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BOUNDARY TRACE OF POSITIVE SOLUTIONS OF SUPERCRITICAL SEMILINEAR ELLIPTIC EQUATIONS IN DIHEDRAL DOMAINS

MOSHE MARCUS AND LAURENT VERON

ABSTRACT. We study the generalized boundary value problem for (E) $-\Delta u + |u|^{q-1}u = 0$ in a dihedral domain Ω , when $q > 1$ is supercritical. The value of the critical exponent can take only a finite number of values depending on the geometry of Ω . When μ is a bounded Borel measure in a k -wedge, we give necessary and sufficient conditions in order it be the boundary value of a solution of (E). We also give conditions which ensure that a boundary compact subset is removable. These conditions are expressed in terms of Bessel capacities $B_{s,q}$ in \mathbb{R}^{N-k} where s depends on the characteristics of the wedge. This allows us to describe the boundary trace of a positive solution of (E).

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Key words. Laplacian; Poisson potential; Borel measures; Besov spaces; harmonic lifting; Bessel capacities.

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1. INTRODUCTION

Let Ω be a bounded Lipschitz domain in \mathbb{R}^N , ρ the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ with supremum 1 and λ the corresponding eigenvalue, and let $q > 1$. A long-term research on the equation

$$(1.1) \quad -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega,$$

has been carried out for more than twenty years by probabilistic and/or analytic methods. Much of the research was focused on three main problems in domains of class C^2 :

- (i) The Dirichlet problem for (1.1) with boundary data given by a finite Borel measure on $\partial\Omega$.
- (ii) The characterization of removable singular subsets of $\partial\Omega$ relative to positive solutions of (1.1).
- (iii) The characterization of arbitrary positive solutions of (1.1) via an appropriate notion of boundary trace.

Consider the Dirichlet problem

$$(1.2) \quad -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \quad u = \mu \quad \text{in } \partial\Omega$$

where $\mu \in \mathfrak{M}(\partial\Omega)$ (= space of finite Borel measures on $\partial\Omega$). Following [23], a (weak) solution $u := u_\mu$ of (1.2) is a function $u \in L_\rho^q(\Omega)$ such that,

$$(1.3) \quad \int_\Omega (-u\Delta\eta + \eta|u|^{q-1}u) dx = - \int_\Omega \mathbb{K}[\mu]\Delta\eta dx,$$

for every $\eta \in X(\Omega)$, where

$$(1.4) \quad X(\Omega) = \left\{ \eta \in W_0^{1,2}(\Omega) : \rho^{-1}\Delta\eta \in L^\infty(\Omega) \right\}.$$

Here $\mathbb{K}[\mu]$ is the harmonic function in Ω with boundary trace μ and ρ is the first eigenfunction of $-\Delta$ in Ω normalized so that $\max_\Omega \rho = 1$. We recall that, if Ω is Lipschitz $\mathbb{K}[\mu] \in L_\rho^1(\Omega)$; if Ω is of class C^2 , $\mathbb{K}[\mu] \in L^1(\Omega)$.

A measure μ is a *q-good measure* if (1.2) has a solution. The space of *q-good measures* is denoted by $\mathfrak{M}_q(\partial\Omega)$. It is known that, if μ is *q-good*, the solution is unique. Furthermore, if μ satisfies the condition

$$(1.5) \quad \int_\Omega \mathbb{K}[|\mu|]^q \rho dx < \infty,$$

then it is *q-good*. When μ satisfies this condition we say that it is a *q-admissible measure*.

When Ω is a domain of class C^2 , $\mathbb{K}[\mu] \in L_\rho^q$ for every $q \in (1, \frac{N+1}{N-1})$ and every $\mu \in \mathfrak{M}(\partial\Omega)$. Therefore, for q in this range, every measure in $\mathfrak{M}(\partial\Omega)$ is *q-good* and there is no removable boundary set (except for the empty set). Problem (iii), for q in this range, was resolved by Le Gall [15] (for $N = q = 2$) and Marcus and Véron [18] (for $1 < q < \frac{N+1}{N-1}$, $N \geq 3$).

The number $q_c = \frac{N+1}{N-1}$ is called the *critical value* for (1.1). If q is supercritical, i.e. $q \geq q_c$, point singularities are removable. In particular there

is no solution of (1.2) when $\mu = \delta_y$ (= a Dirac measure concentrated at a point $y \in \partial\Omega$).

In the supercritical case, problems (i) - (iii), Ω of class C^2 , have been resolved in several stages. We say that a compact set $E \subset \partial\Omega$ is removable relative to equation (1.1) if there exists no positive solution vanishing on $\partial\Omega \setminus E$. We say that E is conditionally removable if any solution u of (1.2), with $\mu \in \mathfrak{M}(\partial\Omega)$, such that $u = 0$ on $\partial\Omega \setminus E$ must vanish in Ω .

With respect to problem (ii) it was shown that a compact set $E \subset \partial\Omega$ is removable if and only if $C_{\frac{2}{q}, q'}(E) = 0$, $q' = q/(q-1)$. Here $C_{\alpha, p}$ denotes the Bessel capacity, with the indicated indexes on $\partial\Omega$. (see Section 4.2 for an overview of Bessel capacities). This result was obtained by Le Gall [15] for $q = 2$, Dynkin and Kuznetsov [8] for $1 < q \leq 2$, Marcus and Véron [19] for $q > 2$. For a unified analytic proof, covering all $q \geq q_c$ see [20].

The above result implies that every q -good measure μ must vanish on sets of $C_{\frac{2}{q}, q'}$ capacity zero. On the other hand a result of Baras and Pierre [3] implies that every positive measure $\mu \in \mathfrak{M}(\partial\Omega)$ that vanishes on sets of $C_{\frac{2}{q}, q'}$ capacity zero is the limit of an increasing sequence of admissible measures and therefore q -good. In conclusion: a measure $\mu \in \mathfrak{M}(\partial\Omega)$ is q -good if and only if it vanishes on sets of $C_{\frac{2}{q}, q'}$ capacity zero. This takes care of problem (i).

Problem (iii) has been treated in several papers, with various definitions of a generalized boundary trace for positive solutions of (1.1), see [9] and [22]. Finally a full characterization of positive solutions was obtained by Mselati [24] for $q = 2$, Dynkin [7] for $1 < q < 2$ and Marcus [17] for every $q \geq q_c$. In [24, 7] the restriction to $q \leq 2$ was dictated by their use of probabilistic techniques that do not apply to $q > 2$. In [17] the proof is purely analytic.

If Ω is Lipschitz, $\xi \in \partial\Omega$, we say that q_ξ is the critical value for (1.1) at ξ if, for $1 < q < q_\xi$, problem (1.2) with $\mu = \delta_\xi$ has a solution, but for $q > q_\xi$ no such solution exists.

In contrast to the case of smooth domains, when Ω is Lipschitz, q_ξ may vary with the point. For every compact set $F \subset \partial\Omega$ there exists a number $q(F) > 1$ such that, for $1 < q < q(F)$, every measure in $\mathfrak{M}(\partial\Omega)$ supported in F is q -good. Obviously $q(F) \leq \min\{q_\xi : \xi \in F\}$ but it is not clear if equality holds.

In the special case when Ω is a polyhedron, the function $\xi \rightarrow q_\xi$ obtains only a finite number of values (in fact, it is constant on each open face and each open edge) and, if $q \geq q_\xi$, an isolated singularity at ξ is removable. Furthermore, the assumption $1 < q < \min\{q_\xi : \xi \in \partial\Omega\}$ implies that every measure in $\mathfrak{M}(\partial\Omega)$ is q -good. For this and related results see [23].

In the present paper we study problem (1.2) when Ω is a polyhedron and q is supercritical, i.e. $q \geq \min\{q_\xi : \xi \in \partial\Omega\}$. Following is a description of the main results.

A. *On the action of Poisson type kernels with fractional dimension.*

In preparation for the study of supercritical boundary value problems we establish an harmonic analytic result, extending a well known result on the action of Poisson kernels on Besov spaces with negative index (see [27, 1.14.4.] and [4]). We first quote the classical result for comparison purposes.

Proposition 1.1. *Let $1 < q < \infty$ and $s > 0$. Then, for any bounded Borel measure μ in \mathbb{R}^{n-1} ,*

$$(1.6) \quad I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|_{B^{-s,q}(\mathbb{R}^{n-1})}^q.$$

Here $\mathbb{K}_n[\mu]$ denotes the Poisson potential of μ in $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$, namely,

$$(1.7) \quad \mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{n-1}} \frac{d\mu(z)}{(y_1^2 + |\zeta - z|^2)^{n/2}} \quad \forall y = (y_1, \zeta) \in \mathbb{R}_+^n$$

where γ_n is a constant depending only on n .

Notation. Let m be a positive integer and let ν be a real number, $\nu \geq m+1$. Denote,

$$(1.8) \quad \mathbb{K}_{\nu,m}[\mu](\tau, \zeta) := \int_{\mathbb{R}^m} \frac{\tau^{\nu-m} d\mu(z)}{(\tau^2 + |\zeta - z|^2)^{\nu/2}} \quad \forall \tau \in (0, \infty), \zeta \in \mathbb{R}^m.$$

Note that

$$\mathbb{K}_n[\mu] = \gamma_n \mathbb{K}_{n,n-1}[\mu].$$

Theorem 1.2. *Let m and ν be as above. Then, for every $q > 1$ and every $s \in (0, m/q')$, $q' = q/(q-1)$, there exists a positive constant c such that, for every positive measure $\mu \in \mathfrak{M}(\mathbb{R}^m)$ supported in $B_{R/2}(0)$ for some $R > 1$,*

$$(1.9) \quad \frac{1}{c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq \int_0^R \left(\int_{|\zeta| < R} |\mathbb{K}_{\nu,m}[\mu](\tau, \zeta)|^q d\zeta \right) \tau^{sq-1} d\tau \\ \leq c R^{(s+\nu-m)q+1} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q.$$

This also holds when $s = m/q'$, provided that the diameter of $\text{supp } \mu$ is sufficiently small.

This is proved in Section 3 (see Theorem 3.8) using a slightly different notation.

B. *The critical value and the characterization of q -good measures in a k -wedge.*

The next step towards the study of boundary value problems in a polyhedron is the treatment of such problems in a k -wedge (or k -dihedron) i.e., the domain defined by the intersection of k hyperplanes in \mathbb{R}^N , $1 < k < N$. The edge is an $(N-k)$ dimensional space.

We note that if $k = N$ the 'edge' is a point and the corresponding wedge is a cone with vertex at this point. If $k = 1$ the wedge is a half space. Both of these cases have been treated in [23].

Let A be a Lipschitz domain in S^{k-1} . If

$$(1.10) \quad S_A := \{x \in \mathbb{R}^N : |x| = 1, x \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}$$

then

$$D_A := \{x = (r, \sigma) : r > 0, \sigma \in S_A\}$$

is a k -wedge in \mathbb{R}^N whose ‘edge’ d_A may be identified with \mathbb{R}^{N-k} and its ‘opening’ is A .

Let λ_A be the first eigenvalue of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S_A)$ and denote by κ_{\pm} the roots of the equation,

$$(1.11) \quad \kappa^2 + (N-2)\kappa - \lambda_A = 0.$$

Put

$$(1.12) \quad q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}$$

and

$$(1.13) \quad q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}.$$

Let $C_{\alpha,p}^{N-k}$ denote the Bessel capacity with the indicated indices in \mathbb{R}^{N-k} . The next theorem provides a characterization of q -good measures supported on d_A .

Theorem 1.3. (a) *If $1 < q < q_c$ every measure in $\mathfrak{M}(d_A)$ is q -good relative to D_A . In fact every such measure is q -admissible.*

(b) *If $q \geq q_c^*$, the only q -good measure in $\mathfrak{M}(d_A)$ is the zero measure.*

(c) *If $q_c \leq q < q_c^*$, a measure $\mu \in \mathfrak{M}(d_A)$ is q -good relative to D_A if and only if μ vanishes on every Borel set $E \subset d_A$ such that $C_{s,q'}^{N-k}(E) = 0$, $s = 2 - \frac{k+\kappa_+}{q}$.*

The characterization of q -good measures in a polyhedron follows as an easy consequence of the above theorem (see Theorem 4.6 below).

C. Characterization of removable sets.

Let Ω be an N -dimensional polyhedron. Theorem 1.3 provides a necessary and sufficient condition for the removability of a singular set E relative to the family of solutions u such that

$$\int_{\Omega} |u|^q \rho dx < \infty.$$

The next result provides a necessary and sufficient condition for *removability* in the sense that the only non-negative solution $u \in C(\bar{\Omega} \setminus E)$ which vanishes on $\bar{\Omega} \setminus E$ is the trivial solution $u = 0$.

Let L denote a face or edge or vertex of Ω and put $k := \text{codim } L$. If $1 < k < N$ let d_L denote the linear space spanned by L , such that L is an open subset of d_L . Let Q_L denote the k -wedge with boundary d_L such that,

for some neighborhood M of L , $\Omega \cap M = Q_L \cap M$ and let A_L denote the opening of Q_L . If $k = N$, Q_L is a cone with vertex L . Let $q_c(L)$ and $q_c^*(L)$ be defined as in (1.12) and (1.13) for $A = A_L$. Finally let

$$s(L) = 2 - \frac{k + \kappa_+}{q'}$$

where κ_{\pm} are the roots of (1.11) for $A = A_L$. If $k = N$, Q_L is a cone with vertex L . In this case $q_c(L) = q_c^*(L) = 1 - \frac{2}{\kappa_-}$. If $k = 1$ $q_c(L) = q_c^*(L) = (N + 1)/(N - 1)$.

Theorem 1.4. *Let Ω be a polyhedron in \mathbb{R}^N . A compact set $E \subset \partial\Omega$ is removable if and only if, for every L as above such that $E \cap L \neq \emptyset$ the following conditions hold.*

If $1 \leq k < N$: either $q_c(L) \leq q < q_c^(L)$ and $C_{s(L), q'}^{N-k}(E \cap L) = 0$ or $q \geq q_c^*(L)$.
If $k = N$: $q \geq q_c(L)$.*

The present paper is part of an article, ‘Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains’ arXiv:0907.1006 (2009). The first part of this article was published in [23]. The second and last part are presented here. The characterization of q -good measures, here established in polyhedrons, was recently established in [2], for arbitrary Lipschitz domains and a general family of nonlinearities. However the full removability result, Theorem 4.11, has not been superseded. (In [2] the authors provided - in the generality mentioned above - a characterization of *conditional removability* but not of full removability.) The methods of proof in the two papers are completely different. In the present paper, the characterization of q -good measures is based on an extension of a result of [4] and [27, 1.14.4.] on the action of Poisson kernels on Besov spaces with negative index. In [2] the proof relies on a relation between elliptic semilinear equations with absorption and linear Schrödinger equations.

2. THE MARTIN KERNEL AND CRITICAL VALUES IN A K -DIMENSIONAL DIHEDRON.

2.1. The geometric framework. An N -dim polyhedra P is a bounded domain bordered by a finite number of hyperplanes. Thus the boundary of P is the union of a finite number of sets $\{L_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$ where $\{L_{1,j}\}$ is the set of open faces of P , $\{L_{k,j}\}$ for $k = 2, \dots, N - 1$, is the family of relatively open $N - k$ -dimensional edges and $\{L_{N,j}\}$ is the family of vertices of P . An $N - k$ dimensional edge is a relatively open set in the intersection of k hyperplanes; it will be described by the characteristic angles of these hyperplanes.

We recall that the spherical coordinates in $\mathbb{R}^N = \{x = (x_1, \dots, x_N)\}$ are expressed by

$$(2.1) \quad \begin{cases} x_1 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \sin \theta_1 \\ x_2 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \cos \theta_1 \\ x_3 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \cos \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{N-1} = r \sin \theta_{N-1} \cos \theta_{N-2}, \\ x_N = r \cos \theta_{N-1} \end{cases}$$

where, $r = |x|$, $\theta_1 \in [0, 2\pi]$ and $\theta_\ell \in [0, \pi]$ for $\ell = 2, 3, \dots, N-1$. We denote $\sigma = (\theta_1, \dots, \theta_{N-1})$. Thus in spherical coordinates $x = (r, \sigma)$.

We consider an unbounded *non-degenerate k-dihedron*, $2 \leq k \leq N$ defined as follows. Let A be given by

$$A = \begin{cases} (0, \alpha_1) \times \prod_{j=2}^{k-1} (\alpha_j, \alpha'_j) & \text{if } k > 2 \\ (0, \alpha_1) & \text{if } k = 2 \end{cases}$$

where

$$0 < \alpha_1 < 2\pi, \quad 0 \leq \alpha_j < \alpha'_j < \pi \quad j = 2, \dots, k-1.$$

We denote by S_A the spherical domain

$$(2.2) \quad S_A = \{x \in \mathbb{R}^N : |x| = 1, \sigma \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}$$

and by D_A the corresponding k-dihedron,

$$D_A = \{x = (r, \sigma) : r > 0, \sigma \in S_A\}.$$

The *edge* of D_A is the $(N-k)$ -dimensional space

$$(2.3) \quad d_A = \{x : x_1 = x_2 = \dots = x_k = 0\}.$$

2.2. On the Martin kernel and critical values in a cone. We recall here some elements of local analysis when $\Omega = C_A \cap B_1$, A is a Lipschitz domain in S^{N-1} and C_A is the cone with vertex 0 and opening A .

Denote by λ_A the first eigenvalue and by ϕ_A the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(A)$ (normalized by $\max \phi_A = 1$). Let κ_- be the negative root of (1.11) and put

$$\Phi_1(x) := \frac{1}{\gamma} |x|^{\kappa_-} \phi_A(x/|x|)$$

where γ is a positive number. Then Φ_1 is a harmonic function in C_A vanishing on $\partial C_A \setminus \{0\}$. We choose $\gamma = \gamma_A$ so that the boundary trace of Φ_1 is δ_0 (=Dirac measure on with mass 1 at the origin).

(i) If $q \geq 1 - \frac{2}{\kappa_-}$, there is no solution of (1.1) in Ω_S with isolated singularity at 0. (See [10].)

(ii) If $1 < q < 1 - \frac{2}{\kappa_-}$, then for any $k > 0$ there exists a unique solution $u := u_k$ to problem (1.2) with $\mu = k\delta_0$ and

$$(2.4) \quad u_k(x) = k\Phi_1(x)(1 + o(1)) \quad \text{as } x \rightarrow 0.$$

The function $u_\infty = \lim_{k \rightarrow \infty} u_k$ is a positive solution of (1.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies

$$(2.5) \quad u_\infty(x) = |x|^{-\frac{2}{q-1}} \omega_A(x/|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0$$

where ω_A is the (unique) positive solution of

$$(2.6) \quad -\Delta' \omega - a_{N,q} \omega + |\omega|^{q-1} \omega = 0$$

on S^{N-1} . Here Δ' is the Laplace - Beltrami operator and

$$(2.7) \quad a_{N,q} = \frac{2}{q-1} \left(\frac{2q}{q-1} - N \right).$$

(iii) If $u \in C(\bar{\Omega}_A \setminus \{0\})$ is a positive solution of (1.1) vanishing on $(\partial C_A \cap B_{r_0}(0)) \setminus \{0\}$, then either u satisfies (2.4) for some $k > 0$ or u satisfies (2.5). In particular there exists a unique positive solution vanishing on $(\partial C_A \cap B_{r_0}(0)) \setminus \{0\}$ with strong singularity at 0. (For (ii) and (iii) see [23, Theorem 5.7].)

2.3. Separable harmonic functions and the Martin kernel in a k-dihedron, $2 \leq k < N$. In the system of spherical coordinates, the Laplacian takes the form

$$\Delta u = \partial_{rr} u + \frac{N-1}{r} \partial_r u + \frac{1}{r^2} \Delta_{S^{N-1}} u$$

where the Laplace-Beltrami operator $\Delta_{S^{N-1}}$ is expressed by induction by

$$(2.8) \quad \begin{aligned} \Delta_{S^{N-1}} u &= \frac{1}{(\sin \theta_{N-1})^{N-2}} \frac{\partial}{\partial \theta_{N-1}} \left((\sin \theta_{N-1})^{N-2} \frac{\partial u}{\partial \theta_{N-1}} \right) \\ &+ \frac{1}{(\sin \theta_{N-1})^2} \Delta_{S^{N-2}} u. \end{aligned}$$

and

$$(2.9) \quad \Delta_{S^1} u = \partial_{\theta_1 \theta_1} u$$

If we compute the positive harmonic functions in the k-dihedron D_A of the form

$$v(x) = v(r, \sigma) = r^\kappa \omega(\sigma) \quad \text{in } D_A, \quad v = 0 \quad \text{in } \partial D_A \setminus \{0\}.$$

we find that ω must be a positive eigenfunction corresponding to the first eigenvalue, λ_A , of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S_A)$,

$$(2.10) \quad \begin{cases} \Delta_{S^{N-1}} \omega + \lambda_A \omega = 0 & \text{in } S_A \\ \omega = 0 & \text{on } \partial S_A \end{cases}$$

and κ must be a root of the algebraic equation (1.11) with λ_A as above. Thus $\kappa = \kappa_{\pm}$ where

$$(2.11) \quad \begin{aligned} \kappa_+ &= \frac{1}{2} \left(2 - N + \sqrt{(N-2)^2 + 4\lambda_A} \right) \\ \kappa_- &= \frac{1}{2} \left(2 - N - \sqrt{(N-2)^2 + 4\lambda_A} \right). \end{aligned}$$

Since

$$S^{N-1} = \{ \sigma = (\sigma_2 \sin \theta_{N-1}, \cos \theta_{N-1}) : \sigma_2 \in S^{N-2}, \theta_{N-1} \in (0, \pi) \},$$

we look for a solution $\omega = \omega^{\{1\}}$ of (2.10) of the form

$$\omega^{\{1\}}(\sigma) = (\sin \theta_{N-1})^{\kappa_+} \omega^{\{2\}}(\sigma_2), \quad \theta_{N-1} \in (0, \pi), \quad \sigma_2 \in S^{N-2}.$$

Here $S^{N-2} = S^{N-1} \cap \{x_N = 0\}$ and we denote

$$S_A^{\{N-2\}} = S_A \cap \{x_N = 0\}, \quad D_A^{\{N-2\}} := D_A \cap \{x_N = 0\} \subset \mathbb{R}^{N-1}.$$

Then (2.11) jointly with relation (2.8) implies

$$(2.12) \quad \begin{cases} \Delta_{S^{N-2}} \omega^{\{2\}} + (\lambda_A - \kappa_+) \omega^{\{2\}} = 0 & \text{on } S_A^{\{N-2\}} \\ \omega^{\{2\}} = 0 & \text{on } \partial S_A^{\{N-2\}}. \end{cases}$$

Since we are interested in $\omega^{\{2\}}$ positive, $\lambda_A^{\{2\}} := \lambda_A - \kappa_+$ must be the first eigenvalue of $-\Delta_{S^{N-2}}$ in $W_0^{1,2}(S_A^{\{N-2\}})$.

Next we look for positive harmonic functions \tilde{u} in $D_A^{\{N-2\}}$ such that

$$\tilde{u}(x_1, \dots, x_{N-1}) = r^{\kappa'} \omega(\sigma_2), \quad \tilde{u} = 0 \text{ on } \partial D_A^{\{N-2\}}$$

The algebraic equation which gives the exponents is

$$(\kappa')^2 + (N-3)\kappa' - \lambda_A^{\{2\}} = 0.$$

Denote by κ'_+ the positive root of this equation. By the definition of $\lambda_A^{\{2\}}$,

$$\kappa'_+{}^2 + (N-3)\kappa'_+ - \lambda_A^{\{2\}} = \kappa'_+{}^2 + (N-2)\kappa'_+ - \lambda_A = 0.$$

Therefore $\kappa'_+ = \kappa_+$. Accordingly, if $k \geq 3$, we set

$$\omega^{\{2\}}(\sigma_2) = (\sin \theta_{N-2})^{\kappa_+} \omega^{\{3\}}(\sigma_3),$$

and find that $\omega^{\{3\}}$ satisfies

$$(2.13) \quad \begin{cases} \Delta_{S^{N-3}} \omega^{\{3\}} + (\lambda_A - 2\kappa_+) \omega^{\{3\}} = 0 & \text{in } S_A^{\{N-3\}} \\ \omega^{\{3\}} = 0 & \text{on } \partial S_A^{\{N-3\}}, \end{cases}$$

where

$$S_A^{\{N-3\}} = S_A \cap \{x_N = x_{N-1} = 0\}.$$

Performing this reduction process (N-k) times, we obtain the following results.

(i) If $k > 2$ then $\omega = \omega^{N-k}(\sigma)$ is given by

$$(2.14) \quad \omega(\sigma) = (\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k)^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})$$

where

$$\sigma_{N-k+1} \in S^{k-1} = S^{N-1} \cap \{x_N = x_{N-1} = \dots = x_{k+1} = 0\}$$

and $\omega' := \omega^{\{N-k+1\}}$ satisfies

$$(2.15) \quad \begin{cases} \Delta_{S^{k-1}} \omega' + (\lambda_A - (N-k)\kappa_+) \omega' = 0, & \text{in } S_A^{\{k-1\}} \\ \omega' = 0, & \text{on } \partial S_A^{\{k-1\}}, \end{cases}$$

where $S_A^{\{k-1\}} = S_A \cap \{x_N = x_{N-1} = \dots = x_{k+1} = 0\} \approx A$ and $\lambda_A - (N-k)\kappa_+$ is the first eigenvalue of the problem.

(ii) If $k = 2$ then

$$(2.16) \quad \omega(\sigma) = (\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2)^{\kappa_+} \omega^{\{N-1\}}(\theta_1)$$

where $\sigma_{N-1} \in S^1 \approx \theta_1 \in (0, 2\pi)$, and $\omega^{\{N-1\}}$ satisfies

$$(2.17) \quad \begin{cases} \Delta_{S^1} \omega^{\{N-1\}} + (\lambda_A - (N-2)\kappa_+) \omega^{\{N-1\}} = 0 & \text{on } S_A^{\{1\}} \\ \omega^{\{N-1\}} = 0 & \text{on } \partial S_A^{\{1\}}, \end{cases}$$

with $\partial S_A^{\{1\}} \approx (0, \alpha)$. In this case

$$(2.18) \quad \kappa_+ = \frac{\pi}{\alpha}, \quad \omega^{\{N-1\}}(\theta_1) = \sin(\pi\theta_1/\alpha),$$

and, by (1.11),

$$(2.19) \quad \lambda_A - (N-2)\kappa_+ = \frac{\pi^2}{\alpha^2} \implies \lambda_A = \frac{\pi^2}{\alpha^2} + (N-2)\frac{\pi}{\alpha}.$$

Observe that $\frac{1}{2} \leq \kappa_+$ with equality holding only in the degenerate case $\alpha = 2\pi$ (which we exclude).

In either case, we find a positive harmonic function v_A in D_A , vanishing on ∂D_A , of the form

$$(2.20) \quad v_A(x) = |x|^{\kappa_+} \omega(x/|x|)$$

with ω as in (2.14) (for $k > 2$) or (2.18) (for $k=2$). Furthermore, if Ω is a domain in \mathbb{R}^N such that, for some $R > 0$, $\Omega \cap B_R(0) = D_A \cap B_R(0)$ and w is a positive harmonic function in Ω vanishing on $d_A \cap B_R(0)$ then $w \sim v_A$ in $\Omega \cap B_{R'}(0)$ for every $R' \in (0, R)$.

Similarly we find a positive harmonic function in D_A vanishing on $\partial D_A \setminus \{0\}$, singular at the origin, of the form

$$K'_A(x) = |x|^{\kappa_-} \omega(x/|x|).$$

If Ω is a domain as above and z is a positive harmonic function in Ω vanishing on $d_A \cap B_R(0) \setminus \{0\}$ then $z \sim K'_A$ in $\Omega \cap B_{R'}(0) \setminus \{0\}$ for every $R' \in (0, R)$.

As K'_A is a kernel function of $-\Delta$ at 0 it follows that K'_A is, up to a multiplicative constant c_A , the Martin kernel of $-\Delta$ in D_A , with singularity at 0. The Martin kernel, with singularity at a point $z \in d_A$, is given by

$$(2.21) \quad K_A(x, z) = c_A \frac{(\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k)^{\kappa+\omega} \{\sigma_{N-k+1}\}}{|x-z|^{N-2+\kappa+}}$$

for every $x \in D_A$. From (2.1)

$$\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k = |x-z|^{-1} \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}.$$

Therefore, if we write $x \in \mathbb{R}^N$ in the form $x = (x', x'')$, $x' = (x_1, \dots, x_k)$, $x'' = (x_{k+1}, \dots, x_N)$, we obtain the formula,

$$(2.22) \quad \begin{aligned} K_A(x, z) &= c_A \frac{|x'|^{\kappa+\omega} \{\sigma_{N-k+1}\}}{|x-z|^{(N-2+2\kappa+)}} \\ &= c_A \frac{|x'|^{\kappa+\omega} \{\sigma_{N-k+1}\}}{(|x'|^2 + |x''-z|^2)^{(N-2+2\kappa+)/2}}. \end{aligned}$$

Therefore, the Poisson potential of a measure $\mu \in \mathfrak{M}(d_A)$ is expressed by

$$(2.23) \quad \begin{aligned} \mathbb{K}_A[\mu](x) &= c_A |x'|^{\kappa+\omega} \{\sigma_{N-k+1}\} \\ &\quad \times \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(|x'|^2 + |x''-z|^2)^{(N-2+2\kappa+)/2}}. \end{aligned}$$

2.4. The admissibility condition. Consider the boundary value problem

$$(2.24) \quad \begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } D_A \\ u = \mu \in \mathfrak{M}(\partial D_A). \end{cases}$$

Let

$$(2.25) \quad \Gamma_R = \{x = (x', x'') : |x'| \leq R, |x''| \leq R\}, \quad D_{A,R} := D_A \cap \Gamma_R$$

and let $\rho_{R,A}$ denote the first (positive) eigenfunction in $D_{A,R} := D_A \cap \Gamma_R$. In the rest of this section we drop the index A in K_A , $\rho_{A,R}$ etc., except for D_A , $D_{A,R}$ and d_A .

First we observe that a positive Radon measure on d_A is q -good relative to D_A if and only if, for every compact set $F \subset d_A$, $\mu \chi_F$ is q -good in D_A .

Now suppose that μ is compactly supported in d_A and denote its support by F . We claim that μ is q -good in D_A if and only if it is q -good relative to $D_{A,R}$ for all sufficiently large R . Let R be such that $F \subset B_{R/2}^{N-k}(0)$. Assume that μ is q -good in $D_{A,R}$. Let v_R be the solution of (1.1) in $D_{A,R}$ such that $v_R = \mu$ on $d_A \cap \Gamma_R$, $v_R = 0$ on $\partial D_{A,R} \setminus d_A$. Then v_R increases with R and $v = \lim_{R \rightarrow \infty} v_R$ is a solution of (1.1) in D_A with boundary data μ . This proves our claim in one direction; the other direction is obvious.

The condition for μ to be q -admissible in $D_{A,R}$ is

$$(2.26) \quad \int_{D_{A,R}} \mathbb{K}^R[|\mu|](x)^q \rho_R(x) dx < \infty.$$

where K^R is the Martin kernel of $-\Delta$ in $D_{A,R}$. If R is sufficiently large then, in a neighborhood of F , $K^R \sim K$ and $\rho^R \sim \rho \sim v_A$. Therefore, a sufficient condition for μ to be q -good in D_A is

$$(2.27) \quad \int_{\Gamma_R \cap D_A} \mathbb{K}[|\mu|](x)^q \rho(x) dx < \infty \quad \forall R > 0.$$

By the first observation in this subsection, it follows that the previous statement remains valid for any positive Radon measure supported on d_A .

By (2.21),

$$(2.28) \quad \mathbb{K}[|\mu|](x) \leq c_A (r')^{\kappa_+} \int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z)$$

where

$$(2.29) \quad j(x) = |x|^{-N+2-2\kappa_+} \quad \forall x \in \mathbb{R}^N.$$

Therefore, using (2.20), condition (2.27) becomes

$$(2.30) \quad \int_0^R \int_{|x''| < R} \left(\int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z) \right)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty$$

for every $R > 0$.

2.5. The critical values. Relative to the equation

$$(2.31) \quad -\Delta u + |u|^{q-1} u = 0$$

there exist two thresholds of criticality associated with the edge d_A .

The first is the value q_c^* such that, for $q_c^* \leq q$ the whole edge d_A is removable but for $1 < q < q_c^*$ there exist non-trivial solutions in D_A which vanish on $\partial D_A \setminus d_A$. The second $q_c < q_c^*$ corresponds to the removability of points on d_A . For $q \geq q_c$ points on d_A are removable while for $1 < q < q_c$ there exist solutions with isolated point singularities on d_A . In the next two propositions we determine these critical values.

Proposition 2.1. *Assume $q > 1$, $1 \leq k < N$. Then the condition*

$$(2.32) \quad q < q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}$$

is necessary and sufficient for the existence of a non-trivial solution u of (2.31) in D_A which vanishes on $\partial D_A \setminus d_A$. Furthermore, when this condition holds, there exist non-trivial positive bounded measures μ on d_A such that $\mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A)$.

Remark. The statement remains true for $k = N$, which is the case of the cone. In this case $q_c = q_c^* = 1 - (2/\kappa_-)$ and a straightforward computation yields:

$$(2.33) \quad q_c = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}}.$$

Proof. Recall that $\lambda_A - (N - k)\kappa_+$ is the first eigenvalue in $S_A^{\{k-1\}}$ (see (2.15) and the remarks following it). Let κ'_+, κ'_- be the two roots of the equation

$$X^2 + (k - 2)X - (\lambda_A - (N - k)\kappa_+) = 0,$$

i.e.

$$\kappa'_\pm = \frac{1}{2}(2 - k \pm \sqrt{(k - 2)^2 + 4(\lambda_A - (N - k)\kappa_+}).$$

Then, by [23, Theorem 5.7], recalled in subsection 2.2, if $1 < q < 1 - (2/\kappa'_-)$ there exists a unique solution of (2.31) in the cone $C_{S_A^{k-1}}$ i.e. the cone with opening $S_A^{k-1} \subset S^{k-1} \subset \mathbb{R}^k$ with trace $a\delta_0$ (where δ_0 denotes the Dirac measure at the vertex of the cone and $a > 0$). By (2.5) this solution satisfies

$$(2.34) \quad u_a(x) = a|x|^{-\alpha} \phi(x/|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0,$$

where ϕ is the first positive eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_A^{k-1})$ normalized so that u_1 possesses trace δ_0 .

The function u given by

$$\tilde{u}_a(x', x'') = u_a(x') \quad \forall (x', x'') \in D_A = C_{S_A^{k-1}} \times \mathbb{R}^{N-k},$$

is a nonzero solution of (2.31) in D_A which vanishes on $\partial D_A \setminus d_A$ and has bounded trace on d_A .

A simple calculation shows that $1 - (2/\kappa'_-)$ equals q_c^* as given in (2.32).

Next, assume that $q \geq q_c^*$ and let u be a solution of (2.31) in D_A which vanishes on $\partial D_A \setminus d_A$.

Given $\epsilon > 0$ let v_ϵ be the solution of (2.31) in $D_A^{\{N-k-1\}} \setminus \{x' \in \mathbb{R}^k : |x'| \leq \epsilon\}$ such that

$$v_\epsilon(x') = \begin{cases} 0, & \text{if } x' \in \partial D_A^{\{N-k-1\}}, |x'| > \epsilon, \\ \infty, & \text{if } |x'| = \epsilon. \end{cases}$$

Given $R > 0$ let w_R be the maximal solution in $\{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$.

Then the function u^* given by

$$u^*(x', x'') = v_\epsilon(x') + w_R(x'')$$

is a supersolution of (2.31) in $D_A \setminus \{(x', x'') : |x'| > \epsilon, |x''| < R\}$ and it dominates u in this domain. But $w_R(x'') \rightarrow 0$ as $R \rightarrow \infty$ and, by [10], $v_\epsilon(x') \rightarrow 0$ as $\epsilon \rightarrow 0$. Therefore $u_+ = 0$ and, by the same token, $u_- = 0$. \square

Proposition 2.2. *Let A be defined as before. Then*

$$(2.35) \quad \mathbb{K}[\mu] \in L_\rho^q(\Gamma_R \cap D_A) \quad \forall \mu \in \mathfrak{M}(d_A), \quad \forall R > 0$$

if and only if

$$(2.36) \quad 1 < q < q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}.$$

This statement is equivalent to the following:

Condition (2.36) is necessary and sufficient in order that the Dirac measure $\mu = \delta_P$, supported at a point $P \in d_A$, satisfy (2.35).

Proof. It is sufficient to prove the result relative to the family of measures μ such that μ is positive, has compact support and $\mu(d_A) = 1$. Let $R > 1$ be sufficiently large so that the support of μ is contained in $\Gamma_{R/2}$. The measure μ can be approximated (in the sense of weak convergence of measures) by a sequence $\{\mu_n\}$ of convex combinations of Dirac measures supported in $d_A \cap \Gamma_{R/2}$. For such a sequence $\mathbb{K}[\mu_n] \rightarrow \mathbb{K}[\mu]$ pointwise and $\{\mathbb{K}[\mu_n]\}$ is uniformly bounded in $D_A \setminus \Gamma_{3R/4}$. Therefore it is sufficient to prove the result when $\mu = \delta_0$. In this case the admissibility condition (1.5) is

$$\int_0^R \int_{|x''| < R} j(x)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty,$$

i.e.,

$$\int_0^R \int_0^R |x|^{q(2-N-2\kappa_+)} (r')^{(q+1)\kappa_+ + k - 1} (r'')^{N-k-1} dr'' dr' < \infty.$$

Substituting $\tau := r''/r'$ the condition becomes

$$\int_0^R \int_0^{R/r'} (1 + \tau^2)^{\frac{q}{2}(2-N-2\kappa_+)} (r')^{q(2-N-\kappa_+) + \kappa_+ + N - 1} \tau^{N-k-1} d\tau dr' < \infty.$$

This holds if and only if $q < (\kappa_+ + N)/(\kappa_+ + N - 2)$. \square

Remark. It is interesting to notice that k does not appear explicitly in (2.36). Furthermore, we observe that

$$(2.37) \quad \frac{2}{q_c - 1} \left(\frac{2q_c}{q_c - 1} - N \right) = \lambda_A \iff \kappa_+ (\kappa_+ + N - 2) = \lambda_A,$$

which follows from (2.11). This implies that there does not exist a nontrivial solution of the nonlinear eigenvalue problem

$$(2.38) \quad \begin{aligned} -\Delta_{S^{N-1}} \psi - \frac{2}{q-1} \left(\frac{2q}{q-1} - N \right) \psi + |\psi|^{q-1} \psi &= 0 \quad \text{in } S_{D_A} \\ \psi &= 0 \quad \text{in } \partial S_{D_A} \end{aligned}$$

which, in turn, implies that there does not exist a nontrivial solution of (2.31) of the form $u(x) = u(r, \sigma) = |x|^{-2/(q-1)} \psi(\sigma)$, and also no solution of this equation in D_A which vanishes on $\partial D_A \setminus \{0\}$. This is the classical ansatz for the removability of isolated singularities in d_A .

3. THE HARMONIC LIFTING OF A BESOV SPACE $B^{-s,p}(d_A)$.

Denote by $W^{\sigma,p}(\mathbb{R}^\ell)$ ($\sigma > 0$, $1 \leq p \leq \infty$) the Sobolev spaces over \mathbb{R}^ℓ . In order to use interpolation, it is useful to introduce the Besov space $B^{\sigma,p}(\mathbb{R}^\ell)$ ($\sigma > 0$). If σ is not an integer then

$$(3.1) \quad B^{\sigma,p}(\mathbb{R}^\ell) = W^{\sigma,p}(\mathbb{R}^\ell).$$

If σ is an integer the space is defined as follows. Put

$$\Delta_{x,y} f = f(x+y) + f(x-y) - 2f(x).$$

Then

$$(3.2) \quad B^{1,p}(\mathbb{R}^\ell) = \left\{ f \in L^p(\mathbb{R}^\ell) : \frac{\Delta_{x,y}f}{|y|^{1+\ell/p}} \in L^p(\mathbb{R}^\ell \times \mathbb{R}^\ell) \right\},$$

with norm

$$(3.3) \quad \|f\|_{B^{1,p}} = \|f\|_{L^p} + \left(\iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|\Delta_{x,y}f|^p}{|y|^{\ell+p}} dx dy \right)^{1/p},$$

(with standard modification if $p = \infty$) and

$$(3.4) \quad B^{m,p}(\mathbb{R}^\ell) = \left\{ f \in W^{m-1,p}(\mathbb{R}^\ell) : \right. \\ \left. D_x^\alpha f \in B^{1,p}(\mathbb{R}^\ell) \forall \alpha \in \mathbb{N}^\ell, |\alpha| = m-1 \right\}$$

with norm

$$(3.5) \quad \|f\|_{B^{m,p}} = \|f\|_{W^{m-1,p}} + \left(\sum_{|\alpha|=m-1} \iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|D_x^\alpha \Delta_{x,y}f|^p}{|y|^{\ell+p}} dx dy \right)^{1/p}.$$

We recall that the following inclusions hold ([26, p 155])

$$(3.6) \quad \begin{aligned} W^{m,p}(\mathbb{R}^\ell) &\subset B^{m,p}(\mathbb{R}^\ell) && \text{if } p \geq 2 \\ B^{m,p}(\mathbb{R}^\ell) &\subset W^{m,p}(\mathbb{R}^\ell) && \text{if } 1 \leq p \leq 2. \end{aligned}$$

When $1 < p < \infty$, the dual spaces of $W^{s,p}$ and $B^{m,p}$ are respectively denoted by $W^{-s,p'}$ and $B^{-m,p'}$.

The following is the main result of this section.

Theorem 3.1. *Suppose that $q_c < q < q_c^*$ and let A be defined as in subsection 2.1. Then there exist positive constants c_1, c_2 , depending on q, N, k, κ_+ , such that for any $R > 1$ and any $\mu \in \mathfrak{M}_+(d_A)$ with support in $B_{R/2}$:*

$$(3.7) \quad \begin{aligned} &c_1 \|\mu\|_{B^{-s,q}(\mathbb{R}^{N-k})}^q \\ &\leq \int_{D_{A,R}} \mathbb{K}[|\mu|^q](x) \rho(x) dx \leq c_2 (1+R)^\beta \|\mu\|_{B^{-s,q}(\mathbb{R}^{N-k})}^q, \end{aligned}$$

where $s = 2 - \frac{\kappa_+ + k}{q}$, $\beta = (q+1)\kappa_+ + k - 1$ and $D_{A,R} = D_A \cap \Gamma_R$. If $q = q_c$ the estimate remains valid for measures μ such that the diameter of $\text{supp } \mu$ is sufficiently small (depending on the parameters mentioned before).

Remark. When $q \geq 2$ the norms in the Besov space may be replaced by the norms in the corresponding Sobolev spaces.

Recall the admissibility condition for a measure $\mu \in \mathfrak{M}_+(d_A)$:

$$\int_{D_{A,R}} \mathbb{K}[|\mu|^q](x) \rho(x) dx < \infty \quad \forall R > 0$$

and the equivalence (see (2.27)–(2.30))

$$(3.8) \quad \int_{D_{A,R}} \mathbb{K}[\mu]^q(x) \rho(x) dx \approx J^{A,R}(\mu) := \int_0^R \int_{B''_R} \left(\int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(\tau^2 + |x'' - z|^2)^{(N-2+2\kappa_+)/2}} \right)^q \tau^{(q+1)\kappa_+ + k - 1} dx'' d\tau,$$

where $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $\tau = |x'|$ and $B''_R = \{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$. We denote,

$$(3.9) \quad \nu = N - 2 + 2\kappa_+.$$

If $2\kappa_+$ is an integer, it is natural to relate (3.8) to the Poisson potential of μ in $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}_{n-1}$ where $n = N - 2 + 2\kappa_+$. We clarify this statement below.

Assuming that $2 \leq n + k - N$, denote

$$y = (y_1, \tilde{y}, y'') \in \mathbb{R}^n, \quad \tilde{y} = (y_2, \dots, y_{n+k-N}), \quad y'' = (y_{n+k-N+1}, \dots, y_n).$$

The Poisson kernel in $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}_{n-1}$ is given by

$$(3.10) \quad P_n(y) = \gamma_n y_1 |y|^{-n} \quad y_1 > 0,$$

for some $\gamma_n > 0$, and the Poisson potential of a bounded Borel measure μ with support in

$$\mathbf{d} := \{y = (0, y'') \in \mathbb{R}^n : y'' \in \mathbb{R}_{N-k}\}$$

is

$$(3.11) \quad \mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2)^{n/2}} \quad \forall y \in \mathbb{R}_+^n.$$

In particular, for $\tilde{y} = 0$,

$$(3.12) \quad \mathbb{K}_n[\mu](y_1, 0, y'') = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(y_1^2 + |y'' - z|^2)^{n/2}}.$$

The integral in (3.12) is precisely the same as the inner integral in (3.8).

In fact, it will be shown that, if we set

$$(3.13) \quad n := \{\nu\} = \inf\{m \in \mathbb{N} : m \geq \nu\},$$

this approach also works when $2\kappa_+$ is not an integer. We note that, for n given by (3.13),

$$(3.14) \quad n - N + k \geq 2,$$

with equality only if $k = 3$ and $\kappa_+ \leq 1/2$ or $k = 2$ and $\kappa_+ \in (1/2, 1]$. Indeed,

$$n - N + k = k + \{2\kappa_+\} - 2$$

and (as $\kappa_+ > 0$) $\{2\kappa_+\} \geq 1$. If $k = 2$ then $\kappa_+ > 1/2$ and consequently $\{2\kappa_+\} \geq 2$. These facts imply our assertion.

We also note that κ_+ is strictly increasing relative to λ_A and

$$(3.15) \quad \kappa_+ \begin{cases} = 1, & \text{if } D_A = \mathbb{R}_+^N, \\ < 1, & \text{if } D_A \subsetneq \mathbb{R}_+^N, \\ > 1, & \text{if } D_A \supsetneq \mathbb{R}_+^N. \end{cases}$$

Finally we observe that $\gamma := \lambda_A - (N - k)\kappa_+ > 0$ (see (2.15)) and, by (2.11) and (2.32):

$$(3.16) \quad \gamma = \kappa_+^2 + (k - 2)\kappa_+, \quad q_c^* = 1 + \frac{-(k - 2) + \sqrt{(k - 2)^2 + 4\gamma}}{\gamma}.$$

Therefore q_c^* is strictly decreasing relative to γ and consequently also relative to κ_+ .

The proof of the theorem is based on the following important result proved in [27, 1.14.4.]

Proposition 3.2. *Let $1 < q < \infty$ and $s > 0$. Then for any bounded Borel measure μ in \mathbb{R}^{n-1} there holds*

$$(3.17) \quad I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|_{B^{-s,q}(\mathbb{R}^{n-1})}^q.$$

In the first part of the proof we derive inequalities comparing $I(\mu)$ and $J^{A,R}(\mu)$. Actually, it is useful to consider a slightly more general expression than $I(\mu)$, namely:

$$(3.18) \quad I_{\nu,\sigma}^{m,j}(\mu) := \int_{\mathbb{R}_+^{m+j}} \left| \int_{\mathbb{R}^m} \frac{y_1 d\mu(z)}{(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2)^{\nu/2}} \right|^q e^{-y_1} y_1^{\sigma q-1} dy,$$

where ν is an arbitrary number such that $\nu > m$, $j \geq 1$ and $\sigma > 0$. A point $y \in \mathbb{R}_+^{m+j}$ is written in the form $y = (y_1, \tilde{y}, y'') \in \mathbb{R}_+ \times \mathbb{R}^{j-1} \times \mathbb{R}^m$. We assume that μ is supported in \mathbb{R}^m . Note that,

$$(3.19) \quad I(\mu) = \gamma_n^q I_{n,s}^{m,j} \quad \text{where } m = N - k, \quad j = n - m = n - N + k.$$

Put

$$(3.20) \quad F_{\nu,m}[\mu](\tau) := \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty).$$

With this notation, if $j \geq 2$ then

$$(3.21) \quad I_{\nu,\sigma}^{m,j}(\mu) := \int_0^\infty \int_{\mathbb{R}^{j-1}} F_{\nu,m}[\mu](\sqrt{y_1^2 + |\tilde{y}|^2}) e^{-y_1} y_1^{(\sigma+1)q-1} d\tilde{y} dy_1$$

and if $j = 1$

$$(3.22) \quad I_{\nu,\sigma}^{m,1}(\mu) := \int_0^\infty F_{\nu,m}[\mu](y_1) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1$$

Lemma 3.3. *Assume that $m < \nu$, $0 < \sigma$, $2 \leq j$ and $1 < q < \infty$. Then there exists a positive constant c , depending on m, j, ν, σ, q , such that, for every bounded Borel measure μ with support in \mathbb{R}^m :*

$$(3.23) \quad \frac{1}{c} \int_0^\infty F_{\nu, m}[\mu](\tau) h_{\sigma, j}(\tau) d\tau \leq I_{\nu, \sigma}^{m, j}(\mu) \leq c \int_0^\infty F_{\nu, m}[\mu](\tau) h_{\sigma, j}(\tau) d\tau,$$

where $F_{\nu, m}$ is given by (3.20) and, for every $\tau > 0$,

$$(3.24) \quad h_{\sigma, j}(\tau) = \begin{cases} \frac{\tau^{(\sigma+1)q+j-2}}{(1+\tau)^{(\sigma+1)q}}, & \text{if } j \geq 2, \\ e^{-\tau} \tau^{(\sigma+1)q-1}, & \text{if } j = 1. \end{cases}$$

Proof. There is nothing to prove in the case $j = 1$. Therefore we assume that $j \geq 2$.

We use the notation $y = (y_1, \tilde{y}, y'') \in \mathbb{R} \times \mathbb{R}^{j-1} \times \mathbb{R}^m$. The integrand in (3.21) depends only on y_1 and $\rho := |\tilde{y}|$. Therefore, $I_{\nu, \sigma}^{m, j}$ can be written in the form

$$I_{\nu, \sigma}^{m, j}(\mu) = c_{m, j} \int_0^\infty \int_0^\infty F_{\nu, m}[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1 \rho^{j-2} d\rho.$$

We substitute $y_1 = (\tau^2 - \rho^2)^{1/2}$, then change the order of integration and finally substitute $\rho = r\tau$. This yields,

$$\begin{aligned} & c_{m, j}^{-1} I_{\nu, \sigma}^{m, j}(\mu) \\ &= \int_0^\infty \int_\rho^\infty F_{\nu, m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2 - \rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau d\tau d\rho \\ &= \int_0^\infty \int_0^\tau F_{\nu, m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2 - \rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau d\rho d\tau \\ &= \int_0^\infty \int_0^1 F_{\nu, m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr d\tau, \end{aligned}$$

where

$$f(r) = r^{j-2} (1 - r^2)^{(\sigma+1)q/2-1}.$$

We denote

$$I_\sigma^j(\tau) = \int_0^1 e^{-\tau\sqrt{1-r^2}} f(r) dr,$$

so that

$$(3.25) \quad I_{\nu, \sigma}^{m, j}(\mu) = c_{m, j} \int_0^\infty F_{\nu, m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} I_\sigma^j(\tau) d\tau.$$

To complete the proof we estimate I_σ^j . Since $j \geq 2$, $f \in L^1(0, 1)$ and I_σ^j is continuous in $[0, \infty)$ and positive everywhere. Hence, for every $\alpha > 0$, there exists a positive constant $c_\alpha = c_\alpha(\sigma)$ such that

$$(3.26) \quad \frac{1}{c_\alpha} \leq I_\sigma^j \leq c_\alpha \quad \text{in } [0, \alpha].$$

Next we estimate I_σ^j for large τ . Since $j \geq 2$,

$$I_\sigma^j \leq 2^{(\sigma+1)q/2-1} \int_0^1 (1-r)^{(\sigma+1)q/2-1} e^{-\tau\sqrt{1-r}} dr.$$

Substituting $r = 1 - t^2$ we obtain,

$$(3.27) \quad I_\sigma^j \leq 2^{(\sigma+1)q/2} \int_0^1 t^{(\sigma+1)q-1} e^{-t\tau} dt = c(\sigma, q) \tau^{-(\sigma+1)q}.$$

On the other hand, if $\tau \geq 2$,

$$(3.28) \quad \begin{aligned} I_\sigma^j(\tau) &= \int_0^1 (1-t^2)^{(j-3)/2} t^{(\sigma+1)q-1} e^{-\tau t} dt \\ &= \tau^{-(\sigma+1)q} \int_0^\tau (1-(s/\tau)^2)^{(j-3)/2} s^{(\sigma+1)q-1} e^{-s} ds \\ &\geq \tau^{-(\sigma+1)q} 2^{-(j-3)} \int_0^1 s^{(\sigma+1)q-1} e^{-s} ds. \end{aligned}$$

Combining (3.25) with (3.26)–(3.28) we obtain (3.23). \square

Next we derive an estimate in which integration over $\mathbb{R}_+^n = \mathbb{R}_+^j \times \mathbb{R}^m$ is replaced by integration over a bounded domain, for measures supported in a fixed bounded subset of \mathbb{R}^m .

Let $B_R^j(0)$ and $B_R^m(0)$ denote the balls of radius R centered at the origin, in \mathbb{R}^j and \mathbb{R}^m respectively. Denote

$$(3.29) \quad F_{\nu, m}^R[\mu](\tau) = \int_{B_R^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty)$$

and, if $j \geq 2$,

$$(3.30) \quad I_{\nu, \sigma}^{m, j}(\mu; R) = \int_{B_R^j \cap \{0 < y_1\}} F_{\nu, m}^R[\mu](\sqrt{y_1^2 + |\tilde{y}|^2}) e^{-y_1} y_1^{\sigma q - 1} d\tilde{y} dy_1.$$

where $(y_1, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^{j-1}$. If $j = 1$ we denote,

$$(3.31) \quad I_{\nu, \sigma}^{m, 1}(\mu; R) = \int_0^R F_{\nu, m}^R[\mu](y_1) e^{-y_1} y_1^{\sigma q - 1} dy_1.$$

Similarly to Lemma 3.3 we obtain,

Lemma 3.4. *If $j \geq 1$, there exists a positive constant c such that, for any bounded Borel measure μ with support in $\mathbb{R}^m \cap B_R$*

$$(3.32) \quad c^{-1} \int_0^R F_{\nu, m}^R[\mu](\tau) h_{\sigma, j}(\tau) d\tau \leq I_{\nu, \sigma}^{m, j}(\mu; R) \leq c \int_0^R F_{\nu, m}^R[\mu](\tau) h_{\sigma, j}(\tau) d\tau$$

with $h_{\sigma, j}$ as in (3.24).

Proof. In the case $j = 1$ there is nothing to prove. Therefore we assume that $j \geq 2$.

From (3.30) we obtain,

$$I_{\nu,\sigma}^{m,j}(\mu; R) = c_{m,j} \int_0^R \int_0^{\sqrt{R^2-\rho^2}} F_{\nu,m}^R[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1 \rho^{j-2} d\rho.$$

Substituting $y_1 = (\tau^2 - \rho^2)^{1/2}$, then changing the order of integration and finally substituting $\rho = r\tau$ we obtain,

$$c_{m,j}^{-1} I_{\nu,\sigma}^{m,j}(\mu; R) = \int_0^R \int_0^1 F_{\nu,\mu}^R[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr d\tau.$$

where

$$f(r) = r^{j-2}(1-r^2)^{(\sigma+1)q/2-1}.$$

The remaining part of the proof is the same as for Lemma 3.3. \square

Lemma 3.5. *Let $1 < q$, $0 < \sigma$ and assume that $m < \nu q$ and $0 \leq j-1 < \nu$. Then there exists a positive constant \bar{c} , depending on j, m, q, σ, ν , such that, for every $R \geq 1$ and every bounded Borel measure μ with support in $B_{R/2}(0) \cap \mathbb{R}^m$,*

$$(3.33) \quad \left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \bar{c} R^{(\sigma+1-\nu)q+m+j-1} \|\mu\|_{\mathfrak{M}}^q$$

with $h_{\sigma,j}$ as in (3.24).

Proof. We estimate,

$$(3.34) \quad \left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \int_R^\infty |F_{\nu,m}[\mu](\tau)| h_{\sigma,j}(\tau) d\tau + \int_0^R |F_{\nu,m}[\mu] - F_{\nu,m}^R[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau.$$

For every $\tau > 0$,

$$(3.35) \quad |F_{\nu,m}[\mu](\tau)| \leq \tau^{-\nu q} \|\mu\|_{\mathfrak{M}}^q.$$

Since $j-1 < \nu q$, it follows that

$$(3.36) \quad \begin{aligned} \int_R^\infty |F_{\nu,m}[\mu](\tau)| h_{\sigma,j}(\tau) d\tau &\leq \|\mu\|_{\mathfrak{M}}^q \int_R^\infty \tau^{-\nu q} h_{\sigma,j}(\tau) d\tau \\ &\leq c(\sigma, q) \|\mu\|_{\mathfrak{M}}^q \int_R^\infty \frac{\tau^{(\sigma+1)q+j-2-\nu q}}{(1+\tau)^{(\sigma+1)q}} d\tau \\ &\leq \frac{c(\sigma, q)}{\nu q - j + 1} \|\mu\|_{\mathfrak{M}}^q R^{j-1-\nu q}. \end{aligned}$$

Since, by assumption, $\text{supp } \mu \subset B_{R/2}$, we have

$$\begin{aligned}
& \int_0^R |F_{\nu,m}[\mu] - F_{\nu,m}^R[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau \\
& \leq \int_0^R \int_{|y''|>R} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' h_{\sigma,j}(\tau) d\tau \\
& \leq \|\mu\|_{\mathfrak{M}}^q \int_0^R \int_{|\zeta|>R/2} (|\tau^2 + |\zeta|^2)^{-\nu q/2} d\zeta h_{\sigma,j} d\tau \\
(3.37) \quad & \leq c(m, q) \|\mu\|_{\mathfrak{M}}^q \int_0^R \int_{R/2}^\infty (\tau^2 + \rho^2)^{-\nu q/2} \rho^{m-1} d\rho h_{\sigma,j} d\tau \\
& \leq c(m, q) \|\mu\|_{\mathfrak{M}}^q \int_0^R \tau^{m-\nu q} \int_{R/2\tau}^\infty (1 + \eta^2)^{-\nu q/2} \eta^{m-1} d\eta h_{\sigma,j} d\tau \\
& \leq \frac{c(m, q)}{\nu q - m} \|\mu\|_{\mathfrak{M}}^q R^{m-\nu q} \int_0^R \tau^{(\sigma+1)q+j-2} d\tau \\
& \leq \frac{c(m, q)}{(\nu q - m)((\sigma+1)q + j - 1)} \|\mu\|_{\mathfrak{M}}^q R^{(\sigma+1)q+j-1+m-\nu q}.
\end{aligned}$$

Combining (3.34)–(3.37) we obtain (3.33). \square

Corollary 3.6. *For every $R > 0$ put*

$$(3.38) \quad J_{\nu,\sigma}^{m,j}(\mu; R) := \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(\sigma+1)q+j-2} d\tau.$$

Then

$$\begin{aligned}
(3.39) \quad & \frac{1}{c} I_{\nu,\sigma}^{m,j}(\mu) - \bar{c} R^\beta \|\mu\|_{\mathfrak{M}}^q \leq J_{\nu,\sigma}^{m,j}(\mu; R) \leq c R^{(\sigma+1)q} I_{\nu,\sigma}^{m,j}(\mu), \\
& \beta = (\sigma + 1 - \nu)q + j + m - 1,
\end{aligned}$$

for every $R > 1$ and every bounded Borel measure μ with support in $B_{R/2}^m(0) := B_{R/2}(0) \cap \mathbb{R}^m$.

Proof. This is an immediate consequence of Lemma 3.5 and Lemma 3.3. \square

Lemma 3.7. *Let m, j be positive integers such that $j \geq 1$ and let $1 < q$, $0 < \sigma$. Put $n := m + j$.*

Then there exist positive constants c, \bar{c} , depending on j, m, q, σ , such that, for every $R > 1$ and every measure $\mu \in \mathfrak{M}_+(B_{R/2}^m(0))$,

$$\begin{aligned}
(3.40) \quad & \frac{1}{c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q - \bar{c} R^{q\left(\sigma - \frac{n-1}{q}\right)} \|\mu\|_{\mathfrak{M}}^q \leq J_{n,\sigma}^{m,j}(\mu; R) \\
& \leq c R^{(\sigma+1)q} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q.
\end{aligned}$$

If $\sigma < \frac{n-1}{q}$, there exists $R_0 > 1$ such that, for all $R > R_0$

$$(3.41) \quad \frac{1}{2c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q \leq J_{n,\sigma}^{m,j}(\mu; R).$$

If $\sigma = \frac{n-1}{q'}$ then, there exists $a > 0$ such that the inequality remains valid for measures μ such that $\text{diam}(\text{supp } \mu) \leq a$.

If, in addition, $\frac{j-1}{q'} < \sigma$ then

$$(3.42) \quad \frac{1}{2c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq J_{n,\sigma}^{m,j}(\mu; R) \leq cR^{(\sigma+1)q} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q,$$

where $s := \sigma - \frac{j-1}{q'}$.

Remark. Assume that $\mu \geq 0$. Then:

(i) If $\mu \in B^{-\sigma,q}(\mathbb{R}^{n-1})$ and $\frac{j-1}{q'} \geq \sigma$ then $\mu(\mathbb{R}^m) = 0$.

(ii) If $\mu \in B^{-s,q}(\mathbb{R}^m)$ and $\sigma > (n-1)/q'$ then $s > m/q'$ and therefore $B^{s,q'}(\mathbb{R}^m)$ can be embedded in $C(\mathbb{R}^m)$.

Proof. Inequality (3.40) follows from (3.39) and Proposition 3.2 (see also (3.19)).

For positive measures μ ,

$$\|\mu\|_{\mathfrak{M}} = \mu(\mathbb{R}^{n-1}) \leq \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q.$$

Therefore, if $\sigma < \frac{n-1}{q'}$, (3.40) implies that there exists $R_0 > 1$ such that (3.41) holds for all $R > R_0$.

If $\sigma = \frac{n-1}{q'}$ (3.40) implies that

$$\frac{1}{c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q - \bar{c} \|\mu\|_{\mathfrak{M}}^q \leq J_{n,\sigma}^{m,j}(\mu; R).$$

But if μ is a positive bounded measure such that $\text{diam}(\text{supp } \mu) \leq a$ then

$$\|\mu\|_{\mathfrak{M}} / \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q \rightarrow 0 \quad \text{as } a \rightarrow 0.$$

The last inequality follows from the imbedding theorem for Besov spaces according to which there exists a continuous trace operator $T : B^{\sigma,q'}(\mathbb{R}^{n-1}) \mapsto B^{s,q'}(\mathbb{R}^m)$ and a continuous lifting $T' : B^{s,q'}(\mathbb{R}^m) \mapsto B^{\sigma,q'}(\mathbb{R}^{n-1})$ where $s = \sigma - \frac{n-m-1}{q'}$. \square

If $\nu \in \mathbb{N}$ and $\sigma = s + \frac{\nu-m-1}{q'}$,

$$\begin{aligned} J_{\nu,\sigma}^{m,\nu-m}(\mu; R) &= \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(\sigma+1)q+\nu-m-2} d\tau \\ &= \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(s+\nu-m)q-1} d\tau. \end{aligned}$$

However, if μ is positive, the expression

$$(3.43) \quad M_{\nu,s}^m(\mu; R) := \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(s+\nu-m)q-1} d\tau,$$

is meaningful for any real $\nu > m$ and $s > 0$. Furthermore, as shown below, the results stated in Lemma 3.7 can be extended to this general case.

Theorem 3.8. *Let $1 < q$, $\nu \in \mathbb{R}$ and m a positive integer. Assume that $1 \leq \nu - m$ and $0 < s < m/q'$. Then there exists a positive constant c such that, for every bounded positive measure μ supported in $\mathbb{R}^m \cap B_{R/2}(0)$, $R > 1$,*

$$(3.44) \quad \frac{1}{c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq M_{\nu,s}^m(\mu; R) \leq cR^{(s+\nu-m)q+1} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q.$$

This also holds when $s = m/q'$, provided that the diameter of $\text{supp } \mu$ is sufficiently small.

Proof. If ν is an integer and $j := \nu - m$ then this statement is part of Lemma 3.7. Indeed the condition $s > 0$ means that $\sigma = s + \frac{j-1}{q'} > \frac{j-1}{q'}$ and the condition $s < m/q'$ means that $\sigma < \frac{n-1}{q'}$.

Therefore we assume that $\nu \notin \mathbb{N}$. Let $n := \{\nu\}$ and $\theta := n - \nu$ so that $0 < \theta < 1$. Our assumptions imply that $1 \leq n - m - 1$ because (as ν is not an integer) $\nu - m > 1$ and consequently $n - m \geq 2$.

If a, b are positive numbers, put

$$A_\nu := \frac{a^{(s+\nu-m)q-1}}{(a^2 + b^2)^{\nu q/2}}.$$

Obviously A_ν decreases as ν increases. Therefore, $A_n \leq A_\nu \leq A_{n-1}$ which in turn implies,

$$M_{n,s}^m \leq M_{\nu,s}^m \leq M_{n-1,s}^m.$$

By Lemma 3.7, the assertions of the theorem are valid in the case that $\nu = n$ or $\nu = n - 1$. Therefore the previous inequality implies that the assertions hold for any real ν subject to the conditions imposed. \square

By (3.8),

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau) \tau^{(q+1)\kappa_+ + k - 1} d\tau,$$

where $m = N - k$ and $\nu = N - 2 + 2\kappa_+$. Consequently, by (3.38),

$$J^{A,R} = M_{\nu,s}^m$$

where s is determined by,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

It follows that

$$sq = -(k - 2 + 2\kappa_+)q + (q + 1)\kappa_+ + k = k(1 - q) + 2q - \kappa_+(q - 1)$$

and therefore

$$s = 2 - \frac{k + \kappa_+}{q'}.$$

Proof of Theorem 3.1.

Put

$$(3.45) \quad \nu := N - 2 + 2\kappa_+, \quad s := 2 - \frac{\kappa_+ + k}{q'}, \quad m := N - k.$$

Recall that in the case $k = 2$ we have $\kappa_+ > 1/2$. Therefore

$$(3.46) \quad \nu - m - 1 = k - 3 + 2\kappa_+ > 0.$$

Furthermore,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

Thus

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau) \tau^{(q+1)\kappa_+ + k - 1} d\tau = M_{\nu,s}^m.$$

Next we show that $0 < s \leq m/q'$. More precisely we prove

$$(3.47) \quad 0 < s \leq m/q' \iff q_c \leq q < q_c^*.$$

Let μ be a bounded non-negative Borel measure in $B^{-s,q}(\mathbb{R}^m)$. If $s \leq 0$, $B^{-s,q}(\mathbb{R}^m) \subset L^q(\mathbb{R}^m)$. Therefore, in this case, every bounded Borel measure on \mathbb{R}^m is admissible i.e. satisfies (2.35). Consequently, by Proposition 2.2, $q < q_c$. As we assume $q \geq q_c$ it follows that $s > 0$.

If, $s > 0$ and $sq' - m \geq 0$ then $C_{s,q'}(K) = 0$ for every compact subset of \mathbb{R}^m and consequently $\mu(K) = 0$ for any such set. Conversely, if $sq' - m < 0$ then there exist non-trivial positive bounded measures in $B^{-s,q}(\mathbb{R}^m)$. Therefore, by Proposition 2.1, $sq' < m$ if and only if $q < q_c^*$.

In conclusion, $0 < s \leq m/q'$ and $\nu - m \geq 1$; therefore Theorem 3.1 is a consequence of Theorem 3.8. \square

Remark. Note that the critical exponent for the imbedding of $B^{2 - \frac{\kappa_+ + k}{q'}, q'}(\mathbb{R}^{N-k})$ into $C(\mathbb{R}^{N-k})$ is again

$$q = q_c = \frac{N + \kappa_+}{N + \kappa_+ - 2}.$$

4. SUPERCRITICAL EQUATIONS IN A POLYHEDRAL DOMAIN

In this section q is a real number larger than 1 and P an N -dim polyhedral domain as described in subsection 6.1. Denote by $\{L_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$ the family of faces, edges and vertices of P . In this notation, $L_{1,j}$ denotes one of the open faces of P ; for $k = 2, \dots, N - 1$, $L_{k,j}$ denotes a relatively open $N - k$ -dimensional edge and $L_{N,j}$ denotes a vertex. For $1 \leq k < N$, the $(N - k)$ dimensional space which contains $L_{k,j}$ is denoted by \mathbb{R}_j^{N-k} . If $1 < k < N$, the cylinder of radius r around the axis \mathbb{R}_j^{N-k} will be denoted by $\Gamma_{k,j,r}^\infty$ and the subset $A_{k,j}$ of S^{k-1} is defined by

$$\lim_{r \rightarrow 0} \frac{1}{r} (\partial \Gamma_{k,j,r}^\infty \cap P) = L_{k,j} \times A_{k,j}.$$

$A_{k,j}$ is the 'opening' of P at the edge $L_{k,j}$. For $k = N$ we replace in this definition the cylinder $\Gamma_{N,j,r}^\infty$ by the ball $B_r(L_{N,j})$. For $1 < k \leq N$ and $A = A_{k,j}$ we use d_A as an alternative notation for \mathbb{R}_j^{N-k} and denote by D_A the k -dihedron with edge d_A and opening A as in subsection 6.1 (with S_A defined as in (2.2)). For $k = 1$, D_A stands for the half space $\mathbb{R}_j^{N-1} \times (0, \infty)$.

4.1. Definitions and auxiliary results. Let Ω be a bounded Lipschitz domain. We say that $\{\Omega_n\}$ is a *Lipschitz exhaustion* of Ω if, for every n , Ω_n is Lipschitz and

$$(4.1) \quad \Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}, \quad \Omega = \cup \Omega_n, \quad \mathbb{H}_{N-1}(\partial\Omega_n) \rightarrow \mathbb{H}_{N-1}(\partial\Omega).$$

If ω_n (respectively ω) is the harmonic measure in Ω_n (respectively Ω) relative to $x_0 \in \Omega_1$, then, for every $Z \in C(\bar{\Omega})$,

$$(4.2) \quad \lim_{n \rightarrow \infty} \int_{\partial\Omega_n} Z d\omega_n = \int_{\partial\Omega} Z d\omega.$$

[23, Lemma 2.1]. Furthermore, if μ is a bounded Borel measure on $\partial\Omega$ and $v := \mathbb{K}^\Omega[\mu]$, there holds

$$(4.3) \quad \lim_{n \rightarrow \infty} \int_{\partial\Omega_n} Z v d\omega_n = \int_{\partial\Omega} Z d\mu,$$

[23, Lemma 2.2]. If v is a positive solution and (4.3) holds we say that μ is the *boundary trace* of v .

The following estimates are proved in [23, Lemma 2.3]

Proposition 4.1. *Let μ be bounded Borel measures on $\partial\Omega$. Then $\mathbb{K}[\mu] \in L^1_\rho(\Omega)$ and there exists a constant $C = C(\Omega)$ such that*

$$(4.4) \quad \|\mathbb{K}[\mu]\|_{L^1_\rho(\Omega)} \leq C \|\mu\|_{\mathfrak{M}(\partial\Omega)}.$$

In particular if $h \in L^1(\partial\Omega; \omega)$ then

$$(4.5) \quad \|\mathbb{P}[h]\|_{L^1_\rho(\Omega)} \leq C \|h\|_{L^1(\partial\Omega; \omega)}.$$

The next result will be used in deriving estimates in a k -dimensional dihedron when the boundary data is concentrated on the edge.

Proposition 4.2. *We denote by G^{Ω_n} (respectively G^Ω) the Green function in Ω_n (respectively Ω). Let v be a positive harmonic function in Ω with boundary trace μ . Let $Z \in C^2(\bar{\Omega})$ and let $\tilde{G} \in C^\infty(\Omega)$ be a function that coincides with $x \mapsto G(x, x_0)$ in $Q \cap \Omega$ for some neighborhood Q of $\partial\Omega$ and some fixed $x_0 \in \Omega$. In addition assume that there exists a constant $c > 0$ such that*

$$(4.6) \quad |\nabla Z \cdot \nabla \tilde{G}| \leq c\rho.$$

Under these assumptions, if $\zeta := Z\tilde{G}$ then

$$(4.7) \quad - \int_{\Omega} v \Delta \zeta dx = \int_{\partial\Omega} Z d\mu.$$

Proof. Let $\{\Omega_n\}$ be a C^1 exhaustion of Ω . We assume that $\partial\Omega_n \subset Q$ for all n and $x_0 \in \Omega_1$. Let $\tilde{G}_n(x)$ be a function in $C^1(\Omega_n)$ such that \tilde{G}_n coincides with $G^{\Omega_n}(\cdot, x_0)$ in $Q \cap \Omega_n$, $\tilde{G}_n(\cdot, x_0) \rightarrow \tilde{G}(\cdot, x_0)$ in $C^2(\Omega \setminus Q)$ and $\tilde{G}_n(\cdot, x_0) \rightarrow \tilde{G}(\cdot, x_0)$ in $\text{Lip}(\Omega)$. If $\zeta_n = Z\tilde{G}_n$ we have,

$$\begin{aligned} - \int_{\Omega_n} v \Delta \zeta_n dx &= \int_{\partial \Omega_n} v \partial_{\mathbf{n}} \zeta dS = \int_{\partial \Omega_n} v Z \partial_{\mathbf{n}} \tilde{G}_n(\xi, x_0) dS \\ &= \int_{\partial \Omega_n} v Z P^{\Omega_n}(x_0, \xi) dS = \int_{\partial \Omega_n} v Z d\omega_n. \end{aligned}$$

By (4.3),

$$\int_{\partial \Omega_n} v Z d\omega_n \rightarrow \int_{\partial \Omega} Z d\mu.$$

On the other hand, in view of (4.6), we have

$$\Delta \zeta_n = \tilde{G}_n \Delta Z + Z \Delta \tilde{G}_n + 2 \nabla Z \cdot \nabla \tilde{G}_n \rightarrow \Delta Z$$

in $L^1_\rho(\Omega)$; therefore,

$$- \int_{\Omega_n} v \Delta \zeta_n dx \rightarrow - \int_{\Omega} v \Delta \zeta dx.$$

□

We denote by $\mathfrak{M}_q = \mathfrak{M}_q(\partial \Omega)$ the set of q -good measures on the boundary. A positive solution u of (1.1) in Ω possesses a boundary trace $\mu \in \mathfrak{M}(\partial \Omega)$ if and only if

$$(4.8) \quad \int_{\Omega} u^q \rho dx < \infty$$

[23, Proposition 4.1]. In this case $\mu \in \mathfrak{M}_q$.

The following statements can be proved in the same way as in the case of smooth domains. For the proof in that case see [19].

I. $\mathfrak{M}_q(\partial \Omega)$ is a linear space and

$$\mu \in \mathfrak{M}_q(\partial \Omega) \iff |\mu| \in \mathfrak{M}_q(\partial \Omega).$$

II. If $\{\mu_n\}$ is an increasing sequence of measures in $\mathfrak{M}_q(\partial \Omega)$ and $\mu := \lim \mu_n$ is a finite measure then $\mu \in \mathfrak{M}_q(\partial \Omega)$.

Proposition 4.3. *Let μ be a bounded measure on ∂P . (μ may be a signed measure.) For $i = 1, \dots, N$, $j = 1, \dots, n_i$, we define the measure $\mu_{k,j}$ on $d_{A_{k,j}}$ by,*

$$\mu_{k,j} = \mu \text{ on } L_{k,j}, \quad \mu_{k,j} = 0 \text{ on } d_{A_{k,j}} \setminus L_{k,j}.$$

Then $\mu \in \mathfrak{M}_q(\partial P)$, i.e., problem

$$(4.9) \quad -\Delta u + u^q = 0 \text{ in } P, \quad u = \mu \text{ on } \partial P$$

possesses a solution, if and only if, $\mu_{k,j}$ is a q -good measure relative to $D_{A_{k,j}}$ for all (k, j) as above.

Proof. In view of statement **I** above, it is sufficient to prove the proposition in the case that μ is non-negative. This is assumed hereafter. If $\mu \in \mathfrak{M}_q(\partial P)$ then any measure ν on ∂P such that $0 \leq \nu \leq \mu$ is a q -good measure relative to P . Therefore

$$\mu \in \mathfrak{M}_q(\partial P) \implies \mu'_{k,j} := \mu \chi_{L_{k,j}} \in \mathfrak{M}_q(\partial P).$$

Assume that $\mu \in \mathfrak{M}_q(\partial P)$ and let $u_{k,j}$ be the solution of (4.9) when μ is replaced by $\mu'_{k,j}$. Denote by $u'_{k,j}$ the extension of $u_{k,j}$ by zero to the k-dihedron $D_{A_{k,j}}$. Then $u'_{k,j}$ is a subsolution of (1.1) in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$. In the present case there always exists a supersolution, e.g. the maximal solution of (1.1) in $D_{A_{k,j}}$ vanishing outside $d_{A_{k,j}} \setminus \bar{L}_{k,j}$. Therefore there exists a solution $v_{k,j}$ of this equation in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$, i.e., $\mu_{k,j}$ is q-good relative to $D_{A_{k,j}}$.

Next assume that $\mu \in \mathfrak{M}(\partial P)$ and that $\mu_{k,j}$ is q-good relative to $D_{A_{k,j}}$ for every (k, j) as above. Let $v_{k,j}$ be the solution of (1.1) in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$. Then $v_{k,j}$ is a supersolution of problem (4.9) with μ replaced by $\mu'_{k,j}$ and consequently there exists a solution $u_{k,j}$ of this problem. It follows that

$$w := \max\{u_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$$

is a subsolution while

$$\bar{w} := \sum_{k=1, \dots, N, j=1, \dots, n_k} u_{k,j}$$

is a supersolution of (4.9). Consequently there exists a solution of this problem, i.e., $\mu \in \mathfrak{M}_q(\partial P)$. \square

4.2. Removable singular sets and 'good measures', I. We first introduce some standard elements associated to the Bessel capacities which are the natural way to characterize good measures or removable sets. For $\alpha \in \mathbb{R}$, we denote by G_α the Bessel kernel of order α , defined by

$$(4.10) \quad G_\alpha(\xi) = \mathcal{F}^{-1} \left((1 + |\cdot|^2)^{-\frac{\alpha}{2}} \right) (\xi),$$

where \mathcal{F} is the Fourier transform in the space $\mathcal{S}'(\mathbb{R}^\ell)$ of moderate distributions in \mathbb{R}^ℓ . For $1 \leq p \leq \infty$, the Bessel space $L_{\alpha,p}(\mathbb{R}^\ell)$ is defined by

$$(4.11) \quad L_{\alpha,p}(\mathbb{R}^\ell) = \{f : f = G_\alpha * g, : g \in L^p(\mathbb{R}^\ell)\},$$

with norm

$$\|f\|_{L_{\alpha,p}} = \|g\|_{L^p} = \|G_{-\alpha} * f\|_{L^p}.$$

For $\alpha, \beta \in \mathbb{R}$ and $1 < p < \infty$, the mapping $f \mapsto G_\beta * f$ is an isomorphism from $L_{\alpha,p}(\mathbb{R}^\ell)$ into $L_{\alpha+\beta,p}(\mathbb{R}^\ell)$. Finally the Bessel spaces are connected to Besov and Sobolev spaces: when $\alpha > 0$ and $1 < p < \infty$, it is known that if $\alpha \in \mathbb{N}$, $L_{\alpha,p}(\mathbb{R}^\ell) = W^{\alpha,p}(\mathbb{R}^\ell)$ and if $\alpha \notin \mathbb{N}$, then $L_{\alpha,p}(\mathbb{R}^\ell) = B^{\alpha,p}(\mathbb{R}^\ell)$, with equivalent norms (see e.g. [5], [26]).

The Bessel capacity $C_{\alpha,p}^{\mathbb{R}^\ell}$ ($\alpha > 0, p \geq 1$) is defined by the following rules: if $K \subset \mathbb{R}^\ell$ is compact

$$(4.12) \quad C_{\alpha,p}^{\mathbb{R}^\ell}(K) = \inf\{\|f\|_{L_{\alpha,p}}^p : f \in \mathcal{S}(\mathbb{R}^\ell), f \geq \chi_K\}.$$

If G is open

$$(4.13) \quad C_{\alpha,p}^{\mathbb{R}^\ell}(G) = \sup\{C_{\alpha,p}^{\mathbb{R}^\ell}(K) : K \subset G, K \text{ compact}\}.$$

If A is any set

$$(4.14) \quad C_{\alpha,p}^{\mathbb{R}^\ell}(A) = \inf\{C_{\alpha,p}^{\mathbb{R}^\ell}(G) : A \subset G, G \text{ open}\}.$$

Note that the capacity of any non-empty set is positive if and only if $\alpha > \frac{\ell}{p}$ because of Sobolev-Besov imbedding theorem.

Proposition 4.4. *Let A be a Lipschitz domain on S^{k-1} , $2 \leq k \leq N-1$, and let D_A be the k -dihedron with opening A . Let $\mu \in \mathfrak{M}(\partial D_A)$ be a positive measure with compact support contained in d_A (= the edge of D_A). Assume that μ is q -good relative to D_A . Let $R > 1$ be large enough so that $\text{supp } \mu \subset B_R^{N-k}(0)$ and let u be the solution of (1.1) in D_A^R with trace μ on d_A^R and trace zero on $\partial D_A^R \setminus d_A^R$. Then:*

(i) For every non-negative $\eta \in C_0^\infty(B_{3R/4}^{N-k}(0))$,

$$(4.15) \quad \begin{aligned} \left(\int_{d_A^R} \eta^{q'} d\mu \right) &\leq cM^{q'} \int_{D_A^R} u^q \rho dx + \\ &+ cM^{q'} \left(\int_{D_A^R} u^q \rho dx \right)^{\frac{1}{q}} \left(1 + M^{-1} \|\eta\|_{L^{q'}(d_A^R)} \right). \end{aligned}$$

where $M = \|\eta\|_{L^\infty}$ and ρ is the first eigenfunction of $-\Delta$ in D_A^R normalized by $\rho(x_0) = 1$ at some point $x_0 \in D_A^R$. The constant c depends only on $N, q, k, x_0, \lambda_1, R$ where λ_1 is the first eigenvalue.

(ii) For any compact set $E \subset d_A$,

$$(4.16) \quad C_{s,q}^{N-k}(E) = 0 \implies \mu(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'},$$

where $C_{s,q}^{N-k}$ denotes the Bessel capacity with the indicated indices in \mathbb{R}^{N-k} .

Remark. If we replace D_A^R by $D_A \cap B_R^k(0) \cap B_R^{N-k}(0)$, $\tilde{R} > 1$, then the constant c in (i) depends on \tilde{R} but not on R .

Proof. We identify d_A with \mathbb{R}^{N-k} and use the notation

$$x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \quad y = |x'|.$$

Let $\eta \in C_0^\infty(\mathbb{R}^{N-k})$ and let R be large enough so that $\text{supp } \eta \subset B_{R/2}^{N-k}(0)$.

Let $w = w_R(t, x'')$ be the solution of the following problem in $\mathbb{R}_+ \times B_R^{N-k}(0)$:

$$(4.17) \quad \begin{aligned} \partial_t w - \Delta_{x''} w &= 0 && \text{in } \mathbb{R}_+ \times B_R^{N-k}(0), \\ w(0, x'') &= \eta(x'') && \text{in } B_R^{N-k}, \\ w(t, x'') &= 0 && \text{on } \partial B_R^{N-k}(0). \end{aligned}$$

Thus $w_R(t, \cdot) = S_R(t)[\eta]$ where $S_R(t)$ is the semi-group operator corresponding to the above problem. Denote,

$$(4.18) \quad H_R[\eta](x', x'') = w_R(|x'|^2, x'') = S_R(y^2)[\eta](x''), \quad y := |x'|.$$

We assume, as we may, that $R > 1$. Let ρ^R be the first eigenfunction of $-\Delta_{x''}$ in the ball $B_R^{N-k}(0)$ normalized by $\rho^R(0) = 1$ and let ρ_A be the first eigenfunction of $-\Delta_{x'}$ in C_A (where C_A denotes the cone with opening A in \mathbb{R}^k) normalized so that $\rho_A(x'_0) = 1$ at some point $x'_0 \in S_A$. Then $\rho^R \rho_A$ is the first eigenfunction of $-\Delta$ in $\{x \in D_A : |x''| < R\}$. Note that $\rho^R \leq 1$ and $\rho^R \rightarrow 1$ as $R \rightarrow \infty$ in $C^2(I)$ for any bounded set $I \subset \mathbb{R}^{N-k}$.

Let $h \in C^\infty(\mathbb{R})$ be a monotone decreasing function such that $h(t) = 1$ for $t < 1/2$ and $h(t) = 0$ for $t > 3/4$. Put

$$\psi_R(x') = h(|x'|/R)$$

and

$$(4.19) \quad \zeta_R := \rho_A \psi_R H_R[\eta]^{q'}.$$

If ρ_A^R is the first eigenfunction (normalized at x_0) of $D_A^R := D_A \cap \Gamma_R$ (Γ_R as in (2.25)) then

$$(4.20) \quad \rho_A \psi_R \leq c \rho_A^R$$

and $\rho^R \rho_A^R$ is the first eigenfunction in D_A^R .

Hereafter we shall drop the index R in ζ_R, H_R, w_R but keep it in the other notations in order to avoid confusion.

We shall verify that $\zeta \in D_A^R$. To this purpose we compute,

$$(4.21) \quad \begin{aligned} \Delta \zeta &= -\lambda_1(\rho_A \psi_R) H[\eta]^{q'} + (\rho_A \psi_R) \Delta H[\eta]^{q'} + 2\nabla(\rho_A \psi_R) \cdot \nabla H[\eta]^{q'} \\ &= -\lambda_1 \zeta + q'(\rho_A \psi_R)(H[\eta])^{q'-1} \Delta H[\eta] \\ &\quad + q'(q' - 1)(\rho_A \psi_R)(H[\eta])^{q'-2} |\nabla H[\eta]|^2 \\ &\quad + 2q'(H[\eta])^{q'-1} \nabla(\rho_A \psi_R) \cdot \nabla H[\eta]. \end{aligned}$$

In addition,

$$\begin{aligned} \nabla H[\eta] &= \nabla_{x'} H[\eta] + \nabla_{x''} H[\eta] = \partial_y H[\eta] \frac{x'}{y} + \nabla_{x''} H[\eta] \\ &= 2y \partial_t w(y^2, x'') \frac{x'}{y} + \nabla_{x''} H[\eta](x', x'') \end{aligned}$$

and consequently (recall that y stands for $|x'|$),

$$\begin{aligned} &\nabla H[\eta] \cdot \nabla(\rho_A \psi_R) \\ &= 2\partial_t w(y^2, x'') x' \cdot \left(\psi_R(|x'|^{\kappa_+ - 1} (\kappa_+ \frac{x'}{y} \omega_k(x'/y) + |x'| \nabla \omega_k(x'/y))) + \rho_A \nabla \psi_R \right) \\ &= 2\kappa_+ \partial_t w(y^2, x'') |x'|^{\kappa_+} \omega_k(x'/y) = 2\partial_t w(y^2, x'') (\kappa_+ \rho_A \psi_R + \rho_A x' \cdot \nabla \psi_R). \end{aligned}$$

Since $w = w_R$ vanishes for $|x''| = R$ and $\eta = 0$ in a neighborhood of this sphere, $|\partial_t w(y^2, x'')| \leq c \rho^R$. As ψ_R vanishes for $|x'| > 3R/4$ we have $\rho_A \nabla \psi_R \leq c \rho_A^R$. Therefore

$$|\nabla H[\eta] \cdot \nabla \rho_A| \leq c \rho^R \rho_A^R$$

and, in view of (4.21),

$$(4.22) \quad |\Delta\zeta| \leq c\rho^R \rho_A^R.$$

Thus $\zeta \in X(D_A^R)$ and consequently

$$(4.23) \quad \int_{D_A^R} (-u\Delta\zeta + u^q\zeta) dx = - \int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta dx.$$

Since $q(q' - 1)\rho_A(H[\eta])^{q'-2}|\nabla H[\eta]|^2 \geq 0$, we have

$$(4.24) \quad \begin{aligned} & \left| \int_{D_A^R} u\Delta\zeta dx \right| \\ & \leq \int_{D_A^R} u \left(\lambda_1\zeta + q'(H[\eta])^{q'-1} (\rho|\Delta H[\eta]| + 2|\nabla\rho \cdot \nabla H[\eta]|) \right) dx \\ & \leq \int_{D_A^R} u \left(\lambda_1\zeta + q'\zeta^{1/q} \left(\rho^{1/q'} |\Delta H[\eta]| + 2\rho^{-1/q} |\nabla\rho \cdot \nabla H[\eta]| \right) \right) dx \\ & \leq \left(\int_{D_A^R} u^q \zeta dx \right)^{\frac{1}{q}} \left(\lambda_1 \left(\int_{D_A^R} \zeta dx \right)^{\frac{1}{q'}} + q' \|L[\eta]\|_{L^{q'}(D_A^R)} \right) \end{aligned}$$

where

$$(4.25) \quad L[\eta] = \rho^{1/q'} |\Delta H[\eta]| + 2\rho^{-1/q} |\nabla\rho \cdot \nabla H[\eta]|.$$

By Proposition 4.2

$$(4.26) \quad - \int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta dx = \int_{d_A^R} \eta^{q'} d\mu.$$

Therefore

$$(4.27) \quad \begin{aligned} & \left(\int_{d_A^R} \eta^{q'} d\mu \right) \leq \int_{D_A^R} u^q \zeta dx + \\ & + \left(\int_{D_A^R} u^q \zeta dx \right)^{\frac{1}{q}} \left(\lambda_1 \left(\int_{D_A^R} \zeta dx \right)^{\frac{1}{q'}} + q' \|L[\eta]\|_{L^{q'}(D_A^R)} \right). \end{aligned}$$

Next we prove that

$$(4.28) \quad \|L[\eta]\|_{L^{q'}(D_A^R)} \leq C \|\eta\|_{W^{s,q'}(\mathbb{R}^{N-k})}$$

starting with the estimate of the first term on the right hand side of (4.25).

$$\begin{aligned} \Delta H[\eta] &= \Delta_{x'} H[\eta] + \Delta_{x''} H[\eta] = \partial_y^2 H[\eta] + \frac{k-1}{y} \partial_y H[\eta] + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w(y^2, x'') + k \partial_t w(y^2, x'') + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w(y^2, x'') + (k+1) \partial_t w(y^2, x''). \end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^N} \rho |\Delta H[\eta]|^{q'} dx &\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_{tt} w(y^2, x'')|^{q'} dx'' y^{\kappa_++2q'+k-1} dy \\
&+ c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(y^2, x'')|^{q'} dx'' y^{\kappa_++k-1} dy \\
&\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_{tt} w(t, x'')|^{q'} dx'' t^{(\kappa_++k)/2+q'} \frac{dt}{t} \\
&+ c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(t, x'')|^{q'} dx'' t^{(\kappa_++k)/2} \frac{dt}{t} \\
&\leq c \int_0^1 \left\| t^{2-(1-\frac{\kappa_++k}{2q'})} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\
&+ c \int_0^1 \left\| t^{1-(1-\frac{\kappa_++k}{2q'})} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t}.
\end{aligned}$$

Put $\beta = \frac{\kappa_++k}{2q'}$ and note that $0 < \beta = \frac{1}{2}(2-s) < 1$. By standard interpolation theory,

$$\begin{aligned}
&\int_0^1 \left\| t^{1-(1-\beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\
&\approx \|\eta\|_{[W^{2,q'}, L^{q'}]_{1-\beta, q'}}^{q'} \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left\| t^{2-(1-\beta)} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\
&\approx \|\eta\|_{[W^{4,q'}, L^{q'}]_{\frac{1}{2}(1-\beta), q'}}^{q'} \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}.
\end{aligned}$$

The second term on the right hand side of (4.25) is estimated in a similar way:

$$\begin{aligned}
\int_{\mathbb{R}^N} \rho^{-q'/q} |\nabla H[\eta] \cdot \nabla \rho|^{q'} dx &\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(y^2, x'')|^{q'} dx'' y^{\kappa_++k-1} dy \\
&\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(t, x'')|^{q'} dx'' t^{\frac{\kappa_++k}{2}} \frac{dt}{t} \\
&\leq c \int_0^1 \left\| t^{1-(\frac{1}{2}-\beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\
&\approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}.
\end{aligned}$$

This proves (4.28). Further, (4.27) and (4.28) imply (4.15).

We turn to the proof of part (ii). Let E be a closed subset of $B_{R/2}^{N-k}(0)$ such that $C_{s, q'}^{N-k}(E) = 0$. Then there exists a sequence $\{\eta_m\}$ in $C_0^\infty(d_A)$ such

that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of E (which may depend on n), $\text{supp } \eta_n \subset B_{3R/4}^{N-k}(0)$ and $\|\eta_n\|_{W^{s,q'}} \rightarrow 0$. Then, by (4.28),

$$\|L[\eta_n]\|_{L^{q'}(D_A^R)} \rightarrow 0.$$

Furthermore

$$\|w\|_{L^{q'}((0,R) \times B_R^{N-k}(0))} \leq c \|\eta_n\|_{L^{q'}(B_R^{N-k}(0))}$$

and consequently

$$H[\eta_n] \rightarrow 0 \text{ in } L^{q'}(D_A^R).$$

(Here we use the fact that $k \geq 2$.) In addition

$$0 \leq H[\eta_n] \leq 1, \quad H[\eta_n] \leq c(R - |x'|)$$

with a constant c independent of n . Hence (see (4.20))

$$\zeta_{n,R} := \rho_A \psi_R H[\eta_n]^{q'} \leq \rho^R \rho_A \psi_R H[\eta_n]^{q'-1} \leq \rho^R \rho_A^R H[\eta_n]^{q'-1}.$$

As $u^q \rho^R \rho_A^R \in L^1(D_A^R)$ we obtain,

$$\lim_{n \rightarrow \infty} \int_{D_A} u^q \zeta_n dx = 0.$$

This fact and (4.27) imply that

$$\int_{d_A^R} \eta_n^{q'} d\mu \rightarrow 0.$$

As $\eta_n = 1$ on a neighborhood of E in \mathbb{R}^{N-k} it follows that $\mu(E) = 0$. \square

Proposition 4.5. *Let D_A be a k -dihedron, $1 \leq k < N$. Let k_+ be as in (2.11) and let q_c^* and q_c be as in Proposition 2.1 and Proposition 2.2 respectively. Assume that $q_c \leq q < q_c^*$. A measure $\mu \in \mathfrak{M}(\partial D_A)$, with compact support contained in d_A , is q -good relative to D_A if and only if μ vanishes on every Borel set $E \subset d_A$ such that $C_{s,q}(E) = 0$, where $s = 2 - \frac{k+\kappa_+}{q'}$.*

Remark. We shall use the notation $\mu \prec C_{s,q'}$ to say that μ vanishes on any Borel set $E \subset (d_A)$ such that $C_{s,q'}(E) = 0$.

In the case $k = N$: $D_A = C_A$ (= the cone with vertex 0 and opening A in \mathbb{R}^k) and $q_c = q_c^*$. By [23] (specifically the results quoted in subsection 2.2) $q_c = 1 - \frac{2}{\kappa_-} = \frac{N+\kappa_+}{N+\kappa_+-2}$ and if $1 < q < q_c$ then there exist solutions for every measure $\mu = k\delta_P$, $P \in d_A$.

In the case $k = 1$, $q_c^* = \infty$, $\kappa_+ = 1$ and $q_c = \frac{N+1}{N-1}$. Thus $s = 2/q$ and the statement of the theorem is well known (see [20]).

Proof. In view of the last remark, it remains to deal only with $2 \leq k \leq N-1$. We shall identify d_A with \mathbb{R}^{N-k} .

It is sufficient to prove the result for positive measures because $\mu \prec C_{s,q'}$ if and only if $|\mu| \prec C_{s,q'}$. In addition, if $|\mu|$ is a q -good measure then μ is a q -good measure.

First we show that if μ is non-negative and q-good then $\mu \prec C_{s,q'}$. If E is a Borel subset of $\partial\Omega$ then $\mu\chi_E$ is q-good. If E is compact and $C_{s,q'}(E) = 0$ then, by Proposition 4.4, E is a removable set. This means that the only positive solution of (1.1) in D_A such that $\mu(\partial\Omega \setminus E) = 0$ is the zero solution. This implies that $\mu\chi_E = 0$, i.e., $\mu(E) = 0$. If $C_{s,q'}(E) = 0$ but E is not compact then $\mu(E') = 0$ for every compact set $E' \subset E$. Therefore, we conclude again that $\mu(E) = 0$.

Next, assume that μ is a positive measure in $\mathfrak{M}(\partial D_A)$ supported in a compact subset of \mathbb{R}^{N-k} .

If $\mu \in B^{-s,q}(\mathbb{R}^{N-k})$ then, by Theorem 3.1, μ is admissible relative to $D_A \cap \Gamma_{k,R}$, for every $R > 0$. (As before $\Gamma_{k,R}$ is the cylinder with radius R around the 'axis' \mathbb{R}^{N-k} .) This implies that μ is q-good relative to D_A .

If $\mu \prec C_{s,q'}$ then, by a theorem of Feyel and de la Pradelle [11] (see also [3]), there exists a sequence $\{\mu_n\} \subset (B^{-s,q}(\mathbb{R}^{N-k}))_+$ such that $\mu_n \uparrow \mu$. As μ_n is q-good, it follows that μ is q-good. \square

Theorem 4.6. *Let P be an N -dimensional polyhedron as described in Proposition 4.3. Let μ be a bounded measure on ∂P , (may be a signed measure). Let $k = 1, \dots, N$, $j = 1, \dots, n_k$, and let $L_{k,j}$ and $A_{k,j}$ be defined as at the beginning of this section. Further, put*

$$(4.29) \quad s(k, j) = 2 - \frac{k + (\kappa_+)_{k,j}}{q'},$$

where $(\kappa_+)_{k,j}$ is defined as in (2.11) with $A = A_{k,j}$. Then $\mu \in \mathfrak{M}_q(\partial P)$, i.e., μ is a good measure for (1.1) relative to P , if and only if, for every pair (k, j) as above and every Borel set $E \subset L_{k,j}$:

If $1 \leq k < N$ then

$$(4.30) \quad \begin{aligned} (q_c)_{k,j} \leq q < (q_c^*)_{k,j}, \quad C_{s(k,j),q'}^{N-k}(E) = 0 &\implies \mu(E) = 0 \\ q \geq (q_c^*)_{k,j} &\implies \mu(L_{N,j}) = 0 \end{aligned}$$

and if $k = N$, i.e., L is a vertex,

$$(4.31) \quad q \geq (q_c)_{k,j} = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}} \implies \mu(L) = 0.$$

Here $(q_c^*)_{k,j}$ and $(q_c)_{k,j}$ are defined as in (2.32) and (2.36) respectively, with $A = A_{k,j}$.

If $1 < q < (q_c)_{k,j}$ then there is no restriction on $\mu\chi_{L_{k,j}}$.

Proof. This is an immediate consequence of Proposition 4.3 and Proposition 4.5 (see also the Remark following it). In the case $k = N$, $L_{N,j}$ is a vertex and the condition says merely that for $q \geq q_c(L_{N,j})$, μ does not charge the vertex. \square

4.3. Removable singular sets II.

Proposition 4.7. *Let A be a Lipschitz domain on S^{k-1} , $2 \leq k \leq N - 1$, and let D_A be the k -dihedron with opening A . Let u be a positive solution of*

(1.1) in D_A^R , for some $R > 0$. Suppose that $F = \mathcal{S}(u) \subset d_A^R$ and let Q be an open neighborhood of F such that $\bar{Q} \subset d_A^R$. (Recall that $d_A^R = d_A \cap B_R^{N-k}(0)$ is an open subset of d_A .) Let μ be the trace of u on $\mathcal{R}(u)$.

Let $\eta \in W_0^{s,q'}(d_A^R)$ such that

$$(4.32) \quad 0 \leq \eta \leq 1, \quad \eta = 0 \quad \text{on } Q.$$

Employing the notation in the proof of Proposition 4.4, put

$$(4.33) \quad \zeta := \rho_A \psi_R H_R[\eta]^{q'}.$$

Then

$$(4.34) \quad \int_{D_A^R} u^q \zeta \, dx \leq c(1 + \|\eta\|_{W^{s,q'}(d_A)})^{q'} + \mu(d_A^R \setminus Q)^q,$$

c independent of u and η .

Proof. First we prove (4.34) for $\eta \in C_0^\infty(d_A^R)$. Let σ_0 be a point in A and let $\{A_n\}$ be a Lipschitz exhaustion of A . If $0 < \epsilon < \text{dist}(\partial A, \partial A_n) = \bar{\epsilon}_n$ then

$$\epsilon \sigma_0 + C_{A_n} \subset C_A.$$

Denote

$$D_A^{R',R''} = D_A \cap [|x'| < R'] \cap [|x''| < R''].$$

Pick a sequence $\{\epsilon_n\}$ decreasing to zero such that $0 < \epsilon_n < \min(\bar{\epsilon}_n/2^n, R/8)$. Let u_n be the function given by

$$u_n(x'x'') = u(x' + \epsilon_n \sigma_0, x'') \quad \forall x \in D_{A_n}^{R_n, R}, \quad R_n = R - \epsilon_n.$$

Then u_n is a solution of (1.1) in $D_{A_n}^{R_n, R}$ belonging to $C^2(\bar{D}_{A_n}^{R_n, R})$ and we denote its boundary trace by h_n . Let

$$\zeta_n := \rho_{A_n} \psi_R H_R[\eta]^{q'},$$

with ψ_R and $H_R[\eta]$ as in the proof of Proposition 4.4. By Proposition 4.2

$$(4.35) \quad - \int_{D_{A_n}^{R_n, R}} \mathbb{P}[h_n] \Delta \zeta_n \, dx = \int_{B_R^{N-k}(0)} \eta^{q'} h_n \, d\omega_n$$

where ω_n is the harmonic measure on $d_{A_n}^R$ relative to $D_{A_n}^{R_n, R}$. (Note that $d_{A_n}^R = d_A^R$ and we may identify it with $B_R^{N-k}(0)$.) Hence

$$(4.36) \quad \int_{D_{A_n}^{R_n, R}} (-u_n \Delta \zeta_n + u_n^q \zeta_n) \, dx = - \int_{B_R^{N-k}(0)} \eta^{q'} h_n \, d\omega_n.$$

Further,

$$\int_{B_R^{N-k}(0)} \eta^{q'} h_n \, d\omega_n \rightarrow \int_{B_R^{N-k}(0)} \eta^{q'} \, d\mu \leq \mu(d_A^R \setminus Q),$$

because $\eta = 0$ in Q . By (4.24), (4.28) we obtain,

$$(4.37) \quad \left| \int_{D_{A_n}^{R_n, R}} u_n \Delta \zeta_n dx \right| \leq c \left(\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{\frac{1}{q}} \left(\left(\int_{D_{A_n}^{R_n, R}} \zeta_n dx \right)^{\frac{1}{q'}} + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))} \right).$$

From the definition of ζ_n it follows that

$$\int_{D_{A_n}^{R_n, R}} \zeta_n dx \leq \int_{D_{A_n}^{R_n, R}} \rho_n dx \rightarrow \int_{D_A^R} \rho dx,$$

where ρ (resp. ρ_n) is the first eigenfunction of $-\Delta$ in D_A^R (resp. $D_{A_n}^{R_n, R}$) normalized by 1 at some $x_0 \in D_{A_1}^{R_1, R}$. Therefore, by (4.36),

$$\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \leq c \left(\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{\frac{1}{q}} (1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))}) + \mu(d_A^R \setminus Q).$$

This implies

$$(4.38) \quad \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \leq c(1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})^{q'} + \mu(d_A^R \setminus Q)^q.$$

To verify this fact, put

$$m = \left(\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{1/q}, \quad b = \mu(d_A^R \setminus Q), \quad a = c(1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})$$

so that (4.38) becomes

$$m^q - am - b \leq 0.$$

If $b \leq m$ then

$$m^{q-1} - a - 1 \leq 0.$$

Therefore,

$$m \leq (a+1)^{\frac{1}{q-1}} + b$$

which implies (4.38). Finally, by the lemma of Fatou we obtain (4.34) for $\eta \in C_0^\infty$. By continuity we obtain the inequality for any $\eta \in W_0^{s, q'}$ satisfying (4.32). \square

Theorem 4.8. *Let A be a Lipschitz domain on S^{k-1} , $2 \leq k \leq N-1$, and let D_A be the k -dihedron with opening A . Let E be a compact subset of d_A^R and let u be a non-negative solution of (1.1) in D_A^R (for some $R > 0$) such that u vanishes on $\partial D_A^R \setminus E$. Then*

$$(4.39) \quad C_{s, q'}^{N-k}(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'} \implies u = 0,$$

where $C_{s, q'}^{N-k}$ denotes the Bessel capacity with the indicated indices in \mathbb{R}^{N-k} .

Proof. By Proposition 4.4, (4.39) holds under the additional assumption

$$(4.40) \quad \int_{D_A^R} u^q \rho_R \rho_A^R dx < \infty.$$

Indeed, by [23, Proposition 4.1], (4.40) implies that the solution u possesses a boundary trace μ on ∂D_A^R . By assumption, $\mu(\partial D_A^R \setminus E) = 0$. Therefore, by Proposition 4.5, the fact that $C_{s,q'}^{N-k}(E) = 0$ implies that $\mu(E) = 0$. Thus $\mu = 0$ and hence $u = 0$.

We show that, under the conditions of the theorem, if $C_{s,q'}^{N-k}(E) = 0$ then (4.40) holds.

By Proposition 4.7, for every $\eta \in W_0^{s,q'}(d_A^R)$ such that $0 \leq \eta \leq 1$ and $\eta = 0$ in a neighborhood of E ,

$$(4.41) \quad \int_{D_A^R} u^q \zeta dx \leq c(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))})^{q'},$$

for ζ as in (4.33). (Here we use the assumption that $u = 0$ on $\partial D_A^R \setminus E$.)

Let $a > 0$ be sufficiently small so that $E \subset B_{(1-4a)R}^{N-k}(0)$. Pick a sequence $\{\phi_n\}$ in $C_0^\infty(\mathbb{R}^{N-k})$ such that, for each n , there exists a neighborhood Q_n of E , $\tilde{Q}_n \subset B_{(1-3a)R}^{N-k}(0)$ and

$$(4.42) \quad \begin{aligned} 0 &\leq \phi_n \leq 1 \text{ everywhere, } \phi_n = 1 \text{ in } Q_n, \\ \tilde{\phi}_n &:= \phi_n \chi_{\{|x''| < (1-2a)R\}} \in C_0^\infty(\mathbb{R}^{N-k}), \\ \|\tilde{\phi}_n\|_{W^{s,q'}(\mathbb{R}^{N-k})} &\rightarrow 0 \text{ as } n \rightarrow \infty \\ \eta_n &:= (1 - \phi_n)|_{\{|x''| < R\}} \in C_0^\infty(d_A^R), \\ \eta_n &= 0 \text{ in } [(1-a)R < |x''| < R]. \end{aligned}$$

Such a sequence exists because $C_{s,q'}^{N-k}(E) = 0$. Applying (4.41) to η_n we obtain,

$$(4.43) \quad \sup \int_{D_A^R} u^q \zeta_n dx \leq c < \infty,$$

where $\zeta_n = \rho_A \psi_R H_R^{q'}[\eta_n]$ (see (4.33)). By taking a subsequence we may assume that $\{\eta_n\}$ converges (say to η) in $L^{q'}(B_R^{N-k}(0))$ and consequently $H[\eta_n] \rightarrow H[\eta]$ in the sense that

$$H_R[\eta_n](x', \cdot) = w_{n,R}(y^2, \cdot) \rightarrow w_R(y^2, \cdot) = H_R[\eta](x', \cdot) \text{ in } L^{q'}$$

uniformly with respect to $y = |x'|$. It follows that

$$(4.44) \quad \int_{D_A^R} u^q \zeta dx < \infty, \quad \zeta = \rho_A \psi_R H_R^{q'}[\eta].$$

As $\tilde{\phi}_n \rightarrow 0$ in $W^{s,q'}(\mathbb{R}^{N-k})$ it follows that $\phi_n \rightarrow 0$ and hence $\eta_n \rightarrow 1$ a.e. in $B_{(1-2a)R}^{N-k}(0)$. Thus $\eta = 1$ in this ball, $\eta = 0$ in $[(1-a)R < |x''| < R]$ and $0 \leq \eta \leq 1$ everywhere.

Consequently, given $\delta > 0$, there exists an N -dimensional neighborhood O of $d_A \cap B_{(1-2a)R}^{N-k}(0)$ such that

$$1 - \delta < H_R[\eta] < 1 \quad \text{and} \quad 1 - \delta < \psi_R/\rho_A^R < 1 \quad \text{in } O.$$

Therefore (4.44) implies that

$$(4.45) \quad \int_{D_A^{(1-3a)R}} u^q \rho^R \rho_A^R dx \leq c < \infty.$$

Recall that the trace of u on $\partial D_A^R \setminus d_A^{(1-4a)R}$ is zero. Therefore u is bounded in $D_A^R \setminus D_A^{(1-3a)R}$. This fact and (4.45) imply (4.40). \square

Definition 4.9. Let Ω be a bounded Lipschitz domain. Denote by ρ the first eigenfunction of $-\Delta$ in Ω normalized by $\rho(x_0) = 1$ for a fixed point $x_0 \in \Omega$.

For every compact set $K \subset \partial\Omega$ we define

$$M_{\rho,q}(K) = \{\mu \in \mathfrak{M}(\partial\Omega) : \mu \geq 0, \mu(\partial\Omega \setminus K) = 0, \mathbb{K}[\mu] \in L_\rho^q(\Omega)\}$$

and

$$\tilde{C}_{\rho,q'}(K) = \sup\{\mu(K)^q : \mu \in M_{\rho,q}(K), \int_\Omega \mathbb{K}[\mu]^q \rho dx = 1\}.$$

Finally we denote by $C_{\rho,q'}$ the outer measure generated by the above functional.

The following statement is verified by standard arguments:

Lemma 4.10. For every compact $K \subset \partial\Omega$, $C_{\rho,q'}(K) = \tilde{C}_{\rho,q'}(K)$. Thus $C_{\rho,q'}$ is a capacity and,

$$(4.46) \quad C_{\rho,q'}(K) = 0 \iff M_{\rho,q}(K) = \{0\}.$$

Theorem 4.11. Let Ω be a bounded polyhedron in \mathbb{R}^N . A compact set $K \subset \partial\Omega$ is removable if and only if

$$(4.47) \quad C_{s(k,j),q'}(K \cap L_{k,j}) = 0,$$

for $k = 1, \dots, N$ $j = 1, \dots, n_k$, where $s(k,j)$ is defined as in (4.29). This condition is equivalent to

$$(4.48) \quad C_{\rho,q'}(K) = 0.$$

A measure $\mu \in \mathfrak{M}(\partial\Omega)$ is q -good if and only if it does not charge sets with $C_{\rho,q'}$ -capacity zero.

Proof. The first assertion is an immediate consequence of Proposition 4.3 and Theorem 4.8. The second assertion follows from the fact that

$$C_{\rho,q'}(K \cap L_{k,j}) = C_{s(k,j),q'}(K \cap L_{k,j}).$$

The third assertion follows from Theorem 4.6 and the previous statement. \square

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