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# BOUNDARY TRACE OF POSITIVE SOLUTIONS OF SUPERCRITICAL SEMILINEAR ELLIPTIC EQUATIONS IN DIHEDRAL DOMAINS

MOSHE MARCUS AND LAURENT VERON

ABSTRACT. We study the generalized boundary value problem for (E)  $-\Delta u + |u|^{q-1}u = 0$  in a dihedral domain  $\Omega$ , when  $q > 1$  is supercritical. The value of the critical exponent can take only a finite number of values depending on the geometry of  $\Omega$ . When  $\mu$  is a bounded Borel measure in a  $k$ -wedge, we give necessary and sufficient conditions in order it be the boundary value of a solution of (E). We also give conditions which ensure that a boundary compact subset is removable. These conditions are expressed in terms of Bessel capacities  $B_{s,q}$  in  $\mathbb{R}^{N-k}$  where  $s$  depends on the characteristics of the wedge. This allows us to describe the boundary trace of a positive solution of (E).

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*Key words.* Laplacian; Poisson potential; Borel measures; Besov spaces; harmonic lifting; Bessel capacities.

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## 1. INTRODUCTION

Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^N$ ,  $\rho$  the first eigenfunction of  $-\Delta$  in  $W_0^{1,2}(\Omega)$  with supremum 1 and  $\lambda$  the corresponding eigenvalue, and let  $q > 1$ . A long-term research on the equation

$$(1.1) \quad -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega,$$

has been carried out for more than twenty years by probabilistic and/or analytic methods. Much of the research was focused on three main problems in domains of class  $C^2$ :

- (i) The Dirichlet problem for (1.1) with boundary data given by a finite Borel measure on  $\partial\Omega$ .
- (ii) The characterization of removable singular subsets of  $\partial\Omega$  relative to positive solutions of (1.1).
- (iii) The characterization of arbitrary positive solutions of (1.1) via an appropriate notion of boundary trace.

Consider the Dirichlet problem

$$(1.2) \quad -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \quad u = \mu \quad \text{in } \partial\Omega$$

where  $\mu \in \mathfrak{M}(\partial\Omega)$  (= space of finite Borel measures on  $\partial\Omega$ ). Following [23], a (weak) solution  $u := u_\mu$  of (1.2) is a function  $u \in L_\rho^q(\Omega)$  such that,

$$(1.3) \quad \int_\Omega (-u\Delta\eta + \eta|u|^{q-1}u) \, dx = - \int_\Omega \mathbb{K}[\mu]\Delta\eta \, dx,$$

for every  $\eta \in X(\Omega)$ , where

$$(1.4) \quad X(\Omega) = \left\{ \eta \in W_0^{1,2}(\Omega) : \rho^{-1}\Delta\eta \in L^\infty(\Omega) \right\}.$$

Here  $\mathbb{K}[\mu]$  is the harmonic function in  $\Omega$  with boundary trace  $\mu$  and  $\rho$  is the first eigenfunction of  $-\Delta$  in  $\Omega$  normalized so that  $\max_\Omega \rho = 1$ . We recall that, if  $\Omega$  is Lipschitz  $\mathbb{K}[\mu] \in L_\rho^1(\Omega)$ ; if  $\Omega$  is of class  $C^2$ ,  $\mathbb{K}[\mu] \in L^1(\Omega)$ .

A measure  $\mu$  is a *q-good measure* if (1.2) has a solution. The space of *q-good measures* is denoted by  $\mathfrak{M}_q(\partial\Omega)$ . It is known that, if  $\mu$  is *q-good*, the solution is unique. Furthermore, if  $\mu$  satisfies the condition

$$(1.5) \quad \int_\Omega \mathbb{K}[|\mu|]^q \rho \, dx < \infty,$$

then it is *q-good*. When  $\mu$  satisfies this condition we say that it is a *q-admissible measure*.

When  $\Omega$  is a domain of class  $C^2$ ,  $\mathbb{K}[\mu] \in L_\rho^q$  for every  $q \in (1, \frac{N+1}{N-1})$  and every  $\mu \in \mathfrak{M}(\partial\Omega)$ . Therefore, for  $q$  in this range, every measure in  $\mathfrak{M}(\partial\Omega)$  is *q-good* and there is no removable boundary set (except for the empty set). Problem (iii), for  $q$  in this range, was resolved by Le Gall [15] (for  $N = q = 2$ ) and Marcus and Véron [18] (for  $1 < q < \frac{N+1}{N-1}$ ,  $N \geq 3$ ).

The number  $q_c = \frac{N+1}{N-1}$  is called the *critical value* for (1.1). If  $q$  is supercritical, i.e.  $q \geq q_c$ , point singularities are removable. In particular there

is no solution of (1.2) when  $\mu = \delta_y$  (= a Dirac measure concentrated at a point  $y \in \partial\Omega$ ).

In the supercritical case, problems (i) - (iii),  $\Omega$  of class  $C^2$ , have been resolved in several stages. We say that a compact set  $E \subset \partial\Omega$  is removable relative to equation (1.1) if there exists no positive solution vanishing on  $\partial\Omega \setminus E$ . We say that  $E$  is conditionally removable if any solution  $u$  of (1.2), with  $\mu \in \mathfrak{M}(\partial\Omega)$ , such that  $u = 0$  on  $\partial\Omega \setminus E$  must vanish in  $\Omega$ .

With respect to problem (ii) it was shown that a compact set  $E \subset \partial\Omega$  is removable if and only if  $C_{\frac{2}{q}, q'}(E) = 0$ ,  $q' = q/(q-1)$ . Here  $C_{\alpha, p}$  denotes the Bessel capacity, with the indicated indexes on  $\partial\Omega$ . (see Section 4.2 for an overview of Bessel capacities). This result was obtained by Le Gall [15] for  $q = 2$ , Dynkin and Kuznetsov [8] for  $1 < q \leq 2$ , Marcus and Véron [19] for  $q > 2$ . For a unified analytic proof, covering all  $q \geq q_c$  see [20].

The above result implies that every  $q$ -good measure  $\mu$  must vanish on sets of  $C_{\frac{2}{q}, q'}$  capacity zero. On the other hand a result of Baras and Pierre [3] implies that every positive measure  $\mu \in \mathfrak{M}(\partial\Omega)$  that vanishes on sets of  $C_{\frac{2}{q}, q'}$  capacity zero is the limit of an increasing sequence of admissible measures and therefore  $q$ -good. In conclusion: a measure  $\mu \in \mathfrak{M}(\partial\Omega)$  is  $q$ -good if and only if it vanishes on sets of  $C_{\frac{2}{q}, q'}$  capacity zero. This takes care of problem (i).

Problem (iii) has been treated in several papers, with various definitions of a generalized boundary trace for positive solutions of (1.1), see [9] and [22]. Finally a full characterization of positive solutions was obtained by Mselati [24] for  $q = 2$ , Dynkin [7] for  $1 < q < 2$  and Marcus [17] for every  $q \geq q_c$ . In [24, 7] the restriction to  $q \leq 2$  was dictated by their use of probabilistic techniques that do not apply to  $q > 2$ . In [17] the proof is purely analytic.

If  $\Omega$  is Lipschitz,  $\xi \in \partial\Omega$ , we say that  $q_\xi$  is the critical value for (1.1) at  $\xi$  if, for  $1 < q < q_\xi$ , problem (1.2) with  $\mu = \delta_\xi$  has a solution, but for  $q > q_\xi$  no such solution exists.

In contrast to the case of smooth domains, when  $\Omega$  is Lipschitz,  $q_\xi$  may vary with the point. For every compact set  $F \subset \partial\Omega$  there exists a number  $q(F) > 1$  such that, for  $1 < q < q(F)$ , every measure in  $\mathfrak{M}(\partial\Omega)$  supported in  $F$  is  $q$ -good. Obviously  $q(F) \leq \min\{q_\xi : \xi \in F\}$  but it is not clear if equality holds.

In the special case when  $\Omega$  is a polyhedron, the function  $\xi \rightarrow q_\xi$  obtains only a finite number of values (in fact, it is constant on each open face and each open edge) and, if  $q \geq q_\xi$ , an isolated singularity at  $\xi$  is removable. Furthermore, the assumption  $1 < q < \min\{q_\xi : \xi \in \partial\Omega\}$  implies that every measure in  $\mathfrak{M}(\partial\Omega)$  is  $q$ -good. For this and related results see [23].

In the present paper we study problem (1.2) when  $\Omega$  is a polyhedron and  $q$  is supercritical, i.e.  $q \geq \min\{q_\xi : \xi \in \partial\Omega\}$ . Following is a description of the main results.

**A.** *On the action of Poisson type kernels with fractional dimension.*

In preparation for the study of supercritical boundary value problems we establish an harmonic analytic result, extending a well known result on the action of Poisson kernels on Besov spaces with negative index (see [27, 1.14.4.] and [4]). We first quote the classical result for comparison purposes.

**Proposition 1.1.** *Let  $1 < q < \infty$  and  $s > 0$ . Then, for any bounded Borel measure  $\mu$  in  $\mathbb{R}^{n-1}$ ,*

$$(1.6) \quad I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|_{B^{-s,q}(\mathbb{R}^{n-1})}^q.$$

Here  $\mathbb{K}_n[\mu]$  denotes the Poisson potential of  $\mu$  in  $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ , namely,

$$(1.7) \quad \mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{n-1}} \frac{d\mu(z)}{(y_1^2 + |\zeta - z|^2)^{n/2}} \quad \forall y = (y_1, \zeta) \in \mathbb{R}_+^n$$

where  $\gamma_n$  is a constant depending only on  $n$ .

*Notation.* Let  $m$  be a positive integer and let  $\nu$  be a real number,  $\nu \geq m+1$ . Denote,

$$(1.8) \quad \mathbb{K}_{\nu,m}[\mu](\tau, \zeta) := \int_{\mathbb{R}^m} \frac{\tau^{\nu-m} d\mu(z)}{(\tau^2 + |\zeta - z|^2)^{\nu/2}} \quad \forall \tau \in (0, \infty), \zeta \in \mathbb{R}^m.$$

Note that

$$\mathbb{K}_n[\mu] = \gamma_n \mathbb{K}_{n,n-1}[\mu].$$

**Theorem 1.2.** *Let  $m$  and  $\nu$  be as above. Then, for every  $q > 1$  and every  $s \in (0, m/q')$ ,  $q' = q/(q-1)$ , there exists a positive constant  $c$  such that, for every positive measure  $\mu \in \mathfrak{M}(\mathbb{R}^m)$  supported in  $B_{R/2}(0)$  for some  $R > 1$ ,*

$$(1.9) \quad \frac{1}{c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq \int_0^R \left( \int_{|\zeta| < R} |\mathbb{K}_{\nu,m}[\mu](\tau, \zeta)|^q d\zeta \right) \tau^{sq-1} d\tau \\ \leq c R^{(s+\nu-m)q+1} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q.$$

*This also holds when  $s = m/q'$ , provided that the diameter of  $\text{supp } \mu$  is sufficiently small.*

This is proved in Section 3 (see Theorem 3.8) using a slightly different notation.

**B.** *The critical value and the characterization of  $q$ -good measures in a  $k$ -wedge.*

The next step towards the study of boundary value problems in a polyhedron is the treatment of such problems in a  $k$ -wedge (or  $k$ -dihedron) i.e., the domain defined by the intersection of  $k$  hyperplanes in  $\mathbb{R}^N$ ,  $1 < k < N$ . The edge is an  $(N-k)$  dimensional space.

We note that if  $k = N$  the 'edge' is a point and the corresponding wedge is a cone with vertex at this point. If  $k = 1$  the wedge is a half space. Both of these cases have been treated in [23].

Let  $A$  be a Lipschitz domain in  $S^{k-1}$ . If

$$(1.10) \quad S_A := \{x \in \mathbb{R}^N : |x| = 1, x \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}$$

then

$$D_A := \{x = (r, \sigma) : r > 0, \sigma \in S_A\}$$

is a  $k$ -wedge in  $\mathbb{R}^N$  whose ‘edge’  $d_A$  may be identified with  $\mathbb{R}^{N-k}$  and its ‘opening’ is  $A$ .

Let  $\lambda_A$  be the first eigenvalue of  $-\Delta_{S^{N-1}}$  in  $W_0^{1,2}(S_A)$  and denote by  $\kappa_{\pm}$  the roots of the equation,

$$(1.11) \quad \kappa^2 + (N-2)\kappa - \lambda_A = 0.$$

Put

$$(1.12) \quad q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}$$

and

$$(1.13) \quad q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}.$$

Let  $C_{\alpha,p}^{N-k}$  denote the Bessel capacity with the indicated indices in  $\mathbb{R}^{N-k}$ . The next theorem provides a characterization of  $q$ -good measures supported on  $d_A$ .

**Theorem 1.3.** (a) *If  $1 < q < q_c$  every measure in  $\mathfrak{M}(d_A)$  is  $q$ -good relative to  $D_A$ . In fact every such measure is  $q$ -admissible.*

(b) *If  $q \geq q_c^*$ , the only  $q$ -good measure in  $\mathfrak{M}(d_A)$  is the zero measure.*

(c) *If  $q_c \leq q < q_c^*$ , a measure  $\mu \in \mathfrak{M}(d_A)$  is  $q$ -good relative to  $D_A$  if and only if  $\mu$  vanishes on every Borel set  $E \subset d_A$  such that  $C_{s,q'}^{N-k}(E) = 0$ ,  $s = 2 - \frac{k+\kappa_+}{q}$ .*

The characterization of  $q$ -good measures in a polyhedron follows as an easy consequence of the above theorem (see Theorem 4.6 below).

### C. Characterization of removable sets.

Let  $\Omega$  be an  $N$ -dimensional polyhedron. Theorem 1.3 provides a necessary and sufficient condition for the removability of a singular set  $E$  relative to the family of solutions  $u$  such that

$$\int_{\Omega} |u|^q \rho dx < \infty.$$

The next result provides a necessary and sufficient condition for *removability* in the sense that the only non-negative solution  $u \in C(\bar{\Omega} \setminus E)$  which vanishes on  $\bar{\Omega} \setminus E$  is the trivial solution  $u = 0$ .

Let  $L$  denote a face or edge or vertex of  $\Omega$  and put  $k := \text{codim } L$ . If  $1 < k < N$  let  $d_L$  denote the linear space spanned by  $L$ , such that  $L$  is an open subset of  $d_L$ . Let  $Q_L$  denote the  $k$ -wedge with boundary  $d_L$  such that,

for some neighborhood  $M$  of  $L$ ,  $\Omega \cap M = Q_L \cap M$  and let  $A_L$  denote the opening of  $Q_L$ . If  $k = N$ ,  $Q_L$  is a cone with vertex  $L$ . Let  $q_c(L)$  and  $q_c^*(L)$  be defined as in (1.12) and (1.13) for  $A = A_L$ . Finally let

$$s(L) = 2 - \frac{k + \kappa_+}{q'}$$

where  $\kappa_{\pm}$  are the roots of (1.11) for  $A = A_L$ . If  $k = N$ ,  $Q_L$  is a cone with vertex  $L$ . In this case  $q_c(L) = q_c^*(L) = 1 - \frac{2}{\kappa_-}$ . If  $k = 1$   $q_c(L) = q_c^*(L) = (N + 1)/(N - 1)$ .

**Theorem 1.4.** *Let  $\Omega$  be a polyhedron in  $\mathbb{R}^N$ . A compact set  $E \subset \partial\Omega$  is removable if and only if, for every  $L$  as above such that  $E \cap L \neq \emptyset$  the following conditions hold.*

*If  $1 \leq k < N$ : either  $q_c(L) \leq q < q_c^*(L)$  and  $C_{s(L), q'}^{N-k}(E \cap L) = 0$  or  $q \geq q_c^*(L)$ .  
If  $k = N$ :  $q \geq q_c(L)$ .*

The present paper is part of an article, ‘Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains’ arXiv:0907.1006 (2009). The first part of this article was published in [23]. The second and last part are presented here. The characterization of  $q$ -good measures, here established in polyhedrons, was recently established in [2], for arbitrary Lipschitz domains and a general family of nonlinearities. However the full removability result, Theorem 4.11, has not been superseded. (In [2] the authors provided - in the generality mentioned above - a characterization of *conditional removability* but not of full removability.) The methods of proof in the two papers are completely different. In the present paper, the characterization of  $q$ -good measures is based on an extension of a result of [4] and [27, 1.14.4.] on the action of Poisson kernels on Besov spaces with negative index. In [2] the proof relies on a relation between elliptic semilinear equations with absorption and linear Schrödinger equations.

## 2. THE MARTIN KERNEL AND CRITICAL VALUES IN A $K$ -DIMENSIONAL DIHEDRON.

**2.1. The geometric framework.** An  $N$ -dim polyhedra  $P$  is a bounded domain bordered by a finite number of hyperplanes. Thus the boundary of  $P$  is the union of a finite number of sets  $\{L_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$  where  $\{L_{1,j}\}$  is the set of open faces of  $P$ ,  $\{L_{k,j}\}$  for  $k = 2, \dots, N - 1$ , is the family of relatively open  $N - k$ -dimensional edges and  $\{L_{N,j}\}$  is the family of vertices of  $P$ . An  $N - k$  dimensional edge is a relatively open set in the intersection of  $k$  hyperplanes; it will be described by the characteristic angles of these hyperplanes.

We recall that the spherical coordinates in  $\mathbb{R}^N = \{x = (x_1, \dots, x_N)\}$  are expressed by

$$(2.1) \quad \begin{cases} x_1 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \sin \theta_1 \\ x_2 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \cos \theta_1 \\ x_3 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \cos \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{N-1} = r \sin \theta_{N-1} \cos \theta_{N-2}, \\ x_N = r \cos \theta_{N-1} \end{cases}$$

where,  $r = |x|$ ,  $\theta_1 \in [0, 2\pi]$  and  $\theta_\ell \in [0, \pi]$  for  $\ell = 2, 3, \dots, N-1$ . We denote  $\sigma = (\theta_1, \dots, \theta_{N-1})$ . Thus in spherical coordinates  $x = (r, \sigma)$ .

We consider an unbounded *non-degenerate k-dihedron*,  $2 \leq k \leq N$  defined as follows. Let  $A$  be given by

$$A = \begin{cases} (0, \alpha_1) \times \prod_{j=2}^{k-1} (\alpha_j, \alpha'_j) & \text{if } k > 2 \\ (0, \alpha_1) & \text{if } k = 2 \end{cases}$$

where

$$0 < \alpha_1 < 2\pi, \quad 0 \leq \alpha_j < \alpha'_j < \pi \quad j = 2, \dots, k-1.$$

We denote by  $S_A$  the spherical domain

$$(2.2) \quad S_A = \{x \in \mathbb{R}^N : |x| = 1, \sigma \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}$$

and by  $D_A$  the corresponding k-dihedron,

$$D_A = \{x = (r, \sigma) : r > 0, \sigma \in S_A\}.$$

The *edge* of  $D_A$  is the  $(N-k)$ -dimensional space

$$(2.3) \quad d_A = \{x : x_1 = x_2 = \dots = x_k = 0\}.$$

**2.2. On the Martin kernel and critical values in a cone.** We recall here some elements of local analysis when  $\Omega = C_A \cap B_1$ ,  $A$  is a Lipschitz domain in  $S^{N-1}$  and  $C_A$  is the cone with vertex 0 and opening  $A$ .

Denote by  $\lambda_A$  the first eigenvalue and by  $\phi_A$  the first eigenfunction of  $-\Delta'$  in  $W_0^{1,2}(A)$  (normalized by  $\max \phi_A = 1$ ). Let  $\kappa_-$  be the negative root of (1.11) and put

$$\Phi_1(x) := \frac{1}{\gamma} |x|^{\kappa_-} \phi_A(x/|x|)$$

where  $\gamma$  is a positive number. Then  $\Phi_1$  is a harmonic function in  $C_A$  vanishing on  $\partial C_A \setminus \{0\}$ . We choose  $\gamma = \gamma_A$  so that the boundary trace of  $\Phi_1$  is  $\delta_0$  (=Dirac measure on with mass 1 at the origin).

(i) If  $q \geq 1 - \frac{2}{\kappa_-}$ , there is no solution of (1.1) in  $\Omega_S$  with isolated singularity at 0. (See [10].)



(ii) If  $1 < q < 1 - \frac{2}{\kappa_-}$ , then for any  $k > 0$  there exists a unique solution  $u := u_k$  to problem (1.2) with  $\mu = k\delta_0$  and

$$(2.4) \quad u_k(x) = k\Phi_1(x)(1 + o(1)) \quad \text{as } x \rightarrow 0.$$

The function  $u_\infty = \lim_{k \rightarrow \infty} u_k$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies

$$(2.5) \quad u_\infty(x) = |x|^{-\frac{2}{q-1}} \omega_A(x/|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0$$

where  $\omega_A$  is the (unique) positive solution of

$$(2.6) \quad -\Delta' \omega - a_{N,q} \omega + |\omega|^{q-1} \omega = 0$$

on  $S^{N-1}$ . Here  $\Delta'$  is the Laplace - Beltrami operator and

$$(2.7) \quad a_{N,q} = \frac{2}{q-1} \left( \frac{2q}{q-1} - N \right).$$

(iii) If  $u \in C(\bar{\Omega}_A \setminus \{0\})$  is a positive solution of (1.1) vanishing on  $(\partial C_A \cap B_{r_0}(0)) \setminus \{0\}$ , then either  $u$  satisfies (2.4) for some  $k > 0$  or  $u$  satisfies (2.5). In particular there exists a unique positive solution vanishing on  $(\partial C_A \cap B_{r_0}(0)) \setminus \{0\}$  with strong singularity at 0. (For (ii) and (iii) see [23, Theorem 5.7].)

**2.3. Separable harmonic functions and the Martin kernel in a k-dihedron,  $2 \leq k < N$ .** In the system of spherical coordinates, the Laplacian takes the form

$$\Delta u = \partial_{rr} u + \frac{N-1}{r} \partial_r u + \frac{1}{r^2} \Delta_{S^{N-1}} u$$

where the Laplace-Beltrami operator  $\Delta_{S^{N-1}}$  is expressed by induction by

$$(2.8) \quad \begin{aligned} \Delta_{S^{N-1}} u &= \frac{1}{(\sin \theta_{N-1})^{N-2}} \frac{\partial}{\partial \theta_{N-1}} \left( (\sin \theta_{N-1})^{N-2} \frac{\partial u}{\partial \theta_{N-1}} \right) \\ &+ \frac{1}{(\sin \theta_{N-1})^2} \Delta_{S^{N-2}} u. \end{aligned}$$

and

$$(2.9) \quad \Delta_{S^1} u = \partial_{\theta_1 \theta_1} u$$

If we compute the positive harmonic functions in the k-dihedron  $D_A$  of the form

$$v(x) = v(r, \sigma) = r^\kappa \omega(\sigma) \quad \text{in } D_A, \quad v = 0 \quad \text{in } \partial D_A \setminus \{0\}.$$

we find that  $\omega$  must be a positive eigenfunction corresponding to the first eigenvalue,  $\lambda_A$ , of  $-\Delta_{S^{N-1}}$  in  $W_0^{1,2}(S_A)$ ,

$$(2.10) \quad \begin{cases} \Delta_{S^{N-1}} \omega + \lambda_A \omega = 0 & \text{in } S_A \\ \omega = 0 & \text{on } \partial S_A \end{cases}$$

and  $\kappa$  must be a root of the algebraic equation (1.11) with  $\lambda_A$  as above. Thus  $\kappa = \kappa_{\pm}$  where

$$(2.11) \quad \begin{aligned} \kappa_+ &= \frac{1}{2} \left( 2 - N + \sqrt{(N-2)^2 + 4\lambda_A} \right) \\ \kappa_- &= \frac{1}{2} \left( 2 - N - \sqrt{(N-2)^2 + 4\lambda_A} \right). \end{aligned}$$

Since

$$S^{N-1} = \{ \sigma = (\sigma_2 \sin \theta_{N-1}, \cos \theta_{N-1}) : \sigma_2 \in S^{N-2}, \theta_{N-1} \in (0, \pi) \},$$

we look for a solution  $\omega = \omega^{\{1\}}$  of (2.10) of the form

$$\omega^{\{1\}}(\sigma) = (\sin \theta_{N-1})^{\kappa_+} \omega^{\{2\}}(\sigma_2), \quad \theta_{N-1} \in (0, \pi), \quad \sigma_2 \in S^{N-2}.$$

Here  $S^{N-2} = S^{N-1} \cap \{x_N = 0\}$  and we denote

$$S_A^{\{N-2\}} = S_A \cap \{x_N = 0\}, \quad D_A^{\{N-2\}} := D_A \cap \{x_N = 0\} \subset \mathbb{R}^{N-1}.$$

Then (2.11) jointly with relation (2.8) implies

$$(2.12) \quad \begin{cases} \Delta_{S^{N-2}} \omega^{\{2\}} + (\lambda_A - \kappa_+) \omega^{\{2\}} = 0 & \text{on } S_A^{\{N-2\}} \\ \omega^{\{2\}} = 0 & \text{on } \partial S_A^{\{N-2\}}. \end{cases}$$

Since we are interested in  $\omega^{\{2\}}$  positive,  $\lambda_A^{\{2\}} := \lambda_A - \kappa_+$  must be the first eigenvalue of  $-\Delta_{S^{N-2}}$  in  $W_0^{1,2}(S_A^{\{N-2\}})$ .

Next we look for positive harmonic functions  $\tilde{u}$  in  $D_A^{\{N-2\}}$  such that

$$\tilde{u}(x_1, \dots, x_{N-1}) = r^{\kappa'} \omega(\sigma_2), \quad \tilde{u} = 0 \text{ on } \partial D_A^{\{N-2\}}$$

The algebraic equation which gives the exponents is

$$(\kappa')^2 + (N-3)\kappa' - \lambda_A^{\{2\}} = 0.$$

Denote by  $\kappa'_+$  the positive root of this equation. By the definition of  $\lambda_A^{\{2\}}$ ,

$$\kappa_+^2 + (N-3)\kappa_+ - \lambda_A^{\{2\}} = \kappa_+^2 + (N-2)\kappa_+ - \lambda_A = 0.$$

Therefore  $\kappa'_+ = \kappa_+$ . Accordingly, if  $k \geq 3$ , we set

$$\omega^{\{2\}}(\sigma_2) = (\sin \theta_{N-2})^{\kappa_+} \omega^{\{3\}}(\sigma_3),$$

and find that  $\omega^{\{3\}}$  satisfies

$$(2.13) \quad \begin{cases} \Delta_{S^{N-3}} \omega^{\{3\}} + (\lambda_A - 2\kappa_+) \omega^{\{3\}} = 0 & \text{in } S_A^{\{N-3\}} \\ \omega^{\{3\}} = 0 & \text{on } \partial S_A^{\{N-3\}}, \end{cases}$$

where

$$S_A^{\{N-3\}} = S_A \cap \{x_N = x_{N-1} = 0\}.$$

Performing this reduction process (N-k) times, we obtain the following results.

(i) If  $k > 2$  then  $\omega = \omega^{N-k}(\sigma)$  is given by

$$(2.14) \quad \omega(\sigma) = (\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k)^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})$$

where

$$\sigma_{N-k+1} \in S^{k-1} = S^{N-1} \cap \{x_N = x_{N-1} = \dots = x_{k+1} = 0\}$$

and  $\omega' := \omega^{\{N-k+1\}}$  satisfies

$$(2.15) \quad \begin{cases} \Delta_{S^{k-1}} \omega' + (\lambda_A - (N-k)\kappa_+) \omega' = 0, & \text{in } S_A^{\{k-1\}} \\ \omega' = 0, & \text{on } \partial S_A^{\{k-1\}}, \end{cases}$$

where  $S_A^{\{k-1\}} = S_A \cap \{x_N = x_{N-1} = \dots = x_{k+1} = 0\} \approx A$  and  $\lambda_A - (N-k)\kappa_+$  is the first eigenvalue of the problem.

(ii) If  $k = 2$  then

$$(2.16) \quad \omega(\sigma) = (\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2)^{\kappa_+} \omega^{\{N-1\}}(\theta_1)$$

where  $\sigma_{N-1} \in S^1 \approx \theta_1 \in (0, 2\pi)$ , and  $\omega^{\{N-1\}}$  satisfies

$$(2.17) \quad \begin{cases} \Delta_{S^1} \omega^{\{N-1\}} + (\lambda_A - (N-2)\kappa_+) \omega^{\{N-1\}} = 0 & \text{on } S_A^{\{1\}} \\ \omega^{\{N-1\}} = 0 & \text{on } \partial S_A^{\{1\}}, \end{cases}$$

with  $\partial S_A^{\{1\}} \approx (0, \alpha)$ . In this case

$$(2.18) \quad \kappa_+ = \frac{\pi}{\alpha}, \quad \omega^{\{N-1\}}(\theta_1) = \sin(\pi\theta_1/\alpha),$$

and, by (1.11),

$$(2.19) \quad \lambda_A - (N-2)\kappa_+ = \frac{\pi^2}{\alpha^2} \implies \lambda_A = \frac{\pi^2}{\alpha^2} + (N-2)\frac{\pi}{\alpha}.$$

Observe that  $\frac{1}{2} \leq \kappa_+$  with equality holding only in the degenerate case  $\alpha = 2\pi$  (which we exclude).

In either case, we find a positive harmonic function  $v_A$  in  $D_A$ , vanishing on  $\partial D_A$ , of the form

$$(2.20) \quad v_A(x) = |x|^{\kappa_+} \omega(x/|x|)$$

with  $\omega$  as in (2.14) (for  $k > 2$ ) or (2.18) (for  $k=2$ ). Furthermore, if  $\Omega$  is a domain in  $\mathbb{R}^N$  such that, for some  $R > 0$ ,  $\Omega \cap B_R(0) = D_A \cap B_R(0)$  and  $w$  is a positive harmonic function in  $\Omega$  vanishing on  $d_A \cap B_R(0)$  then  $w \sim v_A$  in  $\Omega \cap B_{R'}(0)$  for every  $R' \in (0, R)$ .

Similarly we find a positive harmonic function in  $D_A$  vanishing on  $\partial D_A \setminus \{0\}$ , singular at the origin, of the form

$$K'_A(x) = |x|^{\kappa_-} \omega(x/|x|).$$

If  $\Omega$  is a domain as above and  $z$  is a positive harmonic function in  $\Omega$  vanishing on  $d_A \cap B_R(0) \setminus \{0\}$  then  $z \sim K'_A$  in  $\Omega \cap B_{R'}(0) \setminus \{0\}$  for every  $R' \in (0, R)$ .

As  $K'_A$  is a kernel function of  $-\Delta$  at 0 it follows that  $K'_A$  is, up to a multiplicative constant  $c_A$ , the Martin kernel of  $-\Delta$  in  $D_A$ , with singularity at 0. The Martin kernel, with singularity at a point  $z \in d_A$ , is given by

$$(2.21) \quad K_A(x, z) = c_A \frac{(\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k)^{\kappa+\omega} \{\sigma_{N-k+1}\}}{|x-z|^{N-2+\kappa+}}$$

for every  $x \in D_A$ . From (2.1)

$$\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k = |x-z|^{-1} \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}.$$

Therefore, if we write  $x \in \mathbb{R}^N$  in the form  $x = (x', x'')$ ,  $x' = (x_1, \dots, x_k)$ ,  $x'' = (x_{k+1}, \dots, x_N)$ , we obtain the formula,

$$(2.22) \quad \begin{aligned} K_A(x, z) &= c_A \frac{|x'|^{\kappa+\omega} \{\sigma_{N-k+1}\}}{|x-z|^{(N-2+2\kappa+)}} \\ &= c_A \frac{|x'|^{\kappa+\omega} \{\sigma_{N-k+1}\}}{(|x'|^2 + |x''-z|^2)^{(N-2+2\kappa+)/2}}. \end{aligned}$$

Therefore, the Poisson potential of a measure  $\mu \in \mathfrak{M}(d_A)$  is expressed by

$$(2.23) \quad \begin{aligned} \mathbb{K}_A[\mu](x) &= c_A |x'|^{\kappa+\omega} \{\sigma_{N-k+1}\} \\ &\quad \times \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(|x'|^2 + |x''-z|^2)^{(N-2+2\kappa+)/2}}. \end{aligned}$$

**2.4. The admissibility condition.** Consider the boundary value problem

$$(2.24) \quad \begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } D_A \\ u = \mu \in \mathfrak{M}(\partial D_A). \end{cases}$$

Let

$$(2.25) \quad \Gamma_R = \{x = (x', x'') : |x'| \leq R, |x''| \leq R\}, \quad D_{A,R} := D_A \cap \Gamma_R$$

and let  $\rho_{R,A}$  denote the first (positive) eigenfunction in  $D_{A,R} := D_A \cap \Gamma_R$ . In the rest of this section we drop the index  $A$  in  $K_A$ ,  $\rho_{A,R}$  etc., except for  $D_A$ ,  $D_{A,R}$  and  $d_A$ .

First we observe that a positive Radon measure on  $d_A$  is  $q$ -good relative to  $D_A$  if and only if, for every compact set  $F \subset d_A$ ,  $\mu \chi_F$  is  $q$ -good in  $D_A$ .

Now suppose that  $\mu$  is compactly supported in  $d_A$  and denote its support by  $F$ . We claim that  $\mu$  is  $q$ -good in  $D_A$  if and only if it is  $q$ -good relative to  $D_{A,R}$  for all sufficiently large  $R$ . Let  $R$  be such that  $F \subset B_{R/2}^{N-k}(0)$ . Assume that  $\mu$  is  $q$ -good in  $D_{A,R}$ . Let  $v_R$  be the solution of (1.1) in  $D_{A,R}$  such that  $v_R = \mu$  on  $d_A \cap \Gamma_R$ ,  $v_R = 0$  on  $\partial D_{A,R} \setminus d_A$ . Then  $v_R$  increases with  $R$  and  $v = \lim_{R \rightarrow \infty} v_R$  is a solution of (1.1) in  $D_A$  with boundary data  $\mu$ . This proves our claim in one direction; the other direction is obvious.

The condition for  $\mu$  to be  $q$ -admissible in  $D_{A,R}$  is

$$(2.26) \quad \int_{D_{A,R}} \mathbb{K}^R[|\mu|](x)^q \rho_R(x) dx < \infty.$$

where  $K^R$  is the Martin kernel of  $-\Delta$  in  $D_{A,R}$ . If  $R$  is sufficiently large then, in a neighborhood of  $F$ ,  $K^R \sim K$  and  $\rho^R \sim \rho \sim v_A$ . Therefore, a sufficient condition for  $\mu$  to be  $q$ -good in  $D_A$  is

$$(2.27) \quad \int_{\Gamma_R \cap D_A} \mathbb{K}[|\mu|](x)^q \rho(x) dx < \infty \quad \forall R > 0.$$

By the first observation in this subsection, it follows that the previous statement remains valid for any positive Radon measure supported on  $d_A$ .

By (2.21),

$$(2.28) \quad \mathbb{K}[|\mu|](x) \leq c_A (r')^{\kappa_+} \int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z)$$

where

$$(2.29) \quad j(x) = |x|^{-N+2-2\kappa_+} \quad \forall x \in \mathbb{R}^N.$$

Therefore, using (2.20), condition (2.27) becomes

$$(2.30) \quad \int_0^R \int_{|x''| < R} \left( \int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z) \right)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty$$

for every  $R > 0$ .

## 2.5. The critical values. Relative to the equation

$$(2.31) \quad -\Delta u + |u|^{q-1} u = 0$$

there exist two thresholds of criticality associated with the edge  $d_A$ .

The first is the value  $q_c^*$  such that, for  $q_c^* \leq q$  the whole edge  $d_A$  is removable but for  $1 < q < q_c^*$  there exist non-trivial solutions in  $D_A$  which vanish on  $\partial D_A \setminus d_A$ . The second  $q_c < q_c^*$  corresponds to the removability of points on  $d_A$ . For  $q \geq q_c$  points on  $d_A$  are removable while for  $1 < q < q_c$  there exist solutions with isolated point singularities on  $d_A$ . In the next two propositions we determine these critical values.

**Proposition 2.1.** *Assume  $q > 1$ ,  $1 \leq k < N$ . Then the condition*

$$(2.32) \quad q < q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}$$

*is necessary and sufficient for the existence of a non-trivial solution  $u$  of (2.31) in  $D_A$  which vanishes on  $\partial D_A \setminus d_A$ . Furthermore, when this condition holds, there exist non-trivial positive bounded measures  $\mu$  on  $d_A$  such that  $\mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A)$ .*

*Remark.* The statement remains true for  $k = N$ , which is the case of the cone. In this case  $q_c = q_c^* = 1 - (2/\kappa_-)$  and a straightforward computation yields:

$$(2.33) \quad q_c = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}}.$$

*Proof.* Recall that  $\lambda_A - (N - k)\kappa_+$  is the first eigenvalue in  $S_A^{\{k-1\}}$  (see (2.15) and the remarks following it). Let  $\kappa'_+, \kappa'_-$  be the two roots of the equation

$$X^2 + (k - 2)X - (\lambda_A - (N - k)\kappa_+) = 0,$$

i.e.

$$\kappa'_\pm = \frac{1}{2}(2 - k \pm \sqrt{(k - 2)^2 + 4(\lambda_A - (N - k)\kappa_+)}).$$

Then, by [23, Theorem 5.7], recalled in subsection 2.2, if  $1 < q < 1 - (2/\kappa'_-)$  there exists a unique solution of (2.31) in the cone  $C_{S_A^{k-1}}$  i.e. the cone with opening  $S_A^{k-1} \subset S^{k-1} \subset \mathbb{R}^k$  with trace  $a\delta_0$  (where  $\delta_0$  denotes the Dirac measure at the vertex of the cone and  $a > 0$ ). By (2.5) this solution satisfies

$$(2.34) \quad u_a(x) = a|x|^{-\alpha} \phi(x/|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0,$$

where  $\phi$  is the first positive eigenfunction of  $-\Delta'$  in  $W_0^{1,2}(S_A^{k-1})$  normalized so that  $u_1$  possesses trace  $\delta_0$ .

The function  $u$  given by

$$\tilde{u}_a(x', x'') = u_a(x') \quad \forall (x', x'') \in D_A = C_{S_A^{k-1}} \times \mathbb{R}^{N-k},$$

is a nonzero solution of (2.31) in  $D_A$  which vanishes on  $\partial D_A \setminus d_A$  and has bounded trace on  $d_A$ .

A simple calculation shows that  $1 - (2/\kappa'_-)$  equals  $q_c^*$  as given in (2.32).

Next, assume that  $q \geq q_c^*$  and let  $u$  be a solution of (2.31) in  $D_A$  which vanishes on  $\partial D_A \setminus d_A$ .

Given  $\epsilon > 0$  let  $v_\epsilon$  be the solution of (2.31) in  $D_A^{\{N-k-1\}} \setminus \{x' \in \mathbb{R}^k : |x'| \leq \epsilon\}$  such that

$$v_\epsilon(x') = \begin{cases} 0, & \text{if } x' \in \partial D_A^{\{N-k-1\}}, |x'| > \epsilon, \\ \infty, & \text{if } |x'| = \epsilon. \end{cases}$$

Given  $R > 0$  let  $w_R$  be the maximal solution in  $\{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$ .

Then the function  $u^*$  given by

$$u^*(x', x'') = v_\epsilon(x') + w_R(x'')$$

is a supersolution of (2.31) in  $D_A \setminus \{(x', x'') : |x'| > \epsilon, |x''| < R\}$  and it dominates  $u$  in this domain. But  $w_R(x'') \rightarrow 0$  as  $R \rightarrow \infty$  and, by [10],  $v_\epsilon(x') \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Therefore  $u_+ = 0$  and, by the same token,  $u_- = 0$ .  $\square$

**Proposition 2.2.** *Let  $A$  be defined as before. Then*

$$(2.35) \quad \mathbb{K}[\mu] \in L_\rho^q(\Gamma_R \cap D_A) \quad \forall \mu \in \mathfrak{M}(d_A), \quad \forall R > 0$$

*if and only if*

$$(2.36) \quad 1 < q < q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}.$$

*This statement is equivalent to the following:*

*Condition (2.36) is necessary and sufficient in order that the Dirac measure  $\mu = \delta_P$ , supported at a point  $P \in d_A$ , satisfy (2.35).*

*Proof.* It is sufficient to prove the result relative to the family of measures  $\mu$  such that  $\mu$  is positive, has compact support and  $\mu(d_A) = 1$ . Let  $R > 1$  be sufficiently large so that the support of  $\mu$  is contained in  $\Gamma_{R/2}$ . The measure  $\mu$  can be approximated (in the sense of weak convergence of measures) by a sequence  $\{\mu_n\}$  of convex combinations of Dirac measures supported in  $d_A \cap \Gamma_{R/2}$ . For such a sequence  $\mathbb{K}[\mu_n] \rightarrow \mathbb{K}[\mu]$  pointwise and  $\{\mathbb{K}[\mu_n]\}$  is uniformly bounded in  $D_A \setminus \Gamma_{3R/4}$ . Therefore it is sufficient to prove the result when  $\mu = \delta_0$ . In this case the admissibility condition (1.5) is

$$\int_0^R \int_{|x''| < R} j(x)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty,$$

i.e.,

$$\int_0^R \int_0^R |x|^{q(2-N-2\kappa_+)} (r')^{(q+1)\kappa_+ + k - 1} (r'')^{N-k-1} dr'' dr' < \infty.$$

Substituting  $\tau := r''/r'$  the condition becomes

$$\int_0^R \int_0^{R/r'} (1 + \tau^2)^{\frac{q}{2}(2-N-2\kappa_+)} (r')^{q(2-N-\kappa_+) + \kappa_+ + N - 1} \tau^{N-k-1} d\tau dr' < \infty.$$

This holds if and only if  $q < (\kappa_+ + N)/(\kappa_+ + N - 2)$ .  $\square$

*Remark.* It is interesting to notice that  $k$  does not appear explicitly in (2.36). Furthermore, we observe that

$$(2.37) \quad \frac{2}{q_c - 1} \left( \frac{2q_c}{q_c - 1} - N \right) = \lambda_A \iff \kappa_+ (\kappa_+ + N - 2) = \lambda_A,$$

which follows from (2.11). This implies that there does not exist a nontrivial solution of the nonlinear eigenvalue problem

$$(2.38) \quad \begin{aligned} -\Delta_{S^{N-1}} \psi - \frac{2}{q-1} \left( \frac{2q}{q-1} - N \right) \psi + |\psi|^{q-1} \psi &= 0 \quad \text{in } S_{D_A} \\ \psi &= 0 \quad \text{in } \partial S_{D_A} \end{aligned}$$

which, in turn, implies that there does not exist a nontrivial solution of (2.31) of the form  $u(x) = u(r, \sigma) = |x|^{-2/(q-1)} \psi(\sigma)$ , and also no solution of this equation in  $D_A$  which vanishes on  $\partial D_A \setminus \{0\}$ . This is the classical ansatz for the removability of isolated singularities in  $d_A$ .

### 3. THE HARMONIC LIFTING OF A BESOV SPACE $B^{-s,p}(d_A)$ .

Denote by  $W^{\sigma,p}(\mathbb{R}^\ell)$  ( $\sigma > 0$ ,  $1 \leq p \leq \infty$ ) the Sobolev spaces over  $\mathbb{R}^\ell$ . In order to use interpolation, it is useful to introduce the Besov space  $B^{\sigma,p}(\mathbb{R}^\ell)$  ( $\sigma > 0$ ). If  $\sigma$  is not an integer then

$$(3.1) \quad B^{\sigma,p}(\mathbb{R}^\ell) = W^{\sigma,p}(\mathbb{R}^\ell).$$

If  $\sigma$  is an integer the space is defined as follows. Put

$$\Delta_{x,y} f = f(x+y) + f(x-y) - 2f(x).$$

Then

$$(3.2) \quad B^{1,p}(\mathbb{R}^\ell) = \left\{ f \in L^p(\mathbb{R}^\ell) : \frac{\Delta_{x,y}f}{|y|^{1+\ell/p}} \in L^p(\mathbb{R}^\ell \times \mathbb{R}^\ell) \right\},$$

with norm

$$(3.3) \quad \|f\|_{B^{1,p}} = \|f\|_{L^p} + \left( \iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|\Delta_{x,y}f|^p}{|y|^{\ell+p}} dx dy \right)^{1/p},$$

(with standard modification if  $p = \infty$ ) and

$$(3.4) \quad B^{m,p}(\mathbb{R}^\ell) = \left\{ f \in W^{m-1,p}(\mathbb{R}^\ell) : \right. \\ \left. D_x^\alpha f \in B^{1,p}(\mathbb{R}^\ell) \forall \alpha \in \mathbb{N}^\ell, |\alpha| = m-1 \right\}$$

with norm

$$(3.5) \quad \|f\|_{B^{m,p}} = \|f\|_{W^{m-1,p}} + \left( \sum_{|\alpha|=m-1} \iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|D_x^\alpha \Delta_{x,y}f|^p}{|y|^{\ell+p}} dx dy \right)^{1/p}.$$

We recall that the following inclusions hold ([26, p 155])

$$(3.6) \quad \begin{aligned} W^{m,p}(\mathbb{R}^\ell) &\subset B^{m,p}(\mathbb{R}^\ell) && \text{if } p \geq 2 \\ B^{m,p}(\mathbb{R}^\ell) &\subset W^{m,p}(\mathbb{R}^\ell) && \text{if } 1 \leq p \leq 2. \end{aligned}$$

When  $1 < p < \infty$ , the dual spaces of  $W^{s,p}$  and  $B^{m,p}$  are respectively denoted by  $W^{-s,p'}$  and  $B^{-m,p'}$ .

The following is the main result of this section.

**Theorem 3.1.** *Suppose that  $q_c < q < q_c^*$  and let  $A$  be defined as in subsection 2.1. Then there exist positive constants  $c_1, c_2$ , depending on  $q, N, k, \kappa_+$ , such that for any  $R > 1$  and any  $\mu \in \mathfrak{M}_+(d_A)$  with support in  $B_{R/2}$ :*

$$(3.7) \quad \begin{aligned} &c_1 \|\mu\|_{B^{-s,q}(\mathbb{R}^{N-k})}^q \\ &\leq \int_{D_{A,R}} \mathbb{K}[|\mu|^q(x)] \rho(x) dx \leq c_2 (1+R)^\beta \|\mu\|_{B^{-s,q}(\mathbb{R}^{N-k})}^q, \end{aligned}$$

where  $s = 2 - \frac{\kappa_+ + k}{q}$ ,  $\beta = (q+1)\kappa_+ + k - 1$  and  $D_{A,R} = D_A \cap \Gamma_R$ . If  $q = q_c$  the estimate remains valid for measures  $\mu$  such that the diameter of  $\text{supp } \mu$  is sufficiently small (depending on the parameters mentioned before).

*Remark.* When  $q \geq 2$  the norms in the Besov space may be replaced by the norms in the corresponding Sobolev spaces.

Recall the admissibility condition for a measure  $\mu \in \mathfrak{M}_+(d_A)$ :

$$\int_{D_{A,R}} \mathbb{K}[\mu]^q(x) \rho(x) dx < \infty \quad \forall R > 0$$



and the equivalence (see (2.27)–(2.30))

$$(3.8) \quad \int_{D_{A,R}} \mathbb{K}[\mu]^q(x) \rho(x) dx \approx J^{A,R}(\mu) := \int_0^R \int_{B_R''} \left( \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(\tau^2 + |x'' - z|^2)^{(N-2+2\kappa_+)/2}} \right)^q \tau^{(q+1)\kappa_+ + k - 1} dx'' d\tau,$$

where  $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ ,  $\tau = |x'|$  and  $B_R'' = \{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$ . We denote,

$$(3.9) \quad \nu = N - 2 + 2\kappa_+.$$

If  $2\kappa_+$  is an integer, it is natural to relate (3.8) to the Poisson potential of  $\mu$  in  $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}_{n-1}$  where  $n = N - 2 + 2\kappa_+$ . We clarify this statement below.

Assuming that  $2 \leq n + k - N$ , denote

$$y = (y_1, \tilde{y}, y'') \in \mathbb{R}^n, \quad \tilde{y} = (y_2, \dots, y_{n+k-N}), \quad y'' = (y_{n+k-N+1}, \dots, y_n).$$

The Poisson kernel in  $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}_{n-1}$  is given by

$$(3.10) \quad P_n(y) = \gamma_n y_1 |y|^{-n} \quad y_1 > 0,$$

for some  $\gamma_n > 0$ , and the Poisson potential of a bounded Borel measure  $\mu$  with support in

$$\mathbf{d} := \{y = (0, y'') \in \mathbb{R}^n : y'' \in \mathbb{R}_{N-k}\}$$

is

$$(3.11) \quad \mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2)^{n/2}} \quad \forall y \in \mathbb{R}_+^n.$$

In particular, for  $\tilde{y} = 0$ ,

$$(3.12) \quad \mathbb{K}_n[\mu](y_1, 0, y'') = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(y_1^2 + |y'' - z|^2)^{n/2}}.$$

The integral in (3.12) is precisely the same as the inner integral in (3.8).

In fact, it will be shown that, if we set

$$(3.13) \quad n := \{\nu\} = \inf\{m \in \mathbb{N} : m \geq \nu\},$$

this approach also works when  $2\kappa_+$  is not an integer. We note that, for  $n$  given by (3.13),

$$(3.14) \quad n - N + k \geq 2,$$

with equality only if  $k = 3$  and  $\kappa_+ \leq 1/2$  or  $k = 2$  and  $\kappa_+ \in (1/2, 1]$ . Indeed,

$$n - N + k = k + \{2\kappa_+\} - 2$$

and (as  $\kappa_+ > 0$ )  $\{2\kappa_+\} \geq 1$ . If  $k = 2$  then  $\kappa_+ > 1/2$  and consequently  $\{2\kappa_+\} \geq 2$ . These facts imply our assertion.

We also note that  $\kappa_+$  is strictly increasing relative to  $\lambda_A$  and

$$(3.15) \quad \kappa_+ \begin{cases} = 1, & \text{if } D_A = \mathbb{R}_+^N, \\ < 1, & \text{if } D_A \subsetneq \mathbb{R}_+^N, \\ > 1, & \text{if } D_A \supsetneq \mathbb{R}_+^N. \end{cases}$$

Finally we observe that  $\gamma := \lambda_A - (N - k)\kappa_+ > 0$  (see (2.15)) and, by (2.11) and (2.32):

$$(3.16) \quad \gamma = \kappa_+^2 + (k - 2)\kappa_+, \quad q_c^* = 1 + \frac{-(k - 2) + \sqrt{(k - 2)^2 + 4\gamma}}{\gamma}.$$

Therefore  $q_c^*$  is strictly decreasing relative to  $\gamma$  and consequently also relative to  $\kappa_+$ .

The proof of the theorem is based on the following important result proved in [27, 1.14.4.]

**Proposition 3.2.** *Let  $1 < q < \infty$  and  $s > 0$ . Then for any bounded Borel measure  $\mu$  in  $\mathbb{R}^{n-1}$  there holds*

$$(3.17) \quad I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|_{B^{-s,q}(\mathbb{R}^{n-1})}^q.$$

In the first part of the proof we derive inequalities comparing  $I(\mu)$  and  $J^{A,R}(\mu)$ . Actually, it is useful to consider a slightly more general expression than  $I(\mu)$ , namely:

$$(3.18) \quad I_{\nu,\sigma}^{m,j}(\mu) := \int_{\mathbb{R}_+^{m+j}} \left| \int_{\mathbb{R}^m} \frac{y_1 d\mu(z)}{(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2)^{\nu/2}} \right|^q e^{-y_1} y_1^{\sigma q-1} dy,$$

where  $\nu$  is an arbitrary number such that  $\nu > m$ ,  $j \geq 1$  and  $\sigma > 0$ . A point  $y \in \mathbb{R}_+^{m+j}$  is written in the form  $y = (y_1, \tilde{y}, y'') \in \mathbb{R}_+ \times \mathbb{R}^{j-1} \times \mathbb{R}^m$ . We assume that  $\mu$  is supported in  $\mathbb{R}^m$ . Note that,

$$(3.19) \quad I(\mu) = \gamma_n^q I_{n,s}^{m,j} \quad \text{where } m = N - k, \quad j = n - m = n - N + k.$$

Put

$$(3.20) \quad F_{\nu,m}[\mu](\tau) := \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty).$$

With this notation, if  $j \geq 2$  then

$$(3.21) \quad I_{\nu,\sigma}^{m,j}(\mu) := \int_0^\infty \int_{\mathbb{R}^{j-1}} F_{\nu,m}[\mu](\sqrt{y_1^2 + |\tilde{y}|^2}) e^{-y_1} y_1^{(\sigma+1)q-1} d\tilde{y} dy_1$$

and if  $j = 1$

$$(3.22) \quad I_{\nu,\sigma}^{m,1}(\mu) := \int_0^\infty F_{\nu,m}[\mu](y_1) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1$$

**Lemma 3.3.** *Assume that  $m < \nu$ ,  $0 < \sigma$ ,  $2 \leq j$  and  $1 < q < \infty$ . Then there exists a positive constant  $c$ , depending on  $m, j, \nu, \sigma, q$ , such that, for every bounded Borel measure  $\mu$  with support in  $\mathbb{R}^m$ :*

$$(3.23) \quad \frac{1}{c} \int_0^\infty F_{\nu, m}[\mu](\tau) h_{\sigma, j}(\tau) d\tau \leq I_{\nu, \sigma}^{m, j}(\mu) \leq c \int_0^\infty F_{\nu, m}[\mu](\tau) h_{\sigma, j}(\tau) d\tau,$$

where  $F_{\nu, m}$  is given by (3.20) and, for every  $\tau > 0$ ,

$$(3.24) \quad h_{\sigma, j}(\tau) = \begin{cases} \frac{\tau^{(\sigma+1)q+j-2}}{(1+\tau)^{(\sigma+1)q}}, & \text{if } j \geq 2, \\ e^{-\tau} \tau^{(\sigma+1)q-1}, & \text{if } j = 1. \end{cases}$$

*Proof.* There is nothing to prove in the case  $j = 1$ . Therefore we assume that  $j \geq 2$ .

We use the notation  $y = (y_1, \tilde{y}, y'') \in \mathbb{R} \times \mathbb{R}^{j-1} \times \mathbb{R}^m$ . The integrand in (3.21) depends only on  $y_1$  and  $\rho := |\tilde{y}|$ . Therefore,  $I_{\nu, \sigma}^{m, j}$  can be written in the form

$$I_{\nu, \sigma}^{m, j}(\mu) = c_{m, j} \int_0^\infty \int_0^\infty F_{\nu, m}[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1 \rho^{j-2} d\rho.$$

We substitute  $y_1 = (\tau^2 - \rho^2)^{1/2}$ , then change the order of integration and finally substitute  $\rho = r\tau$ . This yields,

$$\begin{aligned} & c_{m, j}^{-1} I_{\nu, \sigma}^{m, j}(\mu) \\ &= \int_0^\infty \int_\rho^\infty F_{\nu, m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2 - \rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau d\tau d\rho \\ &= \int_0^\infty \int_0^\tau F_{\nu, m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2 - \rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau d\rho d\tau \\ &= \int_0^\infty \int_0^1 F_{\nu, m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr d\tau, \end{aligned}$$

where

$$f(r) = r^{j-2} (1 - r^2)^{(\sigma+1)q/2-1}.$$

We denote

$$I_\sigma^j(\tau) = \int_0^1 e^{-\tau\sqrt{1-r^2}} f(r) dr,$$

so that

$$(3.25) \quad I_{\nu, \sigma}^{m, j}(\mu) = c_{m, j} \int_0^\infty F_{\nu, m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} I_\sigma^j(\tau) d\tau.$$

To complete the proof we estimate  $I_\sigma^j$ . Since  $j \geq 2$ ,  $f \in L^1(0, 1)$  and  $I_\sigma^j$  is continuous in  $[0, \infty)$  and positive everywhere. Hence, for every  $\alpha > 0$ , there exists a positive constant  $c_\alpha = c_\alpha(\sigma)$  such that

$$(3.26) \quad \frac{1}{c_\alpha} \leq I_\sigma^j \leq c_\alpha \quad \text{in } [0, \alpha].$$

Next we estimate  $I_\sigma^j$  for large  $\tau$ . Since  $j \geq 2$ ,

$$I_\sigma^j \leq 2^{(\sigma+1)q/2-1} \int_0^1 (1-r)^{(\sigma+1)q/2-1} e^{-\tau\sqrt{1-r}} dr.$$

Substituting  $r = 1 - t^2$  we obtain,

$$(3.27) \quad I_\sigma^j \leq 2^{(\sigma+1)q/2} \int_0^1 t^{(\sigma+1)q-1} e^{-t\tau} dt = c(\sigma, q) \tau^{-(\sigma+1)q}.$$

On the other hand, if  $\tau \geq 2$ ,

$$(3.28) \quad \begin{aligned} I_\sigma^j(\tau) &= \int_0^1 (1-t^2)^{(j-3)/2} t^{(\sigma+1)q-1} e^{-\tau t} dt \\ &= \tau^{-(\sigma+1)q} \int_0^\tau (1-(s/\tau)^2)^{(j-3)/2} s^{(\sigma+1)q-1} e^{-s} ds \\ &\geq \tau^{-(\sigma+1)q} 2^{-(j-3)} \int_0^1 s^{(\sigma+1)q-1} e^{-s} ds. \end{aligned}$$

Combining (3.25) with (3.26)–(3.28) we obtain (3.23).  $\square$

Next we derive an estimate in which integration over  $\mathbb{R}_+^n = \mathbb{R}_+^j \times \mathbb{R}^m$  is replaced by integration over a bounded domain, for measures supported in a fixed bounded subset of  $\mathbb{R}^m$ .

Let  $B_R^j(0)$  and  $B_R^m(0)$  denote the balls of radius  $R$  centered at the origin, in  $\mathbb{R}^j$  and  $\mathbb{R}^m$  respectively. Denote

$$(3.29) \quad F_{\nu, m}^R[\mu](\tau) = \int_{B_R^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty)$$

and, if  $j \geq 2$ ,

$$(3.30) \quad I_{\nu, \sigma}^{m, j}(\mu; R) = \int_{B_R^j \cap \{0 < y_1\}} F_{\nu, m}^R[\mu](\sqrt{y_1^2 + |\tilde{y}|^2}) e^{-y_1} y_1^{\sigma q - 1} d\tilde{y} dy_1.$$

where  $(y_1, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^{j-1}$ . If  $j = 1$  we denote,

$$(3.31) \quad I_{\nu, \sigma}^{m, 1}(\mu; R) = \int_0^R F_{\nu, m}^R[\mu](y_1) e^{-y_1} y_1^{\sigma q - 1} dy_1.$$

Similarly to Lemma 3.3 we obtain,

**Lemma 3.4.** *If  $j \geq 1$ , there exists a positive constant  $c$  such that, for any bounded Borel measure  $\mu$  with support in  $\mathbb{R}^m \cap B_R$*

$$(3.32) \quad c^{-1} \int_0^R F_{\nu, m}^R[\mu](\tau) h_{\sigma, j}(\tau) d\tau \leq I_{\nu, \sigma}^{m, j}(\mu; R) \leq c \int_0^R F_{\nu, m}^R[\mu](\tau) h_{\sigma, j}(\tau) d\tau$$

with  $h_{\sigma, j}$  as in (3.24).

*Proof.* In the case  $j = 1$  there is nothing to prove . Therefore we assume that  $j \geq 2$ .

From (3.30) we obtain,

$$I_{\nu,\sigma}^{m,j}(\mu; R) = c_{m,j} \int_0^R \int_0^{\sqrt{R^2-\rho^2}} F_{\nu,m}^R[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1 \rho^{j-2} d\rho.$$

Substituting  $y_1 = (\tau^2 - \rho^2)^{1/2}$ , then changing the order of integration and finally substituting  $\rho = r\tau$  we obtain,

$$c_{m,j}^{-1} I_{\nu,\sigma}^{m,j}(\mu; R) = \int_0^R \int_0^1 F_{\nu,\mu}^R[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr d\tau.$$

where

$$f(r) = r^{j-2}(1-r^2)^{(\sigma+1)q/2-1}.$$

The remaining part of the proof is the same as for Lemma 3.3.  $\square$

**Lemma 3.5.** *Let  $1 < q$ ,  $0 < \sigma$  and assume that  $m < \nu q$  and  $0 \leq j-1 < \nu$ . Then there exists a positive constant  $\bar{c}$ , depending on  $j, m, q, \sigma, \nu$ , such that, for every  $R \geq 1$  and every bounded Borel measure  $\mu$  with support in  $B_{R/2}(0) \cap \mathbb{R}^m$ ,*

$$(3.33) \quad \left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \bar{c} R^{(\sigma+1-\nu)q+m+j-1} \|\mu\|_{\mathfrak{M}}^q$$

with  $h_{\sigma,j}$  as in (3.24).

*Proof.* We estimate,

$$(3.34) \quad \left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \int_R^\infty |F_{\nu,m}[\mu](\tau)| h_{\sigma,j}(\tau) d\tau + \int_0^R |F_{\nu,m}[\mu] - F_{\nu,m}^R[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau.$$

For every  $\tau > 0$ ,

$$(3.35) \quad |F_{\nu,m}[\mu](\tau)| \leq \tau^{-\nu q} \|\mu\|_{\mathfrak{M}}^q.$$

Since  $j-1 < \nu q$ , it follows that

$$(3.36) \quad \begin{aligned} \int_R^\infty |F_{\nu,m}[\mu](\tau)| h_{\sigma,j}(\tau) d\tau &\leq \|\mu\|_{\mathfrak{M}}^q \int_R^\infty \tau^{-\nu q} h_{\sigma,j}(\tau) d\tau \\ &\leq c(\sigma, q) \|\mu\|_{\mathfrak{M}}^q \int_R^\infty \frac{\tau^{(\sigma+1)q+j-2-\nu q}}{(1+\tau)^{(\sigma+1)q}} d\tau \\ &\leq \frac{c(\sigma, q)}{\nu q - j + 1} \|\mu\|_{\mathfrak{M}}^q R^{j-1-\nu q}. \end{aligned}$$

Since, by assumption,  $\text{supp } \mu \subset B_{R/2}$ , we have

$$\begin{aligned}
& \int_0^R |F_{\nu,m}[\mu] - F_{\nu,m}^R[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau \\
& \leq \int_0^R \int_{|y''|>R} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' h_{\sigma,j}(\tau) d\tau \\
& \leq \|\mu\|_{\mathfrak{M}}^q \int_0^R \int_{|\zeta|>R/2} (|\tau^2 + |\zeta|^2)^{-\nu q/2} d\zeta h_{\sigma,j} d\tau \\
(3.37) \quad & \leq c(m, q) \|\mu\|_{\mathfrak{M}}^q \int_0^R \int_{R/2}^\infty (\tau^2 + \rho^2)^{-\nu q/2} \rho^{m-1} d\rho h_{\sigma,j} d\tau \\
& \leq c(m, q) \|\mu\|_{\mathfrak{M}}^q \int_0^R \tau^{m-\nu q} \int_{R/2\tau}^\infty (1 + \eta^2)^{-\nu q/2} \eta^{m-1} d\eta h_{\sigma,j} d\tau \\
& \leq \frac{c(m, q)}{\nu q - m} \|\mu\|_{\mathfrak{M}}^q R^{m-\nu q} \int_0^R \tau^{(\sigma+1)q+j-2} d\tau \\
& \leq \frac{c(m, q)}{(\nu q - m)((\sigma+1)q + j - 1)} \|\mu\|_{\mathfrak{M}}^q R^{(\sigma+1)q+j-1+m-\nu q}.
\end{aligned}$$

Combining (3.34)–(3.37) we obtain (3.33).  $\square$

**Corollary 3.6.** *For every  $R > 0$  put*

$$(3.38) \quad J_{\nu,\sigma}^{m,j}(\mu; R) := \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(\sigma+1)q+j-2} d\tau.$$

Then

$$\begin{aligned}
(3.39) \quad & \frac{1}{c} I_{\nu,\sigma}^{m,j}(\mu) - \bar{c} R^\beta \|\mu\|_{\mathfrak{M}}^q \leq J_{\nu,\sigma}^{m,j}(\mu; R) \leq c R^{(\sigma+1)q} I_{\nu,\sigma}^{m,j}(\mu), \\
& \beta = (\sigma + 1 - \nu)q + j + m - 1,
\end{aligned}$$

for every  $R > 1$  and every bounded Borel measure  $\mu$  with support in  $B_{R/2}^m(0) := B_{R/2}(0) \cap \mathbb{R}^m$ .

*Proof.* This is an immediate consequence of Lemma 3.5 and Lemma 3.3.  $\square$

**Lemma 3.7.** *Let  $m, j$  be positive integers such that  $j \geq 1$  and let  $1 < q$ ,  $0 < \sigma$ . Put  $n := m + j$ .*

*Then there exist positive constants  $c, \bar{c}$ , depending on  $j, m, q, \sigma$ , such that, for every  $R > 1$  and every measure  $\mu \in \mathfrak{M}_+(B_{R/2}^m(0))$ ,*

$$\begin{aligned}
(3.40) \quad & \frac{1}{c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q - \bar{c} R^{q\left(\sigma - \frac{n-1}{q}\right)} \|\mu\|_{\mathfrak{M}}^q \leq J_{n,\sigma}^{m,j}(\mu; R) \\
& \leq c R^{(\sigma+1)q} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q.
\end{aligned}$$

If  $\sigma < \frac{n-1}{q}$ , there exists  $R_0 > 1$  such that, for all  $R > R_0$

$$(3.41) \quad \frac{1}{2c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q \leq J_{n,\sigma}^{m,j}(\mu; R).$$

If  $\sigma = \frac{n-1}{q'}$  then, there exists  $a > 0$  such that the inequality remains valid for measures  $\mu$  such that  $\text{diam}(\text{supp } \mu) \leq a$ .

If, in addition,  $\frac{j-1}{q'} < \sigma$  then

$$(3.42) \quad \frac{1}{2c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq J_{n,\sigma}^{m,j}(\mu; R) \leq cR^{(\sigma+1)q} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q,$$

where  $s := \sigma - \frac{j-1}{q'}$ .

*Remark.* Assume that  $\mu \geq 0$ . Then:

(i) If  $\mu \in B^{-\sigma,q}(\mathbb{R}^{n-1})$  and  $\frac{j-1}{q'} \geq \sigma$  then  $\mu(\mathbb{R}^m) = 0$ .

(ii) If  $\mu \in B^{-s,q}(\mathbb{R}^m)$  and  $\sigma > (n-1)/q'$  then  $s > m/q'$  and therefore  $B^{s,q'}(\mathbb{R}^m)$  can be embedded in  $C(\mathbb{R}^m)$ .

*Proof.* Inequality (3.40) follows from (3.39) and Proposition 3.2 (see also (3.19)).

For positive measures  $\mu$ ,

$$\|\mu\|_{\mathfrak{M}} = \mu(\mathbb{R}^{n-1}) \leq \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q.$$

Therefore, if  $\sigma < \frac{n-1}{q'}$ , (3.40) implies that there exists  $R_0 > 1$  such that (3.41) holds for all  $R > R_0$ .

If  $\sigma = \frac{n-1}{q'}$  (3.40) implies that

$$\frac{1}{c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q - \bar{c} \|\mu\|_{\mathfrak{M}}^q \leq J_{n,\sigma}^{m,j}(\mu; R).$$

But if  $\mu$  is a positive bounded measure such that  $\text{diam}(\text{supp } \mu) \leq a$  then

$$\|\mu\|_{\mathfrak{M}} / \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q \rightarrow 0 \quad \text{as } a \rightarrow 0.$$

The last inequality follows from the imbedding theorem for Besov spaces according to which there exists a continuous trace operator  $T : B^{\sigma,q'}(\mathbb{R}^{n-1}) \mapsto B^{s,q'}(\mathbb{R}^m)$  and a continuous lifting  $T' : B^{s,q'}(\mathbb{R}^m) \mapsto B^{\sigma,q'}(\mathbb{R}^{n-1})$  where  $s = \sigma - \frac{n-m-1}{q'}$ .  $\square$

If  $\nu \in \mathbb{N}$  and  $\sigma = s + \frac{\nu-m-1}{q'}$ ,

$$\begin{aligned} J_{\nu,\sigma}^{m,\nu-m}(\mu; R) &= \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(\sigma+1)q+\nu-m-2} d\tau \\ &= \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(s+\nu-m)q-1} d\tau. \end{aligned}$$

However, if  $\mu$  is positive, the expression

$$(3.43) \quad M_{\nu,s}^m(\mu; R) := \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(s+\nu-m)q-1} d\tau,$$

is meaningful for any real  $\nu > m$  and  $s > 0$ . Furthermore, as shown below, the results stated in Lemma 3.7 can be extended to this general case.

**Theorem 3.8.** *Let  $1 < q$ ,  $\nu \in \mathbb{R}$  and  $m$  a positive integer. Assume that  $1 \leq \nu - m$  and  $0 < s < m/q'$ . Then there exists a positive constant  $c$  such that, for every bounded positive measure  $\mu$  supported in  $\mathbb{R}^m \cap B_{R/2}(0)$ ,  $R > 1$ ,*

$$(3.44) \quad \frac{1}{c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq M_{\nu,s}^m(\mu; R) \leq cR^{(s+\nu-m)q+1} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q.$$

*This also holds when  $s = m/q'$ , provided that the diameter of  $\text{supp } \mu$  is sufficiently small.*

*Proof.* If  $\nu$  is an integer and  $j := \nu - m$  then this statement is part of Lemma 3.7. Indeed the condition  $s > 0$  means that  $\sigma = s + \frac{j-1}{q'} > \frac{j-1}{q'}$  and the condition  $s < m/q'$  means that  $\sigma < \frac{n-1}{q'}$ .

Therefore we assume that  $\nu \notin \mathbb{N}$ . Let  $n := \{\nu\}$  and  $\theta := n - \nu$  so that  $0 < \theta < 1$ . Our assumptions imply that  $1 \leq n - m - 1$  because (as  $\nu$  is not an integer)  $\nu - m > 1$  and consequently  $n - m \geq 2$ .

If  $a, b$  are positive numbers, put

$$A_\nu := \frac{a^{(s+\nu-m)q-1}}{(a^2 + b^2)^{\nu q/2}}.$$

Obviously  $A_\nu$  decreases as  $\nu$  increases. Therefore,  $A_n \leq A_\nu \leq A_{n-1}$  which in turn implies,

$$M_{n,s}^m \leq M_{\nu,s}^m \leq M_{n-1,s}^m.$$

By Lemma 3.7, the assertions of the theorem are valid in the case that  $\nu = n$  or  $\nu = n - 1$ . Therefore the previous inequality implies that the assertions hold for any real  $\nu$  subject to the conditions imposed.  $\square$

By (3.8),

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau) \tau^{(q+1)\kappa_+ + k - 1} d\tau,$$

where  $m = N - k$  and  $\nu = N - 2 + 2\kappa_+$ . Consequently, by (3.38),

$$J^{A,R} = M_{\nu,s}^m$$

where  $s$  is determined by,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

It follows that

$$sq = -(k - 2 + 2\kappa_+)q + (q + 1)\kappa_+ + k = k(1 - q) + 2q - \kappa_+(q - 1)$$

and therefore

$$s = 2 - \frac{k + \kappa_+}{q'}.$$

*Proof of Theorem 3.1.*

Put

$$(3.45) \quad \nu := N - 2 + 2\kappa_+, \quad s := 2 - \frac{\kappa_+ + k}{q'}, \quad m := N - k.$$



Recall that in the case  $k = 2$  we have  $\kappa_+ > 1/2$ . Therefore

$$(3.46) \quad \nu - m - 1 = k - 3 + 2\kappa_+ > 0.$$

Furthermore,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

Thus

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau) \tau^{(q+1)\kappa_+ + k - 1} d\tau = M_{\nu,s}^m.$$

Next we show that  $0 < s \leq m/q'$ . More precisely we prove

$$(3.47) \quad 0 < s \leq m/q' \iff q_c \leq q < q_c^*.$$

Let  $\mu$  be a bounded non-negative Borel measure in  $B^{-s,q}(\mathbb{R}^m)$ . If  $s \leq 0$ ,  $B^{-s,q}(\mathbb{R}^m) \subset L^q(\mathbb{R}^m)$ . Therefore, in this case, every bounded Borel measure on  $\mathbb{R}^m$  is admissible i.e. satisfies (2.35). Consequently, by Proposition 2.2,  $q < q_c$ . As we assume  $q \geq q_c$  it follows that  $s > 0$ .

If,  $s > 0$  and  $sq' - m \geq 0$  then  $C_{s,q'}(K) = 0$  for every compact subset of  $\mathbb{R}^m$  and consequently  $\mu(K) = 0$  for any such set. Conversely, if  $sq' - m < 0$  then there exist non-trivial positive bounded measures in  $B^{-s,q}(\mathbb{R}^m)$ . Therefore, by Proposition 2.1,  $sq' < m$  if and only if  $q < q_c^*$ .

In conclusion,  $0 < s \leq m/q'$  and  $\nu - m \geq 1$ ; therefore Theorem 3.1 is a consequence of Theorem 3.8.  $\square$

*Remark.* Note that the critical exponent for the imbedding of  $B^{2 - \frac{\kappa_+ + k}{q'}, q'}(\mathbb{R}^{N-k})$  into  $C(\mathbb{R}^{N-k})$  is again

$$q = q_c = \frac{N + \kappa_+}{N + \kappa_+ - 2}.$$

#### 4. SUPERCRITICAL EQUATIONS IN A POLYHEDRAL DOMAIN

In this section  $q$  is a real number larger than 1 and  $P$  an  $N$ -dim polyhedral domain as described in subsection 6.1. Denote by  $\{L_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$  the family of faces, edges and vertices of  $P$ . In this notation,  $L_{1,j}$  denotes one of the open faces of  $P$ ; for  $k = 2, \dots, N - 1$ ,  $L_{k,j}$  denotes a relatively open  $N - k$ -dimensional edge and  $L_{N,j}$  denotes a vertex. For  $1 \leq k < N$ , the  $(N - k)$  dimensional space which contains  $L_{k,j}$  is denoted by  $\mathbb{R}_j^{N-k}$ . If  $1 < k < N$ , the cylinder of radius  $r$  around the axis  $\mathbb{R}_j^{N-k}$  will be denoted by  $\Gamma_{k,j,r}^\infty$  and the subset  $A_{k,j}$  of  $S^{k-1}$  is defined by

$$\lim_{r \rightarrow 0} \frac{1}{r} (\partial \Gamma_{k,j,r}^\infty \cap P) = L_{k,j} \times A_{k,j}.$$

$A_{k,j}$  is the 'opening' of  $P$  at the edge  $L_{k,j}$ . For  $k = N$  we replace in this definition the cylinder  $\Gamma_{N,j,r}^\infty$  by the ball  $B_r(L_{N,j})$ . For  $1 < k \leq N$  and  $A = A_{k,j}$  we use  $d_A$  as an alternative notation for  $\mathbb{R}_j^{N-k}$  and denote by  $D_A$  the  $k$ -dihedron with edge  $d_A$  and opening  $A$  as in subsection 6.1 (with  $S_A$  defined as in (2.2)). For  $k = 1$ ,  $D_A$  stands for the half space  $\mathbb{R}_j^{N-1} \times (0, \infty)$ .

**4.1. Definitions and auxiliary results.** Let  $\Omega$  be a bounded Lipschitz domain. We say that  $\{\Omega_n\}$  is a *Lipschitz exhaustion* of  $\Omega$  if, for every  $n$ ,  $\Omega_n$  is Lipschitz and

$$(4.1) \quad \Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}, \quad \Omega = \cup \Omega_n, \quad \mathbb{H}_{N-1}(\partial\Omega_n) \rightarrow \mathbb{H}_{N-1}(\partial\Omega).$$

If  $\omega_n$  (respectively  $\omega$ ) is the harmonic measure in  $\Omega_n$  (respectively  $\Omega$ ) relative to  $x_0 \in \Omega_1$ , then, for every  $Z \in C(\bar{\Omega})$ ,

$$(4.2) \quad \lim_{n \rightarrow \infty} \int_{\partial\Omega_n} Z d\omega_n = \int_{\partial\Omega} Z d\omega.$$

[23, Lemma 2.1]. Furthermore, if  $\mu$  is a bounded Borel measure on  $\partial\Omega$  and  $v := \mathbb{K}^\Omega[\mu]$ , there holds

$$(4.3) \quad \lim_{n \rightarrow \infty} \int_{\partial\Omega_n} Z v d\omega_n = \int_{\partial\Omega} Z d\mu,$$

[23, Lemma 2.2]. If  $v$  is a positive solution and (4.3) holds we say that  $\mu$  is the *boundary trace* of  $v$ .

The following estimates are proved in [23, Lemma 2.3]

**Proposition 4.1.** *Let  $\mu$  be bounded Borel measures on  $\partial\Omega$ . Then  $\mathbb{K}[\mu] \in L^1_\rho(\Omega)$  and there exists a constant  $C = C(\Omega)$  such that*

$$(4.4) \quad \|\mathbb{K}[\mu]\|_{L^1_\rho(\Omega)} \leq C \|\mu\|_{\mathfrak{M}(\partial\Omega)}.$$

In particular if  $h \in L^1(\partial\Omega; \omega)$  then

$$(4.5) \quad \|\mathbb{P}[h]\|_{L^1_\rho(\Omega)} \leq C \|h\|_{L^1(\partial\Omega; \omega)}.$$

The next result will be used in deriving estimates in a  $k$ -dimensional dihedron when the boundary data is concentrated on the edge.

**Proposition 4.2.** *We denote by  $G^{\Omega_n}$  (respectively  $G^\Omega$ ) the Green function in  $\Omega_n$  (respectively  $\Omega$ ). Let  $v$  be a positive harmonic function in  $\Omega$  with boundary trace  $\mu$ . Let  $Z \in C^2(\bar{\Omega})$  and let  $\tilde{G} \in C^\infty(\Omega)$  be a function that coincides with  $x \mapsto G(x, x_0)$  in  $Q \cap \Omega$  for some neighborhood  $Q$  of  $\partial\Omega$  and some fixed  $x_0 \in \Omega$ . In addition assume that there exists a constant  $c > 0$  such that*

$$(4.6) \quad |\nabla Z \cdot \nabla \tilde{G}| \leq c\rho.$$

Under these assumptions, if  $\zeta := Z\tilde{G}$  then

$$(4.7) \quad - \int_{\Omega} v \Delta \zeta dx = \int_{\partial\Omega} Z d\mu.$$

*Proof.* Let  $\{\Omega_n\}$  be a  $C^1$  exhaustion of  $\Omega$ . We assume that  $\partial\Omega_n \subset Q$  for all  $n$  and  $x_0 \in \Omega_1$ . Let  $\tilde{G}_n(x)$  be a function in  $C^1(\Omega_n)$  such that  $\tilde{G}_n$  coincides with  $G^{\Omega_n}(\cdot, x_0)$  in  $Q \cap \Omega_n$ ,  $\tilde{G}_n(\cdot, x_0) \rightarrow \tilde{G}(\cdot, x_0)$  in  $C^2(\Omega \setminus Q)$  and  $\tilde{G}_n(\cdot, x_0) \rightarrow \tilde{G}(\cdot, x_0)$  in  $\text{Lip}(\Omega)$ . If  $\zeta_n = Z\tilde{G}_n$  we have,

$$\begin{aligned} - \int_{\Omega_n} v \Delta \zeta_n dx &= \int_{\partial \Omega_n} v \partial_{\mathbf{n}} \zeta dS = \int_{\partial \Omega_n} v Z \partial_{\mathbf{n}} \tilde{G}_n(\xi, x_0) dS \\ &= \int_{\partial \Omega_n} v Z P^{\Omega_n}(x_0, \xi) dS = \int_{\partial \Omega_n} v Z d\omega_n. \end{aligned}$$

By (4.3),

$$\int_{\partial \Omega_n} v Z d\omega_n \rightarrow \int_{\partial \Omega} Z d\mu.$$

On the other hand, in view of (4.6), we have

$$\Delta \zeta_n = \tilde{G}_n \Delta Z + Z \Delta \tilde{G}_n + 2 \nabla Z \cdot \nabla \tilde{G}_n \rightarrow \Delta Z$$

in  $L^1_\rho(\Omega)$ ; therefore,

$$- \int_{\Omega_n} v \Delta \zeta_n dx \rightarrow - \int_{\Omega} v \Delta \zeta dx.$$

□

We denote by  $\mathfrak{M}_q = \mathfrak{M}_q(\partial \Omega)$  the set of  $q$ -good measures on the boundary. A positive solution  $u$  of (1.1) in  $\Omega$  possesses a boundary trace  $\mu \in \mathfrak{M}(\partial \Omega)$  if and only if

$$(4.8) \quad \int_{\Omega} u^q \rho dx < \infty$$

[23, Proposition 4.1]. In this case  $\mu \in \mathfrak{M}_q$ .

The following statements can be proved in the same way as in the case of smooth domains. For the proof in that case see [19].

**I.**  $\mathfrak{M}_q(\partial \Omega)$  is a linear space and

$$\mu \in \mathfrak{M}_q(\partial \Omega) \iff |\mu| \in \mathfrak{M}_q(\partial \Omega).$$

**II.** If  $\{\mu_n\}$  is an increasing sequence of measures in  $\mathfrak{M}_q(\partial \Omega)$  and  $\mu := \lim \mu_n$  is a finite measure then  $\mu \in \mathfrak{M}_q(\partial \Omega)$ .

**Proposition 4.3.** *Let  $\mu$  be a bounded measure on  $\partial P$ . ( $\mu$  may be a signed measure.) For  $i = 1, \dots, N$ ,  $j = 1, \dots, n_i$ , we define the measure  $\mu_{k,j}$  on  $d_{A_{k,j}}$  by,*

$$\mu_{k,j} = \mu \text{ on } L_{k,j}, \quad \mu_{k,j} = 0 \text{ on } d_{A_{k,j}} \setminus L_{k,j}.$$

*Then  $\mu \in \mathfrak{M}_q(\partial P)$ , i.e., problem*

$$(4.9) \quad -\Delta u + u^q = 0 \text{ in } P, \quad u = \mu \text{ on } \partial P$$

*possesses a solution, if and only if,  $\mu_{k,j}$  is a  $q$ -good measure relative to  $D_{A_{k,j}}$  for all  $(k, j)$  as above.*

*Proof.* In view of statement **I** above, it is sufficient to prove the proposition in the case that  $\mu$  is non-negative. This is assumed hereafter. If  $\mu \in \mathfrak{M}_q(\partial P)$  then any measure  $\nu$  on  $\partial P$  such that  $0 \leq \nu \leq \mu$  is a  $q$ -good measure relative to  $P$ . Therefore

$$\mu \in \mathfrak{M}_q(\partial P) \implies \mu'_{k,j} := \mu \chi_{L_{k,j}} \in \mathfrak{M}_q(\partial P).$$

Assume that  $\mu \in \mathfrak{M}_q(\partial P)$  and let  $u_{k,j}$  be the solution of (4.9) when  $\mu$  is replaced by  $\mu'_{k,j}$ . Denote by  $u'_{k,j}$  the extension of  $u_{k,j}$  by zero to the k-dihedron  $D_{A_{k,j}}$ . Then  $u'_{k,j}$  is a subsolution of (1.1) in  $D_{A_{k,j}}$  with boundary data  $\mu_{k,j}$ . In the present case there always exists a supersolution, e.g. the maximal solution of (1.1) in  $D_{A_{k,j}}$  vanishing outside  $d_{A_{k,j}} \setminus \bar{L}_{k,j}$ . Therefore there exists a solution  $v_{k,j}$  of this equation in  $D_{A_{k,j}}$  with boundary data  $\mu_{k,j}$ , i.e.,  $\mu_{k,j}$  is q-good relative to  $D_{A_{k,j}}$ .

Next assume that  $\mu \in \mathfrak{M}(\partial P)$  and that  $\mu_{k,j}$  is q-good relative to  $D_{A_{k,j}}$  for every  $(k, j)$  as above. Let  $v_{k,j}$  be the solution of (1.1) in  $D_{A_{k,j}}$  with boundary data  $\mu_{k,j}$ . Then  $v_{k,j}$  is a supersolution of problem (4.9) with  $\mu$  replaced by  $\mu'_{k,j}$  and consequently there exists a solution  $u_{k,j}$  of this problem. It follows that

$$w := \max\{u_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$$

is a subsolution while

$$\bar{w} := \sum_{k=1, \dots, N, j=1, \dots, n_k} u_{k,j}$$

is a supersolution of (4.9). Consequently there exists a solution of this problem, i.e.,  $\mu \in \mathfrak{M}_q(\partial P)$ .  $\square$

**4.2. Removable singular sets and 'good measures', I.** We first introduce some standard elements associated to the Bessel capacities which are the natural way to characterize good measures or removable sets. For  $\alpha \in \mathbb{R}$ , we denote by  $G_\alpha$  the Bessel kernel of order  $\alpha$ , defined by

$$(4.10) \quad G_\alpha(\xi) = \mathcal{F}^{-1} \left( (1 + |\cdot|^2)^{-\frac{\alpha}{2}} \right) (\xi),$$

where  $\mathcal{F}$  is the Fourier transform in the space  $\mathcal{S}'(\mathbb{R}^\ell)$  of moderate distributions in  $\mathbb{R}^\ell$ . For  $1 \leq p \leq \infty$ , the Bessel space  $L_{\alpha,p}(\mathbb{R}^\ell)$  is defined by

$$(4.11) \quad L_{\alpha,p}(\mathbb{R}^\ell) = \{f : f = G_\alpha * g, : g \in L^p(\mathbb{R}^\ell)\},$$

with norm

$$\|f\|_{L_{\alpha,p}} = \|g\|_{L^p} = \|G_{-\alpha} * f\|_{L^p}.$$

For  $\alpha, \beta \in \mathbb{R}$  and  $1 < p < \infty$ , the mapping  $f \mapsto G_\beta * f$  is an isomorphism from  $L_{\alpha,p}(\mathbb{R}^\ell)$  into  $L_{\alpha+\beta,p}(\mathbb{R}^\ell)$ . Finally the Bessel spaces are connected to Besov and Sobolev spaces: when  $\alpha > 0$  and  $1 < p < \infty$ , it is known that if  $\alpha \in \mathbb{N}$ ,  $L_{\alpha,p}(\mathbb{R}^\ell) = W^{\alpha,p}(\mathbb{R}^\ell)$  and if  $\alpha \notin \mathbb{N}$ , then  $L_{\alpha,p}(\mathbb{R}^\ell) = B^{\alpha,p}(\mathbb{R}^\ell)$ , with equivalent norms (see e.g. [5], [26]).

The Bessel capacity  $C_{\alpha,p}^{\mathbb{R}^\ell}$  ( $\alpha > 0$ ,  $p \geq 1$ ) is defined by the following rules: if  $K \subset \mathbb{R}^\ell$  is compact

$$(4.12) \quad C_{\alpha,p}^{\mathbb{R}^\ell}(K) = \inf\{\|f\|_{L_{\alpha,p}}^p : f \in \mathcal{S}(\mathbb{R}^\ell), f \geq \chi_K\}.$$

If  $G$  is open

$$(4.13) \quad C_{\alpha,p}^{\mathbb{R}^\ell}(G) = \sup\{C_{\alpha,p}^{\mathbb{R}^\ell}(K) : K \subset G, K \text{ compact}\}.$$

If  $A$  is any set

$$(4.14) \quad C_{\alpha,p}^{\mathbb{R}^\ell}(A) = \inf\{C_{\alpha,p}^{\mathbb{R}^\ell}(G) : A \subset G, G \text{ open}\}.$$

Note that the capacity of any non-empty set is positive if and only if  $\alpha > \frac{\ell}{p}$  because of Sobolev-Besov imbedding theorem.

**Proposition 4.4.** *Let  $A$  be a Lipschitz domain on  $S^{k-1}$ ,  $2 \leq k \leq N-1$ , and let  $D_A$  be the  $k$ -dihedron with opening  $A$ . Let  $\mu \in \mathfrak{M}(\partial D_A)$  be a positive measure with compact support contained in  $d_A$  (= the edge of  $D_A$ ). Assume that  $\mu$  is  $q$ -good relative to  $D_A$ . Let  $R > 1$  be large enough so that  $\text{supp } \mu \subset B_R^{N-k}(0)$  and let  $u$  be the solution of (1.1) in  $D_A^R$  with trace  $\mu$  on  $d_A^R$  and trace zero on  $\partial D_A^R \setminus d_A^R$ . Then:*

(i) For every non-negative  $\eta \in C_0^\infty(B_{3R/4}^{N-k}(0))$ ,

$$(4.15) \quad \begin{aligned} \left( \int_{d_A^R} \eta^{q'} d\mu \right) &\leq cM^{q'} \int_{D_A^R} u^q \rho dx + \\ &+ cM^{q'} \left( \int_{D_A^R} u^q \rho dx \right)^{\frac{1}{q}} \left( 1 + M^{-1} \|\eta\|_{L^{q'}(d_A^R)} \right). \end{aligned}$$

where  $M = \|\eta\|_{L^\infty}$  and  $\rho$  is the first eigenfunction of  $-\Delta$  in  $D_A^R$  normalized by  $\rho(x_0) = 1$  at some point  $x_0 \in D_A^R$ . The constant  $c$  depends only on  $N, q, k, x_0, \lambda_1, R$  where  $\lambda_1$  is the first eigenvalue.

(ii) For any compact set  $E \subset d_A$ ,

$$(4.16) \quad C_{s,q}^{N-k}(E) = 0 \implies \mu(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'},$$

where  $C_{s,q}^{N-k}$  denotes the Bessel capacity with the indicated indices in  $\mathbb{R}^{N-k}$ .

*Remark.* If we replace  $D_A^R$  by  $D_A \cap B_R^k(0) \cap B_R^{N-k}(0)$ ,  $\tilde{R} > 1$ , then the constant  $c$  in (i) depends on  $\tilde{R}$  but not on  $R$ .

*Proof.* We identify  $d_A$  with  $\mathbb{R}^{N-k}$  and use the notation

$$x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \quad y = |x'|.$$

Let  $\eta \in C_0^\infty(\mathbb{R}^{N-k})$  and let  $R$  be large enough so that  $\text{supp } \eta \subset B_{R/2}^{N-k}(0)$ . Let  $w = w_R(t, x'')$  be the solution of the following problem in  $\mathbb{R}_+ \times B_R^{N-k}(0)$ :

$$(4.17) \quad \begin{aligned} \partial_t w - \Delta_{x''} w &= 0 && \text{in } \mathbb{R}_+ \times B_R^{N-k}(0), \\ w(0, x'') &= \eta(x'') && \text{in } B_R^{N-k}, \\ w(t, x'') &= 0 && \text{on } \partial B_R^{N-k}(0). \end{aligned}$$

Thus  $w_R(t, \cdot) = S_R(t)[\eta]$  where  $S_R(t)$  is the semi-group operator corresponding to the above problem. Denote,

$$(4.18) \quad H_R[\eta](x', x'') = w_R(|x'|^2, x'') = S_R(y^2)[\eta](x''), \quad y := |x'|.$$

We assume, as we may, that  $R > 1$ . Let  $\rho^R$  be the first eigenfunction of  $-\Delta_{x''}$  in the ball  $B_R^{N-k}(0)$  normalized by  $\rho^R(0) = 1$  and let  $\rho_A$  be the first eigenfunction of  $-\Delta_{x'}$  in  $C_A$  (where  $C_A$  denotes the cone with opening  $A$  in  $\mathbb{R}^k$ ) normalized so that  $\rho_A(x'_0) = 1$  at some point  $x'_0 \in S_A$ . Then  $\rho^R \rho_A$  is the first eigenfunction of  $-\Delta$  in  $\{x \in D_A : |x''| < R\}$ . Note that  $\rho^R \leq 1$  and  $\rho^R \rightarrow 1$  as  $R \rightarrow \infty$  in  $C^2(I)$  for any bounded set  $I \subset \mathbb{R}^{N-k}$ .

Let  $h \in C^\infty(\mathbb{R})$  be a monotone decreasing function such that  $h(t) = 1$  for  $t < 1/2$  and  $h(t) = 0$  for  $t > 3/4$ . Put

$$\psi_R(x') = h(|x'|/R)$$

and

$$(4.19) \quad \zeta_R := \rho_A \psi_R H_R[\eta]^{q'}.$$

If  $\rho_A^R$  is the first eigenfunction (normalized at  $x_0$ ) of  $D_A^R := D_A \cap \Gamma_R$  ( $\Gamma_R$  as in (2.25)) then

$$(4.20) \quad \rho_A \psi_R \leq c \rho_A^R$$

and  $\rho^R \rho_A^R$  is the first eigenfunction in  $D_A^R$ .

Hereafter we shall drop the index  $R$  in  $\zeta_R, H_R, w_R$  but keep it in the other notations in order to avoid confusion.

We shall verify that  $\zeta \in D_A^R$ . To this purpose we compute,

$$(4.21) \quad \begin{aligned} \Delta \zeta &= -\lambda_1(\rho_A \psi_R) H[\eta]^{q'} + (\rho_A \psi_R) \Delta H[\eta]^{q'} + 2\nabla(\rho_A \psi_R) \cdot \nabla H[\eta]^{q'} \\ &= -\lambda_1 \zeta + q'(\rho_A \psi_R)(H[\eta])^{q'-1} \Delta H[\eta] \\ &\quad + q'(q' - 1)(\rho_A \psi_R)(H[\eta])^{q'-2} |\nabla H[\eta]|^2 \\ &\quad + 2q'(H[\eta])^{q'-1} \nabla(\rho_A \psi_R) \cdot \nabla H[\eta]. \end{aligned}$$

In addition,

$$\begin{aligned} \nabla H[\eta] &= \nabla_{x'} H[\eta] + \nabla_{x''} H[\eta] = \partial_y H[\eta] \frac{x'}{y} + \nabla_{x''} H[\eta] \\ &= 2y \partial_t w(y^2, x'') \frac{x'}{y} + \nabla_{x''} H[\eta](x', x'') \end{aligned}$$

and consequently (recall that  $y$  stands for  $|x'|$ ),

$$\begin{aligned} &\nabla H[\eta] \cdot \nabla(\rho_A \psi_R) \\ &= 2\partial_t w(y^2, x'') x' \cdot \left( \psi_R(|x'|^{\kappa_+ - 1} (\kappa_+ \frac{x'}{y} \omega_k(x'/y) + |x'| \nabla \omega_k(x'/y))) + \rho_A \nabla \psi_R \right) \\ &= 2\kappa_+ \partial_t w(y^2, x'') |x'|^{\kappa_+} \omega_k(x'/y) = 2\partial_t w(y^2, x'') (\kappa_+ \rho_A \psi_R + \rho_A x' \cdot \nabla \psi_R). \end{aligned}$$

Since  $w = w_R$  vanishes for  $|x''| = R$  and  $\eta = 0$  in a neighborhood of this sphere,  $|\partial_t w(y^2, x'')| \leq c \rho^R$ . As  $\psi_R$  vanishes for  $|x'| > 3R/4$  we have  $\rho_A \nabla \psi_R \leq c \rho_A^R$ . Therefore

$$|\nabla H[\eta] \cdot \nabla \rho_A| \leq c \rho^R \rho_A^R$$

and, in view of (4.21),

$$(4.22) \quad |\Delta\zeta| \leq c\rho^R \rho_A^R.$$

Thus  $\zeta \in X(D_A^R)$  and consequently

$$(4.23) \quad \int_{D_A^R} (-u\Delta\zeta + u^q\zeta) dx = - \int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta dx.$$

Since  $q(q' - 1)\rho_A(H[\eta])^{q'-2}|\nabla H[\eta]|^2 \geq 0$ , we have

$$(4.24) \quad \begin{aligned} & \left| \int_{D_A^R} u\Delta\zeta dx \right| \\ & \leq \int_{D_A^R} u \left( \lambda_1\zeta + q'(H[\eta])^{q'-1} (\rho|\Delta H[\eta]| + 2|\nabla\rho \cdot \nabla H[\eta]|) \right) dx \\ & \leq \int_{D_A^R} u \left( \lambda_1\zeta + q'\zeta^{1/q} \left( \rho^{1/q'} |\Delta H[\eta]| + 2\rho^{-1/q} |\nabla\rho \cdot \nabla H[\eta]| \right) \right) dx \\ & \leq \left( \int_{D_A^R} u^q \zeta dx \right)^{\frac{1}{q}} \left( \lambda_1 \left( \int_{D_A^R} \zeta dx \right)^{\frac{1}{q'}} + q' \|L[\eta]\|_{L^{q'}(D_A^R)} \right) \end{aligned}$$

where

$$(4.25) \quad L[\eta] = \rho^{1/q'} |\Delta H[\eta]| + 2\rho^{-1/q} |\nabla\rho \cdot \nabla H[\eta]|.$$

By Proposition 4.2

$$(4.26) \quad - \int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta dx = \int_{d_A^R} \eta^{q'} d\mu.$$

Therefore

$$(4.27) \quad \begin{aligned} & \left( \int_{d_A^R} \eta^{q'} d\mu \right) \leq \int_{D_A^R} u^q \zeta dx + \\ & + \left( \int_{D_A^R} u^q \zeta dx \right)^{\frac{1}{q}} \left( \lambda_1 \left( \int_{D_A^R} \zeta dx \right)^{\frac{1}{q'}} + q' \|L[\eta]\|_{L^{q'}(D_A^R)} \right). \end{aligned}$$

Next we prove that

$$(4.28) \quad \|L[\eta]\|_{L^{q'}(D_A^R)} \leq C \|\eta\|_{W^{s,q'}(\mathbb{R}^{N-k})}$$

starting with the estimate of the first term on the right hand side of (4.25).

$$\begin{aligned} \Delta H[\eta] &= \Delta_{x'} H[\eta] + \Delta_{x''} H[\eta] = \partial_y^2 H[\eta] + \frac{k-1}{y} \partial_y H[\eta] + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w(y^2, x'') + k \partial_t w(y^2, x'') + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w(y^2, x'') + (k+1) \partial_t w(y^2, x''). \end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^N} \rho |\Delta H[\eta]|^{q'} dx &\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_{tt} w(y^2, x'')|^{q'} dx'' y^{\kappa_++2q'+k-1} dy \\
&+ c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(y^2, x'')|^{q'} dx'' y^{\kappa_++k-1} dy \\
&\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_{tt} w(t, x'')|^{q'} dx'' t^{(\kappa_++k)/2+q'} \frac{dt}{t} \\
&+ c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(t, x'')|^{q'} dx'' t^{(\kappa_++k)/2} \frac{dt}{t} \\
&\leq c \int_0^1 \left\| t^{2-(1-\frac{\kappa_++k}{2q'})} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\
&+ c \int_0^1 \left\| t^{1-(1-\frac{\kappa_++k}{2q'})} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t}.
\end{aligned}$$

Put  $\beta = \frac{\kappa_++k}{2q'}$  and note that  $0 < \beta = \frac{1}{2}(2-s) < 1$ . By standard interpolation theory,

$$\begin{aligned}
&\int_0^1 \left\| t^{1-(1-\beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\
&\approx \|\eta\|_{[W^{2,q'}, L^{q'}]_{1-\beta, q'}}^{q'} \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left\| t^{2-(1-\beta)} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\
&\approx \|\eta\|_{[W^{4,q'}, L^{q'}]_{\frac{1}{2}(1-\beta), q'}}^{q'} \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}.
\end{aligned}$$

The second term on the right hand side of (4.25) is estimated in a similar way:

$$\begin{aligned}
\int_{\mathbb{R}^N} \rho^{-q'/q} |\nabla H[\eta] \cdot \nabla \rho|^{q'} dx &\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(y^2, x'')|^{q'} dx'' y^{\kappa_++k-1} dy \\
&\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(t, x'')|^{q'} dx'' t^{\frac{\kappa_++k}{2}} \frac{dt}{t} \\
&\leq c \int_0^1 \left\| t^{1-(\frac{1}{2}-\beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\
&\approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}.
\end{aligned}$$

This proves (4.28). Further, (4.27) and (4.28) imply (4.15).

We turn to the proof of part (ii). Let  $E$  be a closed subset of  $B_{R/2}^{N-k}(0)$  such that  $C_{s, q'}^{N-k}(E) = 0$ . Then there exists a sequence  $\{\eta_m\}$  in  $C_0^\infty(d_A)$  such



that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood of  $E$  (which may depend on  $n$ ),  $\text{supp } \eta_n \subset B_{3R/4}^{N-k}(0)$  and  $\|\eta_n\|_{W^{s,q'}} \rightarrow 0$ . Then, by (4.28),

$$\|L[\eta_n]\|_{L^{q'}(D_A^R)} \rightarrow 0.$$

Furthermore

$$\|w\|_{L^{q'}((0,R) \times B_R^{N-k}(0))} \leq c \|\eta_n\|_{L^{q'}(B_R^{N-k}(0))}$$

and consequently

$$H[\eta_n] \rightarrow 0 \text{ in } L^{q'}(D_A^R).$$

(Here we use the fact that  $k \geq 2$ .) In addition

$$0 \leq H[\eta_n] \leq 1, \quad H[\eta_n] \leq c(R - |x'|)$$

with a constant  $c$  independent of  $n$ . Hence (see (4.20))

$$\zeta_{n,R} := \rho_A \psi_R H[\eta_n]^{q'} \leq \rho^R \rho_A \psi_R H[\eta_n]^{q'-1} \leq \rho^R \rho_A^R H[\eta_n]^{q'-1}.$$

As  $u^q \rho^R \rho_A^R \in L^1(D_A^R)$  we obtain,

$$\lim_{n \rightarrow \infty} \int_{D_A} u^q \zeta_n dx = 0.$$

This fact and (4.27) imply that

$$\int_{d_A^R} \eta_n^{q'} d\mu \rightarrow 0.$$

As  $\eta_n = 1$  on a neighborhood of  $E$  in  $\mathbb{R}^{N-k}$  it follows that  $\mu(E) = 0$ .  $\square$

**Proposition 4.5.** *Let  $D_A$  be a  $k$ -dihedron,  $1 \leq k < N$ . Let  $k_+$  be as in (2.11) and let  $q_c^*$  and  $q_c$  be as in Proposition 2.1 and Proposition 2.2 respectively. Assume that  $q_c \leq q < q_c^*$ . A measure  $\mu \in \mathfrak{M}(\partial D_A)$ , with compact support contained in  $d_A$ , is  $q$ -good relative to  $D_A$  if and only if  $\mu$  vanishes on every Borel set  $E \subset d_A$  such that  $C_{s,q}(E) = 0$ , where  $s = 2 - \frac{k+\kappa_+}{q'}$ .*

*Remark.* We shall use the notation  $\mu \prec C_{s,q'}$  to say that  $\mu$  vanishes on any Borel set  $E \subset (d_A)$  such that  $C_{s,q'}(E) = 0$ .

In the case  $k = N$ :  $D_A = C_A$  (= the cone with vertex 0 and opening  $A$  in  $\mathbb{R}^k$ ) and  $q_c = q_c^*$ . By [23] (specifically the results quoted in subsection 2.2)  $q_c = 1 - \frac{2}{\kappa_-} = \frac{N+\kappa_+}{N+\kappa_+-2}$  and if  $1 < q < q_c$  then there exist solutions for every measure  $\mu = k\delta_P$ ,  $P \in d_A$ .

In the case  $k = 1$ ,  $q_c^* = \infty$ ,  $\kappa_+ = 1$  and  $q_c = \frac{N+1}{N-1}$ . Thus  $s = 2/q$  and the statement of the theorem is well known (see [20]).

*Proof.* In view of the last remark, it remains to deal only with  $2 \leq k \leq N-1$ . We shall identify  $d_A$  with  $\mathbb{R}^{N-k}$ .

It is sufficient to prove the result for positive measures because  $\mu \prec C_{s,q'}$  if and only if  $|\mu| \prec C_{s,q'}$ . In addition, if  $|\mu|$  is a  $q$ -good measure then  $\mu$  is a  $q$ -good measure.

First we show that if  $\mu$  is non-negative and q-good then  $\mu \prec C_{s,q'}$ . If  $E$  is a Borel subset of  $\partial\Omega$  then  $\mu\chi_E$  is q-good. If  $E$  is compact and  $C_{s,q'}(E) = 0$  then, by Proposition 4.4,  $E$  is a removable set. This means that the only positive solution of (1.1) in  $D_A$  such that  $\mu(\partial\Omega \setminus E) = 0$  is the zero solution. This implies that  $\mu\chi_E = 0$ , i.e.,  $\mu(E) = 0$ . If  $C_{s,q'}(E) = 0$  but  $E$  is not compact then  $\mu(E') = 0$  for every compact set  $E' \subset E$ . Therefore, we conclude again that  $\mu(E) = 0$ .

Next, assume that  $\mu$  is a positive measure in  $\mathfrak{M}(\partial D_A)$  supported in a compact subset of  $\mathbb{R}^{N-k}$ .

If  $\mu \in B^{-s,q}(\mathbb{R}^{N-k})$  then, by Theorem 3.1,  $\mu$  is admissible relative to  $D_A \cap \Gamma_{k,R}$ , for every  $R > 0$ . (As before  $\Gamma_{k,R}$  is the cylinder with radius  $R$  around the 'axis'  $\mathbb{R}^{N-k}$ .) This implies that  $\mu$  is q-good relative to  $D_A$ .

If  $\mu \prec C_{s,q'}$  then, by a theorem of Feyel and de la Pradelle [11] (see also [3]), there exists a sequence  $\{\mu_n\} \subset (B^{-s,q}(\mathbb{R}^{N-k}))_+$  such that  $\mu_n \uparrow \mu$ . As  $\mu_n$  is q-good, it follows that  $\mu$  is q-good.  $\square$

**Theorem 4.6.** *Let  $P$  be an  $N$ -dimensional polyhedron as described in Proposition 4.3. Let  $\mu$  be a bounded measure on  $\partial P$ , (may be a signed measure). Let  $k = 1, \dots, N$ ,  $j = 1, \dots, n_k$ , and let  $L_{k,j}$  and  $A_{k,j}$  be defined as at the beginning of this section. Further, put*

$$(4.29) \quad s(k, j) = 2 - \frac{k + (\kappa_+)_{k,j}}{q'},$$

where  $(\kappa_+)_{k,j}$  is defined as in (2.11) with  $A = A_{k,j}$ . Then  $\mu \in \mathfrak{M}_q(\partial P)$ , i.e.,  $\mu$  is a good measure for (1.1) relative to  $P$ , if and only if, for every pair  $(k, j)$  as above and every Borel set  $E \subset L_{k,j}$ :

If  $1 \leq k < N$  then

$$(4.30) \quad \begin{aligned} (q_c)_{k,j} \leq q < (q_c^*)_{k,j}, \quad C_{s(k,j),q'}^{N-k}(E) = 0 &\implies \mu(E) = 0 \\ q \geq (q_c^*)_{k,j} &\implies \mu(L_{N,j}) = 0 \end{aligned}$$

and if  $k = N$ , i.e.,  $L$  is a vertex,

$$(4.31) \quad q \geq (q_c)_{k,j} = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}} \implies \mu(L) = 0.$$

Here  $(q_c^*)_{k,j}$  and  $(q_c)_{k,j}$  are defined as in (2.32) and (2.36) respectively, with  $A = A_{k,j}$ .

If  $1 < q < (q_c)_{k,j}$  then there is no restriction on  $\mu\chi_{L_{k,j}}$ .

*Proof.* This is an immediate consequence of Proposition 4.3 and Proposition 4.5 (see also the Remark following it). In the case  $k = N$ ,  $L_{N,j}$  is a vertex and the condition says merely that for  $q \geq q_c(L_{N,j})$ ,  $\mu$  does not charge the vertex.  $\square$

### 4.3. Removable singular sets II.

**Proposition 4.7.** *Let  $A$  be a Lipschitz domain on  $S^{k-1}$ ,  $2 \leq k \leq N - 1$ , and let  $D_A$  be the  $k$ -dihedron with opening  $A$ . Let  $u$  be a positive solution of*

(1.1) in  $D_A^R$ , for some  $R > 0$ . Suppose that  $F = \mathcal{S}(u) \subset d_A^R$  and let  $Q$  be an open neighborhood of  $F$  such that  $\bar{Q} \subset d_A^R$ . (Recall that  $d_A^R = d_A \cap B_R^{N-k}(0)$  is an open subset of  $d_A$ .) Let  $\mu$  be the trace of  $u$  on  $\mathcal{R}(u)$ .

Let  $\eta \in W_0^{s,q'}(d_A^R)$  such that

$$(4.32) \quad 0 \leq \eta \leq 1, \quad \eta = 0 \quad \text{on } Q.$$

Employing the notation in the proof of Proposition 4.4, put

$$(4.33) \quad \zeta := \rho_A \psi_R H_R[\eta]^{q'}.$$

Then

$$(4.34) \quad \int_{D_A^R} u^q \zeta \, dx \leq c(1 + \|\eta\|_{W^{s,q'}(d_A)})^{q'} + \mu(d_A^R \setminus Q)^q,$$

$c$  independent of  $u$  and  $\eta$ .

*Proof.* First we prove (4.34) for  $\eta \in C_0^\infty(d_A^R)$ . Let  $\sigma_0$  be a point in  $A$  and let  $\{A_n\}$  be a Lipschitz exhaustion of  $A$ . If  $0 < \epsilon < \text{dist}(\partial A, \partial A_n) = \bar{\epsilon}_n$  then

$$\epsilon \sigma_0 + C_{A_n} \subset C_A.$$

Denote

$$D_A^{R',R''} = D_A \cap [|x'| < R'] \cap [|x''| < R''].$$

Pick a sequence  $\{\epsilon_n\}$  decreasing to zero such that  $0 < \epsilon_n < \min(\bar{\epsilon}_n/2^n, R/8)$ . Let  $u_n$  be the function given by

$$u_n(x'x'') = u(x' + \epsilon_n \sigma_0, x'') \quad \forall x \in D_{A_n}^{R_n, R}, \quad R_n = R - \epsilon_n.$$

Then  $u_n$  is a solution of (1.1) in  $D_{A_n}^{R_n, R}$  belonging to  $C^2(\bar{D}_{A_n}^{R_n, R})$  and we denote its boundary trace by  $h_n$ . Let

$$\zeta_n := \rho_{A_n} \psi_R H_R[\eta]^{q'},$$

with  $\psi_R$  and  $H_R[\eta]$  as in the proof of Proposition 4.4. By Proposition 4.2

$$(4.35) \quad - \int_{D_{A_n}^{R_n, R}} \mathbb{P}[h_n] \Delta \zeta_n \, dx = \int_{B_R^{N-k}(0)} \eta^{q'} h_n \, d\omega_n$$

where  $\omega_n$  is the harmonic measure on  $d_{A_n}^R$  relative to  $D_{A_n}^{R_n, R}$ . (Note that  $d_{A_n}^R = d_A^R$  and we may identify it with  $B_R^{N-k}(0)$ .) Hence

$$(4.36) \quad \int_{D_{A_n}^{R_n, R}} (-u_n \Delta \zeta_n + u_n^q \zeta_n) \, dx = - \int_{B_R^{N-k}(0)} \eta^{q'} h_n \, d\omega_n.$$

Further,

$$\int_{B_R^{N-k}(0)} \eta^{q'} h_n \, d\omega_n \rightarrow \int_{B_R^{N-k}(0)} \eta^{q'} \, d\mu \leq \mu(d_A^R \setminus Q),$$

because  $\eta = 0$  in  $Q$ . By (4.24), (4.28) we obtain,

$$(4.37) \quad \left| \int_{D_{A_n}^{R_n, R}} u_n \Delta \zeta_n dx \right| \leq c \left( \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{\frac{1}{q}} \left( \left( \int_{D_{A_n}^{R_n, R}} \zeta_n dx \right)^{\frac{1}{q'}} + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))} \right).$$

From the definition of  $\zeta_n$  it follows that

$$\int_{D_{A_n}^{R_n, R}} \zeta_n dx \leq \int_{D_{A_n}^{R_n, R}} \rho_n dx \rightarrow \int_{D_A^R} \rho dx,$$

where  $\rho$  (resp.  $\rho_n$ ) is the first eigenfunction of  $-\Delta$  in  $D_A^R$  (resp.  $D_{A_n}^{R_n, R}$ ) normalized by 1 at some  $x_0 \in D_{A_1}^{R_1, R}$ . Therefore, by (4.36),

$$\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \leq c \left( \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{\frac{1}{q}} (1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))}) + \mu(d_A^R \setminus Q).$$

This implies

$$(4.38) \quad \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \leq c(1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})^{q'} + \mu(d_A^R \setminus Q)^q.$$

To verify this fact, put

$$m = \left( \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{1/q}, \quad b = \mu(d_A^R \setminus Q), \quad a = c(1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})$$

so that (4.38) becomes

$$m^q - am - b \leq 0.$$

If  $b \leq m$  then

$$m^{q-1} - a - 1 \leq 0.$$

Therefore,

$$m \leq (a+1)^{\frac{1}{q-1}} + b$$

which implies (4.38). Finally, by the lemma of Fatou we obtain (4.34) for  $\eta \in C_0^\infty$ . By continuity we obtain the inequality for any  $\eta \in W_0^{s, q'}$  satisfying (4.32).  $\square$

**Theorem 4.8.** *Let  $A$  be a Lipschitz domain on  $S^{k-1}$ ,  $2 \leq k \leq N-1$ , and let  $D_A$  be the  $k$ -dihedron with opening  $A$ . Let  $E$  be a compact subset of  $d_A^R$  and let  $u$  be a non-negative solution of (1.1) in  $D_A^R$  (for some  $R > 0$ ) such that  $u$  vanishes on  $\partial D_A^R \setminus E$ . Then*

$$(4.39) \quad C_{s, q'}^{N-k}(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'} \implies u = 0,$$

where  $C_{s, q'}^{N-k}$  denotes the Bessel capacity with the indicated indices in  $\mathbb{R}^{N-k}$ .

*Proof.* By Proposition 4.4, (4.39) holds under the additional assumption

$$(4.40) \quad \int_{D_A^R} u^q \rho_R \rho_A^R dx < \infty.$$

Indeed, by [23, Proposition 4.1], (4.40) implies that the solution  $u$  possesses a boundary trace  $\mu$  on  $\partial D_A^R$ . By assumption,  $\mu(\partial D_A^R \setminus E) = 0$ . Therefore, by Proposition 4.5, the fact that  $C_{s,q'}^{N-k}(E) = 0$  implies that  $\mu(E) = 0$ . Thus  $\mu = 0$  and hence  $u = 0$ .

We show that, under the conditions of the theorem, if  $C_{s,q'}^{N-k}(E) = 0$  then (4.40) holds.

By Proposition 4.7, for every  $\eta \in W_0^{s,q'}(d_A^R)$  such that  $0 \leq \eta \leq 1$  and  $\eta = 0$  in a neighborhood of  $E$ ,

$$(4.41) \quad \int_{D_A^R} u^q \zeta dx \leq c(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))})^{q'},$$

for  $\zeta$  as in (4.33). (Here we use the assumption that  $u = 0$  on  $\partial D_A^R \setminus E$ .)

Let  $a > 0$  be sufficiently small so that  $E \subset B_{(1-4a)R}^{N-k}(0)$ . Pick a sequence  $\{\phi_n\}$  in  $C_0^\infty(\mathbb{R}^{N-k})$  such that, for each  $n$ , there exists a neighborhood  $Q_n$  of  $E$ ,  $\tilde{Q}_n \subset B_{(1-3a)R}^{N-k}(0)$  and

$$(4.42) \quad \begin{aligned} 0 &\leq \phi_n \leq 1 \text{ everywhere, } \phi_n = 1 \text{ in } Q_n, \\ \tilde{\phi}_n &:= \phi_n \chi_{\{|x''| < (1-2a)R\}} \in C_0^\infty(\mathbb{R}^{N-k}), \\ \|\tilde{\phi}_n\|_{W^{s,q'}(\mathbb{R}^{N-k})} &\rightarrow 0 \text{ as } n \rightarrow \infty \\ \eta_n &:= (1 - \phi_n)|_{\{|x''| < R\}} \in C_0^\infty(d_A^R), \\ \eta_n &= 0 \text{ in } [(1-a)R < |x''| < R]. \end{aligned}$$

Such a sequence exists because  $C_{s,q'}^{N-k}(E) = 0$ . Applying (4.41) to  $\eta_n$  we obtain,

$$(4.43) \quad \sup \int_{D_A^R} u^q \zeta_n dx \leq c < \infty,$$

where  $\zeta_n = \rho_A \psi_R H_R^{q'}[\eta_n]$  (see (4.33)). By taking a subsequence we may assume that  $\{\eta_n\}$  converges (say to  $\eta$ ) in  $L^{q'}(B_R^{N-k}(0))$  and consequently  $H[\eta_n] \rightarrow H[\eta]$  in the sense that

$$H_R[\eta_n](x', \cdot) = w_{n,R}(y^2, \cdot) \rightarrow w_R(y^2, \cdot) = H_R[\eta](x', \cdot) \text{ in } L^{q'}$$

uniformly with respect to  $y = |x'|$ . It follows that

$$(4.44) \quad \int_{D_A^R} u^q \zeta dx < \infty, \quad \zeta = \rho_A \psi_R H_R^{q'}[\eta].$$

As  $\tilde{\phi}_n \rightarrow 0$  in  $W^{s,q'}(\mathbb{R}^{N-k})$  it follows that  $\phi_n \rightarrow 0$  and hence  $\eta_n \rightarrow 1$  a.e. in  $B_{(1-2a)R}^{N-k}(0)$ . Thus  $\eta = 1$  in this ball,  $\eta = 0$  in  $[(1-a)R < |x''| < R]$  and  $0 \leq \eta \leq 1$  everywhere.

Consequently, given  $\delta > 0$ , there exists an  $N$ -dimensional neighborhood  $O$  of  $d_A \cap B_{(1-2a)R}^{N-k}(0)$  such that

$$1 - \delta < H_R[\eta] < 1 \quad \text{and} \quad 1 - \delta < \psi_R / \rho_A^R < 1 \quad \text{in } O.$$

Therefore (4.44) implies that

$$(4.45) \quad \int_{D_A^{(1-3a)R}} u^q \rho^R \rho_A^R dx \leq c < \infty.$$

Recall that the trace of  $u$  on  $\partial D_A^R \setminus d_A^{(1-4a)R}$  is zero. Therefore  $u$  is bounded in  $D_A^R \setminus D_A^{(1-3a)R}$ . This fact and (4.45) imply (4.40).  $\square$

**Definition 4.9.** Let  $\Omega$  be a bounded Lipschitz domain. Denote by  $\rho$  the first eigenfunction of  $-\Delta$  in  $\Omega$  normalized by  $\rho(x_0) = 1$  for a fixed point  $x_0 \in \Omega$ .

For every compact set  $K \subset \partial\Omega$  we define

$$M_{\rho,q}(K) = \{\mu \in \mathfrak{M}(\partial\Omega) : \mu \geq 0, \mu(\partial\Omega \setminus K) = 0, \mathbb{K}[\mu] \in L_\rho^q(\Omega)\}$$

and

$$\tilde{C}_{\rho,q'}(K) = \sup\{\mu(K)^q : \mu \in M_{\rho,q}(K), \int_\Omega \mathbb{K}[\mu]^q \rho dx = 1\}.$$

Finally we denote by  $C_{\rho,q'}$  the outer measure generated by the above functional.

The following statement is verified by standard arguments:

**Lemma 4.10.** For every compact  $K \subset \partial\Omega$ ,  $C_{\rho,q'}(K) = \tilde{C}_{\rho,q'}(K)$ . Thus  $C_{\rho,q'}$  is a capacity and,

$$(4.46) \quad C_{\rho,q'}(K) = 0 \iff M_{\rho,q}(K) = \{0\}.$$

**Theorem 4.11.** Let  $\Omega$  be a bounded polyhedron in  $\mathbb{R}^N$ . A compact set  $K \subset \partial\Omega$  is removable if and only if

$$(4.47) \quad C_{s(k,j),q'}(K \cap L_{k,j}) = 0,$$

for  $k = 1, \dots, N$   $j = 1, \dots, n_k$ , where  $s(k,j)$  is defined as in (4.29). This condition is equivalent to

$$(4.48) \quad C_{\rho,q'}(K) = 0.$$

A measure  $\mu \in \mathfrak{M}(\partial\Omega)$  is  $q$ -good if and only if it does not charge sets with  $C_{\rho,q'}$ -capacity zero.

*Proof.* The first assertion is an immediate consequence of Proposition 4.3 and Theorem 4.8. The second assertion follows from the fact that

$$C_{\rho,q'}(K \cap L_{k,j}) = C_{s(k,j),q'}(K \cap L_{k,j}).$$

The third assertion follows from Theorem 4.6 and the previous statement.  $\square$

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