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Patrick Maheux, Ivan Gentil. Super Poincaré and Nash-type inequalities for Subordinated Semigroups. 2011. hal-00709358v1

**HAL Id: hal-00709358**

**<https://hal.science/hal-00709358v1>**

Preprint submitted on 18 Jun 2012 (v1), last revised 20 Oct 2014 (v2)

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# Super Poincaré and Nash-type inequalities for Subordinated Semigroups

Ivan Gentil <sup>\*</sup>, Patrick Maheux <sup>†</sup>

June 16, 2012

## Abstract

We prove that if super Poincaré inequality is satisfied by an infinitesimal generator  $-A$  of a symmetric sub-markovian semigroup then it implies a corresponding super Poincaré inequality for  $-g(A)$  with any Bernstein function  $g$ . We also study the converse statement. We deduce similar results when the assumption of super Poincaré inequality is changed by a Nash-type inequality. In particular, we prove that if  $D$  is a Nash function for  $A$  then  $g \circ D$  is essentially a Nash function for  $g(A)$ . Our results apply to fractional powers of  $A$  and  $\log(I + A)$  generalizing results of [B-M] and [W1]. We provide several examples and settings of applications.

**Key words:** Super Poincaré inequality, Nash-type inequality, Sub-Markovian semigroup, Subordination in the sense of Bochner, Bernstein function, Super Poincaré Profile.

**AMS 2010:** 39B62, 47D60, 26A33, 46T12.

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## 1 Introduction and main results

Let  $(T_t)_{t>0}$  be a symmetric sub-markovian semigroup on  $L^2(X, \mu)$  with  $(X, \mu)$  a  $\sigma$ -finite measure space i.e  $(T_t)_{t>0}$  is a strongly continuous symmetric contraction semigroup on  $L^2$  such that for any  $t > 0$  and  $f \in L^2$ ,

$$0 \leq f \leq 1 \Rightarrow 0 \leq T_t f \leq 1.$$

The semigroup  $(T_t)_{t>0}$  is said markovian if  $T_t(1) = 1$ . This semigroup extends as a contraction semigroup on each  $L^p$ ,  $1 \leq p \leq +\infty$ . The infinitesimal generator  $-A$  on  $L^2(\mu)$  of  $(T_t)_{t>0}$  is defined by

$$-Af = \lim_{t \rightarrow 0^+} \frac{T_t f - f}{t}$$

on the domain

$$\mathcal{D}(A) := \{f \in L^2 : \lim_{t \rightarrow 0^+} \frac{T_t f - f}{t} \text{ exists in } L^2\}.$$

The quadratic form  $(Af, f)$  associated to  $A$  is a Dirichlet form (see for instance [J]). The operator  $A$  is self-adjoint and non-negative with dense domain in  $L^2$ . Let us denote by  $(\cdot, \cdot)$  the inner product on  $L^2(X, \mu)$  and  $\|\cdot\|_p$  the  $L^p$ -norm with respect to  $\mu$ .

We shall say that  $A$  satisfies a super Poincaré inequality if, for any  $f \in \mathcal{D}(A) \cap L^1$ ,

$$\|f\|_2^2 \leq r(Af, f) + \beta(r)\|f\|_1^2, \quad r > 0, \quad (1)$$

holds true for some function  $\beta : (0, +\infty) \rightarrow (0, +\infty)$ . More generally, we say that  $A$  satisfies a  $(r_0, r_1)$ -super Poincaré inequality if (1) holds for  $r \in (r_0, r_1)$  with  $0 \leq r_0 < r_1 \leq +\infty$  (see examples below Theorem 1.2). Note that we can always assume that  $\beta$  is non-increasing by considering what we shall call the *super Poincaré profile*  $\beta_p (\leq \beta)$  of  $A$  defined, for any  $r > 0$ , by

$$\beta_p(r) := \sup\{\|f\|_2^2 - r(Af, f) : f \in \mathcal{D}(A) \cap L^1, \|f\|_1 \leq 1\}. \quad (2)$$

See applications of this notion of profile below Proposition 2.1. The assumption  $\beta$  is non-increasing is not essential for some of our results.

We shall say that  $A$  satisfies a Nash-type inequality if, for any  $f \in \mathcal{D}(A) \cap L^1$ ,

$$\|f\|_2^2 D(\|f\|_2^2) \leq (Af, f), \quad \|f\|_1 \leq 1, \quad (3)$$

holds true for some non-decreasing function  $D : [0, +\infty) \rightarrow [0, +\infty)$ . We shall call  $D$  a Nash function for  $A$ . It is well known that the inequalities (1) and (3) are essentially

equivalent (see Proposition 2.2).

We now briefly recall some definitions and some facts about the subordination of semigroups in the sense of Bochner.

A Bernstein function  $g$  is a smooth function  $g : (0, +\infty) \rightarrow (0, +\infty)$  such that the derivatives  $g^{(n)}$  satisfy, for any  $n \in \mathbb{N}$ ,  $(-1)^{n-1}g^{(n)}(s) \geq 0, s > 0$ . There exists a convolution semigroup of sub-probability measures  $(\nu_t^g)_{t>0}$  on  $(0, +\infty)$  with density  $(\eta_t^g)_{t>0}$  with respect to the Lebesgue measure  $ds$  such that the Laplace transform of  $\nu_t^g$  is given by:

$$\int_0^{+\infty} e^{-sx} d\nu_t^g(s) = \int_0^{+\infty} e^{-sx} \eta_t^g(s) ds = e^{-tg(x)}, \quad x > 0. \quad (4)$$

There is one-to-one correspondence between  $g$  and  $(\eta_t^g)_{t>0}$ , see ([J] p.177, [B-F]).

Despite the fact that we shall not use this representation, we recall that all Bernstein functions  $g$  can be written by Lévy-Khintchine formula

$$g(x) = a + bx + \int_0^\infty (1 - e^{-\lambda x}) d\nu(\lambda) = a + bx + J(x) \quad (5)$$

with  $a, b \geq 0$  and  $\nu$  a positive measure on  $(0, +\infty)$  such that  $\int_0^\infty \frac{\lambda}{1+\lambda} d\nu(\lambda) < \infty$ . The triplet  $(a, b, \nu)$  is uniquely determined by  $g$  (See Theorem 3.9.4 in [J] p.174). We have  $a = g(0) = 0$  if and only if  $\nu_t^g$  is a probability measure for any  $t > 0$ . For instance with  $a = b = 0$ , the Lévy measure associated to  $g(x) = x^\alpha$  with  $\alpha \in (0, 1)$  (resp.  $g(x) = \ln(1+x)$ ) is given by  $d\nu(\lambda) = \frac{\alpha}{\Gamma(1-\alpha)} \lambda^{-1-\alpha} d\lambda$  (resp.  $d\nu(\lambda) = \frac{e^{-\lambda}}{\lambda} d\lambda$ ).

Now, let  $(E_\lambda)_{\lambda \in [0, +\infty)}$  be the spectral resolution of the non-negative self-adjoint operator  $A$  and  $\Psi : [0, +\infty) \rightarrow \mathbb{R}$  any measurable function. The operator  $\Psi(A)$  is defined on  $L^2$  by the formula

$$\Psi(A)f = \int_0^{+\infty} \Psi(\lambda) dE(\lambda)f$$

with domain

$$\mathcal{D}(\Psi(A)) = \{f \in L^2 : \int_0^{+\infty} |\Psi(\lambda)|^2 d(E(\lambda)f, f) < \infty\}.$$

We shall set  $(\Psi(A)f, f) = +\infty$  when  $f \notin \mathcal{D}(\Psi(A))$ . When  $\Psi$  has real non-negative values, the operator  $\Psi(A)$  is non-negative and self-adjoint on  $L^2$ . It defines a symmetric semigroup of contractions on  $L^2$  by the spectral formula:

$$e^{-t\Psi(A)}f = \int_0^{+\infty} e^{-t\Psi(\lambda)} dE(\lambda)f, \quad f \in L^2.$$

Let us denote  $e^{-t\Psi(A)}$  by  $T_t^\Psi$  for a fixed operator  $A$ . When  $\Psi = g$  is a Bernstein function, it can be easily shown that the semigroup  $(T_t^g)_{t>0}$  satisfies also the so-called subordination formula

$$T_t^g = \int_0^{+\infty} T_s d\nu_t^g(s) = \int_0^{+\infty} T_s \eta_t^g(s) ds. \quad (6)$$

Among many examples of Bernstein functions, we are interested at least by the following ones:

- i. The fractional subordinator (one-sided  $\alpha$ -stable process):  $g(x) = x^\alpha$ ,  $x > 0$ , ( $0 < \alpha < 1$ ). Then  $g(A) = A^\alpha$ .
- ii. The Gamma subordinator:  $g(x) = \log(1 + x)$ . Then  $g(A) = \log(I + A)$  where  $I$  denotes the identity operator on  $L^2$ .
- iii. The generalization of the previous example:  $g(x) = [\log(1 + x^\alpha)]^\gamma$  with  $0 < \alpha, \gamma \leq 1$ . So,  $g(A) = [\log(I + A^\alpha)]^\gamma$ . When  $\gamma = 1$ ,  $g$  is called the geometric  $\alpha$ -stable subordinator (see [S-S-V]).
- iv. Elementary functions  $g_\lambda(x) = 1 - e^{-\lambda x}$ ,  $\lambda > 0$ . Then  $g_\lambda(A) = I - T_\lambda$ .

For a recent study of the case (ii): see [S-S-V].

We have the inclusion  $\mathcal{D}(A) \subset \mathcal{D}(g(A))$  and, for any  $f \in \mathcal{D}(A)$ ,

$$g(A)f = af + bAf + \int_0^\infty (f - T_\lambda f) d\nu(\lambda). \quad (7)$$

(see [Sc-S-V]. Example 11.6 where our  $A$  is their  $-A$ ). We set  $J(A)f = \int_0^\infty (f - T_\lambda f) d\nu(\lambda)$ .

Throughout all the paper, we shall always assume implicitly that the functions  $f$  are in the domain of the operator under consideration. If not, we set  $(g(A)f, f) = +\infty$ .

We now state the main results of this paper.

The first main result says that if super Poincaré inequality is satisfied by an infinitesimal generator  $-A$  of a symmetric sub-markovian semigroup then it implies a corresponding super Poincaré inequality for  $-g(A)$  with any Bernstein function  $g$ . More precisely, we have:

**Theorem 1.1** *Let  $(T_t)_{t>0}$  be a symmetric sub-markovian semigroup with  $-A$  as infinitesimal generator. Assume that  $A$  satisfies the following super Poincaré inequality*

$$\|f\|_2^2 \leq r(Af, f) + \beta(r)\|f\|_1^2, \quad r > 0, \quad (8)$$

for some function  $\beta : (0, +\infty) \rightarrow (0, +\infty)$ . Then for any Bernstein function  $g$ , the infinitesimal generator  $g(A)$  satisfies the following  $(r_0, r_1)$ -super Poincaré inequality

$$\|f\|_2^2 \leq r(g(A)f, f) + \beta_g(r)\|f\|_1^2, \quad (9)$$

where  $\beta_g(r) = \beta\left(\frac{1}{g^{-1}(1/r)}\right)$ ,  $r \in (r_0, r_1)$  with  $r_0 = \frac{1}{g(+\infty)}$  and  $r_1 = \frac{1}{g(0^+)}$ .

Note that by (5),  $g$  is either strictly increasing and  $g^{-1}$  is defined from  $(g(0^+), g(+\infty))$  into  $(0, +\infty)$  or  $g$  is constant and  $(r_0, r_1)$  is empty. The Bernstein function  $g$  is bounded if and only if  $b = 0$  and  $\nu$  is a bounded measure (see [J] p.174). If  $g$  is not bounded i.e.  $g(+\infty) = +\infty$  then we have  $r_0 = 0$ . If  $g(0^+) = a = 0$  then  $r_1 = +\infty$ . But if  $a = g(0^+) > 0$  then we have the obvious spectral gap inequality:

$$\|f\|_2^2 \leq \frac{1}{a}(g(A)f, f).$$

In other words,

$$\|f\|_2^2 \leq r(g(A)f, f) + \beta_g(r)\|f\|_1^2 \quad \text{with} \quad \beta_g(r) = 0, \quad r \in [1/a, +\infty).$$

So, in any case we can consider that  $r_1 = +\infty$ .

The second main result is similar with Nash-type inequality assumption. Moreover, we prove that if  $D$  is a Nash function for  $A$  then  $g \circ D$  is essentially a Nash function for  $g(A)$ . More precisely, we have:

**Theorem 1.2** *Let  $(T_t)_{t>0}$  be a symmetric sub-markovian semigroup with  $-A$  as infinitesimal generator and  $g$  be any Bernstein function. Let  $D$  be a function from  $[0, +\infty)$  to  $[0, +\infty)$  and set*

$$\beta(r) = \sup_{x>0} (x - rxD(x)) \in (-\infty, +\infty].$$

*Assume that  $\beta(r)$  is finite for any  $r > 0$ . If the following Nash-type inequality holds true for  $A$ ,*

$$\|f\|_2^2 D(\|f\|_2^2) \leq (Af, f), \quad \|f\|_1 \leq 1. \quad (10)$$

*i. Then  $g(A)$  satisfies a Nash-type inequality of the form*

$$\|f\|_2^2 D_g(\|f\|_2^2) \leq (g(A)f, f), \quad \|f\|_1 \leq 1, \quad (11)$$

*where  $D_g(x) = \sup_{r \in (r_0, r_1)} \left( \frac{1}{r} - \frac{1}{rx} \beta \left( \frac{1}{g^{-1}(\frac{1}{r})} \right) \right)$ ,  $x > 0$  and  $(r_0, r_1) = \left( \frac{1}{g(+\infty)}, \frac{1}{g(0^+)} \right)$ .*

*ii. Moreover, assume that  $g$  is a bijection from  $[0, +\infty)$  to itself and  $\beta : (0, +\infty) \rightarrow (0, +\infty)$  is a decreasing differentiable bijection. Then we have*

$$\|f\|_2^2 \sup_{\rho>1} (1 - \rho^{-1}) (g \circ D)(\rho^{-1} \|f\|_2^2) \leq (g(A)f, f), \quad \|f\|_1 \leq 1, \quad (12)$$

*and, for any  $x > 0$ ,*

$$\sup_{\rho>1} (1 - \rho^{-1}) (g \circ D)(\rho^{-1} x) \leq D_g(x) \leq g \circ D(x),$$

*where  $g \circ D$  denotes the composition of the functions  $g$  and  $D$ .*

For a discussion of properties of  $D$  as a function of  $\beta$  and conversely  $\beta$  as a function of  $D$ , independently of the operator  $A$ : see Appendix (Section 8).

Our approach simplifies and generalizes the proofs of the main results of [B-M] and [W1]. The inequality (12) also clarifies the constants obtained in [B-M] for the fractional powers  $A^\alpha$ . With the same arguments of proof, we can replace  $\|f\|_1$  in Theorem 1.1 and 1.2 by any non-negative functional  $\Phi(f)$  satisfying  $\Phi(T_t f) \leq \Phi(f)$ ,  $t > 0$ . We can also generalize our results in Hilbert spaces as in Wang's paper [W1]. But we shall not give details.

In particular, our results apply to several important examples. We provide a short list of couples  $(g(A), \beta_g)$  just below:

- i. *Fractional powers.* If  $g(A) = A^\alpha$  with  $0 < \alpha < 1$  then  $\beta_\alpha(r) = \beta(r^{\frac{1}{\alpha}})$ ,  $r > 0$ , (improving constants given in [W1]).
- ii. *Gamma subordinator.* If  $g(A) = \log(I + A)$  then  $\beta_{\log}(r) = \beta((e^{1/r} - 1)^{-1})$ ,  $r > 0$ .
- iii. *Generalized geometric stable subordinators.* If  $g(A) = [\log(I + A^\alpha)]^\gamma$  with  $0 < \alpha, \gamma \leq 1$  then

$$\beta_g(r) = \beta \left( \left[ e^{(\frac{1}{r})^{\frac{1}{\alpha}}} - 1 \right]^{\frac{-1}{\alpha}} \right), \quad r > 0.$$

iv. *Random walks.* If  $g(A) = I - T_\lambda = I - e^{-\lambda A}$  with  $\lambda > 0$  then

$$\beta^{(\lambda)}(r) = \beta \left( \frac{\lambda}{\log(1 + \frac{1}{r-1})} \right), \quad r > 1.$$

The same super Poincaré inequality for the generator  $B_\lambda = I - T_\lambda$  ( $\lambda > 0$ ) can also be deduced by a different route: see (iii) of Proposition 2.1 below.

Many others Bernstein functions can be considered. For instance,  $g(x) = \frac{x}{x+\lambda}$ ,  $g(x) = \sqrt{x} \arctan(\frac{\lambda}{\sqrt{x}})$  with  $\lambda > 0$  (See p.204 of [J]). Already with these functions, we can construct infinitely many Bernstein functions because if  $g$  and  $h$  are Bernstein functions then  $g \circ h$  is a Bernstein function (Cor. 3.9.36 of [J]). For instance,  $g_1(x) = [\log(1+x)]^\alpha$ ,  $g_2(x) = \log(1+x^\alpha)$ ,  $g_3(x) = [\log(1+x^\alpha)]^{\alpha'}$  with  $\alpha, \alpha' \in (0,1)$  (see Chapter 15 of the recent book [Sc-S-V] for a long list of examples of Bernstein functions).

The paper is organized as follows:

In Section 1, we describe the setting of our study and we state the main theorems.

In Section 2, we prove the main theorems of Section 1. More precisely:

In Section 2.1, we first recall that the super Poincaré inequality for  $A$  is equivalent to the decay for the corresponding semigroup  $(T_t)_{t>0}$ . We also show with (iii) of Proposition 2.1 that the decay of the semigroup is nothing else than the super Poincaré inequality for  $I - T_t$ . This is exactly the inequality obtained by Theorem 1.1 for elementary functions  $g(x) = 1 - e^{-tx}$ . We apply this fact to determine the super Poincaré *profile* of the generator  $I - e^{-\lambda \Delta}$  with  $\Delta$  the Laplacian on  $\mathbb{R}^n$ .

Also in Section 2.1, we recall that a Nash-type inequality is essentially an optimized version of a super Poincaré inequality. This allows us to deduce Theorem 1.2 from Theorem 1.1.

Sections 2.2 and 2.3 are devoted respectively to the proof of Theorem 1.1 and 1.2 for  $g(A)$  with  $g$  a Bernstein function using the preparatory Section 2.1.

In Section 3, we briefly apply our results to study the eventual ultracontractivity property of subordinated semigroups.

In Section 4, we provide several examples of settings where our results apply: 1) the Laplacian on the Euclidean space, 2) the Laplace-Beltrami operator on some complete Riemannian manifolds, 3) some hypoelliptic operators on Lie groups and 4) the Ornstein-Uhlenbeck operator on the Euclidean space.

In Section 5, we study the same results as in Theorem 1.1 and 1.2 concerning Nash-type and super Poincaré inequalities but for  $\Psi(A)$  with  $\Psi$  convex. For these two type of inequalities, we use spectral representation of the generator. From these results, we deduce converse implications of Theorem 1.1 and 1.2 by noting that the inverse of a Bernstein function is a convex function.

In Section 6, we revisit the spectral gap in  $L^p$  for  $g(A)$  using the approach by subordination as in Theorem 1.1.

In Section 7, we prove super Poincaré inequalities for  $g(\Delta)$  with  $\Delta$  the Laplacian on  $\mathbb{R}^n$  for a larger class of functions than Bernstein functions. Our tool is Fourier analysis as used in the original paper by J.Nash [N]. The afferent Nash-type inequalities can be deduced. The results are similar to Theorem 1.1 and 1.2 .

We conclude this paper by an Appendix in Section 8. It concerns the Legendre transform which is underlying in the equivalence between Nash-type inequalities and super Poincaré inequalities through the functions  $\beta$  and  $D$ . We weaken the usual conditions on  $\beta$  and  $D$  of the N-functions theory, (see [R-R] p.13), more adapted to our situation. We provide examples used in Section 4.

## 2 Proof of main Theorems

### 2.1 Super Poincaré inequality versus semigroup decay and Nash-type inequality

We recall some known result used in the proof of Theorem 1.1 and 1.2.

For the proof of Theorem 1.1, we use in a crucial way the following result of F-Y. Wang namely the equivalence between Super Poincaré inequality for  $A$  and the exponential decay of the associated semigroup  $(T_t)_{t>0}$  (see [W2] p.230 or [W4] Lemma 3.3.5, see also [W3] p.3 and [W4] p.50 for extended results). This is the analogue of the equivalence between the usual exponential decay of a semigroup and Poincaré inequality. We recall the proof of this proposition for completeness and, additionally, we show (iii) that is the exponential decay of  $T_t$  turns out to be the super Poincaré inequality for the operator  $I - T_t$ . This operator is related to the elementary Bernstein function  $g_t(x) = 1 - e^{-tx}$ .

**Proposition 2.1** *Let  $(T_t)_{t>0}$  be a symmetric sub-markovian semigroup with infinitesimal generator  $-A$ . Let  $\beta : (0, +\infty) \rightarrow (0, +\infty)$  be some function. Then the three following inequalities are equivalent:*

*i. For any  $f \in \mathcal{D}(A) \cap L^1$  and  $r > 0$ ,*

$$\|f\|_2^2 \leq r(Af, f) + \beta(r)\|f\|_1^2. \quad (13)$$

*ii. For any  $f \in L^2 \cap L^1$ ,  $t > 0$  and  $r > 0$ ,*

$$\|T_t f\|_2^2 \leq e^{-2t/r}\|f\|_2^2 + (1 - e^{-2t/r})\beta(r)\|f\|_1^2. \quad (14)$$

*iii. For any  $f \in L^2 \cap L^1$ ,  $t > 0$  and  $r > 1$ ,*

$$\|f\|_2^2 \leq r((I - T_t)f, f) + \beta\left(\frac{t}{\log(1 + \frac{1}{r-1})}\right)\|f\|_1^2. \quad (15)$$

The exponential  $e^{-2t/r}$  in (14) is suitable to deal with Laplace transforms (4) and this is the key point of our paper. This allows us to easily transfer (14) from  $A$  to  $g(A)$ .

During the proof, we can notice that the equivalence between (13) and (14) holds for any fixed  $a = r > 0$  and fixed  $b = \beta(r) \geq 0$ . In particular, if (13) holds on some interval



$(r_0, r_1)$  then (14) also holds on the same interval  $(r_0, r_1)$  and conversely.

The inequality (15) corresponds exactly to (9) for the elementary functions  $g_\lambda(x) = 1 - e^{-\lambda x}$  ( $t = \lambda$ ) for any  $\lambda > 0$ . The equivalence between (13) and (15) is particularly interesting in terms of relationships between the super Poincaré profiles for the generators  $A$  and  $B_\lambda = I - T_\lambda$  for any fixed  $\lambda > 0$  (see (2) above for the definition of the profile). If  $\beta_p(s), s > 0$  is the super Poincaré profile of  $A$  and  $\gamma_p^{(\lambda)}(r), r > 1$  is the super Poincaré profile of  $B_\lambda$  then they correspond by the formulas

$$\gamma_p^{(\lambda)}(r) = \beta_p \left( \frac{\lambda}{\log(1 + \frac{1}{r-1})} \right), \quad r > 1$$

or equivalently

$$\beta_p(s) = \gamma_p^{(\lambda)} \left( 1 + (e^{\lambda/s} - 1)^{-1} \right), \quad s > 0.$$

For instance in the Euclidean setting, the optimal Nash inequality (21) below provides the super Poincaré profile for the Laplacian  $\Delta$ , namely  $\beta_p(s) = C_n s^{-n/2}$  with the best constant  $C_n$  given by (22). Thus, the super Poincaré profile of  $B_\lambda$  is explicit and given by

$$\gamma_p^{(\lambda)}(r) = C_n \lambda^{-n/2} \left( \log \left[ 1 + \frac{1}{r-1} \right] \right)^{n/2}, \quad r > 1.$$

We have the following interpretation in terms of *random walks*.

For fixed  $\lambda > 0$ , the kernel  $h_\lambda$  of the operator  $T_\lambda$  can be seen as a probability transition of a discrete random walk  $(X_k)_k$  on  $\mathbb{R}^n$  given by  $\mathbb{P}(X_{k+1} = x, X_k = y) = h_\lambda(x - y) = \frac{1}{(4\pi\lambda)^{n/2}} \exp(-\frac{|x-y|^2}{4\lambda})$ ,  $x, y \in \mathbb{R}^n$  and  $k \in \mathbb{N} \cup \{0\}$ . The operator  $B_\lambda = I - T_\lambda$  is the generator of the continuous-time Markov semigroup  $Q_t^{(\lambda)} = e^{-tB_\lambda} = e^{-t} \sum_{k \geq 0} \frac{t^k}{k!} T_{k\lambda}$  obtained by convolution with the following probability transition

$$q_t^{(\lambda)} = e^{-t} \sum_{k \geq 0} \frac{t^k}{k!} h_{k\lambda}$$

where  $h_0 = \delta_0$  ( $\delta_a$  is the Dirac mass at  $a \in [0, +\infty)$ ). This semigroup  $Q_t^{(\lambda)}$  is subordinated to the heat semigroup  $e^{-t\Delta}$  by the Poisson semigroup with jumps of size  $\lambda$  defined on  $[0, +\infty)$  by  $\nu_t = \sum_{k \geq 0} \frac{t^k}{k!} e^{-t} \delta_{k\lambda}$  in (6) (see [J] p.180).

**Proof:** *Equivalence between (i) and (ii).* Let  $H(t) = e^{2t/r} \|T_t f\|_2^2$  for  $t > 0$ , fixed  $r > 0$  and  $f \in \mathcal{D}(A) \cap L^1$ . We have

$$H(t) - H(0) = \int_0^t H'(u) du = \int_0^t 2e^{2u/r} \left( \frac{1}{r} \|T_u f\|_2^2 - (AT_u f, T_u f) \right) du.$$

By applying (13) to  $T_u f$  and because  $T_u$  is a contraction on  $L^1$ , we deduce

$$H(t) - H(0) \leq \frac{2}{r} \beta(r) \|f\|_1^2 \left( \int_0^t e^{2u/r} du \right).$$

This proves (14) for  $f \in \mathcal{D}(A) \cap L^1$ . For the general case, let  $f \in L^2 \cap L^1$  then there exists  $f_u$  ( $u > 0$ ) such that  $f_u \in \mathcal{D}(A)$  and  $f_u$  converges to  $f$  in  $L^1$  and  $L^2$  as  $u \rightarrow 0^+$  (e.g.  $f_u = \frac{1}{u} \int_0^u T_s f ds$ ).

Conversely, let  $r > 0$  be fixed and  $f \in \mathcal{D}(A) \cap L^1$ . The inequality (14) can be rewritten as

$$\frac{\|T_t f\|_2^2 - \|f\|_2^2}{2t} \leq \left( \frac{e^{-2t/r} - 1}{2t} \right) \|f\|_2^2 + \beta(r) \|f\|_1^2 \left( \frac{1 - e^{-2t/r}}{2t} \right).$$

We conclude (13) by taking the limit as  $t$  goes to 0.

*Equivalence between (ii) and (iii).* Assume that (ii) holds i.e.

$$(T_t f, f) = \|T_{t/2} f\|_2^2 \leq e^{-t/r} \|f\|_2^2 + (1 - e^{-t/r}) \beta(r) \|f\|_1^2$$

for any  $t, r > 0$ . It is equivalent to

$$(1 - e^{-t/r}) \|f\|_2^2 \leq (f - T_t f, f) + (1 - e^{-t/r}) \beta(r) \|f\|_1^2.$$

Let  $g(x) = g_t(x) = 1 - e^{-tx}$ . The last inequality reads as

$$\|f\|_2^2 \leq \frac{1}{g_t(1/r)} (g_t(A)f, f) + \beta(r) \|f\|_1^2.$$

Fix  $t > 0$ . Let  $\rho > 1$ , choose  $r > 0$  such that  $\rho = \frac{1}{g_t(1/r)}$  i.e.  $r = \frac{1}{g_t^{-1}(1/\rho)} = \frac{t}{\log(1 + \frac{1}{\rho-1})}$ . This yields  $(1, \infty)$ -super Poincaré (15) for the operator  $I - T_t$  as expected. The converse is clear. ■

Now we recall that super Poincaré and Nash-type inequalities are essentially equivalent under natural conditions on  $\beta$  in (1) and on  $D$  in (3). This result is more or less well known but we formulate the relations between  $\beta$  and  $D$  implicitly in terms of Legendre transforms (for a detailed discussion: see appendix Section 8).

**Proposition 2.2** *Let  $A$  be a non-negative self-adjoint operator on  $L^2$ .*

*i. Assume that the following super Poincaré inequality holds true*

$$\|f\|_2^2 \leq r(Af, f) + \beta(r) \|f\|_1^2, \quad f \in \mathcal{D}(A), r > 0. \quad (16)$$

*for some function  $\beta : (0, +\infty) \rightarrow (0, +\infty)$ . Then*

$$\|f\|_2^2 D(\|f\|_2^2) \leq (Af, f), \quad f \in \mathcal{D}(A), f \neq 0, \|f\|_1 \leq 1 \quad (17)$$

*where  $D(x) = \sup_{t>0} \left( t - \frac{t\beta(1/t)}{x} \right) \in (-\infty, +\infty]$ ,  $x > 0$ . The function  $D$  is non-decreasing, finite on the set  $(0, \sup \mathcal{F})$  where  $\mathcal{F} = \{\|f\|_2^2, f \in \mathcal{D}(A) \cap L^1, \|f\|_1 \leq 1\}$  and  $D(+\infty) = +\infty$ .*

*ii. Conversely, suppose Nash-type inequality (17) holds true for  $A$  and for some function  $D : [0, +\infty) \rightarrow \mathbb{R}$ . Set*

$$\beta(r) = \sup_{x>0} (x - rx D(x)) \in (-\infty, +\infty]. \quad (18)$$

*Then super Poincaré inequality (16) is satisfied.*

This proposition is in the spirit of Theorem 3.1 and Section 5 of [W2], see also Proposition 3.3.16 of [W4]. A much closer formulation can be found in [Bi-M], Proposition 2.1.

Note that if  $\mathcal{F}$  of Proposition 2.2 is unbounded above as a subset of  $\mathbb{R}$  then  $D(x)$  is finite for any  $x > 0$ . For many examples,  $\beta$  is a non-negative non-increasing function and satisfies  $\beta(0^+) = +\infty$ ,  $\lim_{t \rightarrow 0^+} t\beta(1/t) = 0$  which implies that  $D(x)$  is finite for any  $x > 0$ , non-negative, continuous and non-decreasing (see appendix Section 8 for details). Note also that we can always consider in (17) that  $D$  is non-negative by replacing  $D$  by  $D_+ = \sup(0, D)$  because  $A$  is a non-negative operator.

**Remark 2.3** *i. Usually Nash-type inequality is written in the following form*

$$\Theta(\|f\|_2^2) \leq (Af, f), \quad \|f\|_1 \leq 1$$

*(see [C]) but the equivalent expression (17) is more appropriate to deal with the Bernstein functions  $g(x) = x^\alpha$ ,  $\alpha \in (0, 1)$  as shown in [B-M] and more generally for any Bernstein functions by Theorem 1.2 above.*

- ii. In the second statement, the assumption on the functional  $(Af, f)$  can certainly be relaxed because only the existence of the function  $\beta$  (which depends of this functional) is crucial for the proof.*
- iii. In some cases, it appears that we can pass from  $D$  to  $\beta$  and  $\beta$  to  $D$  with no loss (up to multiplicative constants). For instance with the classical example  $D(x) = cx^\gamma$ ,  $\gamma > 0$  (see appendix Section 8). In general, this is not the case.*

**Proof:** (i) Assume super Poincaré inequality (16) holds true. For any  $\|f\|_1 \leq 1$ ,  $f \neq 0$ ,  $f \in \mathcal{D}(A)$  and any  $r > 0$ , we easily deduce

$$\|f\|_2^2 \left( \frac{1}{r} - \frac{\beta(r)}{r\|f\|_2^2} \right) \leq (Af, f).$$

Taking the supremum over  $r > 0$ , we conclude (17). Note that  $D$  is automatically finite on the subset  $\mathcal{F} \setminus \{0\}$  of  $\mathbb{R}$  since in that case  $(Af, f) < +\infty$  for any  $f \in \mathcal{D}(A)$ . On one hand, the set  $\mathcal{F}$  is not empty since it contains 0. On the other hand, if  $f \in \mathcal{F}$  and  $\lambda \in (0, 1)$  then  $\lambda f \in \mathcal{F}$ . Hence  $(0, \sup \mathcal{F}) \subset \mathcal{F}$  and  $D$  is finite on  $(0, \sup \mathcal{F})$ . It is easily proved that  $D$  is non-decreasing using the fact that  $\beta > 0$ . Moreover, for any  $x, t > 0$ , we have

$$D(x) \geq t - \frac{t\beta(1/t)}{x}.$$

Therefore,  $\liminf_{x \rightarrow +\infty} D(x) \geq t$  for any  $t > 0$  that implies  $\lim_{x \rightarrow +\infty} D(x) = +\infty$ .

(ii) By definition of  $\beta$ , one has for any  $x, r > 0$ ,

$$\frac{x}{r} - \frac{\beta(r)}{r} \leq xD(x)$$

Let  $x = \|f\|_2^2$  (with  $f \neq 0$ ). So, for fixed  $r > 0$ ,

$$\frac{\|f\|_2^2}{r} - \frac{1}{r}\beta(r) \leq \|f\|_2^2 D(\|f\|_2^2).$$

By (17), the last term is bounded by  $(Af, f)$  when  $\|f\|_1 \leq 1$ . Hence,

$$\frac{\|f\|_2^2}{r} - \frac{1}{r}\beta(r) \leq (Af, f).$$

which is super Poincaré inequality (16). This concludes the proof of the proposition. ■

Note that when  $\beta(r) = +\infty$  for some  $r > 0$  then (16) is trivially satisfied and the proof is also valid. For properties of  $\beta$  defined by (18): see appendix Section 8.

## 2.2 Proof of Theorem 1.1

We suppose that  $g$  is non-constant (if not there is nothing to prove since  $(r_0, r_1) = \emptyset$ ). Assume super Poincaré inequality (8) holds true. By Proposition 2.1, the inequality (14) is satisfied. Now by symmetry and semigroup property of  $(T_t)_{t>0}$ , this inequality (14) can be written as

$$\|T_t f\|_2^2 = (T_{2t} f, f) \leq e^{-2t/r} \|f\|_2^2 + \beta(r) \|f\|_1^2 (1 - e^{-2t/r}), \quad t > 0.$$

Let  $s > 0$  and set  $t = s/2$ . We deduce for any  $r > 0$  and  $s > 0$ ,

$$(T_s f, f) \|f\|_2^2 \leq e^{-s/r} \|f\|_2^2 + \beta(r) \|f\|_1^2 (1 - e^{-s/r}).$$

By the subordination formula (6) and Fubini, we get for any  $t, r > 0$ ,  $f \in L^1 \cap L^2$  and any Bernstein function  $g$ ,

$$\begin{aligned} (T_t^g f, f) &= \int_0^{+\infty} (T_s f, f) d\nu_t^g(s) \leq \left( \int_0^{+\infty} e^{-s/r} d\nu_t^g(s) \right) \|f\|_2^2 \\ &\quad + \beta(r) \|f\|_1^2 \left( \int_0^{+\infty} (1 - e^{-s/r}) d\nu_t^g(s) \right). \end{aligned}$$

By the Laplace transform of the sub-probability  $\nu_t^g$ , we get

$$(T_t^g f, f) \leq e^{-tg(1/r)} \|f\|_2^2 + \beta(r) \|f\|_1^2 (1 - e^{-tg(1/r)}), \quad t > 0, r > 0.$$

Changing  $t$  by  $2t$  and using symmetry and semigroup properties of  $(T_t^g)$ , we obtain

$$\|T_t^g f\|_2^2 \leq e^{-2tg(1/r)} \|f\|_2^2 + \beta(r) \|f\|_1^2 (1 - e^{-2tg(1/r)}), \quad t > 0, r > 0.$$

Now let  $\rho \in (r_0, r_1) := (\frac{1}{g(+\infty)}, \frac{1}{g(0^+)})$ . Since  $r \rightarrow \frac{1}{g(1/r)}$  is a bijection from  $(0, +\infty)$  onto  $(r_0, r_1)$ , there exists a (unique)  $r > 0$  such that  $\rho = \frac{1}{g(1/r)}$  i.e.  $r = \frac{1}{g^{-1}(1/\rho)}$  and

$$\|T_t^g f\|_2^2 \leq e^{-2t/\rho} \|f\|_2^2 + \beta \left( \frac{1}{g^{-1}(1/\rho)} \right) \|f\|_1^2 (1 - e^{-2t/\rho}).$$

We conclude (9) by applying (ii)  $\Rightarrow$  (i) of Proposition 2.1 with  $g(A)$ . Theorem 1.1 is proved. ■

Note that we do not need the existence of the density of the measures  $\nu_t^g$  nor additional properties of the function  $\beta$ .

An attempt have been made by the authors to use Lévy-Khintchine formula (7) with (iii) of Proposition 2.1 to proved Theorem 1.1 and 1.2. But the authors fails in this task. Furthermore, our method shows that this approach can be avoided by direct subordination.

## 2.3 Proof of Theorem 1.2

*Proof of (i).* We assume that  $A$  satisfies Nash-type inequality (10). By (ii) of Proposition 2.2 and the definition of  $\beta$ , we get super Poincaré inequality: for any  $f \in \mathcal{D}(A) \cap L^1$ ,

$$\|f\|_2^2 \leq r(Af, f) + \beta(r)\|f\|_1^2, \quad r > 0,$$

We now apply Theorem 1.1 and deduce super Poincaré inequality (9) for  $g(A)$  i.e.

$$\|f\|_2^2 \leq r(g(A)f, f) + \beta_g(r)\|f\|_1^2,$$

with  $\beta_g(r) = \beta\left(\frac{1}{g^{-1}(1/r)}\right)$ ,  $r \in (r_0, r_1) = \left(\frac{1}{g(+\infty)}, \frac{1}{g(0^+)}\right)$ . Now, we conclude by applying (i) of Proposition 2.2.

*Proof of (ii).*

From the next lemma which compares  $D_g$  and  $g \circ D$  and (i), we immediately deduce the inequality (12).

**Lemma 2.1** *Let  $g : (0, +\infty) \rightarrow (0, +\infty)$  be a bijective continuous increasing concave function (e.g. bijective Bernstein function),  $D$  and  $D_g$  defined as in Theorem 1.2 with  $\beta$  a decreasing differentiable bijection from  $(0, +\infty)$  to itself. Then for any  $x > 0$  and  $\rho > 1$ ,*

$$(1 - \rho^{-1})(g \circ D)(\rho^{-1}x) \leq D_g(x) \leq g \circ D(x).$$

**Proof:** To simplify our discussion, we set  $V(t) := \beta(1/t)$ ,  $V_g(t) := \beta_g(1/t) = \beta\left(\frac{1}{g^{-1}(t)}\right)$  for  $t > 0$ . So,  $D(x) = \sup_{t>0} \left(t - \frac{t}{x}V(t)\right)$  and  $D_g(x) = \sup_{t>0} \left(t - \frac{t}{x}V_g(t)\right)$  for  $x > 0$ . As a consequence of the assumptions  $\beta(0^+) = +\infty$ , we have that  $D$  and  $D_g$  are well defined and finite on  $(0, +\infty)$ .

Let  $u > 0$ . Since  $g$  is a bijection from  $(0, +\infty)$  to itself, there exists a unique  $t > 0$  such that  $\frac{1}{u} = \frac{1}{g^{-1}(t)}$  i.e  $t = g(u)$ . Thus  $D_g$  can be written as

$$D_g(x) = \sup_{u>0} g(u) \left(1 - \frac{V(u)}{x}\right).$$

Since  $D = D_{id}$  and by continuity of  $g$ , we get

$$g \circ D(x) = \sup_{\{u>0:V(u)\leq x\}} g\left(u \left[1 - \frac{V(u)}{x}\right]\right).$$

Let  $a = 1 - V(u)/x$ . Because  $V \geq 0$ , it is sufficient to consider  $a \in (0, 1)$ . By concavity of  $g$  and  $g(0) = 0$ , we have  $ag(u) = ag(u) + (1 - a)g(0) \leq g(au)$ . Thus we conclude  $D_g(x) \leq g \circ D(x)$  for any  $x > 0$ .

Now, we prove the lower bound on  $D_g$ . From the very definition, we have for any  $x, u > 0$ ,

$$D_g(x) \geq g(u) \left(1 - \frac{V(u)}{x}\right).$$

By the assumptions on  $\beta$ , the function  $V$  is a differentiable increasing bijection from  $(0, +\infty)$  to itself. Fix  $x > 0$ . For  $\rho > 1$ , we set  $u = V^{-1}(\rho^{-1}x)$ . It yields

$$D_g(x) \geq (1 - \rho^{-1})g(V^{-1}(\rho^{-1}x)).$$

Fix  $y > 0$ . The supremum defining  $D(y) = t_0 - \frac{t_0}{y}V(t_0)$  exists and it is attained at some point  $t_0 > 0$  which is characterized by  $1 - \frac{1}{y}V(t_0) - \frac{t_0}{y}V'(t_0) = 0$ . It implies that  $y = V(t_0) + t_0V'(t_0) \geq V(t_0)$  because  $V' \geq 0$ . Finally, we get  $t_0 \leq V^{-1}(y)$ . Since  $V \geq 0$ , we deduce  $D(y) \leq t_0 \leq V^{-1}(y)$ . Thus  $g(D(y)) \leq g(V^{-1}(y))$  for any  $y > 0$ . Now set  $y = \rho^{-1}x$  and obtain the expected lower bound

$$D_g(x) \geq (1 - \rho^{-1})(g \circ D)(\rho^{-1}x).$$

The proof is completed. ■

### 3 Application to ultracontractivity of subordinated semigroups

Recall that a symmetric semigroup  $(T_t)_{t>0}$  of contraction on  $L^2$  and  $L^1$  is ultracontractive if for any  $t > 0$ ,

$$\|T_t f\|_2 \leq b(t)\|f\|_1 \quad (19)$$

for some non-increasing function  $b : (0, +\infty) \rightarrow (0, +\infty)$  with  $b(0^+) = +\infty$ . We have also mentioned in the preceding section that ultracontractivity implies super Poincaré (1) with  $\beta(r) = b^2(r/2)$ .

By interpolation and duality, the property of ultracontractivity is equivalent to

$$\|T_t f\|_\infty \leq a(t)\|f\|_1, \quad t > 0, \quad (20)$$

for some non-increasing function  $a : (0, +\infty) \rightarrow (0, +\infty)$  with  $a(0^+) = +\infty$ . More precisely, from (19) we get  $a(t) \leq b^2(t/2)$  and from (20) we obtain  $b(t) \leq \sqrt{a(t)}$ . In a general framework, it also implies the existence of a heat kernel  $h_t(x, y)$  uniformly bounded in  $x$  and  $y$  by  $a(t)$  (see for instance [D]).

If  $b_g(t) := \int_0^{+\infty} b(s)\eta_t^g(s) ds < +\infty$  for any  $t > 0$  then the semigroup  $(T_t^g)$  is ultracontractive because

$$\|e^{-tg(A)} f\|_2 \leq \int_0^{+\infty} \eta_t^g(s)\|T_s f\|_2 ds \leq \left( \int_0^{+\infty} b(s)\eta_t^g(s) ds \right) \|f\|_1.$$

But unfortunately, to check this condition is rather hard because the densities  $\eta_t^g$  are not well known apart from the case of  $g(x) = \sqrt{x}$  (see [J] p.181). A way to overcome this difficulty is by considering Nash-type inequalities. For that purpose, we recall a result due to T.Couhlon. The author deduces ultracontractivity bounds from Nash-type inequality under some integrability condition (see [C] and also [M1]). For applications we have in mind, we restrict his result to our setting.

**Theorem 3.1** *Let  $(T_t)_{t>0}$  be a sub-markovian semigroup with (non-negative) infinitesimal generator  $A$ . Assume that there exists a non-decreasing function  $\Theta : (0, +\infty) \rightarrow (0, +\infty)$  satisfying the following Nash-type inequality*

$$\Theta(\|f\|_2^2) \leq (Af, f), \quad \|f\|_1 \leq 1,$$

If  $\int^\infty \frac{dx}{\Theta(x)} < +\infty$  then  $(T_t)_{t>0}$  is ultracontractive and for any  $t > 0$ ,

$$\|T_t f\|_\infty \leq a(t) \|f\|_1, \quad t > 0,$$

where  $a(t)$  is the inverse of the function  $s \rightarrow \int_s^\infty \frac{dx}{\Theta(x)}$ .

We apply this result to the eventual ultracontractivity of the subordinated semigroup  $(T_t^g)$  and give a sufficient condition on  $D_g$  of (11) to get ultracontractivity from a Nash-type inequality satisfied by  $A$ .

**Corollary 3.2** *Under the assumptions and notations of Theorem 1.2, let's assume that  $\int^\infty \frac{dx}{x D_g(x)} < +\infty$ . Then  $(T_t^g)$  is ultracontractive and for any  $t > 0$ ,*

$$\|T_t^g f\|_\infty \leq a_g(t) \|f\|_1$$

where  $a_g$  is the inverse function of  $s \rightarrow \int_s^\infty \frac{dx}{x D_g(x)}$ .

**Proof:** Apply Theorems 1.2 and 3.1.

It is clear that  $(T_t)_{t>0}$  can be ultracontractive but not  $(T_t^g)$  for some Bernstein functions  $g$ . For instance, let  $A = \Delta$  be the usual Laplacian on  $\mathbb{R}^n$ . In that case, Nash inequality has the form (10) with  $D(x) = cx^{2/n}$ . Let  $g(r) = \log(1+r)$ . Then  $\int^\infty \frac{dx}{x D_g(x)} \geq \int^\infty \frac{dx}{x \ln(1+cx^{2/n})} = +\infty$ . This is obtained from the inequality  $g \circ D(x) \geq D_g(x)$  of Theorem 1.2 (ii). Now by a direct computation, we can show that  $(T_t^g)$  is not ultracontractive for small  $t > 0$ , see (23) below for details.

## 4 Examples of Settings

Here, we give examples of settings where our main results apply. The list is clearly not exhaustive.

### 4.1 The Euclidean space

Let  $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$  be the usual Laplacian on  $\mathbb{R}^n$ . The profile of the super Poincaré inequality can be deduced from the optimal Nash inequality obtained in [C-L]. Let  $N_n$  be the best constant in Nash inequality,

$$\frac{1}{N_n} \|f\|_2^{2+4/n} \leq (\Delta f, f), \quad \|f\|_1 \leq 1.$$

By Proposition 2.2, this is equivalent to the following super Poincaré inequality,

$$\|f\|_2^2 \leq r(\Delta f, f) + C_n r^{-n/2} \|f\|_1^2, \quad r > 0, \quad (21)$$

with

$$C_n = \frac{2(nN_n)^{n/2}}{(n+2)^{1+n/2}}. \quad (22)$$

Thus (1) is satisfied with the super Poincaré profile  $\beta_p(r) = C_n r^{-n/2}$  and (3) with  $D(x) = \frac{1}{N_n} x^{2/n}$ . See [C-L] for a description of the constant  $N_n$ .

To simplify the presentation of our results, we shall assume that the Bernstein function  $g$  is a bijection from  $(0, +\infty)$  to itself. By applying (ii) of Theorem 1.2, we get for any  $\rho > 1$ ,

$$\frac{1}{2}(1 - \rho^{-1})\|f\|_2^2 g\left(2N_n^{-1}\rho^{-2/n}\|f\|_2^{4/n}\right) \leq (g(\Delta)f, f), \quad \|f\|_1 \leq 1.$$

*Examples of Bernstein functions.*

i. Let  $g(x) = x^\alpha$ . We obtain for the fractional power of the Laplacian  $\Delta^\alpha$ ,  $0 < \alpha < 1$ ,

$$\frac{1}{2(N_n\rho)^\alpha} \left(1 - \frac{1}{\rho}\right) \|f\|_2^{2+4\alpha/n} \leq (\Delta^\alpha f, f), \quad \|f\|_1 \leq 1.$$

By optimizing over  $\rho > 1$ , we deduce

$$L_{n,\alpha} \|f\|_2^{2+4\alpha/n} \leq (\Delta^\alpha f, f), \quad \|f\|_1 \leq 1$$

with  $L_{n,\alpha} = 2^{\alpha-1} N_n^{-\alpha} \frac{n(2\alpha)^{2\alpha/n}}{(2\alpha+n)^{1+2\alpha/n}}$ .

See [VSC] for such a result in the general setting of sub-markovian symmetric semi-groups but with a different approach. Note that the constant  $L_{n,\alpha}$  is explicit but probably not optimal. Indeed, we get a better constant if we apply Theorem 1.1 with  $g(x) = x^\alpha$ :

$$\|f\|_2^2 \leq r(\Delta^\alpha f, f) + C_n 2^{\frac{n}{2}(\frac{1}{\alpha}-1)} r^{-\frac{n}{2\alpha}} \|f\|_1^2, \quad r > 0,$$

with  $C_n$  as above. By applying (i) of Proposition 2.2, we get

$$K_{n,\alpha} \|f\|_2^{2+4\alpha/n} \leq (\Delta^\alpha f, f), \quad \|f\|_1 \leq 1.$$

with

$$K_{n,\alpha} = \left(\frac{n}{n+2\alpha}\right) 2^{\alpha-1} \left(\left(\frac{n}{2\alpha} + 1\right) C_n\right)^{\frac{-2\alpha}{n}}.$$

By the relationships connecting  $N_n$  and  $C_n$ , we have that  $L_{n,\alpha} < K_{n,\alpha}$  (equivalent to the trivial inequality  $n 2^{\frac{2}{n}} < (n+2)^{\frac{2}{n}+1}$ ). We postpone to Section 7 the study of super Poincaré inequalities for a larger class of functions of the Laplacian using Fourier analysis tools. But with this approach, the best constants are lost.

ii. Let  $g(x) = \log(1+x)$ . The geometrically stable operator  $\log(I + \Delta)$  satisfies

$$\frac{1}{2}\left(1 - \frac{1}{\rho}\right)\|f\|_2^2 \log\left(1 + N_n^{-1}\rho^{-1}\|f\|_2^{4/n}\right) \leq (\log(I + \Delta)f, f), \quad \|f\|_1 \leq 1.$$

To estimate  $D_g$  with  $g(x) = \log(1+x)$  is not a pleasant task. So, we prefer to state this explicit inequality for each parameter  $\rho > 1$ .

Note that, in general, the eventual ultracontractivity can be proved for  $e^{-tg(\Delta)}$  directly by the formula,

$$\|e^{-tg(\Delta)}\|_{1 \rightarrow 2}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-2tg(|y|^2)} dy.$$

Applied to  $g(x) = \log(1+x)$ , this leads us to

$$\|e^{-t\log(I+\Delta)}\|_{1 \rightarrow 2}^2 = \frac{|S_{n-1}|}{(2\pi)^n} \int_0^\infty (1+r^2)^{-2t} r^{n-1} dr < +\infty \quad \text{iff} \quad t > \frac{n}{4}. \quad (23)$$

Thus this semigroup is not ultracontractive for  $0 < t \leq n/4$ . But note that it satisfies super Poincaré and Nash-type inequalities.



## 4.2 The Riemannian setting

The following example is taken from [W2] Cor.2.5. Let  $M$  be a connected complete Riemannian manifold with Ricci curvature bounded from below. Assume that the boundary  $\partial M$  is convex or empty. For  $V \in C^1(M)$ , we assume  $Z = \int_M e^{-V(x)} dx$  is finite and define the probability measure  $\mu$  by  $d\mu(x) = Z^{-1}e^{-V(x)} dx$  where  $dx$  is the Riemannian volume measure. Let  $A = \Delta + \nabla V$ ,  $A$  is (essentially) self-adjoint on  $L^2(\mu)$  (with Neumann boundary condition whenever  $\partial M$  is nonempty). Set  $\rho(x) = \rho(x, o)$  the Riemannian distance function to a fixed point  $o \in M$ . Consider  $V = -\alpha\rho^\delta$ ,  $\alpha > 0$  and  $\delta > 1$  then super Poincaré (1) holds true with

$$\beta(r) = \exp \left[ c(1 + r^{-\lambda}) \right] \quad (24)$$

with  $\lambda = \delta/[2(\delta - 1)]$  and some constant  $c > 0$ . Moreover, super Poincaré (1) also holds if  $V = -\exp[\alpha\rho]$ ,  $\alpha > 0$  with  $\lambda = 1/2$  in (24). Theorem 1.1 applies: for any Bernstein function  $g$ , super Poincaré (9) holds true for  $g(A)$  with  $\beta_g(r)$  given in Theorem 1.1. Nash-type inequalities can be deduced for  $g(A)$  from super Poincaré by Theorem 1.2. Our results generalize the particular case of the fractional powers  $g(x) = x^\alpha$ ,  $0 < \alpha < 1$  treated in [W1].

## 4.3 The hypoelliptic setting

Here, we consider sub-laplacians on Lie groups of polynomial growth. Let  $G$  be a connected Lie group of polynomial growth of index  $D$  and  $(X_1, X_2, \dots, X_m)$  be a system of left-invariant vector fields satisfying Hörmander's condition with local dimension  $d$ . We assume  $d \leq D$ . The sub-laplacian  $L = -\sum_{i=1}^m X_i^2$  generates a semigroup  $e^{-tL}$  with kernel  $p_t$  satisfying for all  $n$  satisfying  $d \leq n \leq D$ ,

$$\sup_{x, y \in G} p_t(x, y) = \|e^{-tL}\|_{1 \rightarrow \infty} \leq \frac{c_1}{t^{n/2}}.$$

(See [VSC]). Hence, super Poincaré inequality holds true

$$\|f\|_2^2 \leq t(Lf, f) + \frac{c_0}{t^{n/2}} \|f\|_1^2, \quad t > 0.$$

and our results applies to  $g(L)$  for any Bernstein function  $g$  with  $\beta(t) = c_0 t^{-n/2}$ . We now discuss in this context the four examples of Bernstein function  $g$  introduced below Theorem 1.1. We provide asymptotic behaviors of  $\beta_g(r)$  when  $r$  tends to 0 and  $r$  tends to  $+\infty$ .

- i. If  $g(x) = x^\alpha$ ,  $0 < \alpha \leq 1$  then  $\beta_g(r) = \frac{c_0}{r^{n/2\alpha}}$ ,  $r > 0$ .
- ii. If  $g(x) = \ln(1 + x)$  then  $\beta_g(r) = c_0 (e^{1/r} - 1)^{n/2}$ ,  $r > 0$ .

$$\beta_g(r) \sim \begin{cases} c_0 e^{n/2r} & \text{as } r \rightarrow 0^+, \\ c_0 \frac{1}{r^{n/2}} & \text{as } r \rightarrow +\infty. \end{cases}$$

- iii. If  $g(x) = [\ln(1 + x^\alpha)]^\gamma$ ,  $0 < \alpha, \gamma \leq 1$  then  $\beta_g(r) = c_0 \left[ e^{(1/r)^{1/\gamma}} - 1 \right]^{n/2\alpha}$ ,  $r > 0$ .

$$\beta_g(r) \sim \begin{cases} c_0 e^{\frac{n}{2\alpha}(1/r)^{1/\gamma}} & \text{as } r \rightarrow 0^+, \\ c_0 \frac{1}{r^{n/2\alpha\gamma}} & \text{as } r \rightarrow +\infty. \end{cases}$$

iv. Let  $t > 0$ . If  $g(x) = 1 - e^{-tx}$  then  $\beta_g(r) = \frac{c_0}{t^{n/2}} \left[ \ln\left(1 + \frac{1}{r-1}\right) \right]^{n/2}$ ,  $r > 1$ .

$$\beta_g(r) \sim \begin{cases} \frac{c_0}{t^{n/2}} \left[ \ln\left(\frac{1}{r-1}\right) \right]^{n/2} & \text{as } r \rightarrow 1^+, \\ \frac{c_0}{(rt)^{n/2}} & \text{as } r \rightarrow +\infty. \end{cases}$$

Note that this discussion with this family of Bernstein functions is always valid when  $A$  satisfies super Poincaré inequality with  $\beta(t) = c_0 t^{-n/2}$ .

#### 4.4 The Ornstein-Uhlenbeck operator

Let  $A = \mathcal{L} = \Delta + x \cdot \nabla$  be the Ornstein-Uhlenbeck operator define on  $L^2(\mathbb{R}^n, \gamma)$  with the gaussian measure  $\gamma(dx) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}} dx$ . It is well known that Gross' inequality is satisfied

$$\int_{\mathbb{R}^n} f^2 \log(|f|/\|f\|_2) d\gamma \leq (\mathcal{L}f, f)$$

with  $(\mathcal{L}f, f) = \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma$ .

It is also well known that we can deduce super Poincaré inequality. We recall the arguments. For any  $f \in L^1 \cap L^2$  such that  $\|f\|_1 = 1$ , we have by Jensen's inequality:

$$\|f\|_2^2 \log \|f\|_2 \leq \int_{\mathbb{R}^n} f^2 \log(|f|/\|f\|_2) d\gamma.$$

By renormalization, this inequality is also satisfied when  $\|f\|_1 \leq 1$ . This yields

$$\|f\|_2^2 \log \|f\|_2 \leq (\mathcal{L}f, f).$$

Using the relation  $xy - e^{y-1} \leq x \log x$  for any  $y \in \mathbb{R}$ , we deduce

$$\|f\|_2^2 \leq t(\mathcal{L}f, f) + \frac{t}{2e} e^{\frac{2}{t}} \|f\|_1^2, \quad t > 0.$$

On the other hand, Poincaré inequality deduced from Gross inequality, trivially implies  $\|f\|_2^2 \leq (\mathcal{L}f, f) + \|f\|_1^2$ . Together, the preceding inequalities leads to the following formulation of super Poincaré inequality

$$\|f\|_2^2 \leq t(\mathcal{L}f, f) + \beta(t) \|f\|_1^2, \quad t > 0, \tag{25}$$

with  $\beta(t) = \frac{t}{2e} e^{\frac{2}{t}}$ ,  $0 < t \leq 1$  and  $\beta(t) = 1, t \geq 1$ .

Our Theorems 1.1 and 1.2 apply in this case. For instance, we obtain by Theorem 1.1

i. For any  $0 < \alpha < 1$ ,

$$\|f\|_2^2 \leq t(\mathcal{L}^\alpha f, f) + \frac{t^\frac{1}{\alpha}}{2e} e^{2t^{-1/\alpha}} \|f\|_1^2, \quad 0 < t < 1,$$

and

$$\|f\|_2^2 \leq t(\mathcal{L}^\alpha f, f) + \|f\|_1^2, \quad t \geq 1.$$

ii.

$$\|f\|_2^2 \leq t(\log(I + \mathcal{L})f, f) + \frac{1}{2e^3} \frac{1}{(e^{1/t} - 1)} e^{2e^{1/t}} \|f\|_1^2, \quad 0 < t < \frac{1}{\log 2}$$

and

$$\|f\|_2^2 \leq t(\log(I + \mathcal{L})f, f) + \|f\|_1^2, \quad t \geq \frac{1}{\log 2}.$$

we have  $\beta_{\log}(t) \sim \frac{1}{2e^3} e^{2e^{1/t}-1/t}$  as  $t$  goes to 0.

Similar inequalities can be written for the cases (iii) and (iv) considered in Section 1. This is left as exercises. Of course, the discussion is not limited to these cases just above.

## 5 Study of $\Psi(A)$ with $\Psi$ convex

It is useful to deduce super Poincaré or Nash inequality for  $\Psi(A)$  when  $A$  satisfies such inequality and  $\Psi$  is convex. The reason is that the inverse function of a concave increasing function is convex and increasing. For instance, Bernstein functions. Of course,  $\Psi(A)$  may not generate a sub-markovian semigroup (e.g. with  $\Psi(\Delta) = \Delta^2$  in  $\mathbb{R}^n$ ). But it doesn't harm too much as far as the generator  $\Psi(A)$  is concern. In what follows, we study the following converse implication. Assume that  $g(A)$  satisfies super Poincaré or Nash-type inequality then deduce a similar inequality for  $A$ .

Let  $(E_\lambda), \lambda > 0$  be the spectral resolution associated to  $A$  and  $\psi : [0, \infty) \rightarrow [0, \infty)$  be a measurable function. We define  $\psi(A)$  on its domain  $\mathcal{D}(\psi(A)) \subset L^2$  as in the Section 1. In particular, we have on their respective domains the following representations:

$$(Af, f) = \int_0^{+\infty} \lambda d(E_\lambda f, f), \quad (T_t f, f) = \int_0^{+\infty} e^{-\lambda t} d(E_\lambda f, f), \quad \|f\|_2^2 = \int_0^{+\infty} d(E_\lambda f, f),$$

(See [Sc-S-V] Thm.11.4).

**Proposition 5.1** *Assume that  $A$  is a non-negative self-adjoint operator satisfying Nash-type inequality (3). Then for any non-negative non-decreasing convex function  $\Psi$  with  $\Psi(0) = 0$ , we have*

$$\|f\|_2^2 (\Psi \circ D)(\|f\|_2^2) \leq (\Psi(A)f, f), \quad \|f\|_1 \leq 1. \quad (26)$$

Note that such result can be generalized in the framework of Hilbert space  $H$  with the norm  $\|f\|_1$  replaced by another control  $\Phi(f)$ ,  $f$  in some subspace of  $H$ , satisfying properties as defined in [W4].

**Proof:** Write equivalently Nash-type inequality (3) as follows. For any  $f \in \mathcal{D}(A) \cap L^1$  and  $\|f\|_2 = 1$ ,

$$D\left(\frac{1}{\|f\|_1^2}\right) \leq (Af, f)$$

Since  $\Psi$  is non-decreasing, we get

$$(\Psi \circ D)\left(\frac{1}{\|f\|_1^2}\right) \leq \Psi[(Af, f)].$$

Now, by functional calculus and Jensen inequality applied to the probability measure  $d(E_\lambda f, f)$  (i.e.  $\|f\|_2 = 1$ ), we get

$$\Psi[(Af, f)] = \Psi\left(\int_0^{+\infty} \lambda d(E_\lambda f, f)\right) \leq \int_0^{+\infty} \Psi(\lambda) d(E_\lambda f, f) = (\Psi(A)f, f).$$

Thus

$$(\Psi \circ D)\left(\frac{1}{\|f\|_1^2}\right) \leq (\Psi(A)f, f), \quad \|f\|_2 = 1, f \in \mathcal{D}(A) \cap L^1.$$

Under the assumptions on  $\Psi$ , there exists  $\lambda_0 > 0$  and  $k > 0$  such that, for any  $\lambda > \lambda_0$ :  $k\lambda \leq \Psi(\lambda)$  which implies  $\mathcal{D}(\Psi(A)) \subset \mathcal{D}(A)$ . Now, by reversing the process of normalization from  $L^2$  to  $L^1$ , it yields for any  $f \in \mathcal{D}(\Psi(A)) \cap L^1$ ,

$$\|f\|_2^2 (\Psi \circ D)\left(\frac{\|f\|_2^2}{\|f\|_1^2}\right) \leq (\Psi(A)f, f).$$

Since  $\Psi \circ D$  is non-increasing, we deduce (26) for  $\|f\|_1 \leq 1$  and conclude the proof. ■

In the second part of this section, we deal with the case of super Poincaré inequalities generalizing the arguments of [W1] used for  $A^\alpha$ ,  $\alpha > 1$ .

**Theorem 5.2** *Let  $\Psi : (0, +\infty) \rightarrow (0, +\infty)$  be a non-decreasing convex function. Assume that  $\Psi^*(x) := \sup_{y \in (0, +\infty)}(xy - \Psi(y))$  is a bijection from  $(0, +\infty)$  to  $(0, +\infty)$  and that  $B$  is a non-negative self-adjoint operator satisfying: for any  $f \in \mathcal{D}(B) \cap L^1$  and  $t > 0$ ,*

$$\|f\|_2^2 \leq t(Bf, f) + \gamma(t)\|f\|_1^2 \tag{27}$$

for some function  $\gamma : (0, +\infty) \rightarrow (0, +\infty)$ . Then  $\Psi(B)$  satisfies for any  $f \in \mathcal{D}(\Psi(B)) \cap L^1$  and  $t > 0$ ,

$$\|f\|_2^2 \leq t(\Psi(B)f, f) + \gamma_\Psi(t)\|f\|_1^2 \tag{28}$$

with  $\gamma_\Psi(t) = \inf_{0 < \varepsilon < 1} \frac{1}{\varepsilon} \gamma(\varepsilon t (\Psi^*)^{-1}(\frac{1-\varepsilon}{\varepsilon}))$ .

In particular, if  $\Psi(x) = x^{1/\alpha}$  with  $\alpha \in (0, 1)$  then  $\gamma_\Psi(t) \leq \frac{1}{\alpha} \gamma(t^\alpha)$ .

**Proof:** From the very definition of  $\Psi^*$ , we have Young's inequality i.e. for any  $y, s > 0$ ,  $ys \leq \Psi(y) + \Psi^*(s)$ . Let  $y = (Bf, f)$  with  $f \in \mathcal{D}(B)$ , we obtain for any  $s > 0$ :

$$s(Bf, f) \leq \Psi((Bf, f)) + \Psi^*(s).$$

Assume that  $\|f\|_2 = 1$ . By spectral representation of  $B$  and by Jensen's inequality with  $\Psi$  as convex function, we have already seen that

$$0 \leq \Psi((Bf, f)) \leq (\Psi(B)f, f).$$

By (27), we have for any  $t, s > 0$  with  $\|f\|_2 = 1$ ,

$$1 \leq ts(Bf, f) + \gamma(ts)\|f\|_1^2.$$

Combining the above inequalities, it yields for  $f \in \mathcal{D}(\Psi(B))$ :

$$1 \leq t(\Psi(B)f, f) + t\Psi^*(s) + \gamma(ts)\|f\|_1^2.$$

Let  $\varepsilon \in (0, 1)$ . Since  $\Psi^*$  is a bijection, for any fixed  $t > 0$ , there exists  $s > 0$  such that  $\varepsilon = 1 - t\Psi^*(s)$  i.e.  $s = (\Psi^*)^{-1}(\frac{1-\varepsilon}{t})$ . Thus we obtain

$$\varepsilon \leq t(\Psi(B)f, f) + \gamma(t(\Psi^*)^{-1}((1-\varepsilon)t^{-1}))\|f\|_1^2.$$

Changing  $t$  by  $\varepsilon t$  and dividing by  $\varepsilon$ , we get for any  $t > 0$ ,  $\varepsilon \in (0, 1)$  and  $\|f\|_2 = 1$ ,

$$1 \leq t(\Psi(B)f, f) + \frac{1}{\varepsilon}\gamma\left(\varepsilon t(\Psi^*)^{-1}\left(\frac{1-\varepsilon}{\varepsilon t}\right)\right)\|f\|_1^2.$$

We conclude (28) by changing  $f$  by  $f/\|f\|_2$  and by taking the infimum over  $\varepsilon \in (0, 1)$ .

We now prove the last statement. Let  $\Psi(x) = x^{1/\alpha}$ , we have  $\Psi^*(s) = c_\alpha s^{\frac{1}{1-\alpha}}$  with  $c_\alpha = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$ . By a simple computation, it yields for any  $\varepsilon \in (0, 1)$ ,

$$\gamma_\Psi(t) \leq \frac{1}{\varepsilon}\gamma\left[k_\alpha \varepsilon \left(\frac{1-\varepsilon}{\varepsilon}\right)^{1-\alpha} t^\alpha\right]$$

with  $k_\alpha = \alpha^{-\alpha}(1-\alpha)^{\alpha-1}$ . Choosing  $\varepsilon = \alpha$ , we conclude  $\gamma_\Psi(t) \leq \frac{1}{\alpha}\gamma(t^\alpha)$ .

The proof is completed. ■

For the case  $g(x) = x^\alpha$ , the function obtained in [W1] is given by  $\tilde{\gamma}_\Psi(t) = 2\gamma(\frac{t^\alpha}{2})$ . Since  $\gamma_\Psi$  is usually decreasing and  $\gamma(0^+) = +\infty$ , the result above is sharper up to a multiplicative constant. We notice that  $\|f\|_1^2$  plays no particular role in the proof. So, it can be replaced by some functional  $\Phi(f)$  and  $L^2$  by a general Hilbert space as in [W1].

Now we make the connection between Bernstein functions and convex functions. Assume that  $g$  is a Bernstein function. Since  $g$  is non-decreasing and concave,  $\Psi = g^{-1}$  is non-decreasing and convex. Hence, Theorem 5.2 allows us to prove a converse to Theorem 1.1 about super Poincaré inequalities. Thus applying Theorem 5.2 with  $B = g(A)$ , we get:

**Corollary 5.3** *Let  $g$  be a bijective Bernstein function and  $A$  be a non-negative self-adjoint operator. Assume that  $\Psi = g^{-1}$  satisfies the conditions of Theorem 5.2 and that for any  $f \in \mathcal{D}(g(A)) \cap L^1$  and any  $t > 0$ ,*

$$\|f\|_2^2 \leq t(g(A)f, f) + \gamma(t)\|f\|_1^2. \quad (29)$$

Then for any  $f \in \mathcal{D}(A)$  and any  $t > 0$ ,

$$\|f\|_2^2 \leq t(Af, f) + \gamma_\Psi(t)\|f\|_1^2 \quad (30)$$

with  $\gamma_\Psi$  given as in Theorem 5.2.

Corollary 5.3 is sharp in the particular case  $g(x) = x^\alpha$ ,  $\alpha \in (0, 1)$ . Indeed, assume that  $A$  satisfies super Poincaré inequality (1) for some  $\beta$ . By Theorem 1.1,  $g(A)$  satisfies super Poincaré inequality (29) with  $\gamma = \beta_g$  given by (9). Now take  $\Psi(x) = x^{1/\alpha}$  in Theorem 5.2, it gives back that  $A$  satisfies super Poincaré inequality with  $\tilde{\beta}(t) = \gamma_\Psi(t) \leq \frac{1}{\alpha}\gamma(t^\alpha) = \frac{1}{\alpha}\beta(t)$ . In this case, Corollary 5.3 is essentially an optimal converse of Theorem 1.1 up to the multiplicative constant  $\frac{1}{\alpha}$ .

## 6 Asymptotic behavior of $g(A)$

We briefly revisit the relation between the asymptotic behavior of  $g(A)$  and the asymptotic behavior of  $A$  in terms of Poincaré inequality (equivalent to a bound on the bottom of the spectrum). The proof uses arguments of Theorem 1.1. We obtain Poincaré inequality in  $L^p$  for  $g(A)$  starting from the same inequality for  $A$ . The first result is certainly known for Poincaré inequality in  $L^2$ . See [C-G-R] for recent results on the subject on  $L^p$ .

**Proposition 6.1** *Assume that  $(T_t)_{t>0}$  is a symmetric Markov semigroup and has a spectral gap on  $L^p(X, \mu)$  in the following sense*

$$\|T_t f - \mu(f)\|_p \leq e^{-\lambda t} C(f), \quad t > 0, \quad (31)$$

for some  $\lambda \in [0, +\infty)$  and some positively homogeneous functional  $C(f)$ . Then for any Bernstein function  $g$  with  $g(0) = 0$ , we have

$$\|T_t^g f - \mu(f)\|_p \leq e^{-tg(\lambda)} C(f), \quad t > 0. \quad (32)$$

For instance with  $C(f) = \|f - \mu(f)\|_p$ , see [C-G-R]. For  $p = 2$  and  $C(f) = \|f - \mu(f)\|_2$ , the interpretation of (32) in terms of spectrum is well known. Indeed, (31) is equivalent to Poincaré inequality

$$\|f - \mu(f)\|_2^2 \leq \frac{1}{\lambda} (Af, f)_{L^2}, \quad f \in \mathcal{D}(A).$$

Then similarly for  $g(A)$ , we deduce:

$$\|f - \mu(f)\|_2^2 \leq \frac{1}{g(\lambda)} (g(A)f, f).$$

Similar results can be written in the  $L^p$ -setting for  $1 < p < +\infty$ .

**Proof:** Let  $f$  such that  $\mu(f) = 0$ . Then  $\mu(T_s f) = \mu(T_s^g f) = 0, s > 0$  because  $T_s$  and  $T_s^g$  are symmetric Markov semigroups. Let  $g$  be a Bernstein function. We have for any  $t > 0$ ,

$$\begin{aligned} \|T_t^g f - \mu(f)\|_p &= \|T_t^g f\|_p = \left\| \int_0^{+\infty} \eta_t^g(s) T_s f ds \right\|_p \\ &\leq \int_0^{+\infty} \eta_t^g(s) \|T_s f\|_p ds \leq C(f) \int_0^{+\infty} \eta_t^g(s) e^{-\lambda s} ds \leq e^{-tg(\lambda)} C(f). \end{aligned}$$

This concludes the proof. ■

## 7 Functions of the Laplacian on $\mathbb{R}^n$

Here, we give a direct proof of super Poincaré inequality for  $g(\Delta)$  with  $g$  a Bernstein function and  $\Delta$  the usual Laplacian on  $\mathbb{R}^n$ . In fact,  $g$  need not be a Bernstein function. The constants are certainly not optimal. The proof follows the original idea used for the Laplacian in the paper by J. Nash [N].

We use the following definition of Fourier transform  $\mathcal{F}f(x) = \int_{\mathbb{R}^n} f(y)e^{-ixy} dy$ . Let  $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ . So,  $\mathcal{F}(\Delta f)(x) = |x|^2 \mathcal{F}f(x)$ . The function  $g(\Delta)f$  is defined by its Fourier transform

$$\mathcal{F}(g(\Delta)f)(x) = g(|x|^2)\mathcal{F}f(x), \quad x \in \mathbb{R}^n.$$

The domain of  $g(\Delta)$  is defined by

$$\mathcal{D}(g(\Delta)) = \{f \in L^2 : \int_{\mathbb{R}^n} |g(|x|^2)|^2 |\mathcal{F}f(x)|^2 dx < +\infty\}.$$

We denote by  $\omega_n$  the volume of the unit ball of  $\mathbb{R}^n$  and set  $c_n = (2\pi)^{-n}$ .

**Theorem 7.1** *Let  $g : [0, +\infty) \rightarrow [0, +\infty)$  be a non-decreasing function such that  $g(0) = 0$ . We define the function  $g^\rightarrow$  on  $[0, +\infty)$  by  $g^\rightarrow(u) = \sup\{s \geq 0 : g(s) \leq u\} \in [0, +\infty]$ . Then we have*

i. *For any  $t > 0$  and any  $f \in \mathcal{D}(g(\Delta)) \cap L^1(\mathbb{R}^n)$ ,*

$$\|f\|_2^2 \leq t(g(\Delta)f, f) + \tilde{\beta} \left( \frac{1}{g^\rightarrow(t^{-1})} \right) \|f\|_1^2. \quad (33)$$

*with  $\tilde{\beta}(t) = c_n \omega_n t^{-n/2}$ .*

ii. *For any  $f \in \mathcal{D}(g(\Delta)) \cap L^1(\mathbb{R}^n)$  with  $\|f\|_1 \leq 1$ ,*

$$\|f\|_2^2 \tilde{D}_g(\|f\|_2^2) \leq (g(\Delta)f, f),$$

*with  $\tilde{D}_g(x) = \sup_{t>0} \left( t - \frac{t}{x} c_n \omega_n [g^\rightarrow(t)]^{n/2} \right)$ .*

When  $g$  is unbounded and  $g(0) = 0$ , the function  $g^\rightarrow(t)$  is well defined and finite for any  $t > 0$ . It is clear that  $g^\rightarrow$  is non-decreasing. When  $g$  is an increasing bijection from  $[0, +\infty)$  to itself, we have  $g^\rightarrow = g^{-1}$ . The function  $\tilde{\beta}_g(t) := \tilde{\beta} \left( \frac{1}{g^\rightarrow(t^{-1})} \right)$  in (33) is similar to  $\beta_g(t)$  in (9) when  $g$  is invertible. If we assume that  $g$  is bounded then  $g(\Delta)$  is a bounded operator and  $\tilde{\beta}_g(t) = +\infty$  when  $t \leq 1/\|g\|_\infty$ . In that case, the inequality (33) is meaningful only for  $t > 1/\|g\|_\infty$ . Note that this restriction already appears in Theorem 1.1 when  $g$  is bounded.

**Proof:** Let  $f \in \mathcal{D}(g(\Delta)) \cap L^1$  and  $t > 0$ . By Plancherel formula,

$$\begin{aligned} \|f\|_2^2 &= c_n \int_{\mathbb{R}^n} |\mathcal{F}f|^2(x) dx \\ &= c_n \int_{\{x \in \mathbb{R}^n : 1 \leq tg(|x|^2)\}} |\mathcal{F}f|^2(x) dx + c_n \int_{\{x \in \mathbb{R}^n : 1 > tg(|x|^2)\}} |\mathcal{F}f|^2(x) dx \\ &\leq c_n t \int_{\mathbb{R}^n} g(|x|^2) |\mathcal{F}f|^2(x) dx + c_n \|\mathcal{F}f\|_\infty^2 V(\{x \in \mathbb{R}^n : g(|x|^2) < \frac{1}{t}\}). \end{aligned} \quad (34)$$

where  $V(\Omega)$  is the Lebesgue measure of the set  $\Omega \subset \mathbb{R}^n$ . Now, since  $g(r) \leq u$  implies  $r \leq g^\rightarrow(u)$  and  $\|\mathcal{F}f\|_\infty \leq \|f\|_1$ , we deduce for any  $t > 0$ ,

$$\|f\|_2^2 \leq t(g(\Delta)f, f) + c_n \|f\|_1^2 V(\{x \in \mathbb{R}^n : |x| \leq \sqrt{g^\rightarrow(t^{-1})}\}).$$

This concludes (i).

We prove the second part by applying Proposition 2.2 with  $\tilde{\beta}(t) = c_n \omega_n [g^\rightarrow(t^{-1})]^{n/2}$ . This completes the proof. ■

## 8 Appendix on Legendre transform

In the first part of this section, we are interested to discuss the properties of  $D$  directly from the properties of  $\beta$  independently of the set  $\mathcal{F}$  defined in Proposition 2.2. In the second part of this section, reversing the role of  $\beta$  and  $D$  leads to a similar discussion. The conditions introduced here are usually satisfied in the applications.

**Lemma 8.1** *Let  $\beta$  be a non-negative function defined on  $(0, +\infty)$  and set*

$$D(x) = \sup_{t>0} \left( t - \frac{t}{x} \beta(1/t) \right) \in (-\infty, +\infty], \quad x > 0.$$

- i. If  $\lim_{t \rightarrow 0^+} t\beta(1/t) = 0$  then  $D$  is non-negative. This condition is satisfied if  $\beta$  is bounded above at infinity, in particular if  $\beta$  is non-increasing.*
- ii. If  $\beta(0^+) = +\infty$  then  $D(x)$  is finite for any  $x > 0$ . Moreover, the function  $x \rightarrow xD(x)$  is convex, non-decreasing on  $(0, +\infty)$  and  $D$  is continuous.*

**Proof:**

- i. Assume that  $\lim_{t \rightarrow 0^+} t\beta(1/t) = 0$  then  $D(x) \geq \lim_{t \rightarrow 0^+} (t - \frac{t}{x} \beta(1/t)) = 0$  for any  $x > 0$ . Obviously, if  $\beta$  is bounded above at infinity (e.g. non-increasing) and non-negative then  $\lim_{t \rightarrow 0^+} t\beta(1/t) = 0$ .*
- ii. Assume  $\beta(0^+) = +\infty$ . Fix  $x > 0$  then there exists  $t_x > 0$  such that for any  $t > t_x$ ,  $x < \beta(1/t)$ . So,  $t - \frac{t}{x} \beta(1/t) < 0$  when  $t > t_x$ . If  $0 < t \leq t_x$  then  $t - \frac{t}{x} \beta(1/t) \leq t_x$  since  $\beta$  is non-negative. Therefore  $D(x) \leq t_x$  and  $D(x)$  is finite. Now, the function  $x \rightarrow h^*(x) = xD(x) = \sup_{t>0} (tx - t\beta(1/t))$  is convex on  $(0, +\infty)$ . Consequently,  $h^*$  and  $D$  are continuous.*

The proof is completed. ■

Now, we study properties of  $\beta$  in terms of  $D$  defined as in Theorem 1.2. Natural conditions on  $D$  comes from the previous lemma. The discussion is similar.

**Lemma 8.2** *Let  $D : (0, +\infty) \rightarrow \mathbb{R}$  be a fixed function and set*

$$\beta(r) = \sup_{x>0} (x - rxD(x)) \in (-\infty, +\infty].$$

- i. If  $\lim_{x \rightarrow +\infty} D(x) = +\infty$  and  $D$  is non-negative then  $\beta(r)$  is finite for any  $r > 0$ , convex, continuous and non-increasing.*
- ii. If  $\lim_{x \rightarrow 0} xD(x) = 0$  then  $\beta$  is non-negative.*

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The proof is similar to Proposition 8.1. The reason of this fact is apparent below.

Let  $\beta$  and  $D$  as in Proposition 8.1. Note that  $h^*(x) := xD(x) = \sup_{t>0} (tx - t\beta(1/t))$  is the Legendre transform (or complementary function) of  $h(t) = t\beta(1/t)$  thus the afferent theory applies. See for instance [R-R] p.6 for a discussion about Young functions and p.13 for the specific class of N-functions. Usually  $h$  is obtained from  $h^*$  by the same formula i.e.  $h(t) = \sup_{x>0} (tx - h^*(x)), t > 0$ . In that case, we recover the definition of  $\beta$  in terms of  $D$  in Lemma 8.2 by the formulas  $\beta(t) = th(1/t)$  and  $D(x) = h^*(x)/x$ . Here are some examples of couples of N-functions. They appear as asymptotics of functions  $\beta$  or  $D$  of our examples in Section 4. Let  $1 < p, q < +\infty, 1/p + 1/q = 1$ .



- i.  $(h_1(t), h_1^*(x)) = (t^p/p, x^q/q)$ .
- ii.  $(h_2(t), h_2^*(x)) = (e^t - t - 1, (1+x)\ln(1+x) - x)$ .
- iii.  $(h_3(t), h_3^*(x)) = ((1+t)\ln(1+t) - t, e^x - x - 1)$ .
- iv.  $h_4(t) = e^{t^p} - 1$ ,  $h_4^*$ : no explicit form.

But one can prove that  $h_4^*(x) \sim x(\ln x)^{1/p}$  as  $x \rightarrow +\infty$  and  $h_4^*(x) \sim c_q x^q$  as  $x \rightarrow 0^+$  with  $1/p + 1/q = 1$  and  $c_q = (p-1) \left(\frac{1}{p}\right)^q$ .

Of course, in applications, functions like  $c_1 h(c_2 t)$ ,  $c_i > 0$  should be considered or functions having asymptotics of this type. Indeed, in practice  $h$  is not exactly an N-function but often close to such function. Fortunately, it doesn't cause much trouble in practice. This justifies the interest of both propositions just above. Now, we mentioned some cases where the asymptotics of these functions really appear in our applications. Recall the relations  $\beta(t) = th(1/t)$  and  $D(x) = h^*(x)/x$ .

- i. For  $h_1(t) = c_0 t^p$  then we get  $\beta_1(t) = \frac{c_1}{t^\nu}$  where  $\nu = p - 1 > 0$  and  $D(x) = c_2 x^q$  with  $1/p + 1/q = 1$  and  $q = 1 + \frac{1}{\nu}$ . Such cases correspond in Section 4.1 to the fractional Laplacian  $\Delta^\alpha$  on the Euclidean space and to Section 4.3 with  $L^\alpha$  where  $L$  is a sum of vector fields satisfying Hörmander's condition and  $\nu = \frac{n}{2\alpha}$ . For all these cases:  $D(x) = c_3 x^{\frac{2\alpha}{n}}$ .
- ii. The function  $h_2(t) = e^t - t - 1$  leads to  $\beta_2(t) \sim te^{\frac{1}{t}}$  as  $t \rightarrow 0^+$  and  $\beta_2(t) \sim \frac{1}{2t}$  as  $t \rightarrow +\infty$ . This situation is realized up to multiplicative constants by the Ornstein-Uhlenbeck operator in Section 4.4 as far as the local behavior (i.e.  $t \rightarrow 0^+$ ) is concerned. In that case,  $D(x) \sim \ln x$  as  $x \rightarrow +\infty$ .
- iii. The function  $h_3(t) = (1+t)\ln(1+t) - t$  gives  $\beta_3(t) \sim \ln(1/t)$  as  $t \rightarrow 0^+$ . Using results of Section 7 with  $n = 2$ , we set  $A = g(\Delta) \geq 0$  with  $g(y) = e^{y/4\pi} - 1$ ,  $y \geq 0$ . This provides an example of positive operator such that the super Poincaré inequality (33) is satisfied with  $\beta(t) = \ln(1 + \frac{1}{4\pi^2 t})$ ,  $t > 0$  and  $D(x) \sim \frac{4\pi^2}{x} e^{x-1}$  as  $x \rightarrow +\infty$ . We also have  $D(x) \sim \pi^2 x$  as  $x \rightarrow 0^+$ .
- iv. For the function  $h_4(t) = e^{t^p} - 1$  with  $1 < p < +\infty$ , we deduce  $\beta_4(t) \sim te^{\frac{1}{t^p}} =: \tilde{\beta}(t)$  as  $t \rightarrow 0^+$  and  $e^{(1-\frac{1}{p})\frac{1}{t^p}} \leq \tilde{\beta}(t) \leq e^{\frac{1}{t^p}}$  when  $t \in (0, 1)$ . Examples with such behavior are given in the Riemannian setting of Section 4.2 with  $p = \frac{\delta}{2(\delta-1)}$  where  $1 < \delta < 2$  in (24). In that case,  $D(x) \sim (\ln x)^{1/p}$  as  $x \rightarrow +\infty$ . Note that it is a general fact that the behavior of  $\beta(t)$  as  $t \rightarrow 0^+$  determines the behavior of  $D(x)$  as  $x \rightarrow +\infty$  and conversely.

It will be interesting to know if there exists an operator  $A$  satisfying super Poincaré inequality with  $\beta(t) \sim th_3(1/t) \sim \ln(1/t)$  as  $t \rightarrow 0^+$  and  $(Af, f)$  a Dirichlet form.

*Acknowledgments:* The authors thank the anonymous referee for helpful comments and suggestions to improve the paper. This research was supported in part by the ANR project EVOL. The second author thanks the CNRS for a period of delegation during which this paper has been completed.

*Note:* The article [W1] can be obtained on demand to the author and difficult by other sources.

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