f\|_1, \quad t > 0 \quad (28)$$

for some non-increasing function $a : (0, \infty) \rightarrow (0, \infty)$. More precisely, from (27) we obtain $a(t) \leq b^2(t/2)$ in (28), and from (28) we have $b(t) \leq \sqrt{a(t)}$ in (27).

Now let g be a Bernstein function. The semigroup $(T_t^g)_{t \geq 0}$ is ultracontractive if $b_g(t) := \int_0^\infty b(s)\eta_t^g(s) ds$ is finite for all $t > 0$. Indeed, by the subordination formula (7) we deduce immediately that

$$\|T_t^g f\|_2 \leq \int_0^\infty \eta_t^g(s)\|T_s f\|_2 ds \leq \left(\int_0^\infty b(s)\eta_t^g(s) ds \right) \|f\|_1 \leq b_g(t)\|f\|_1, \quad t > 0.$$

Unfortunately the condition of finiteness of b_g is rather difficult to check because the densities η_t^g are not well known, except in a few cases for example $g(x) = \sqrt{x}$, see [J, p.181]. A possible way to overcome this difficulty consists in considering Nash-type inequalities. For that purpose, we recall a result due to T.Coulhon who deduces ultracontractivity bounds from Nash-type inequalities under some integrability condition, see [C]. For the applications we have in mind, we restrict his result to our setting.

Theorem 3.1 *Let $(T_t)_{t \geq 0}$ be a semigroup with infinitesimal generator $-A$ as in Section 1. Assume that there exists a non-decreasing function $\Theta : (0, \infty) \rightarrow (0, \infty)$ such that*

$$\Theta(\|f\|_2^2) \leq (Af, f)$$

for all $f \in \mathcal{D}(A) \cap L^1$ with $\|f\|_1 \leq 1$ and $\int_0^\infty \frac{dx}{\Theta(x)} < \infty$. Then $(T_t)_{t \geq 0}$ is ultracontractive. More precisely,

$$\|T_t f\|_\infty \leq a(t)\|f\|_1, \quad t > 0$$

where $a(t)$ is the inverse of the function $s \mapsto \int_s^\infty \frac{dx}{\Theta(x)}$.

For a fixed Bernstein function g , we may apply Theorem 3.1 to the subordinated semigroup $(T_t^g)_{t \geq 0}$, assuming that a Nash-type inequality holds for the generator A as follows.

Corollary 3.2 *Let $(T_t)_{t \geq 0}$ be a semigroup with infinitesimal generator $-A$ satisfying the assumptions of Theorem 1.2. Let g be any Bernstein function. Assume that $\int_s^\infty \frac{dx}{xD_{g,\beta}(x)}$ is finite where $D_{g,\beta}$ is given by (11). Then the semigroup $(T_t^g)_{t \geq 0}$ is ultracontractive and*

$$\|T_t^g f\|_\infty \leq a_g(t) \|f\|_1, \quad t > 0$$

where a_g is the inverse function of $s \mapsto \int_s^\infty \frac{dx}{xD_{g,\beta}(x)}$.

Proof Apply Theorems 1.2 and 3.1. \square

In practice, the lower bounds $(1 - \varepsilon)(g \circ D)(\varepsilon x) \leq D_{g,\beta}(x)$ or better $\frac{1}{4}g \circ D \leq D_{g,\beta}$ of Theorem 1.2 are particularly useful to obtain (explicit) bounds of ultracontractivity for the subordinated semigroup $(T_t^g)_{t \geq 0}$. On the other hand, it is clear that $(T_t)_{t \geq 0}$ can be ultracontractive but, for certain Bernstein functions g , $(T_t^g)_{t \geq 0}$ may not be. For instance, let $A = \Delta$ be the usual Laplacian on \mathbb{R}^n . We recall that the operator Δ satisfies the Nash-type inequality with rate function $D(x) = cx^{2/n}$. First note that the assumption of finiteness of Corollary 3.2 fails with $A = \Delta$ and $g(r) = \log(1 + r)$. Indeed, it is easy to see that

$$\int^\infty \frac{dx}{xD_{g,\beta}(x)} \geq \int^\infty \frac{dx}{x \log(1 + cx^{2/n})} = \infty$$

by using the inequality $(g \circ D) \geq D_{g,\beta}$ of Theorem 1.2 (ii). In fact, by a direct computation we can show that $(T_t^g)_{t \geq 0}$ is not ultracontractive for small $t > 0$, see (32) below for details.

Applications to heat kernel bounds.

Let $(T_t)_{t \geq 0}$ be a semigroup on $L^2(X, \mu)$ defined as in Section 1 and g be a Bernstein function. Ultracontractivity of the semigroup $(T_t^g)_{t \geq 0}$ ensures the existence of a heat kernel $k_t^g(x, y)$ with respect to the σ -finite measure μ . So, we have the formula

$$T_t^g f(x) = \int_X k_t^g(x, y) f(y) d\mu(y).$$

Moreover, the kernel satisfies the following uniform bound

$$\operatorname{ess\,sup}_{x,y \in X} k_t^g(x, y) \leq a_g(t), \quad t > 0$$

where $a_g(t)$ is given by Corollary 3.2. See the recent paper [G-H, Lemma 3.7] for a detailed exposition on the existence of the heat kernel (see also [D, Chapter 2]).

4 Examples of Applications

In this section, we provide several examples of settings where our results apply.

4.1 The Euclidean Space

Let $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ be the usual Laplacian on \mathbb{R}^n . The profile of the super-Poincaré inequality for Δ can be deduced from the optimal Nash inequality obtained in [C-L] (for the definition of *profile*, see Section 2.1). Let N_n be the best constant of the Nash inequality for the Laplacian,

$$\frac{1}{N_n} \|f\|_2^{2+4/n} \leq (\Delta f, f), \quad \|f\|_1 \leq 1$$

see [C-L]. This Nash inequality is equivalent to the following super-Poincaré inequality,

$$\|f\|_2^2 \leq r(\Delta f, f) + C_n r^{-n/2} \|f\|_1^2, \quad r > 0 \quad (29)$$

with

$$C_n = \frac{2(nN_n)^{n/2}}{(n+2)^{1+n/2}}. \quad (30)$$

Hence the super-Poincaré profile for Δ has the form $\beta_p(r) = C_n r^{-n/2}$ and the rate function D of the Nash inequality (2) is given by $D(x) = N_n^{-1} x^{2/n}$. The equivalence above is obtained by applying Proposition 2.2 to the Laplace operator. Indeed, both formulas (16) and (17) are satisfied by the couple of rate functions (β_p, D) given above where the constants C_n and N_n are related by (30). Note that the rate function D is concave when $n \geq 2$ which implies that $g \circ D$ is concave for any Bernstein function g . Therefore, the last statement of Theorem 1.2 (ii) applies when $n \geq 2$.

Examples of Bernstein functions.

1. Let $g(x) = x^\alpha$. The fractional power of the Laplacian Δ^α , $0 < \alpha < 1$, satisfies the following super-Poincaré inequality obtained from Theorem 1.1,

$$\|f\|_2^2 \leq r(\Delta^\alpha f, f) + C_n r^{-\frac{n}{2\alpha}} \|f\|_1^2, \quad r > 0.$$

The Nash-type inequality for Δ^α can be deduced from (i) of Proposition 2.2 or from (i) of Theorem 1.2. More precisely, we get

$$K_{n,\alpha} \|f\|_2^{2+4\alpha/n} \leq (\Delta^\alpha f, f), \quad \|f\|_1 \leq 1$$

with the constant

$$K_{n,\alpha} = \frac{n(2\alpha)^{2\alpha/n}}{(2\alpha+n)^{1+2\alpha/n}} C_n^{-\frac{2\alpha}{n}}$$

where C_n is given by (30). We can also apply (ii) of Theorem 1.2 and get another constant

$$K'_{n,\alpha} = \frac{n(2\alpha)^{2\alpha/n}}{(2\alpha+n)^{1+2\alpha/n}} N_n^{-\alpha}$$

instead of $K_{n,\alpha}$. It is easy to see that $K'_{n,\alpha} < K_{n,\alpha}$ from the relationship between the constants N_n and C_n . But the constant $K_{n,\alpha}$ is probably not the best constant of the Nash inequality for the fractional power Δ^α of the Laplacian.

We postpone to Section 7 the study of the super-Poincaré inequality (1) for a larger class of functions of the Laplacian using Fourier theory as originally done in [N]

for the Laplacian itself. Note that this approach does not provide the best constants. We also refer for instance to [VSC] for a study of Nash inequalities (and Hardy-Littlewood-Sobolev inequalities) for the fractional powers of the generator of a sub-markovian symmetric semigroup.

2. Let $g(x) = \log(1+x)$. The geometrically stable operator $\log(I + \Delta)$ satisfies

$$(1 - \varepsilon)\|f\|_2^2 \log\left(1 + N_n^{-1}\varepsilon^{2/n}\|f\|_2^{4/n}\right) \leq (\log(I + \Delta)f, f), \quad \|f\|_1 \leq 1$$

for all $\varepsilon \in (0, 1)$. To estimate $D_{g,\beta}$ of (11) and $H_{g,\beta}$ in (12) with $g(x) = \log(1+x)$ is not a pleasant task. So, we have preferred to state the explicit Nash-type inequality (12) for each parameter $\varepsilon \in (0, 1)$. When the dimension $n \geq 2$, the last statement of Theorem 1.2 (ii) yields by concavity of $g \circ D$,

$$\frac{1}{4}\|f\|_2^2 \log\left(1 + N_n^{-1}\|f\|_2^{4/n}\right) \leq (\log(I + \Delta)f, f), \quad \|f\|_1 \leq 1.$$

As a last remark, note that the possible ultracontractivity (27) can be proved for the semigroup $e^{-tg(\Delta)}$ directly by the formula

$$b^2(t) := \|e^{-tg(\Delta)}\|_{1 \rightarrow 2}^2 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-2tg(|y|^2)} dy \in (0, \infty] \quad (31)$$

obtained by using Fourier analysis. When we apply this formula to $g(x) = \log(1+x)$, this leads to

$$\|e^{-t\log(I+\Delta)}\|_{1 \rightarrow 2}^2 = nv_n(2\pi)^{-n} \int_0^\infty (1+r^2)^{-2t} r^{n-1} dr \quad (32)$$

which is finite if and only if $t > \frac{n}{4}$ (here v_n denotes the volume of the unit ball of \mathbb{R}^n). Therefore, the semigroup with generator $\log(I + \Delta)$ is not ultracontractive for $0 < t \leq n/4$. This example shows that the super-Poincaré inequality and the Nash-type inequality can be satisfied by a generator but without ultracontractivity of the corresponding semigroup.

4.2 The Riemannian setting

Let M be a connected complete Riemannian manifold with Ricci curvature bounded below. The Riemannian volume measure is denoted by dx , the Laplace-Beltrami operator by Δ and the gradient by ∇ . Assume that the boundary ∂M is either convex or empty. Let $V \in W_{loc}^{2,1}(M, dx)$ be given such that $Z = \int_M e^{V(x)} dx$ is finite. We define a probability measure μ by putting $d\mu = Z^{-1}e^V dx$. Let $-A = \Delta + \nabla V$. Then the operator A is non-negative and essentially self-adjoint on $L^2(\mu)$ (with Neumann boundary condition whenever ∂M is non-empty). Put $\rho(x) = \rho(x, o)$ the Riemannian distance function to a fixed point $o \in M$. Consider $V = -\alpha\rho^\delta$, $\alpha > 0$ and $\delta > 1$. Then the super-Poincaré inequality (1) holds true with rate function

$$\beta(r) = \exp\left[c(1+r^{-\lambda})\right] \quad (33)$$

where $\lambda = \delta/[2(\delta-1)]$ and c is some positive constant. If $V = -\exp(\alpha\rho)$, $\alpha > 0$, then the super-Poincaré inequality holds true with β given by (33) with $\lambda = 1/2$. These two

examples are taken from [W2, Corollary 2.5].

In this setting, for any Bernstein function g , the operator $g(A)$ satisfies the super-Poincaré inequality (1) with rate function $\beta_g(r) = \beta\left(\frac{1}{g^{-1}(1/r)}\right)$ by Theorem 1.1. Moreover, the Nash-type inequality (2) can be deduced for $g(A)$ by Theorem 1.2.

4.3 The hypoelliptic setting

In this section, we consider as generators of one-parameter semigroups some sub-Laplacians on Lie groups of polynomial growth.

Let G be a connected Lie group of polynomial growth of index D and (X_1, X_2, \dots, X_m) be a system of left-invariant vector fields satisfying Hörmander's condition with local dimension d . Let us assume that $d \leq D$. The sub-Laplacian $\mathcal{L} = -\sum_{i=1}^m X_i^2$ generates a semigroup $e^{-t\mathcal{L}}$ with density kernel p_t satisfying

$$\sup_{x,y \in G} p_t(x,y) = \|e^{-t\mathcal{L}}\|_{1 \rightarrow \infty} \leq \frac{c_1}{t^{n/2}}, \quad t > 0$$

for all n such that $d \leq n \leq D$, see [VSC]. As a consequence, the following super-Poincaré inequality holds true

$$\|f\|_2^2 \leq r(\mathcal{L}f, f) + c_0 r^{-n/2} \|f\|_1^2, \quad r > 0.$$

Thus the rate function β for the generator \mathcal{L} is given by $\beta(r) = c_0 r^{-n/2}$. The results obtained in this paper apply to $g(\mathcal{L})$ for any Bernstein function g .

We now discuss more specifically the four examples of Bernstein functions g introduced after Theorem 1.1. Below we provide the asymptotic behaviours of $\beta_g(r)$ when r tends to 0 and r tends to ∞ for these examples. Recall that in our case we have

$$\beta_g(r) = \beta\left(\frac{1}{g^{-1}(1/r)}\right) = c_0 [g^{-1}(1/r)]^{n/2}.$$

1. If $g(x) = x^\alpha$, $0 < \alpha \leq 1$, then $\beta_g(r) = c_0 r^{-n/2\alpha}$, $r > 0$.
2. If $g(x) = \log(1+x)$ then $\beta_g(r) = c_0 (e^{1/r} - 1)^{n/2}$, $r > 0$. Moreover

$$\beta_g(r) \sim \begin{cases} c_0 e^{n/2r} & \text{as } r \rightarrow 0^+, \\ c_0 r^{-n/2} & \text{as } r \rightarrow \infty. \end{cases}$$

3. If $g(x) = [\log(1+x^\alpha)]^\gamma$, $0 < \alpha, \gamma \leq 1$, then $\beta_g(r) = c_0 [e^{(1/r)^{1/\gamma}} - 1]^{n/2\alpha}$, $r > 0$. Moreover

$$\beta_g(r) \sim \begin{cases} c_0 e^{\frac{n}{2\alpha}(1/r)^{1/\gamma}} & \text{as } r \rightarrow 0^+, \\ c_0 r^{-n/2\alpha\gamma} & \text{as } r \rightarrow \infty. \end{cases}$$

4. Let $t > 0$. If $g(x) = 1 - e^{-tx}$ then $\beta_g(r) = c_0 t^{-n/2} \left[\log\left(1 + \frac{1}{r-1}\right)\right]^{n/2}$, $r > 1$. Moreover

$$\beta_g(r) \sim \begin{cases} c_0 t^{-n/2} \left[\log\left(\frac{1}{r-1}\right)\right]^{n/2} & \text{as } r \rightarrow 1^+, \\ c_0 (rt)^{-n/2} & \text{as } r \rightarrow \infty. \end{cases}$$

Note that this discussion applies verbatim to the general setting where the generator A satisfies the super-Poincaré inequality (1) with rate function $\beta(r) = c_0 r^{-n/2}$, $r > 0$.

4.4 The Ornstein-Uhlenbeck operator

Let $A = \mathcal{L} = \Delta + x \cdot \nabla$ be the positive Ornstein-Uhlenbeck operator defined on $L^2(\mathbb{R}^n, \gamma)$ where γ is the Gaussian measure $\gamma(dx) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$. It is well known that the logarithmic Sobolev inequality of Gross

$$\int_{\mathbb{R}^n} f^2 \log(|f|/\|f\|_2) d\gamma \leq (\mathcal{L}f, f)$$

holds. By Jensen's inequality, the left hand side of the above inequality is bounded below as follows

$$\|f\|_2^2 \log \|f\|_2 \leq \int_{\mathbb{R}^n} f^2 \log(|f|/\|f\|_2) d\gamma \quad (34)$$

for all $f \in L^1(\gamma) \cap L^2(\gamma)$ with $\|f\|_1 = 1$. By renormalization of the L^1 -norm of f , the inequality (34) is also satisfied when $\|f\|_1 \leq 1$. This yields the following Nash-type inequality

$$\|f\|_2^2 \log \|f\|_2 \leq (\mathcal{L}f, f), \quad \|f\|_1 \leq 1.$$

Using the inequality $xy - e^{y-1} \leq x \log x$ for all $x > 0$ and all $y \in \mathbb{R}$, we obtain the following super-Poincaré inequality

$$\|f\|_2^2 \leq r(\mathcal{L}f, f) + \frac{r}{2e} e^{2/r} \|f\|_1^2, \quad r > 0. \quad (35)$$

On the other hand, it is well-known that Gross' inequality implies the following Poincaré inequality

$$\|f - \gamma(f)\|_2^2 \leq (\mathcal{L}f, f)$$

with $\gamma(f) = \int f d\gamma$, from which we deduce that

$$\|f\|_2^2 \leq (\mathcal{L}f, f) + \|f\|_1^2. \quad (36)$$

The inequalities (35) and (36) lead to the following formulation of the super-Poincaré inequality for the Ornstein-Uhlenbeck operator

$$\|f\|_2^2 \leq r(\mathcal{L}f, f) + \beta(r) \|f\|_1^2, \quad r > 0$$

with $\beta(r) = \frac{r}{2e} e^{2/r}$, $0 < r \leq 1$ and $\beta(r) = 1$, $r \geq 1$.

Theorem 1.1 applies to the Ornstein-Uhlenbeck operator \mathcal{L} . Below, we formulate super-Poincaré inequalities for $g(\mathcal{L})$ with two Bernstein functions g already introduced in this paper.

1. For $g(x) = x^\alpha$, $0 < \alpha < 1$, we have

$$\|f\|_2^2 \leq r(\mathcal{L}^\alpha f, f) + \frac{r^\alpha}{2e} e^{2r^{-1/\alpha}} \|f\|_1^2, \quad 0 < r < 1$$

and

$$\|f\|_2^2 \leq r(\mathcal{L}^\alpha f, f) + \|f\|_1^2, \quad r \geq 1.$$

2. For $g(x) = \log(1+x)$, we have

$$\|f\|_2^2 \leq r(\log(I + \mathcal{L})f, f) + \frac{1}{2e^3} \frac{1}{(e^{1/r} - 1)} e^{2e^{1/r}} \|f\|_1^2, \quad 0 < r < (\log 2)^{-1}$$

and

$$\|f\|_2^2 \leq r(\log(I + \mathcal{L})f, f) + \|f\|_1^2, \quad r \geq (\log 2)^{-1}.$$

In this case, we have the following estimate $\beta_g(r) \sim \frac{1}{2e^3} \exp(2e^{1/r} - \frac{1}{r})$ as r goes to zero.

Similar inequalities can be written for the Bernstein functions of the cases (iii) and (iv) considered in Section 1. Of course, the discussion is not limited to the cases treated above.

5 Study of $\Psi(A)$ with Ψ Convex

It is very useful to obtain a super-Poincaré or Nash-type inequality for $\Psi(A)$ when A itself satisfies such an inequality and Ψ is convex. The reason is that the inverse function of a concave increasing function is a convex increasing function. This argument applies to the class of Bernstein functions which is a particular class of concave functions. The results described in this section allow to obtain a super-Poincaré or a Nash-type inequality for the operator A when assuming that $g(A)$ satisfies such an inequality for some Bernstein function g .

Let $(E_\lambda)_{\lambda \in [0, \infty)}$ be the spectral resolution of A on $L^2(\mu)$ and let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a Borel function. We define $\psi(A)$ on its domain $\mathcal{D}(\psi(A)) \subset L^2(\mu)$, see Section 1. In particular, we have on their respective domains the following representations

$$(Af, f) = \int_0^\infty \lambda d\nu_f(\lambda), \quad (T_t f, f) = \int_0^\infty e^{-\lambda t} d\nu_f(\lambda), \quad \|f\|_2^2 = \int_0^\infty d\nu_f(\lambda) \quad (37)$$

with $d\nu_f(\lambda) = d(E_\lambda f, f)$, see [Sc-S-V, Theorem 11.4].

Proposition 5.1 *Assume that A is a non-negative self-adjoint operator satisfying the Nash-type inequality (2). Then we have*

$$\|f\|_2^2 (\Psi \circ D)(\|f\|_2^2) \leq (\Psi(A)f, f), \quad \|f\|_1 \leq 1 \quad (38)$$

for any non-negative non-decreasing convex function Ψ with $\Psi(0) = 0$.

Proposition 5.1 can be generalized in the framework of a Hilbert space H with the norm $\|f\|_1$ replaced by another control $\Phi(f)$ where f belongs to a subspace of H and where the functional Φ satisfies some properties as described in [W4].

Proof We first note the inclusion of the domains $\mathcal{D}(\Psi(A)) \subset \mathcal{D}(A)$. Indeed, the function Ψ is a non-negative non-decreasing convex function with $\Psi(0) = 0$ which implies that there exist $\lambda_0 > 0$ and $k > 0$ such that $k\lambda \leq \Psi(\lambda)$ for any $\lambda > \lambda_0$. We obtain the inclusion by the representation formula (6) of the domains $\mathcal{D}(\Psi(A))$ and $\mathcal{D}(A)$.

The proof of (38) is as follows. Let $g \in \mathcal{D}(\Psi(A)) \cap L^1(\mu)$ with $\|g\|_2 = 1$ and put $v = \|g\|_1^{-1} g$. We deduce from the above remark that $v \in \mathcal{D}(A) \cap L^1(\mu)$ and $\|v\|_1 = 1$. We apply the Nash-type inequality (2) to v . It implies that the function g satisfies

$$D(\|g\|_1^{-2}) \leq (Ag, g).$$

We compose both sides of this inequality by the non-decreasing function Ψ and obtain

$$(\Psi \circ D) (\|g\|_1^{-2}) \leq \Psi [(Ag, g)].$$

By applying Jensen's inequality to the spectral representation (37) of (Ag, g) with probability measure $d(E_\lambda g, g)$ and convex function Ψ , we obtain

$$\Psi [(Ag, g)] = \Psi \left(\int_0^\infty \lambda d(E_\lambda g, g) \right) \leq \int_0^\infty \Psi(\lambda) d(E_\lambda g, g) = (\Psi(A)g, g).$$

The above rightmost equality comes from the spectral representation (5). Hence

$$(\Psi \circ D) (\|g\|_1^{-2}) \leq (\Psi(A)g, g) \tag{39}$$

holds for all $g \in \mathcal{D}(\Psi(A)) \cap L^1(\mu)$ with $\|g\|_2 = 1$. Now we perform a second normalization to deduce the Nash-type inequality (38) for $\Psi(A)$ as follows. Let $f \in \mathcal{D}(\Psi(A)) \cap L^1(\mu)$ with $\|f\|_1 \leq 1$. Choose $g = \|f\|_2^{-1} f$ in the inequality (39), we obtain

$$\|f\|_2^2 (\Psi \circ D) \left(\frac{\|f\|_2^2}{\|f\|_1^2} \right) \leq (\Psi(A)f, f). \tag{40}$$

We are now in a position to conclude. The function $\Psi \circ D$ is non-decreasing and $\|f\|_1 \leq 1$. This yields

$$(\Psi \circ D) (\|f\|_2^2) \leq (\Psi \circ D) \left(\frac{\|f\|_2^2}{\|f\|_1^2} \right).$$

Multiplying this inequality by $\|f\|_2^2$ and using (40), we conclude that (38) holds true. This completes the proof. \square

Our next task is to prove the analogue of Proposition 5.1 for $\Psi(A)$ assuming that the operator A satisfies a super-Poincaré inequality instead of a Nash-type inequality. Here we generalize the arguments of [W1] used for the fractional powers A^α , $\alpha > 1$.

Theorem 5.2 *Assume that A is a non-negative self-adjoint operator satisfying the super-Poincaré inequality (1) for some rate function γ . Let $\Psi : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing convex function with $\Psi(0) = 0$. We also suppose that*

$$\Psi^*(x) := \sup_{y \in [0, \infty)} (xy - \Psi(y))$$

is a bijection from $[0, \infty)$ onto $[0, \infty)$. Then $\Psi(A)$ satisfies the super-Poincaré inequality (1) with rate function γ_Ψ given by

$$\gamma_\Psi(r) = \inf_{0 < \varepsilon < 1} \varepsilon^{-1} \gamma \left[\varepsilon r (\Psi^*)^{-1} \left[(1 - \varepsilon)(\varepsilon r)^{-1} \right] \right], \quad r > 0.$$

In particular, if $\Psi(x) = x^{1/\alpha}$ with $\alpha \in (0, 1)$, then $\gamma_\Psi(r) \leq \alpha^{-1} \gamma(r^\alpha)$.

Proof The functions Ψ and Ψ^* satisfy Young's inequality, i.e. $sy \leq \Psi(y) + \Psi^*(s)$ for all $y, s \geq 0$. For each $f \in \mathcal{D}(\Psi(A)) \cap L^1(\mu)$, we put $y = (Af, f)$. We deduce that

$$s(Af, f) \leq \Psi((Af, f)) + \Psi^*(s)$$

for all $s > 0$. Now we assume that $\|f\|_2 = 1$. For any convex function Ψ , we have already proved that

$$0 \leq \Psi((Af, f)) \leq (\Psi(A)f, f),$$

see above the equation (39). On the other hand, the super-Poincaré inequality assumption (1) satisfied by the operator A with $r = ts$ implies that

$$1 \leq ts(Af, f) + \gamma(ts)\|f\|_1^2$$

for all $t, s > 0$. By combining the above inequalities this yields

$$1 \leq t(\Psi(A)f, f) + t\Psi^*(s) + \gamma(ts)\|f\|_1^2$$

Let $\varepsilon \in (0, 1)$. Since Ψ^* is a bijection and $\Psi^*(0) = 0$, for each fixed $t > 0$ there exists $s > 0$ such that $\varepsilon = 1 - t\Psi^*(s)$, i.e. $s = (\Psi^*)^{-1}[(1 - \varepsilon)t^{-1}]$. We deduce that

$$\varepsilon \leq t(\Psi(A)f, f) + \gamma[t(\Psi^*)^{-1}[(1 - \varepsilon)t^{-1}]]\|f\|_1^2.$$

By putting $t = \varepsilon r$ with $r > 0$ and dividing this inequality by ε , we obtain

$$1 \leq r(\Psi(A)f, f) + \varepsilon^{-1}\gamma[\varepsilon r(\Psi^*)^{-1}[(1 - \varepsilon)(\varepsilon r)^{-1}]]\|f\|_1^2$$

for all $r > 0$, $\varepsilon \in (0, 1)$ and all $f \in \mathcal{D}(\Psi(A)) \cap L^1(\mu)$ such that $\|f\|_2 = 1$. We now consider $g \in \mathcal{D}(\Psi(A)) \cap L^1(\mu)$, $g \neq 0$. We put $f = \|g\|_2^{-1}g$ in the above inequality and conclude by taking the infimum over $\varepsilon \in (0, 1)$. This proves the first assertion of Theorem 5.2.

We now prove the last statement. Let $\Psi(x) = x^{1/\alpha}$ then we have $\Psi^*(s) = c_\alpha s^{\frac{1}{1-\alpha}}$ with $c_\alpha = (1 - \alpha)\alpha^{\frac{\alpha}{1-\alpha}}$. A simple computation yields

$$\gamma_\Psi(r) \leq \varepsilon^{-1}\gamma(k_\alpha(1 - \varepsilon)^{1-\alpha}\varepsilon^\alpha r^\alpha), \quad r > 0$$

for all $\varepsilon \in (0, 1)$ with $k_\alpha = \alpha^{-\alpha}(1 - \alpha)^{\alpha-1}$. By choosing $\varepsilon = \alpha$, we conclude that $\gamma_\Psi(r) \leq \alpha^{-1}\gamma(r^\alpha)$. The proof of the theorem is complete. \square

In the case $\Psi(x) = x^{1/\alpha}$, $0 < \alpha < 1$, the rate function of the super-Poincaré inequality for $\Psi(A)$ obtained in [W1, Theorem 2.1] (with $p = 1/\alpha$) is given by $\tilde{\gamma}_\Psi(r) = 2\gamma(r^\alpha/2)$. Since γ is usually decreasing, the second result stated in Theorem 5.2 gives a strictly stronger bound for γ_Ψ if $\alpha > 1/2$ (and also stronger up to a multiplicative constant in front of γ if $\alpha \leq 1/2$). Note that $\|f\|_1^2$ plays no particular role in the proof. So, the square of the L^1 -norm can be replaced by some functional $\Phi(f)$ and $L^2(\mu)$ by a general Hilbert space as in [W1].

Now we make the connection between Bernstein functions and convex functions. Assume that g is a Bernstein function. Since g is non-decreasing and concave, $\Psi = g^{-1}$ is non-decreasing and convex. Hence Theorem 5.2 applies and allows us to prove a converse to Theorem 1.1 concerning super-Poincaré inequalities. We apply Theorem 5.2 with $g(A)$ in place of A and get the following result.

Corollary 5.3 *Let g be a bijective Bernstein function and A be a non-negative self-adjoint operator. Assume that $g(A)$ satisfies a super-Poincaré inequality with rate function γ and that $\Psi = g^{-1}$ satisfies the conditions of Theorem 5.2. Then A satisfies the super-Poincaré inequality with rate function γ_Ψ as described in Theorem 5.2.*

We show that Corollary 5.3 is sharp in the particular case $g(x) = x^\alpha$, $\alpha \in (0, 1)$ in the following sense. Assume that A satisfies the super-Poincaré inequality (1) with rate function β . By Theorem 1.1, $g(A)$ satisfies the super-Poincaré inequality (1) with $\gamma(r) = \beta(r^{1/\alpha})$. Now apply Theorem 5.2 with $\Psi(x) = x^{1/\alpha}$, then we get back to a super-Poincaré inequality for A with another rate function $\tilde{\beta}(r) := \gamma_\Psi(r) \leq \frac{1}{\alpha}\gamma(r^\alpha) = \frac{1}{\alpha}\beta(r)$, $r > 0$. To summarize the case of the fractional power, Corollary 5.3 is essentially an optimal converse of Theorem 1.1 up to the multiplicative constant $1/\alpha$.

6 Asymptotic behaviour of $g(A)$

We briefly revisit the relationship between the asymptotic behaviour of $g(A)$ and the asymptotic behaviour of A in terms of Poincaré inequalities in L^p (which is known to be equivalent to a lower bound on the spectrum of A when $p = 2$). Assume that A satisfies a Poincaré inequality in $L^p := L^p(\mu)$, then we deduce that $g(A)$ satisfies a corresponding Poincaré inequality in L^p . For recent results on L^p -Poincaré inequalities, see for instance [C-G-R]. We write $\mu(f)$ to denote the average $\int_X f(x) d\mu(x)$ of $f \in L^1(\mu)$. The proof of the next proposition uses some arguments of Theorem 1.1 without the help of the spectral theorem.

Proposition 6.1 *Let $p \in [1, \infty]$ and $\lambda \in [0, \infty)$ be fixed.*

1. *Let $(T_t)_{t \geq 0}$ be a symmetric Markov semigroup on $L^2(\mu)$ where μ is a probability measure. For a fixed $f \in L^p(\mu)$, assume that we have*

$$\|T_t f - \mu(f)\|_p \leq c e^{-\lambda t}, \quad t > 0 \quad (41)$$

where $c > 0$ and $\lambda \geq 0$ are some constants independent of t . Then we have

$$\|T_t^g f - \mu(f)\|_p \leq c e^{-g(\lambda)t}, \quad t > 0 \quad (42)$$

for any Bernstein function g such that $g(0) = 0$, and where $c > 0$ and $\lambda \geq 0$ are the constants of the inequality (41).

2. *Let $(T_t)_{t \geq 0}$ be a semigroup acting on $L^p(\mu)$ for some measure μ (finite or not). Let $f \in L^p(\mu)$ be fixed such that*

$$\|T_t f\|_p \leq c e^{-\lambda t}, \quad t > 0 \quad (43)$$

where $c > 0$ and $\lambda \geq 0$ are some constants independent of t . Then we have

$$\|T_t^g f\|_p \leq c e^{-g(\lambda)t}, \quad t > 0 \quad (44)$$

for any Bernstein function g , and where $c > 0$ and $\lambda \geq 0$ are the constants of the inequality (43).

The class of symmetric Markov semigroups is stable under subordination for the class of Bernstein functions g satisfying $g(0) = 0$. Similarly, the class of one-parameter contraction semigroups on L^p is preserved under subordination for any Bernstein function and any $p \in [1, \infty]$. For this latter statement, we can apply Proposition 6.1 (ii) above with $c = \|f\|_p$ and $\lambda = 0$.

Proof (i) Let $f \in L^p$. Because μ is a probability measure, we have $L^p \subset L^1$ for $p \geq 1$. Thus $f \in L^1$ and $\mu(f)$ is well-defined. We put $h := f - \mu(f)\mathbf{1}$ which belongs to L^p since $\mathbf{1} \in L^p$. By the Markov property, i.e. $T_t\mathbf{1} = \mathbf{1}$, $t > 0$, it implies that $T_t h = T_t f - \mu(f)\mathbf{1}$ for all $t > 0$. From the assumption (41) and the fact that $\mu(h) = 0$, we deduce that

$$\|T_s h\|_p \leq c e^{-\lambda s}, \quad s > 0.$$

Let $t > 0$ be fixed and let g be a Bernstein function such that $g(0) = 0$. The subordinated semigroup T_t^g satisfies

$$\|T_t^g h\|_p = \left\| \int_0^\infty \eta_t^g(s) T_s h ds \right\|_p \leq \int_0^\infty \eta_t^g(s) \|T_s h\|_p ds.$$

Hence

$$\|T_t^g h\|_p \leq c \int_0^\infty \eta_t^g(s) e^{-\lambda s} ds \leq c e^{-g(\lambda)t}.$$

Because $g(0) = 0$, we have the Markov property of T_t^g , i.e. $T_t^g \mathbf{1} = \mathbf{1}$. We conclude that

$$\|T_t^g h\|_p = \|T_t^g f - \mu(f)\|_p \leq c e^{-g(\lambda)t}.$$

This proves the statement (i).

(ii) The proof of the statement (ii) is similar to that of (i).

The proof of the proposition is completed. \square

We now discuss briefly both cases (i) and (ii) of Proposition 6.1 in some specific situations. In the first case, we recall the connection with the L^2 -Poincaré inequalities. In the second case, we apply our result to functions defined on \mathbb{R}^n for which the support of their Fourier transforms avoids the origin.

Case (i). A classical case for $p = 2$ is $c = \|f - \mu(f)\|_2$. Then the inequality (41) is known to be equivalent to the following classical Poincaré inequality

$$\|f - \mu(f)\|_2^2 \leq \frac{1}{\lambda}(Af, f), \quad f \in \mathcal{D}(A)$$

with the same $\lambda > 0$ as in (41). For instance, this result can be deduced from Proposition 2.1 with $r = \lambda^{-1}$ and β is identically zero with a slight modification of the proof to take into account the average $\mu(f)$. More generally, under the condition (41), we obtain the Poincaré inequality for $g(A)$ from (42) of the form

$$\|f - \mu(f)\|_2^2 \leq \frac{1}{g(\lambda)}(g(A)f, f).$$

Similar results can be reformulated under the assumption (43) with $p = 2$ and $c = \|f\|_2$. For a discussion about how to pass from L^2 to L^p with $c := \|f - \mu(f)\|_p$ for some or all $p \in [1, \infty]$, see [C-G-R].

Case (ii). Now we emphasize that Proposition 6.1 can be applied to an individual function f by considering a simple situation on the Euclidean space \mathbb{R}^n .

Let $f \in L^2(\mathbb{R}^n)$ with its Fourier transform defined by $\mathcal{F}f(x) = \int_{\mathbb{R}^n} f(y) e^{-ixy} dy$. Assume that $\mathcal{F}f = 0$ a.e. on $B(\varepsilon)$ which denotes the Euclidean ball with center 0 and radius $\varepsilon > 0$. We define the heat convolution semigroup on \mathbb{R}^n by $T_t f = p_t * f$ with

$p_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$. For the solution $u(t) = T_t f$ of the heat equation with Cauchy data $u(0) = f \in L^2$, we have an exponential decay. Indeed, by Plancherel's formula we can write

$$\|T_t f\|_2^2 = c_n \int_{B(\varepsilon)^c} e^{-2t|x|^2} |\mathcal{F}f(x)|^2 dx \leq e^{-2t\varepsilon^2} \|f\|_2^2 \quad (45)$$

where $c_n = (2\pi)^{-n}$. Note that the converse also holds true. Indeed, for any $\alpha > 0$ and any $t > 0$ we have

$$c_n e^{-2t\alpha^2} \int_{B(\alpha)} |\mathcal{F}f(x)|^2 dx \leq \|T_t f\|_2^2.$$

Since the assumption on f is

$$\|T_t f\|_2^2 \leq e^{-2t\varepsilon^2} \|f\|_2^2,$$

this yields

$$c_n e^{-2t\alpha^2} \int_{B(\alpha)} |\mathcal{F}f(x)|^2 dx \leq e^{-2t\varepsilon^2} \|f\|_2^2.$$

It implies that $\mathcal{F}f = 0$ a.e. on $B(\alpha)$ for any $\alpha < \varepsilon$ by comparing the growth of the exponential terms on both sides as t goes to infinity. Thus $\mathcal{F}f = 0$ a.e. on $B(\varepsilon)$.

We now apply (ii) of Proposition 6.1 to this specific function f satisfying (45) with $p = 2$, $c = \|f\|_2$ and $\lambda = \varepsilon^2$. For any Bernstein function g , we obtain an exponential decay for the one-parameter family $(T_t^g f)_{t \geq 0}$ of functions of the form

$$\|T_t^g f\|_2 \leq e^{-g(\varepsilon^2)t} \|f\|_2, \quad t > 0.$$

Among many examples of applications of Proposition 6.1 (ii), we can also consider certain sub-Laplacians on non-unimodular groups as generators of semigroups (see, for instance, the $L^2 - L^2$ estimates p.126 of [VSC]).

7 Functions of the Laplacian on \mathbb{R}^n

In this section, we give a direct proof of super-Poincaré inequalities for $g(\Delta)$ when g is any Bernstein function and Δ is the Laplacian on \mathbb{R}^n . The method of proof follows the original idea used in [N] but it does not provide any optimal constants. In fact, the proof is valid well beyond the class of Bernstein functions.

We use the definition of the Fourier transform of $f \in L^1 \cap L^2$ given by $\mathcal{F}f(x) = \int_{\mathbb{R}^n} f(y) e^{-ixy} dy$. We also denote by \mathcal{F} the extension of the Fourier transform on $L^2(\mathbb{R}^n)$. Here, we consider the (positive) Laplacian on \mathbb{R}^n given by $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. The action of the Fourier transform on the Laplacian is described by $\mathcal{F}(\Delta f)(x) = |x|^2 \mathcal{F}f(x)$, $f \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space on \mathbb{R}^n . It leads to the definition of $g(\Delta)$ in terms of the Fourier transform

$$\mathcal{F}(g(\Delta)f)(x) = g(|x|^2) \mathcal{F}f(x), \quad x \in \mathbb{R}^n.$$

The domain of $g(\Delta)$ is given by

$$\mathcal{D}(g(\Delta)) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |g(|x|^2)|^2 |\mathcal{F}f(x)|^2 dx < \infty\}.$$

The following theorem states that the super-Poincaré inequalities (46) and the associated Nash-type inequalities (47) below hold true for $g(\Delta)$ under mild assumptions on g with explicit rate functions compatible with Theorems 1.1 and 1.2. We next put $c_n = (2\pi)^{-n}$ and denote by ω_n the volume of the unit ball of \mathbb{R}^n .

Theorem 7.1 *Let $g : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing function such that $g(0) = 0$. We define the function g^\rightarrow on $[0, \infty)$ by $g^\rightarrow(u) = \sup\{s \geq 0 : g(s) \leq u\} \in [0, \infty]$. Then we have*

1. *For all $f \in \mathcal{D}(g(\Delta)) \cap L^1(\mathbb{R}^n)$ and all $t > 0$,*

$$\|f\|_2^2 \leq t(g(\Delta)f, f) + \tilde{\beta} \left(\frac{1}{g^\rightarrow(t^{-1})} \right) \|f\|_1^2 \quad (46)$$

with $\tilde{\beta}(r) = c_n \omega_n r^{-n/2}$, $r > 0$.

2. *For all $f \in \mathcal{D}(g(\Delta)) \cap L^1(\mathbb{R}^n)$ with $\|f\|_1 \leq 1$,*

$$\|f\|_2^2 \tilde{D}_g(\|f\|_2^2) \leq (g(\Delta)f, f) \quad (47)$$

with $\tilde{D}_g(x) = \sup_{t>0} \left(t - tx^{-1} c_n \omega_n [g^\rightarrow(t)]^{n/2} \right)$, $x > 0$.

Under the assumptions on g of the above theorem, the generalized inverse function g^\rightarrow is always non-decreasing. In addition, if we assume that g is unbounded then the function $g^\rightarrow(u)$ is well defined and finite for all $u \geq 0$. More precisely, if g is an increasing bijection from $[0, \infty)$ to $[0, \infty)$ then $g^\rightarrow = g^{-1}$. The expression of the function $\tilde{\beta}_g(t) := \tilde{\beta} \left(\frac{1}{g^\rightarrow(t^{-1})} \right)$ in (46) and that of the function $\beta_g(t)$ in Theorem 1.1 are similar when g is bijective. If we assume that g is bounded and $\beta(0^+) = \infty$, then $g(\Delta)$ is a bounded operator and $\tilde{\beta}_g(t) = \infty$ when $t \leq 1/\|g\|_\infty$. In that case, the inequality (46) is meaningful only for $t > 1/\|g\|_\infty$. Note that this restriction already appears in Theorem 1.1 when g is bounded.

Proof Let $f \in \mathcal{D}(g(\Delta)) \cap L^1(\mathbb{R}^n)$ and let $t > 0$. By Plancherel's formula,

$$\begin{aligned} \|f\|_2^2 &= c_n \int_{\mathbb{R}^n} |\mathcal{F}f(x)|^2 dx = c_n \int_{\{x \in \mathbb{R}^n : 1 \leq tg(|x|^2)\}} |\mathcal{F}f(x)|^2 dx + c_n \int_{\{x \in \mathbb{R}^n : 1 > tg(|x|^2)\}} |\mathcal{F}f(x)|^2 dx \\ &\leq c_n t \int_{\mathbb{R}^n} g(|x|^2) |\mathcal{F}f(x)|^2 dx + c_n \|\mathcal{F}f\|_\infty^2 V(\{x \in \mathbb{R}^n : g(|x|^2) < t^{-1}\}) \end{aligned}$$

where $V(\Omega)$ is the Lebesgue measure of the set $\Omega \subset \mathbb{R}^n$. Since $g(r) \leq u$ implies $r \leq g^\rightarrow(u)$ and $\|\mathcal{F}f\|_\infty \leq \|f\|_1$ we deduce that

$$\|f\|_2^2 \leq t(g(\Delta)f, f) + c_n \|f\|_1^2 V(\{x \in \mathbb{R}^n : |x| \leq \sqrt{g^\rightarrow(t^{-1})}\}).$$

We finally have

$$\|f\|_2^2 \leq t(g(\Delta)f, f) + \tilde{\beta}_g(t) \|f\|_1^2$$

where $\tilde{\beta}_g(t) := c_n \omega_n [g^\rightarrow(t^{-1})]^{n/2}$ for all $t > 0$. This proves the statement (i) of the theorem.

The second part of the theorem is obtained by applying Proposition 2.2 (i). This completes the proof of the theorem. \square

8 Appendix on the Legendre Transform

In this section, we discuss the properties of D directly from the properties of β independently of the set \mathcal{G} where D and β are related by (16), see Proposition 2.2 and Theorem 1.2. We have a similar discussion when reversing the role of β and D now related by (17), see Proposition 2.2. In applications, the conditions on D and β introduced here are usually satisfied.

Lemma 8.1 *Let β be a non-negative function defined on $(0, \infty)$ and put*

$$D(x) = \sup_{t>0} (t - tx^{-1}\beta(t^{-1})) \in (-\infty, \infty], \quad x > 0. \quad (48)$$

1. *If $\lim_{t \rightarrow 0^+} t\beta(t^{-1}) = 0$ then D is non-negative. This condition is satisfied if β is bounded above at infinity, in particular if β is non-increasing.*
2. *If $\beta(0^+) = \infty$ then $D(x)$ is finite for any $x > 0$. Moreover, the function $x \rightarrow xD(x)$ is convex and non-decreasing on $(0, \infty)$ and D is continuous.*

Proof (i) By assumption $\lim_{t \rightarrow 0^+} t\beta(t^{-1}) = 0$ then $D(x) \geq \lim_{t \rightarrow 0^+} (t - x^{-1}t\beta(t^{-1})) = 0$ for all $x > 0$. So, the function D is non-negative. If β is bounded above at infinity (e.g. non-increasing) and non-negative then clearly $\lim_{t \rightarrow 0^+} t\beta(t^{-1}) = 0$.

(ii) Let $x > 0$ be fixed. Because $\beta(0^+) = \infty$, there exists $t_x > 0$ such that for all $t > t_x$, $x < \beta(t^{-1})$. Thus, $t - tx^{-1}\beta(t^{-1}) < 0$ when $t > t_x$. Now if $0 < t \leq t_x$ then immediately $t - tx^{-1}\beta(t^{-1}) \leq t_x$ since β is non-negative. Therefore $D(x) \leq t_x$ and $D(x)$ is finite for all $x > 0$. To prove (ii), we use the fact that the function $x \mapsto h^*(x) := xD(x) = \sup_{t>0} (tx - t\beta(t^{-1}))$ is clearly convex and non-decreasing on $(0, \infty)$. In particular, h^* and D are continuous. This completes the proof. \square

Now, we study some properties of β in terms of D when these functions are related by (17). The natural conditions to impose on the function D come from the previous lemma. The discussion and the proof are similar to that of Lemma 8.1.

Lemma 8.2 *Let $D : (0, \infty) \rightarrow \mathbb{R}$ be a given function and put*

$$\beta(r) = \sup_{x>0} (x - rxD(x)). \quad (49)$$

1. *If $\lim_{x \rightarrow \infty} D(x) = \infty$ and D is non-negative, then $\beta(r)$ is finite for any $r > 0$, convex, continuous and non-increasing.*
2. *If $\lim_{x \rightarrow 0} xD(x) = 0$ then β is non-negative.*

Note that it is a general fact that the behaviour of $\beta(t)$ as $t \rightarrow 0^+$ determines the behaviour of $D(x)$ as $x \rightarrow \infty$ and conversely. The connection between the formulas (48) and (49) is clarified when introducing the Legendre transform. Indeed, the equation (48) may be rewritten in the form

$$h^*(x) = \sup_{t>0} (tx - h(t))$$

where $h^*(x) := xD(x)$ and $h(t) := t\beta(t^{-1})$. Thus the function h^* is the Legendre transform of h . Note that the transformation $\Phi : h \rightarrow \Phi(h) = h^*$ is an involution on the convex functions. Similarly, the equation (49) may be rewritten in the form

$$h(x) = \sup_{t>0} (tx - h^*(t))$$

and, conversely, h is obtained as the Legendre transform of h^* . In this situation the theory of the Legendre transform applies but not directly. Indeed, in practice the functions β and D of our examples in Section 4 only appear as asymptotics of N-functions. See [R-R, p.6 and p.13] for the definition of N-functions. Below we provide a list of classical couples of N-functions related to our examples described in Section 4.

Let $1 < p, q < \infty$ with $1/p + 1/q = 1$.

1. $(h_1(t), h_1^*(x)) = (t^p/p, x^q/q)$.
2. $(h_2(t), h_2^*(x)) = (e^t - t - 1, (1+x)\log(1+x) - x)$.
3. $(h_3(t), h_3^*(x)) = ((1+t)\log(1+t) - t, e^x - x - 1)$.
4. $h_4(t) = e^{t^p} - 1$, h_4^* : no explicit form.

In case (iv), one can prove that $h_4^*(x) \sim x(\log x)^{1/p}$ as $x \rightarrow \infty$ and that $h_4^*(x) \sim c_q x^q$ as $x \rightarrow 0^+$ with $c_q = (p-1)p^{-q}$.

We now describe the correspondence between the rate functions D and β which appear in applications and also the couples of functions (h, h^*) mentioned above.

1. The function $h_1(t) = c_0 t^p$ is associated with $\beta_1(t) = th_1(t^{-1}) = c_1 t^{-\nu}$ where $\nu = p-1 > 0$ and $D(x) = c_2 x^{q-1}$ with $1/p + 1/q = 1$, i.e. $q = 1 + \frac{1}{p}$. For instance, this situation is realized by the fractional Laplacian Δ^α on the Euclidean space, see Section 4.1. More generally, it is also realized by \mathcal{L}^α where \mathcal{L} is a sum of vector fields satisfying Hörmander's condition on Lie groups of polynomial growth with $\nu = \frac{n}{2\alpha}$, see Section 4.3. In all these cases the rate function D is given by $D(x) = c_3 x^{\frac{2\alpha}{n}}$ for some constant $c_3 > 0$.
2. The function $h_2(t) = e^t - t - 1$ is associated with $\beta_2(t) = th_2(t^{-1}) \sim te^{t^{-1}}$ as $t \rightarrow 0^+$ and $\beta_2(t) \sim (2t)^{-1}$ as $t \rightarrow \infty$. This situation is realized up to multiplicative constants by the Ornstein-Uhlenbeck operator as far as the local behaviour of β_2 (i.e. $t \rightarrow 0^+$) is concerned, see Section 4.4. In that case, $D(x) \sim \log x$ as $x \rightarrow \infty$.
3. The function $h_3(t) = (1+t)\log(1+t) - t$ is associated with $\beta_3(t) = th_3(t^{-1}) \sim \log(1/t)$ as $t \rightarrow 0^+$. Theorem 7.1 provides an example with this behaviour by choosing $g(y) = e^{y/4\pi} - 1$ and the 2-dimensional Laplacian Δ . Then the operator $g(\Delta) = e^{\Delta/4\pi} - I$ satisfies the super-Poincaré inequality (46) with $\beta(t) = \log(1 + \frac{1}{4\pi^2 t})$, $t > 0$. The rate function D of the corresponding Nash-type inequality satisfies $D(x) \sim \frac{4\pi^2}{x} e^{x-1}$ as $x \rightarrow \infty$ and $D(x) \sim \pi^2 x$ as $x \rightarrow 0^+$.
4. Let $1 < p < \infty$. The function $h_4(t) = e^{t^p} - 1$ is associated with $\beta_4(t) = th_4(t^{-1}) \sim te^{t^{-p}}$ as $t \rightarrow 0^+$ and $\beta_4(t) \sim t^{-(p-1)}$ as $t \rightarrow \infty$. Some examples with this behaviour arise from the operators considered in the Riemannian setting of Section 4.2 where p and the exponent $\delta \in (1, 2)$ in the function $V = -\alpha\rho^\delta$, are related by the equation $p = \frac{\delta}{2(\delta-1)}$. In that case, $D(x) \sim (\log x)^{1/p}$ as $x \rightarrow \infty$.

To conclude this discussion, it would be interesting to know whether or not there exists an operator A satisfying a super-Poincaré profile of the form $\beta(t) \sim th_3(1/t) \sim \log(1/t)$ as $t \rightarrow 0^+$ with (Af, f) a possibly Dirichlet form.

Note added in Proof After we have finished this paper, René Schilling informed us, at the early stage of our respective preprints, that he and Jian Wang have, independently, obtained similar results in their paper [Sc-W] now published; although our findings partially overlap, the methods used here and in [Sc-W] are essentially different.

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Note For the paper [W1] the author may be asked directly if other sources are not available.

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