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BOUNDARY TRACE OF POSITIVE SOLUTIONS OF SUPERCRITICAL SEMILINEAR ELLIPTIC EQUATIONS IN DIHEDRAL DOMAINS

MOSHE MARCUS AND LAURENT VERON

ABSTRACT. We study the generalized boundary value problem for (E) $-\Delta u + |u|^{q-1}u = 0$ in a dihedral domain Ω , when $q > 1$ is supercritical. The value of the critical exponent can take only a finite number of values depending on the geometry of Ω . When μ is a bounded Borel measure in a k -wedge, we give necessary and sufficient conditions in order it be the boundary value of a solution of (E). We also give conditions which ensure that a boundary compact subset is removable. These conditions are expressed in terms of Bessel capacities $B_{s,q}$ in \mathbb{R}^{N-k} where s depends on the characteristics of the wedge. This allows us to describe the boundary trace of a positive solution of (E).

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Key words. Laplacian; Poisson potential; Borel measures; Besov spaces; harmonic lifting; Bessel capacities.

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1. INTRODUCTION

Let Ω be a bounded Lipschitz domain in \mathbb{R}^N , ρ the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ with supremum 1 and λ the corresponding eigenvalue, and let $q > 1$. A long-term research on the equation

$$(1.1) \quad -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega,$$

has been carried out for more than twenty years by probabilistic and/or analytic methods. Much of the research was focused on three main problems in domains of class C^2 :

- (i) The Dirichlet problem for (1.1) with boundary data given by a finite Borel measure on $\partial\Omega$.
- (ii) The characterization of removable singular subsets of $\partial\Omega$ relative to positive solutions of (1.1).
- (iii) The characterization of arbitrary positive solutions of (1.1) via an appropriate notion of boundary trace.

Consider the Dirichlet problem

$$(1.2) \quad -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \quad u = \mu \quad \text{in } \partial\Omega$$

where $\mu \in \mathfrak{M}(\partial\Omega)$ (= space of finite Borel measures on $\partial\Omega$). Following [23], a (weak) solution $u := u_\mu$ of (1.2) is a function $u \in L_\rho^q(\Omega)$ such that,

$$(1.3) \quad \int_\Omega (-u\Delta\eta + |u|^{q-1}u)\,dx = - \int_\Omega \mathbb{K}[\mu]\Delta\eta\,dx,$$

for every $\eta \in X(\Omega)$, where

$$(1.4) \quad X(\Omega) = \left\{ \eta \in W_0^{1,2}(\Omega) : \rho^{-1}\Delta\eta \in L^\infty(\Omega) \right\}.$$

Here $\mathbb{K}[\mu]$ is the harmonic function in Ω with boundary trace μ and ρ is the first eigenfunction of $-\Delta$ in Ω normalized so that $\max_\Omega \rho = 1$. We recall that, if Ω is Lipschitz $\mathbb{K}[\mu] \in L_\rho^1(\Omega)$; if Ω is of class C^2 , $\mathbb{K}[\mu] \in L^1(\Omega)$.

A measure μ is a *q-good measure* if (1.2) has a solution. The space of *q-good measures* is denoted by $\mathfrak{M}_q(\partial\Omega)$. It is known that, if μ is *q-good*, the solution is unique. Furthermore, if μ satisfies the condition

$$(1.5) \quad \int_\Omega \mathbb{K}[|\mu|]^q \rho\,dx < \infty,$$

then it is *q-good*. When μ satisfies this condition we say that it is a *q-admissible measure*.

When Ω is a domain of class C^2 , $\mathbb{K}[\mu] \in L_\rho^q$ for every $q \in (1, \frac{N+1}{N-1})$ and every $\mu \in \mathfrak{M}(\partial\Omega)$. Therefore, for q in this range, every measure in $\mathfrak{M}(\partial\Omega)$ is *q-good* and there is no removable boundary set (except for the empty set). Problem (iii), for q in this range, was resolved by Le Gall [14] (for $N = q = 2$) and Marcus and Véron [18] (for $1 < q < \frac{N+1}{N-1}$, $N \geq 3$).

The number $q_c = \frac{N+1}{N-1}$ is called the *critical value* for (1.1). If q is supercritical, i.e. $q \geq q_c$, point singularities are removable. In particular there

is no solution of (1.2) when $\mu = \delta_y$ (= a Dirac measure concentrated at a point $y \in \partial\Omega$).

In the supercritical case, problems (i) - (iii), Ω of class C^2 , have been resolved in several stages. We say that a compact set $E \subset \partial\Omega$ is removable relative to equation (1.1) if there exists no positive solution vanishing on $\partial\Omega \setminus E$. We say that E is conditionally removable if any solution u of (1.2), with $\mu \in \mathfrak{M}(\partial\Omega)$, such that $u = 0$ on $\partial\Omega \setminus E$ must vanish in Ω .

With respect to problem (ii) it was shown that a compact set $E \subset \partial\Omega$ is removable if and only if $C_{\frac{2}{q}, q'}(E) = 0$, $q' = q/(q-1)$. Here $C_{\alpha, p}$ denotes Bessel capacity, with the indicated indexes on $\partial\Omega$. This result was obtained by Le Gall [15] for $q = 2$, Dynkin and Kuznetsov [7] for $1 < q \leq 2$, Marcus and Véron [19] for $q > 2$. For a unified analytic proof, covering all $q \geq q_c$ see [20].

The above result implies that every q -good measure μ must vanish on sets of $C_{\frac{2}{q}, q'}$ capacity zero. On the other hand a result of Baras and Pierre [3] implies that every positive measure $\mu \in \mathfrak{M}(\partial\Omega)$ that vanishes on sets of $C_{\frac{2}{q}, q'}$ capacity zero is the limit of an increasing sequence of admissible measures and therefore q -good. In conclusion: a measure $\mu \in \mathfrak{M}(\partial\Omega)$ is q -good if and only if it vanishes on sets of $C_{\frac{2}{q}, q'}$ capacity zero. This takes care of problem (i).

Problem (iii) has been treated in several papers, with various definitions of a generalized boundary trace for positive solutions of (1.1), see [8] and [22]. Finally a full characterization of positive solutions was obtained by Mselati [24] for $q = 2$, Dynkin [6] for $1 < q < 2$ and Marcus [17] for every $q \geq q_c$. In [24, 6] the restriction to $q \leq 2$ was dictated by their use of probabilistic techniques that do not apply to $q > 2$. In [17] the proof is purely analytic.

If Ω is Lipschitz, $\xi \in \partial\Omega$, we say that q_ξ is the critical value for (1.1) at ξ if, for $1 < q < q_\xi$, problem (1.2) with $\mu = \delta_\xi$ has a solution, but for $q > q_\xi$ no such solution exists.

In contrast to the case of smooth domains, when Ω is Lipschitz, q_ξ may vary with the point. Furthermore the assumption $1 < q < \min\{q_\xi : \xi \in \partial\Omega\}$ does not guarantee that every measure in $\mathfrak{M}(\partial\Omega)$ is q -good, although there always exists $q(\Omega) > 1$ such that, for $1 < q < q(\Omega)$, this result holds.

In the special case when Ω is a polyhedron, the function $\xi \rightarrow q_\xi$ obtains only a finite number of values (in fact, it is constant on each open face and each open edge) and, if $q \geq q_\xi$, an isolated singularity at ξ is removable. Furthermore, the assumption $1 < q < \min\{q_\xi : \xi \in \partial\Omega\}$, implies that every measure in $\mathfrak{M}(\partial\Omega)$ is q -good. For this and related results see [23].

In the present paper we study problem (1.2) when Ω is a polyhedron and q is supercritical, i.e. $q \geq \min\{q_\xi : \xi \in \partial\Omega\}$. Following is a description of the main results.

A. *On the action of Poisson type kernels with fractional dimension.*

In preparation for the study of supercritical boundary value problems we establish an harmonic analytic result, extending a well known result on the action of Poisson kernels on Besov spaces with negative index (see [27, 1.14.4.] and [4]). We first quote the classical result for comparison purposes.

Proposition 1.1. *Let $1 < q < \infty$ and $s > 0$. Then, for any bounded Borel measure μ in \mathbb{R}^{n-1} ,*

$$(1.6) \quad I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|_{B^{-s,q}(\mathbb{R}^{n-1})}^q.$$

Here $\mathbb{K}_n[\mu]$ denotes the Poisson potential of μ in $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$, namely,

$$(1.7) \quad \mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{n-1}} \frac{d\mu(z)}{(y_1^2 + |\zeta - z|^2)^{n/2}} \quad \forall y = (y_1, \zeta) \in \mathbb{R}_+^n$$

where γ_n is a constant depending only on n .

Notation. Let m be a positive integer and let ν be a real number, $\nu \geq m+1$. Denote,

$$(1.8) \quad \mathbb{K}_{\nu,m}[\mu](\tau, \zeta) := \int_{\mathbb{R}^m} \frac{\tau^{\nu-m} d\mu(z)}{(\tau^2 + |\zeta - z|^2)^{\nu/2}} \quad \forall \tau \in (0, \infty), \zeta \in \mathbb{R}^m.$$

Note that

$$\mathbb{K}_n[\mu] = \gamma_n \mathbb{K}_{n,n-1}[\mu].$$

Theorem 1.2. *Let m and ν be as above. Then, for every $q > 1$ and every $s \in (0, m/q')$, $q' = q/(q-1)$, there exists a positive constant c such that, for every positive measure $\mu \in \mathfrak{M}(\mathbb{R}^m)$ supported in $B_{R/2}(0)$ for some $R > 1$,*

$$(1.9) \quad \frac{1}{c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq \int_0^R \left(\int_{|\zeta| < R} |\mathbb{K}_{\nu,m}[\mu](\tau, \zeta)|^q d\zeta \right) \tau^{sq-1} d\tau \\ \leq cR^{(s+\nu-m)q+1} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q.$$

This also holds when $s = m/q'$, provided that the diameter of $\text{supp } \mu$ is sufficiently small.

This is proved in Section 3 (see Theorem 3.8) using a slightly different notation.

B. *The critical value and the characterization of q -good measures in a k -wedge.*

The next step towards the study of boundary value problems in a polyhedron is the treatment of such problems in a k -wedge (or k -dihedron) i.e., the domain defined by the intersection of k hyperplanes in \mathbb{R}^N , $1 < k < N$. The edge is an $(N-k)$ dimensional space.

We note that if $k = N$ the 'edge' is a point and the corresponding wedge is a cone with vertex at this point. If $k = 1$ the wedge is a half space. Both of these cases have been treated in [23].

Let A be a Lipschitz domain in S^{k-1} . If

$$(1.10) \quad S_A := \{x \in \mathbb{R}^N : |x| = 1, x \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}$$

then

$$D_A := \{x = (r, \sigma) : r > 0, \sigma \in S_A\}$$

is a k -wedge in \mathbb{R}^N whose ‘edge’ d_A may be identified with \mathbb{R}^{N-k} and its ‘opening’ is A .

Let λ_A be the first eigenvalue of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S_A)$ and denote by κ_{\pm} the roots of the equation,

$$(1.11) \quad \kappa^2 + (N-2)\kappa - \lambda_A = 0.$$

Put

$$(1.12) \quad q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}$$

and

$$(1.13) \quad q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}.$$

Let $C_{\alpha,p}^{N-k}$ denote the Bessel capacity with the indicated indices in \mathbb{R}^{N-k} . The next theorem provides a characterization of q -good measures supported on d_A .

Theorem 1.3. (a) *If $1 < q < q_c$ every measure in $\mathfrak{M}(d_A)$ is q -good relative to D_A . In fact every such measure is q -admissible.*

(b) *If $q \geq q_c^*$, the only q -good measure in $\mathfrak{M}(d_A)$ is the zero measure.*

(c) *If $q_c \leq q < q_c^*$, a measure $\mu \in \mathfrak{M}(d_A)$ is q -good relative to D_A if and only if μ vanishes on every Borel set $E \subset d_A$ such that $C_{s,q'}^{N-k}(E) = 0$, $s = 2 - \frac{k+\kappa_+}{q}$.*

The characterization of q -good measures in a polyhedron follows as an easy consequence of the above theorem (see Theorem 4.6 below).

C. Characterization of removable sets.

Let Ω be an N -dimensional polyhedron. Theorem 1.3 provides a necessary and sufficient condition for the removability of a singular set E relative to the family of solutions u such that

$$\int_{\Omega} |u|^q \rho dx < \infty.$$

The next result provides a necessary and sufficient condition for *removability* in the sense that the only non-negative solution $u \in C(\bar{\Omega} \setminus E)$ which vanishes on $\bar{\Omega} \setminus E$ is the trivial solution $u = 0$.

Let L denote a face or edge or vertex of Ω and put $k := \text{codim } L$. If $1 < k < N$ let d_L denote the linear space spanned by L , such that L is an open subset of d_L . Let Q_L denote the k -wedge with boundary d_L such that,

for some neighborhood M of L , $\Omega \cap M = Q_L \cap M$ and let A_L denote the opening of Q_L . If $k = N$, Q_L is a cone with vertex L . Let $q_c(L)$ and $q_c^*(L)$ be defined as in (1.12) and (1.13) for $A = A_L$. Finally let

$$s(L) = 2 - \frac{k + \kappa_+}{q'}$$

where κ_{\pm} are the roots of (1.11) for $A = A_L$. If $k = N$, Q_L is a cone with vertex L . In this case $q_c(L) = q_c^*(L) = 1 - \frac{2}{\kappa_-}$. If $k = 1$ $q_c(L) = q_c^*(L) = (N + 1)/(N - 1)$.

Theorem 1.4. *Let Ω be a polyhedron in \mathbb{R}^N . A compact set $E \subset \partial\Omega$ is removable if and only if, for every L as above such that $E \cap L \neq \emptyset$ the following conditions hold.*

If $1 \leq k < N$: either $q_c(L) \leq q < q_c^(L)$ and $C_{s(L), q'}^{N-k}(E \cap L) = 0$ or $q \geq q_c^*(L)$.
If $k = N$: $q \geq q_c(L)$.*

The present paper is part of an article, ‘Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains’ arXiv:0907.1006 (2009). The first part of this article was published in [23]. The second and last part are presented here. The characterization of q -good measures, here established in polyhedrons, was recently established in [2], for arbitrary Lipschitz domains and a general family of nonlinearities. However the full removability result, Theorem 4.11, has not been superseded. (In [2] the authors provided - in the generality mentioned above - a characterization of *conditional removability* but not of full removability.) The methods of proof in the two papers are completely different. In the present paper, the characterization of q -good measures is based on an extension of a result of [4] and [27, 1.14.4.] on the action of Poisson kernels on Besov spaces with negative index. In [2] the proof relies on a relation between elliptic semilinear equations with absorption and linear Schrödinger equations.

2. THE MARTIN KERNEL AND CRITICAL VALUES IN A K -DIMENSIONAL DIHEDRON.

2.1. The geometric framework. An N -dim polyhedra P is a bounded domain bordered by a finite number of hyperplanes. Thus the boundary of P is the union of a finite number of sets $\{L_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$ where $\{L_{1,j}\}$ is the set of open faces of P , $\{L_{k,j}\}$ for $k = 2, \dots, N - 1$, is the family of relatively open $N - k$ -dimensional edges and $\{L_{N,j}\}$ is the family of vertices of P . An $N - k$ dimensional edge is a relatively open set in the intersection of k hyperplanes; it will be described by the characteristic angles of these hyperplanes.

We recall that the spherical coordinates in $\mathbb{R}^N = \{x = (x_1, \dots, x_N)\}$ are expressed by

$$(2.1) \quad \begin{cases} x_1 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \sin \theta_1 \\ x_2 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \cos \theta_1 \\ x_3 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \cos \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{N-1} = r \sin \theta_{N-1} \cos \theta_{N-2}, \\ x_N = r \cos \theta_{N-1} \end{cases}$$

where, $r = |x|$, $\theta_1 \in [0, 2\pi]$ and $\theta_\ell \in [0, \pi]$ for $\ell = 2, 3, \dots, N-1$. We denote $\sigma = (\theta_1, \dots, \theta_{N-1})$. Thus in spherical coordinates $x = (r, \sigma)$.

We consider an unbounded *non-degenerate k-dihedron*, $2 \leq k \leq N$ defined as follows. Let A be given by

$$A = \begin{cases} (0, \alpha_1) \times \prod_{j=2}^{k-1} (\alpha_j, \alpha'_j) & \text{if } k > 2 \\ (0, \alpha_1) & \text{if } k = 2 \end{cases}$$

where

$$0 < \alpha_1 < 2\pi, \quad 0 \leq \alpha_j < \alpha'_j < \pi \quad j = 2, \dots, k-1.$$

We denote by S_A the spherical domain

$$(2.2) \quad S_A = \{x \in \mathbb{R}^N : |x| = 1, \sigma \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}$$

and by D_A the corresponding k-dihedron,

$$D_A = \{x = (r, \sigma) : r > 0, \sigma \in S_A\}.$$

The *edge* of D_A is the $(N-k)$ -dimensional space

$$(2.3) \quad d_A = \{x : x_1 = x_2 = \dots = x_k = 0\}.$$

2.2. On the Martin kernel and critical values in a cone. We recall here some elements of local analysis when $\Omega = C_A \cap B_1$, A is a Lipschitz domain in S^{N-1} and C_A is the cone with vertex 0 and opening A .

Denote by λ_A the first eigenvalue and by ϕ_A the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(A)$ (normalized by $\max \phi_A = 1$). Let κ_- be the negative root of (1.11) and put

$$\Phi_1(x) := \frac{1}{\gamma} |x|^{\kappa_-} \phi_A(x/|x|)$$

where γ is a positive number. Then Φ_1 is a harmonic function in C_A vanishing on $\partial C_A \setminus \{0\}$. We choose $\gamma = \gamma_A$ so that the boundary trace of Φ_1 is δ_0 (=Dirac measure on with mass 1 at the origin).

(i) If $q \geq 1 - \frac{2}{\kappa_-}$, there is no solution of (1.1) in Ω_S with isolated singularity at 0. (See [9].)

(ii) If $1 < q < 1 - \frac{2}{\kappa_-}$, then for any $k > 0$ there exists a unique solution $u := u_k$ to problem (1.2) with $\mu = k\delta_0$ and

$$(2.4) \quad u_k(x) = k\Phi_1(x)(1 + o(1)) \quad \text{as } x \rightarrow 0.$$

The function $u_\infty = \lim_{k \rightarrow \infty} u_k$ is a positive solution of (1.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies

$$(2.5) \quad u_\infty(x) = |x|^{-\frac{2}{q-1}} \omega_A(x/|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0$$

where ω_A is the (unique) positive solution of

$$(2.6) \quad -\Delta' \omega - a_{N,q} \omega + |\omega|^{q-1} \omega = 0$$

on S^{N-1} . Here Δ' is the Laplace - Beltrami operator and

$$(2.7) \quad a_{N,q} = \frac{2}{q-1} \left(\frac{2q}{q-1} - N \right).$$

(iii) If $u \in C(\bar{\Omega}_A \setminus \{0\})$ is a positive solution of (1.1) vanishing on $(\partial C_A \cap B_{r_0}(0)) \setminus \{0\}$, then either u satisfies (2.4) for some $k > 0$ or u satisfies (2.5). In particular there exists a unique positive solution vanishing on $(\partial C_A \cap B_{r_0}(0)) \setminus \{0\}$ with strong singularity at 0. (For (ii) and (iii) see [23, Theorem 5.7].)

2.3. Separable harmonic functions and the Martin kernel in a k-dihedron, $2 \leq k < N$. In the system of spherical coordinates, the Laplacian takes the form

$$\Delta u = \partial_{rr} u + \frac{N-1}{r} \partial_r u + \frac{1}{r^2} \Delta_{S^{N-1}} u$$

where the Laplace-Beltrami operator $\Delta_{S^{N-1}}$ is expressed by induction by

$$(2.8) \quad \begin{aligned} \Delta_{S^{N-1}} u &= \frac{1}{(\sin \theta_{N-1})^{N-2}} \frac{\partial}{\partial \theta_{N-1}} \left((\sin \theta_{N-1})^{N-2} \frac{\partial u}{\partial \theta_{N-1}} \right) \\ &+ \frac{1}{(\sin \theta_{N-1})^2} \Delta_{S^{N-2}} u. \end{aligned}$$

and

$$(2.9) \quad \Delta_{S^1} u = \partial_{\theta_1 \theta_1} u$$

If we compute the positive harmonic functions in the k-dihedron D_A of the form

$$v(x) = v(r, \sigma) = r^\kappa \omega(\sigma) \quad \text{in } D_A, \quad v = 0 \quad \text{in } \partial D_A \setminus \{0\}.$$

we find that ω must be a positive eigenfunction corresponding to the first eigenvalue, λ_A , of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S_A)$,

$$(2.10) \quad \begin{cases} \Delta_{S^{N-1}} \omega + \lambda_A \omega = 0 & \text{in } S_A \\ \omega = 0 & \text{on } \partial S_A \end{cases}$$

and κ must be a root of the algebraic equation (1.11) with λ_A as above. Thus $\kappa = \kappa_{\pm}$ where

$$(2.11) \quad \begin{aligned} \kappa_+ &= \frac{1}{2} \left(2 - N + \sqrt{(N-2)^2 + 4\lambda_A} \right) \\ \kappa_- &= \frac{1}{2} \left(2 - N - \sqrt{(N-2)^2 + 4\lambda_A} \right). \end{aligned}$$

Since

$$S^{N-1} = \{ \sigma = (\sigma_2 \sin \theta_{N-1}, \cos \theta_{N-1}) : \sigma_2 \in S^{N-2}, \theta_{N-1} \in (0, \pi) \},$$

we look for a solution $\omega = \omega^{\{1\}}$ of (2.10) of the form

$$\omega^{\{1\}}(\sigma) = (\sin \theta_{N-1})^{\kappa_+} \omega^{\{2\}}(\sigma_2), \quad \theta_{N-1} \in (0, \pi), \quad \sigma_2 \in S^{N-2}.$$

Here $S^{N-2} = S^{N-1} \cap \{x_N = 0\}$ and we denote

$$S_A^{\{N-2\}} = S_A \cap \{x_N = 0\}, \quad D_A^{\{N-2\}} := D_A \cap \{x_N = 0\} \subset \mathbb{R}^{N-1}.$$

Then (2.11) jointly with relation (2.8) implies

$$(2.12) \quad \begin{cases} \Delta_{S^{N-2}} \omega^{\{2\}} + (\lambda_A - \kappa_+) \omega^{\{2\}} = 0 & \text{on } S_A^{\{N-2\}} \\ \omega^{\{2\}} = 0 & \text{on } \partial S_A^{\{N-2\}}. \end{cases}$$

Since we are interested in $\omega^{\{2\}}$ positive, $\lambda_A^{\{2\}} := \lambda_A - \kappa_+$ must be the first eigenvalue of $-\Delta_{S^{N-2}}$ in $W_0^{1,2}(S_A^{\{N-2\}})$.

Next we look for positive harmonic functions \tilde{u} in $D_A^{\{N-2\}}$ such that

$$\tilde{u}(x_1, \dots, x_{N-1}) = r^{\kappa'} \omega(\sigma_2), \quad \tilde{u} = 0 \text{ on } \partial D_A^{\{N-2\}}$$

The algebraic equation which gives the exponents is

$$(\kappa')^2 + (N-3)\kappa' - \lambda_A^{\{2\}} = 0.$$

Denote by κ'_+ the positive root of this equation. By the definition of $\lambda_A^{\{2\}}$,

$$\kappa_+^2 + (N-3)\kappa_+ - \lambda_A^{\{2\}} = \kappa_+^2 + (N-2)\kappa_+ - \lambda_A = 0.$$

Therefore $\kappa'_+ = \kappa_+$. Accordingly, if $k \geq 3$, we set

$$\omega^{\{2\}}(\sigma_2) = (\sin \theta_{N-2})^{\kappa_+} \omega^{\{3\}}(\sigma_3),$$

and find that $\omega^{\{3\}}$ satisfies

$$(2.13) \quad \begin{cases} \Delta_{S^{N-3}} \omega^{\{3\}} + (\lambda_A - 2\kappa_+) \omega^{\{3\}} = 0 & \text{in } S_A^{\{N-3\}} \\ \omega^{\{3\}} = 0 & \text{on } \partial S_A^{\{N-3\}}, \end{cases}$$

where

$$S_A^{\{N-3\}} = S_A \cap \{x_N = x_{N-1} = 0\}.$$

Performing this reduction process (N-k) times, we obtain the following results.

(i) If $k > 2$ then $\omega = \omega^{N-k}(\sigma)$ is given by

$$(2.14) \quad \omega(\sigma) = (\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k)^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})$$

where

$$\sigma_{N-k+1} \in S^{k-1} = S^{N-1} \cap \{x_N = x_{N-1} = \dots = x_{k+1} = 0\}$$

and $\omega' := \omega^{\{N-k+1\}}$ satisfies

$$(2.15) \quad \begin{cases} \Delta_{S^{k-1}} \omega' + (\lambda_A - (N-k)\kappa_+) \omega' = 0, & \text{in } S_A^{\{k-1\}} \\ \omega' = 0, & \text{on } \partial S_A^{\{k-1\}}, \end{cases}$$

where $S_A^{\{k-1\}} = S_A \cap \{x_N = x_{N-1} = \dots = x_{k+1} = 0\} \approx A$ and $\lambda_A - (N-k)\kappa_+$ is the first eigenvalue of the problem.

(ii) If $k = 2$ then

$$(2.16) \quad \omega(\sigma) = (\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2)^{\kappa_+} \omega^{\{N-1\}}(\theta_1)$$

where $\sigma_{N-1} \in S^1 \approx \theta_1 \in (0, 2\pi)$, and $\omega^{\{N-1\}}$ satisfies

$$(2.17) \quad \begin{cases} \Delta_{S^1} \omega^{\{N-1\}} + (\lambda_A - (N-2)\kappa_+) \omega^{\{N-1\}} = 0 & \text{on } S_A^{\{1\}} \\ \omega^{\{N-1\}} = 0 & \text{on } \partial S_A^{\{1\}}, \end{cases}$$

with $\partial S_A^{\{1\}} \approx (0, \alpha)$. In this case

$$(2.18) \quad \kappa_+ = \frac{\pi}{\alpha}, \quad \omega^{\{N-1\}}(\theta_1) = \sin(\pi\theta_1/\alpha),$$

and, by (1.11),

$$(2.19) \quad \lambda_A - (N-2)\kappa_+ = \frac{\pi^2}{\alpha^2} \implies \lambda_A = \frac{\pi^2}{\alpha^2} + (N-2)\frac{\pi}{\alpha}.$$

Observe that $\frac{1}{2} \leq \kappa_+$ with equality holding only in the degenerate case $\alpha = 2\pi$ (which we exclude).

In either case, we find a positive harmonic function v_A in D_A , vanishing on ∂D_A , of the form

$$(2.20) \quad v_A(x) = |x|^{\kappa_+} \omega(x/|x|)$$

with ω as in (2.14) (for $k > 2$) or (2.18) (for $k=2$). Furthermore, if Ω is a domain in \mathbb{R}^N such that, for some $R > 0$, $\Omega \cap B_R(0) = D_A \cap B_R(0)$ and w is a positive harmonic function in Ω vanishing on $d_A \cap B_R(0)$ then $w \sim v_A$ in $\Omega \cap B_{R'}(0)$ for every $R' \in (0, R)$.

Similarly we find a positive harmonic function in D_A vanishing on $\partial D_A \setminus \{0\}$, singular at the origin, of the form

$$K'_A(x) = |x|^{\kappa_-} \omega(x/|x|).$$

If Ω is a domain as above and z is a positive harmonic function in Ω vanishing on $d_A \cap B_R(0) \setminus \{0\}$ then $z \sim K'_A$ in $\Omega \cap B_{R'}(0) \setminus \{0\}$ for every $R' \in (0, R)$.

As K'_A is a kernel function of $-\Delta$ at 0 it follows that K'_A is, up to a multiplicative constant c_A , the Martin kernel of $-\Delta$ in D_A , with singularity at 0. The Martin kernel, with singularity at a point $z \in d_A$, is given by

$$(2.21) \quad K_A(x, z) = c_A \frac{(\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k)^{\kappa+\omega} \{\sigma_{N-k+1}\}}{|x-z|^{N-2+\kappa+}}$$

for every $x \in D_A$. From (2.1)

$$\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k = |x-z|^{-1} \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}.$$

Therefore, if we write $x \in \mathbb{R}^N$ in the form $x = (x', x'')$, $x' = (x_1, \dots, x_k)$, $x'' = (x_{k+1}, \dots, x_N)$, we obtain the formula,

$$(2.22) \quad \begin{aligned} K_A(x, z) &= c_A \frac{|x'|^{\kappa+\omega} \{\sigma_{N-k+1}\}}{|x-z|^{(N-2+2\kappa+)}} \\ &= c_A \frac{|x'|^{\kappa+\omega} \{\sigma_{N-k+1}\}}{(|x'|^2 + |x''-z|^2)^{(N-2+2\kappa+)/2}}. \end{aligned}$$

Therefore, the Poisson potential of a measure $\mu \in \mathfrak{M}(d_A)$ is expressed by

$$(2.23) \quad \begin{aligned} \mathbb{K}_A[\mu](x) &= c_A |x'|^{\kappa+\omega} \{\sigma_{N-k+1}\} \\ &\quad \times \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(|x'|^2 + |x''-z|^2)^{(N-2+2\kappa+)/2}}. \end{aligned}$$

2.4. The admissibility condition. Consider the boundary value problem

$$(2.24) \quad \begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } D_A \\ u = \mu \in \mathfrak{M}(\partial D_A). \end{cases}$$

Let

$$(2.25) \quad \Gamma_R = \{x = (x', x'') : |x'| \leq R, |x''| \leq R\}, \quad D_{A,R} := D_A \cap \Gamma_R$$

and let $\rho_{R,A}$ denote the first (positive) eigenfunction in $D_{A,R} := D_A \cap \Gamma_R$. In the rest of this section we drop the index A in K_A , $\rho_{A,R}$ etc., except for D_A , $D_{A,R}$ and d_A .

First we observe that a positive Radon measure on d_A is q -good relative to D_A if and only if, for every compact set $F \subset d_A$, $\mu \chi_F$ is q -good in D_A .

Now suppose that μ is compactly supported in d_A and denote its support by F . We claim that μ is q -good in D_A if and only if it is q -good relative to $D_{A,R}$ for all sufficiently large R . Let R be such that $F \subset B_{R/2}^{N-k}(0)$. Assume that μ is q -good in $D_{A,R}$. Let v_R be the solution of (1.1) in $D_{A,R}$ such that $v_R = \mu$ on $d_A \cap \Gamma_R$, $v_R = 0$ on $\partial D_{A,R} \setminus d_A$. Then v_R increases with R and $v = \lim_{R \rightarrow \infty} v_R$ is a solution of (1.1) in D_A with boundary data μ . This proves our claim in one direction; the other direction is obvious.

The condition for μ to be q -admissible in $D_{A,R}$ is

$$(2.26) \quad \int_{D_{A,R}} \mathbb{K}^R[|\mu|](x)^q \rho_R(x) dx < \infty.$$

where K^R is the Martin kernel of $-\Delta$ in $D_{A,R}$. If R is sufficiently large then, in a neighborhood of F , $K^R \sim K$ and $\rho^R \sim \rho \sim v_A$. Therefore, a sufficient condition for μ to be q -good in D_A is

$$(2.27) \quad \int_{\Gamma_R \cap D_A} \mathbb{K}[|\mu|](x)^q \rho(x) dx < \infty \quad \forall R > 0.$$

By the first observation in this subsection, it follows that the previous statement remains valid for any positive Radon measure supported on d_A .

By (2.21),

$$(2.28) \quad \mathbb{K}[|\mu|](x) \leq c_A (r')^{\kappa_+} \int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z)$$

where

$$(2.29) \quad j(x) = |x|^{-N+2-2\kappa_+} \quad \forall x \in \mathbb{R}^N.$$

Therefore, using (2.20), condition (2.27) becomes

$$(2.30) \quad \int_0^R \int_{|x''| < R} \left(\int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z) \right)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty$$

for every $R > 0$.

2.5. The critical values. Relative to the equation

$$(2.31) \quad -\Delta u + |u|^{q-1} u = 0$$

there exist two thresholds of criticality associated with the edge d_A .

The first is the value q_c^* such that, for $q_c^* \leq q$ the whole edge d_A is removable but for $1 < q < q_c^*$ there exist non-trivial solutions in D_A which vanish on $\partial D_A \setminus d_A$. The second $q_c < q_c^*$ corresponds to the removability of points on d_A . For $q \geq q_c$ points on d_A are removable while for $1 < q < q_c$ there exist solutions with isolated point singularities on d_A . In the next two propositions we determine these critical values.

Proposition 2.1. *Assume $q > 1$, $1 \leq k < N$. Then the condition*

$$(2.32) \quad q < q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}$$

is necessary and sufficient for the existence of a non-trivial solution u of (2.31) in D_A which vanishes on $\partial D_A \setminus d_A$. Furthermore, when this condition holds, there exist non-trivial positive bounded measures μ on d_A such that $\mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A)$.

Remark. The statement remains true for $k = N$, which is the case of the cone. In this case $q_c = q_c^* = 1 - (2/\kappa_-)$ and a straightforward computation yields:

$$(2.33) \quad q_c = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}}.$$

Proof. Recall that $\lambda_A - (N - k)\kappa_+$ is the first eigenvalue in $S_A^{\{k-1\}}$ (see (2.15) and the remarks following it). Let κ'_+, κ'_- be the two roots of the equation

$$X^2 + (k - 2)X - (\lambda_A - (N - k)\kappa_+) = 0,$$

i.e.

$$\kappa'_\pm = \frac{1}{2}(2 - k \pm \sqrt{(k - 2)^2 + 4(\lambda_A - (N - k)\kappa_+)}).$$

Then, by [23, Theorem 5.7], recalled in subsection 2.2, if $1 < q < 1 - (2/\kappa'_-)$ there exists a unique solution of (2.31) in the cone $C_{S_A^{k-1}}$ i.e. the cone with opening $S_A^{k-1} \subset S^{k-1} \subset \mathbb{R}^k$ with trace $a\delta_0$ (where δ_0 denotes the Dirac measure at the vertex of the cone and $a > 0$). By (2.5) this solution satisfies

$$(2.34) \quad u_a(x) = a|x|^{-\alpha} \phi(x/|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0,$$

where ϕ is the first positive eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_A^{k-1})$ normalized so that u_1 possesses trace δ_0 .

The function u given by

$$\tilde{u}_a(x', x'') = u_a(x') \quad \forall (x', x'') \in D_A = C_{S_A^{k-1}} \times \mathbb{R}^{N-k},$$

is a nonzero solution of (2.31) in D_A which vanishes on $\partial D_A \setminus d_A$ and has bounded trace on d_A .

A simple calculation shows that $1 - (2/\kappa'_-)$ equals q_c^* as given in (2.32).

Next, assume that $q \geq q_c^*$ and let u be a solution of (2.31) in D_A which vanishes on $\partial D_A \setminus d_A$.

Given $\epsilon > 0$ let v_ϵ be the solution of (2.31) in $D_A^{\{N-k-1\}} \setminus \{x' \in \mathbb{R}^k : |x'| \leq \epsilon\}$ such that

$$v_\epsilon(x') = \begin{cases} 0, & \text{if } x' \in \partial D_A^{\{N-k-1\}}, |x'| > \epsilon, \\ \infty, & \text{if } |x'| = \epsilon. \end{cases}$$

Given $R > 0$ let w_R be the maximal solution in $\{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$.

Then the function u^* given by

$$u^*(x', x'') = v_\epsilon(x') + w_R(x'')$$

is a supersolution of (2.31) in $D_A \setminus \{(x', x'') : |x'| > \epsilon, |x''| < R\}$ and it dominates u in this domain. But $w_R(x'') \rightarrow 0$ as $R \rightarrow \infty$ and, by [9], $v_\epsilon(x') \rightarrow 0$ as $\epsilon \rightarrow 0$. Therefore $u_+ = 0$ and, by the same token, $u_- = 0$. \square

Proposition 2.2. *Let A be defined as before. Then*

$$(2.35) \quad \mathbb{K}[\mu] \in L_\rho^q(\Gamma_R \cap D_A) \quad \forall \mu \in \mathfrak{M}(d_A), \quad \forall R > 0$$

if and only if

$$(2.36) \quad 1 < q < q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}.$$

This statement is equivalent to the following:

Condition (2.36) is necessary and sufficient in order that the Dirac measure $\mu = \delta_P$, supported at a point $P \in d_A$, satisfy (2.35).

Proof. It is sufficient to prove the result relative to the family of measures μ such that μ is positive, has compact support and $\mu(d_A) = 1$. Let $R > 1$ be sufficiently large so that the support of μ is contained in $\Gamma_{R/2}$. The measure μ can be approximated (in the sense of weak convergence of measures) by a sequence $\{\mu_n\}$ of convex combinations of Dirac measures supported in $d_A \cap \Gamma_{R/2}$. For such a sequence $\mathbb{K}[\mu_n] \rightarrow \mathbb{K}[\mu]$ pointwise and $\{\mathbb{K}[\mu_n]\}$ is uniformly bounded in $D_A \setminus \Gamma_{3R/4}$. Therefore it is sufficient to prove the result when $\mu = \delta_0$. In this case the admissibility condition (1.5) is

$$\int_0^R \int_{|x''| < R} j(x)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty,$$

i.e.,

$$\int_0^R \int_0^R |x|^{q(2-N-2\kappa_+)} (r')^{(q+1)\kappa_+ + k - 1} (r'')^{N-k-1} dr'' dr' < \infty.$$

Substituting $\tau := r''/r'$ the condition becomes

$$\int_0^R \int_0^{R/r'} (1 + \tau^2)^{\frac{q}{2}(2-N-2\kappa_+)} (r')^{q(2-N-\kappa_+) + \kappa_+ + N - 1} \tau^{N-k-1} d\tau dr' < \infty.$$

This holds if and only if $q < (\kappa_+ + N)/(\kappa_+ + N - 2)$. \square

Remark. It is interesting to notice that k does not appear explicitly in (2.36). Furthermore, we observe that

$$(2.37) \quad \frac{2}{q_c - 1} \left(\frac{2q_c}{q_c - 1} - N \right) = \lambda_A \iff \kappa_+ (\kappa_+ + N - 2) = \lambda_A,$$

which follows from (2.11). This implies that there does not exist a nontrivial solution of the nonlinear eigenvalue problem

$$(2.38) \quad \begin{aligned} -\Delta_{S^{N-1}} \psi - \frac{2}{q-1} \left(\frac{2q}{q-1} - N \right) \psi + |\psi|^{q-1} \psi &= 0 \quad \text{in } S_{D_A} \\ \psi &= 0 \quad \text{in } \partial S_{D_A} \end{aligned}$$

which, in turn, implies that there does not exist a nontrivial solution of (2.31) of the form $u(x) = u(r, \sigma) = |x|^{-2/(q-1)} \psi(\sigma)$, and also no solution of this equation in D_A which vanishes on $\partial D_A \setminus \{0\}$. This is the classical ansatz for the removability of isolated singularities in d_A .

3. THE HARMONIC LIFTING OF A BESOV SPACE $B^{-s,p}(d_A)$.

Denote by $W^{\sigma,p}(\mathbb{R}^\ell)$ ($\sigma > 0$, $1 \leq p \leq \infty$) the Sobolev spaces over \mathbb{R}^ℓ . In order to use interpolation, it is useful to introduce the Besov space $B^{\sigma,p}(\mathbb{R}^\ell)$ ($\sigma > 0$). If σ is not an integer then

$$(3.1) \quad B^{\sigma,p}(\mathbb{R}^\ell) = W^{\sigma,p}(\mathbb{R}^\ell).$$

If σ is an integer the space is defined as follows. Put

$$\Delta_{x,y} f = f(x+y) + f(x-y) - 2f(x).$$

Then

$$(3.2) \quad B^{1,p}(\mathbb{R}^\ell) = \left\{ f \in L^p(\mathbb{R}^\ell) : \frac{\Delta_{x,y}f}{|y|^{1+\ell/p}} \in L^p(\mathbb{R}^\ell \times \mathbb{R}^\ell) \right\},$$

with norm

$$(3.3) \quad \|f\|_{B^{1,p}} = \|f\|_{L^p} + \left(\iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|\Delta_{x,y}f|^p}{|y|^{\ell+p}} dx dy \right)^{1/p},$$

(with standard modification if $p = \infty$) and

$$(3.4) \quad B^{m,p}(\mathbb{R}^\ell) = \left\{ f \in W^{m-1,p}(\mathbb{R}^\ell) : \right. \\ \left. D_x^\alpha f \in B^{1,p}(\mathbb{R}^\ell) \forall \alpha \in \mathbb{N}^\ell, |\alpha| = m-1 \right\}$$

with norm

$$(3.5) \quad \|f\|_{B^{m,p}} = \|f\|_{W^{m-1,p}} + \left(\sum_{|\alpha|=m-1} \iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|D_x^\alpha \Delta_{x,y}f|^p}{|y|^{\ell+p}} dx dy \right)^{1/p}.$$

We recall that the following inclusions hold ([26, p 155])

$$(3.6) \quad \begin{aligned} W^{m,p}(\mathbb{R}^\ell) &\subset B^{m,p}(\mathbb{R}^\ell) && \text{if } p \geq 2 \\ B^{m,p}(\mathbb{R}^\ell) &\subset W^{m,p}(\mathbb{R}^\ell) && \text{if } 1 \leq p \leq 2. \end{aligned}$$

When $1 < p < \infty$, the dual spaces of $W^{s,p}$ and $B^{m,p}$ are respectively denoted by $W^{-s,p'}$ and $B^{-m,p'}$.

The following is the main result of this section.

Theorem 3.1. *Suppose that $q_c < q < q_c^*$ and let A be defined as in subsection 2.1. Then there exist positive constants c_1, c_2 , depending on q, N, k, κ_+ , such that for any $R > 1$ and any $\mu \in \mathfrak{M}_+(d_A)$ with support in $B_{R/2}$:*

$$(3.7) \quad \begin{aligned} &c_1 \|\mu\|_{B^{-s,q}(\mathbb{R}^{N-k})}^q \\ &\leq \int_{D_{A,R}} \mathbb{K}[|\mu|^q](x) \rho(x) dx \leq c_2 (1+R)^\beta \|\mu\|_{B^{-s,q}(\mathbb{R}^{N-k})}^q, \end{aligned}$$

where $s = 2 - \frac{\kappa_+ + k}{q}$, $\beta = (q+1)\kappa_+ + k - 1$ and $D_{A,R} = D_A \cap \Gamma_R$. If $q = q_c$ the estimate remains valid for measures μ such that the diameter of $\text{supp } \mu$ is sufficiently small (depending on the parameters mentioned before).

Remark. When $q \geq 2$ the norms in the Besov space may be replaced by the norms in the corresponding Sobolev spaces.

Recall the admissibility condition for a measure $\mu \in \mathfrak{M}_+(d_A)$:

$$\int_{D_{A,R}} \mathbb{K}[\mu]^q(x) \rho(x) dx < \infty \quad \forall R > 0$$

and the equivalence (see (2.27)–(2.30))

$$(3.8) \quad \int_{D_{A,R}} \mathbb{K}[\mu]^q(x) \rho(x) dx \approx J^{A,R}(\mu) := \int_0^R \int_{B_R''} \left(\int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(\tau^2 + |x'' - z|^2)^{(N-2+2\kappa_+)/2}} \right)^q \tau^{(q+1)\kappa_+ + k - 1} dx'' d\tau,$$

where $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $\tau = |x'|$ and $B_R'' = \{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$. We denote,

$$(3.9) \quad \nu = N - 2 + 2\kappa_+.$$

If $2\kappa_+$ is an integer, it is natural to relate (3.8) to the Poisson potential of μ in $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}_{n-1}$ where $n = N - 2 + 2\kappa_+$. We clarify this statement below.

Assuming that $2 \leq n + k - N$, denote

$$y = (y_1, \tilde{y}, y'') \in \mathbb{R}^n, \quad \tilde{y} = (y_2, \dots, y_{n+k-N}), \quad y'' = (y_{n+k-N+1}, \dots, y_n).$$

The Poisson kernel in $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}_{n-1}$ is given by

$$(3.10) \quad P_n(y) = \gamma_n y_1 |y|^{-n} \quad y_1 > 0,$$

for some $\gamma_n > 0$, and the Poisson potential of a bounded Borel measure μ with support in

$$\mathbf{d} := \{y = (0, y'') \in \mathbb{R}^n : y'' \in \mathbb{R}_{N-k}\}$$

is

$$(3.11) \quad \mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2)^{n/2}} \quad \forall y \in \mathbb{R}_+^n.$$

In particular, for $\tilde{y} = 0$,

$$(3.12) \quad \mathbb{K}_n[\mu](y_1, 0, y'') = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(y_1^2 + |y'' - z|^2)^{n/2}}.$$

The integral in (3.12) is precisely the same as the inner integral in (3.8).

In fact, it will be shown that, if we set

$$(3.13) \quad n := \{\nu\} = \inf\{m \in \mathbb{N} : m \geq \nu\},$$

this approach also works when $2\kappa_+$ is not an integer. We note that, for n given by (3.13),

$$(3.14) \quad n - N + k \geq 2,$$

with equality only if $k = 3$ and $\kappa_+ \leq 1/2$ or $k = 2$ and $\kappa_+ \in (1/2, 1]$. Indeed,

$$n - N + k = k + \{2\kappa_+\} - 2$$

and (as $\kappa_+ > 0$) $\{2\kappa_+\} \geq 1$. If $k = 2$ then $\kappa_+ > 1/2$ and consequently $\{2\kappa_+\} \geq 2$. These facts imply our assertion.

We also note that κ_+ is strictly increasing relative to λ_A and

$$(3.15) \quad \kappa_+ \begin{cases} = 1, & \text{if } D_A = \mathbb{R}_+^N, \\ < 1, & \text{if } D_A \subsetneq \mathbb{R}_+^N, \\ > 1, & \text{if } D_A \supsetneq \mathbb{R}_+^N. \end{cases}$$

Finally we observe that $\gamma := \lambda_A - (N - k)\kappa_+ > 0$ (see (2.15)) and, by (2.11) and (2.32):

$$(3.16) \quad \gamma = \kappa_+^2 + (k - 2)\kappa_+, \quad q_c^* = 1 + \frac{-(k - 2) + \sqrt{(k - 2)^2 + 4\gamma}}{\gamma}.$$

Therefore q_c^* is strictly decreasing relative to γ and consequently also relative to κ_+ .

The proof of the theorem is based on the following important result proved in [27, 1.14.4.]

Proposition 3.2. *Let $1 < q < \infty$ and $s > 0$. Then for any bounded Borel measure μ in \mathbb{R}^{n-1} there holds*

$$(3.17) \quad I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|_{B^{-s,q}(\mathbb{R}^{n-1})}^q.$$

In the first part of the proof we derive inequalities comparing $I(\mu)$ and $J^{A,R}(\mu)$. Actually, it is useful to consider a slightly more general expression than $I(\mu)$, namely:

$$(3.18) \quad I_{\nu,\sigma}^{m,j}(\mu) := \int_{\mathbb{R}_+^{m+j}} \left| \int_{\mathbb{R}^m} \frac{y_1 d\mu(z)}{(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2)^{\nu/2}} \right|^q e^{-y_1} y_1^{\sigma q-1} dy,$$

where ν is an arbitrary number such that $\nu > m$, $j \geq 1$ and $\sigma > 0$. A point $y \in \mathbb{R}_+^{m+j}$ is written in the form $y = (y_1, \tilde{y}, y'') \in \mathbb{R}_+ \times \mathbb{R}^{j-1} \times \mathbb{R}^m$. We assume that μ is supported in \mathbb{R}^m . Note that,

$$(3.19) \quad I(\mu) = \gamma_n^q I_{n,s}^{m,j} \quad \text{where } m = N - k, \quad j = n - m = n - N + k.$$

Put

$$(3.20) \quad F_{\nu,m}[\mu](\tau) := \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty).$$

With this notation, if $j \geq 2$ then

$$(3.21) \quad I_{\nu,\sigma}^{m,j}(\mu) := \int_0^\infty \int_{\mathbb{R}^{j-1}} F_{\nu,m}[\mu](\sqrt{y_1^2 + |\tilde{y}|^2}) e^{-y_1} y_1^{(\sigma+1)q-1} d\tilde{y} dy_1$$

and if $j = 1$

$$(3.22) \quad I_{\nu,\sigma}^{m,1}(\mu) := \int_0^\infty F_{\nu,m}[\mu](y_1) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1$$

Lemma 3.3. *Assume that $m < \nu$, $0 < \sigma$, $2 \leq j$ and $1 < q < \infty$. Then there exists a positive constant c , depending on m, j, ν, σ, q , such that, for every bounded Borel measure μ with support in \mathbb{R}^m :*

$$(3.23) \quad \frac{1}{c} \int_0^\infty F_{\nu, m}[\mu](\tau) h_{\sigma, j}(\tau) d\tau \leq I_{\nu, \sigma}^{m, j}(\mu) \leq c \int_0^\infty F_{\nu, m}[\mu](\tau) h_{\sigma, j}(\tau) d\tau,$$

where $F_{\nu, m}$ is given by (3.20) and, for every $\tau > 0$,

$$(3.24) \quad h_{\sigma, j}(\tau) = \begin{cases} \frac{\tau^{(\sigma+1)q+j-2}}{(1+\tau)^{(\sigma+1)q}}, & \text{if } j \geq 2, \\ e^{-\tau} \tau^{(\sigma+1)q-1}, & \text{if } j = 1. \end{cases}$$

Proof. There is nothing to prove in the case $j = 1$. Therefore we assume that $j \geq 2$.

We use the notation $y = (y_1, \tilde{y}, y'') \in \mathbb{R} \times \mathbb{R}^{j-1} \times \mathbb{R}^m$. The integrand in (3.21) depends only on y_1 and $\rho := |\tilde{y}|$. Therefore, $I_{\nu, \sigma}^{m, j}$ can be written in the form

$$I_{\nu, \sigma}^{m, j}(\mu) = c_{m, j} \int_0^\infty \int_0^\infty F_{\nu, m}[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1 \rho^{j-2} d\rho.$$

We substitute $y_1 = (\tau^2 - \rho^2)^{1/2}$, then change the order of integration and finally substitute $\rho = r\tau$. This yields,

$$\begin{aligned} & c_{m, j}^{-1} I_{\nu, \sigma}^{m, j}(\mu) \\ &= \int_0^\infty \int_\rho^\infty F_{\nu, m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2 - \rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau d\tau d\rho \\ &= \int_0^\infty \int_0^\tau F_{\nu, m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2 - \rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau d\rho d\tau \\ &= \int_0^\infty \int_0^1 F_{\nu, m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr d\tau, \end{aligned}$$

where

$$f(r) = r^{j-2} (1 - r^2)^{(\sigma+1)q/2-1}.$$

We denote

$$I_\sigma^j(\tau) = \int_0^1 e^{-\tau\sqrt{1-r^2}} f(r) dr,$$

so that

$$(3.25) \quad I_{\nu, \sigma}^{m, j}(\mu) = c_{m, j} \int_0^\infty F_{\nu, m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} I_\sigma^j(\tau) d\tau.$$

To complete the proof we estimate I_σ^j . Since $j \geq 2$, $f \in L^1(0, 1)$ and I_σ^j is continuous in $[0, \infty)$ and positive everywhere. Hence, for every $\alpha > 0$, there exists a positive constant $c_\alpha = c_\alpha(\sigma)$ such that

$$(3.26) \quad \frac{1}{c_\alpha} \leq I_\sigma^j \leq c_\alpha \quad \text{in } [0, \alpha].$$

Next we estimate I_σ^j for large τ . Since $j \geq 2$,

$$I_\sigma^j \leq 2^{(\sigma+1)q/2-1} \int_0^1 (1-r)^{(\sigma+1)q/2-1} e^{-\tau\sqrt{1-r}} dr.$$

Substituting $r = 1 - t^2$ we obtain,

$$(3.27) \quad I_\sigma^j \leq 2^{(\sigma+1)q/2} \int_0^1 t^{(\sigma+1)q-1} e^{-\tau t} dt = c(\sigma, q) \tau^{-(\sigma+1)q}.$$

On the other hand, if $\tau \geq 2$,

$$(3.28) \quad \begin{aligned} I_\sigma^j(\tau) &= \int_0^1 (1-t^2)^{(j-3)/2} t^{(\sigma+1)q-1} e^{-\tau t} dt \\ &= \tau^{-(\sigma+1)q} \int_0^\tau (1-(s/\tau)^2)^{(j-3)/2} s^{(\sigma+1)q-1} e^{-s} ds \\ &\geq \tau^{-(\sigma+1)q} 2^{-(j-3)} \int_0^1 s^{(\sigma+1)q-1} e^{-s} ds. \end{aligned}$$

Combining (3.25) with (3.26)–(3.28) we obtain (3.23). \square

Next we derive an estimate in which integration over $\mathbb{R}_+^n = \mathbb{R}_+^j \times \mathbb{R}^m$ is replaced by integration over a bounded domain, for measures supported in a fixed bounded subset of \mathbb{R}^m .

Let $B_R^j(0)$ and $B_R^m(0)$ denote the balls of radius R centered at the origin, in \mathbb{R}^j and \mathbb{R}^m respectively. Denote

$$(3.29) \quad F_{\nu, m}^R[\mu](\tau) = \int_{B_R^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty)$$

and, if $j \geq 2$,

$$(3.30) \quad I_{\nu, \sigma}^{m, j}(\mu; R) = \int_{B_R^j \cap \{0 < y_1\}} F_{\nu, m}^R[\mu](\sqrt{y_1^2 + |\tilde{y}|^2}) e^{-y_1} y_1^{\sigma q - 1} d\tilde{y} dy_1.$$

where $(y_1, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^{j-1}$. If $j = 1$ we denote,

$$(3.31) \quad I_{\nu, \sigma}^{m, 1}(\mu; R) = \int_0^R F_{\nu, m}^R[\mu](y_1) e^{-y_1} y_1^{\sigma q - 1} dy_1.$$

Similarly to Lemma 3.3 we obtain,

Lemma 3.4. *If $j \geq 1$, there exists a positive constant c such that, for any bounded Borel measure μ with support in $\mathbb{R}^m \cap B_R$*

$$(3.32) \quad c^{-1} \int_0^R F_{\nu, m}^R[\mu](\tau) h_{\sigma, j}(\tau) d\tau \leq I_{\nu, \sigma}^{m, j}(\mu; R) \leq c \int_0^R F_{\nu, m}^R[\mu](\tau) h_{\sigma, j}(\tau) d\tau$$

with $h_{\sigma, j}$ as in (3.24).

Proof. In the case $j = 1$ there is nothing to prove . Therefore we assume that $j \geq 2$.

From (3.30) we obtain,

$$I_{\nu,\sigma}^{m,j}(\mu; R) = c_{m,j} \int_0^R \int_0^{\sqrt{R^2-\rho^2}} F_{\nu,m}^R[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1 \rho^{j-2} d\rho.$$

Substituting $y_1 = (\tau^2 - \rho^2)^{1/2}$, then changing the order of integration and finally substituting $\rho = r\tau$ we obtain,

$$c_{m,j}^{-1} I_{\nu,\sigma}^{m,j}(\mu; R) = \int_0^R \int_0^1 F_{\nu,\mu}^R[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr d\tau.$$

where

$$f(r) = r^{j-2}(1-r^2)^{(\sigma+1)q/2-1}.$$

The remaining part of the proof is the same as for Lemma 3.3. \square

Lemma 3.5. *Let $1 < q$, $0 < \sigma$ and assume that $m < \nu q$ and $0 \leq j-1 < \nu$. Then there exists a positive constant \bar{c} , depending on j, m, q, σ, ν , such that, for every $R \geq 1$ and every bounded Borel measure μ with support in $B_{R/2}(0) \cap \mathbb{R}^m$,*

$$(3.33) \quad \left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \bar{c} R^{(\sigma+1-\nu)q+m+j-1} \|\mu\|_{\mathfrak{M}}^q$$

with $h_{\sigma,j}$ as in (3.24).

Proof. We estimate,

$$(3.34) \quad \left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \int_R^\infty |F_{\nu,m}[\mu](\tau)| h_{\sigma,j}(\tau) d\tau + \int_0^R |F_{\nu,m}[\mu] - F_{\nu,m}^R[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau.$$

For every $\tau > 0$,

$$(3.35) \quad |F_{\nu,m}[\mu](\tau)| \leq \tau^{-\nu q} \|\mu\|_{\mathfrak{M}}^q.$$

Since $j-1 < \nu q$, it follows that

$$(3.36) \quad \begin{aligned} \int_R^\infty |F_{\nu,m}[\mu](\tau)| h_{\sigma,j}(\tau) d\tau &\leq \|\mu\|_{\mathfrak{M}}^q \int_R^\infty \tau^{-\nu q} h_{\sigma,j}(\tau) d\tau \\ &\leq c(\sigma, q) \|\mu\|_{\mathfrak{M}}^q \int_R^\infty \frac{\tau^{(\sigma+1)q+j-2-\nu q}}{(1+\tau)^{(\sigma+1)q}} d\tau \\ &\leq \frac{c(\sigma, q)}{\nu q - j + 1} \|\mu\|_{\mathfrak{M}}^q R^{j-1-\nu q}. \end{aligned}$$

Since, by assumption, $\text{supp } \mu \subset B_{R/2}$, we have

$$\begin{aligned}
& \int_0^R |F_{\nu,m}[\mu] - F_{\nu,m}^R[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau \\
& \leq \int_0^R \int_{|y''|>R} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' h_{\sigma,j}(\tau) d\tau \\
& \leq \|\mu\|_{\mathfrak{M}}^q \int_0^R \int_{|\zeta|>R/2} (|\tau^2 + |\zeta|^2)^{-\nu q/2} d\zeta h_{\sigma,j} d\tau \\
(3.37) \quad & \leq c(m, q) \|\mu\|_{\mathfrak{M}}^q \int_0^R \int_{R/2}^\infty (\tau^2 + \rho^2)^{-\nu q/2} \rho^{m-1} d\rho h_{\sigma,j} d\tau \\
& \leq c(m, q) \|\mu\|_{\mathfrak{M}}^q \int_0^R \tau^{m-\nu q} \int_{R/2\tau}^\infty (1 + \eta^2)^{-\nu q/2} \eta^{m-1} d\eta h_{\sigma,j} d\tau \\
& \leq \frac{c(m, q)}{\nu q - m} \|\mu\|_{\mathfrak{M}}^q R^{m-\nu q} \int_0^R \tau^{(\sigma+1)q+j-2} d\tau \\
& \leq \frac{c(m, q)}{(\nu q - m)((\sigma+1)q + j - 1)} \|\mu\|_{\mathfrak{M}}^q R^{(\sigma+1)q+j-1+m-\nu q}.
\end{aligned}$$

Combining (3.34)–(3.37) we obtain (3.33). \square

Corollary 3.6. *For every $R > 0$ put*

$$(3.38) \quad J_{\nu,\sigma}^{m,j}(\mu; R) := \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(\sigma+1)q+j-2} d\tau.$$

Then

$$\begin{aligned}
(3.39) \quad & \frac{1}{c} I_{\nu,\sigma}^{m,j}(\mu) - \bar{c} R^\beta \|\mu\|_{\mathfrak{M}}^q \leq J_{\nu,\sigma}^{m,j}(\mu; R) \leq c R^{(\sigma+1)q} I_{\nu,\sigma}^{m,j}(\mu), \\
& \beta = (\sigma + 1 - \nu)q + j + m - 1,
\end{aligned}$$

for every $R > 1$ and every bounded Borel measure μ with support in $B_{R/2}^m(0) := B_{R/2}(0) \cap \mathbb{R}^m$.

Proof. This is an immediate consequence of Lemma 3.5 and Lemma 3.3. \square

Lemma 3.7. *Let m, j be positive integers such that $j \geq 1$ and let $1 < q$, $0 < \sigma$. Put $n := m + j$.*

Then there exist positive constants c, \bar{c} , depending on j, m, q, σ , such that, for every $R > 1$ and every measure $\mu \in \mathfrak{M}_+(B_{R/2}^m(0))$,

$$\begin{aligned}
(3.40) \quad & \frac{1}{c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q - \bar{c} R^{q\left(\sigma - \frac{n-1}{q}\right)} \|\mu\|_{\mathfrak{M}}^q \leq J_{n,\sigma}^{m,j}(\mu; R) \\
& \leq c R^{(\sigma+1)q} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q.
\end{aligned}$$

If $\sigma < \frac{n-1}{q}$, there exists $R_0 > 1$ such that, for all $R > R_0$

$$(3.41) \quad \frac{1}{2c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q \leq J_{n,\sigma}^{m,j}(\mu; R).$$

If $\sigma = \frac{n-1}{q'}$ then, there exists $a > 0$ such that the inequality remains valid for measures μ such that $\text{diam}(\text{supp } \mu) \leq a$.

If, in addition, $\frac{j-1}{q'} < \sigma$ then

$$(3.42) \quad \frac{1}{2c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq J_{n,\sigma}^{m,j}(\mu; R) \leq cR^{(\sigma+1)q} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q,$$

where $s := \sigma - \frac{j-1}{q'}$.

Remark. Assume that $\mu \geq 0$. Then:

(i) If $\mu \in B^{-\sigma,q}(\mathbb{R}^{n-1})$ and $\frac{j-1}{q'} \geq \sigma$ then $\mu(\mathbb{R}^m) = 0$.

(ii) If $\mu \in B^{-s,q}(\mathbb{R}^m)$ and $\sigma > (n-1)/q'$ then $s > m/q'$ and therefore $B^{s,q'}(\mathbb{R}^m)$ can be embedded in $C(\mathbb{R}^m)$.

Proof. Inequality (3.40) follows from (3.39) and Proposition 3.2 (see also (3.19)).

For positive measures μ ,

$$\|\mu\|_{\mathfrak{M}} = \mu(\mathbb{R}^{n-1}) \leq \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q.$$

Therefore, if $\sigma < \frac{n-1}{q'}$, (3.40) implies that there exists $R_0 > 1$ such that (3.41) holds for all $R > R_0$.

If $\sigma = \frac{n-1}{q'}$ (3.40) implies that

$$\frac{1}{c} \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q - \bar{c} \|\mu\|_{\mathfrak{M}}^q \leq J_{n,\sigma}^{m,j}(\mu; R).$$

But if μ is a positive bounded measure such that $\text{diam}(\text{supp } \mu) \leq a$ then

$$\|\mu\|_{\mathfrak{M}} / \|\mu\|_{B^{-\sigma,q}(\mathbb{R}^{n-1})}^q \rightarrow 0 \quad \text{as } a \rightarrow 0.$$

The last inequality follows from the imbedding theorem for Besov spaces according to which there exists a continuous trace operator $T : B^{\sigma,q'}(\mathbb{R}^{n-1}) \mapsto B^{s,q'}(\mathbb{R}^m)$ and a continuous lifting $T' : B^{s,q'}(\mathbb{R}^m) \mapsto B^{\sigma,q'}(\mathbb{R}^{n-1})$ where $s = \sigma - \frac{n-m-1}{q'}$. \square

If $\nu \in \mathbb{N}$ and $\sigma = s + \frac{\nu-m-1}{q'}$,

$$\begin{aligned} J_{\nu,\sigma}^{m,\nu-m}(\mu; R) &= \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(\sigma+1)q+\nu-m-2} d\tau \\ &= \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(s+\nu-m)q-1} d\tau. \end{aligned}$$

However, if μ is positive, the expression

$$(3.43) \quad M_{\nu,s}^m(\mu; R) := \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(s+\nu-m)q-1} d\tau,$$

is meaningful for any real $\nu > m$ and $s > 0$. Furthermore, as shown below, the results stated in Lemma 3.7 can be extended to this general case.

Theorem 3.8. *Let $1 < q$, $\nu \in \mathbb{R}$ and m a positive integer. Assume that $1 \leq \nu - m$ and $0 < s < m/q'$. Then there exists a positive constant c such that, for every bounded positive measure μ supported in $\mathbb{R}^m \cap B_{R/2}(0)$, $R > 1$,*

$$(3.44) \quad \frac{1}{c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq M_{\nu,s}^m(\mu; R) \leq cR^{(s+\nu-m)q+1} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q.$$

This also holds when $s = m/q'$, provided that the diameter of $\text{supp } \mu$ is sufficiently small.

Proof. If ν is an integer and $j := \nu - m$ then this statement is part of Lemma 3.7. Indeed the condition $s > 0$ means that $\sigma = s + \frac{j-1}{q'} > \frac{j-1}{q'}$ and the condition $s < m/q'$ means that $\sigma < \frac{n-1}{q'}$.

Therefore we assume that $\nu \notin \mathbb{N}$. Let $n := \{\nu\}$ and $\theta := n - \nu$ so that $0 < \theta < 1$. Our assumptions imply that $1 \leq n - m - 1$ because (as ν is not an integer) $\nu - m > 1$ and consequently $n - m \geq 2$.

If a, b are positive numbers, put

$$A_\nu := \frac{a^{(s+\nu-m)q-1}}{(a^2 + b^2)^{\nu q/2}}.$$

Obviously A_ν decreases as ν increases. Therefore, $A_n \leq A_\nu \leq A_{n-1}$ which in turn implies,

$$M_{n,s}^m \leq M_{\nu,s}^m \leq M_{n-1,s}^m.$$

By Lemma 3.7, the assertions of the theorem are valid in the case that $\nu = n$ or $\nu = n - 1$. Therefore the previous inequality implies that the assertions hold for any real ν subject to the conditions imposed. \square

By (3.8),

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau) \tau^{(q+1)\kappa_+ + k - 1} d\tau,$$

where $m = N - k$ and $\nu = N - 2 + 2\kappa_+$. Consequently, by (3.38),

$$J^{A,R} = M_{\nu,s}^m$$

where s is determined by,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

It follows that

$$sq = -(k - 2 + 2\kappa_+)q + (q + 1)\kappa_+ + k = k(1 - q) + 2q - \kappa_+(q - 1)$$

and therefore

$$s = 2 - \frac{k + \kappa_+}{q'}.$$

Proof of Theorem 3.1.

Put

$$(3.45) \quad \nu := N - 2 + 2\kappa_+, \quad s := 2 - \frac{\kappa_+ + k}{q'}, \quad m := N - k.$$

Recall that in the case $k = 2$ we have $\kappa_+ > 1/2$. Therefore

$$(3.46) \quad \nu - m - 1 = k - 3 + 2\kappa_+ > 0.$$

Furthermore,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

Thus

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau) \tau^{(q+1)\kappa_+ + k - 1} d\tau = M_{\nu,s}^m.$$

Next we show that $0 < s \leq m/q'$. More precisely we prove

$$(3.47) \quad 0 < s \leq m/q' \iff q_c \leq q < q_c^*.$$

Let μ be a bounded non-negative Borel measure in $B^{-s,q}(\mathbb{R}^m)$. If $s \leq 0$, $B^{-s,q}(\mathbb{R}^m) \subset L^q(\mathbb{R}^m)$. Therefore, in this case, every bounded Borel measure on \mathbb{R}^m is admissible i.e. satisfies (2.35). Consequently, by Proposition 2.2, $q < q_c$. As we assume $q \geq q_c$ it follows that $s > 0$.

If, $s > 0$ and $sq' - m \geq 0$ then $C_{s,q'}(K) = 0$ for every compact subset of \mathbb{R}^m and consequently $\mu(K) = 0$ for any such set. Conversely, if $sq' - m < 0$ then there exist non-trivial positive bounded measures in $B^{-s,q}(\mathbb{R}^m)$. Therefore, by Proposition 2.1, $sq' < m$ if and only if $q < q_c^*$.

In conclusion, $0 < s \leq m/q'$ and $\nu - m \geq 1$; therefore Theorem 3.1 is a consequence of Theorem 3.8. \square

Remark. Note that the critical exponent for the imbedding of $B^{2 - \frac{\kappa_+ + k}{q'}, q'}(\mathbb{R}^{N-k})$ into $C(\mathbb{R}^{N-k})$ is again

$$q = q_c = \frac{N + \kappa_+}{N + \kappa_+ - 2}.$$

4. SUPERCRITICAL EQUATIONS IN A POLYHEDRAL DOMAIN

In this section q is a real number larger than 1 and P an N -dim polyhedral domain as described in subsection 6.1. Denote by $\{L_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$ the family of faces, edges and vertices of P . In this notation, $L_{1,j}$ denotes one of the open faces of P ; for $k = 2, \dots, N - 1$, $L_{k,j}$ denotes a relatively open $N - k$ -dimensional edge and $L_{N,j}$ denotes a vertex. For $1 \leq k < N$, the $(N - k)$ dimensional space which contains $L_{k,j}$ is denoted by \mathbb{R}_j^{N-k} . If $1 < k < N$, the cylinder of radius r around the axis \mathbb{R}_j^{N-k} will be denoted by $\Gamma_{k,j,r}^\infty$ and the subset $A_{k,j}$ of S^{k-1} is defined by

$$\lim_{r \rightarrow 0} \frac{1}{r} (\partial \Gamma_{k,j,r}^\infty \cap P) = L_{k,j} \times A_{k,j}.$$

$A_{k,j}$ is the 'opening' of P at the edge $L_{k,j}$. For $k = N$ we replace in this definition the cylinder $\Gamma_{N,j,r}^\infty$ by the ball $B_r(L_{N,j})$. For $1 < k \leq N$ and $A = A_{k,j}$ we use d_A as an alternative notation for \mathbb{R}_j^{N-k} and denote by D_A the k -dihedron with edge d_A and opening A as in subsection 6.1 (with S_A defined as in (2.2)). For $k = 1$, D_A stands for the half space $\mathbb{R}_j^{N-1} \times (0, \infty)$.

4.1. Definitions and auxiliary results. Let Ω be a bounded Lipschitz domain. We say that $\{\Omega_n\}$ is a *Lipschitz exhaustion* of Ω if, for every n , Ω_n is Lipschitz and

$$(4.1) \quad \Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}, \quad \Omega = \cup \Omega_n, \quad \mathbb{H}_{N-1}(\partial\Omega_n) \rightarrow \mathbb{H}_{N-1}(\partial\Omega).$$

If ω_n (respectively ω) is the harmonic measure in Ω_n (respectively Ω) relative to $x_0 \in \Omega_1$, then, for every $Z \in C(\bar{\Omega})$,

$$(4.2) \quad \lim_{n \rightarrow \infty} \int_{\partial\Omega_n} Z d\omega_n = \int_{\partial\Omega} Z d\omega.$$

[23, Lemma 2.1]. Furthermore, if μ is a bounded Borel measure on $\partial\Omega$ and $v := \mathbb{K}^\Omega[\mu]$, there holds

$$(4.3) \quad \lim_{n \rightarrow \infty} \int_{\partial\Omega_n} Z v d\omega_n = \int_{\partial\Omega} Z d\mu,$$

[23, Lemma 2.2]. If v is a positive solution and (4.3) holds we say that μ is the *boundary trace* of v .

The following estimates are proved in [23, Lemma 2.3]

Proposition 4.1. *Let μ be bounded Borel measures on $\partial\Omega$. Then $\mathbb{K}[\mu] \in L^1_\rho(\Omega)$ and there exists a constant $C = C(\Omega)$ such that*

$$(4.4) \quad \|\mathbb{K}[\mu]\|_{L^1_\rho(\Omega)} \leq C \|\mu\|_{\mathfrak{M}(\partial\Omega)}.$$

In particular if $h \in L^1(\partial\Omega; \omega)$ then

$$(4.5) \quad \|\mathbb{P}[h]\|_{L^1_\rho(\Omega)} \leq C \|h\|_{L^1(\partial\Omega; \omega)}.$$

The next result will be used in deriving estimates in a k -dimensional dihedron when the boundary data is concentrated on the edge.

Proposition 4.2. *We denote by G^{Ω_n} (respectively G^Ω) the Green function in Ω_n (respectively Ω). Let v be a positive harmonic function in Ω with boundary trace μ . Let $Z \in C^2(\bar{\Omega})$ and let $\tilde{G} \in C^\infty(\Omega)$ be a function that coincides with $x \mapsto G(x, x_0)$ in $Q \cap \Omega$ for some neighborhood Q of $\partial\Omega$ and some fixed $x_0 \in \Omega$. In addition assume that there exists a constant $c > 0$ such that*

$$(4.6) \quad |\nabla Z \cdot \nabla \tilde{G}| \leq c\rho.$$

Under these assumptions, if $\zeta := Z\tilde{G}$ then

$$(4.7) \quad - \int_{\Omega} v \Delta \zeta dx = \int_{\partial\Omega} Z d\mu.$$

Proof. Let $\{\Omega_n\}$ be a C^1 exhaustion of Ω . We assume that $\partial\Omega_n \subset Q$ for all n and $x_0 \in \Omega_1$. Let $\tilde{G}_n(x)$ be a function in $C^1(\Omega_n)$ such that \tilde{G}_n coincides with $G^{\Omega_n}(\cdot, x_0)$ in $Q \cap \Omega_n$, $\tilde{G}_n(\cdot, x_0) \rightarrow \tilde{G}(\cdot, x_0)$ in $C^2(\Omega \setminus Q)$ and $\tilde{G}_n(\cdot, x_0) \rightarrow \tilde{G}(\cdot, x_0)$ in $\text{Lip}(\Omega)$. If $\zeta_n = Z\tilde{G}_n$ we have,

$$\begin{aligned} - \int_{\Omega_n} v \Delta \zeta_n dx &= \int_{\partial\Omega_n} v \partial_{\mathbf{n}} \zeta dS = \int_{\partial\Omega_n} v Z \partial_{\mathbf{n}} \tilde{G}_n(\xi, x_0) dS \\ &= \int_{\partial\Omega_n} v Z P^{\Omega_n}(x_0, \xi) dS = \int_{\partial\Omega_n} v Z d\omega_n. \end{aligned}$$

By (4.3),

$$\int_{\partial\Omega_n} v Z d\omega_n \rightarrow \int_{\partial\Omega} Z d\mu.$$

On the other hand, in view of (4.6), we have

$$\Delta \zeta_n = \tilde{G}_n \Delta Z + Z \Delta \tilde{G}_n + 2 \nabla Z \cdot \nabla \tilde{G}_n \rightarrow \Delta Z$$

in $L^1_\rho(\Omega)$; therefore,

$$- \int_{\Omega_n} v \Delta \zeta_n dx \rightarrow - \int_{\Omega} v \Delta \zeta dx.$$

□

We denote by $\mathfrak{M}_q = \mathfrak{M}_q(\partial\Omega)$ the set of q-good measures on the boundary. A positive solution u of (1.1) in Ω possesses a boundary trace $\mu \in \mathfrak{M}(\partial\Omega)$ if and only if

$$(4.8) \quad \int_{\Omega} u^q \rho dx < \infty$$

[23, Proposition 4.1]. In this case $\mu \in \mathfrak{M}_q$.

The following statements can be proved in the same way as in the case of smooth domains. For the proof in that case see [19].

I. $\mathfrak{M}_q(\partial\Omega)$ is a linear space and

$$\mu \in \mathfrak{M}_q(\partial\Omega) \iff |\mu| \in \mathfrak{M}_q(\partial\Omega).$$

II. If $\{\mu_n\}$ is an increasing sequence of measures in $\mathfrak{M}_q(\partial\Omega)$ and $\mu := \lim \mu_n$ is a finite measure then $\mu \in \mathfrak{M}_q(\partial\Omega)$.

Proposition 4.3. *Let μ be a bounded measure on ∂P . (μ may be a signed measure.) For $i = 1, \dots, N$, $j = 1, \dots, n_i$, we define the measure $\mu_{k,j}$ on $d_{A_{k,j}}$ by,*

$$\mu_{k,j} = \mu \text{ on } L_{k,j}, \quad \mu_{k,j} = 0 \text{ on } d_{A_{k,j}} \setminus L_{k,j} \text{ .}$$

Then $\mu \in \mathfrak{M}_q(\partial P)$, i.e., problem

$$(4.9) \quad -\Delta u + u^q = 0 \text{ in } P, \quad u = \mu \text{ on } \partial P$$

possesses a solution, if and only if, $\mu_{k,j}$ is a q-good measure relative to $D_{A_{k,j}}$ for all (k, j) as above.

Proof. In view of statement **I** above, it is sufficient to prove the proposition in the case that μ is non-negative. This is assumed hereafter. If $\mu \in \mathfrak{M}_q(\partial P)$ then any measure ν on ∂P such that $0 \leq \nu \leq \mu$ is a q-good measure relative to P . Therefore

$$\mu \in \mathfrak{M}_q(\partial P) \implies \mu'_{k,j} := \mu \chi_{L_{k,j}} \in \mathfrak{M}_q(\partial P).$$

Assume that $\mu \in \mathfrak{M}_q(\partial P)$ and let $u_{k,j}$ be the solution of (4.9) when μ is replaced by $\mu'_{k,j}$. Denote by $u'_{k,j}$ the extension of $u_{k,j}$ by zero to the k -dihedron $D_{A_{k,j}}$. Then $u'_{k,j}$ is a subsolution of (1.1) in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$. In the present case there always exists a supersolution, e.g. the maximal solution of (1.1) in $D_{A_{k,j}}$ vanishing outside $d_{A_{k,j}} \setminus \bar{L}_{k,j}$. Therefore there exists a solution $v_{k,j}$ of this equation in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$, i.e., $\mu_{k,j}$ is q -good relative to $D_{A_{k,j}}$.

Next assume that $\mu \in \mathfrak{M}(\partial P)$ and that $\mu_{k,j}$ is q -good relative to $D_{A_{k,j}}$ for every (k, j) as above. Let $v_{k,j}$ be the solution of (1.1) in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$. Then $v_{k,j}$ is a supersolution of problem (4.9) with μ replaced by $\mu'_{k,j}$ and consequently there exists a solution $u_{k,j}$ of this problem. It follows that

$$w := \max\{u_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$$

is a subsolution while

$$\bar{w} := \sum_{k=1, \dots, N, j=1, \dots, n_k} u_{k,j}$$

is a supersolution of (4.9). Consequently there exists a solution of this problem, i.e., $\mu \in \mathfrak{M}_q(\partial P)$. \square

4.2. Removable singular sets and 'good measures', I.

Proposition 4.4. *Let A be a Lipschitz domain on S^{k-1} , $2 \leq k \leq N-1$, and let D_A be the k -dihedron with opening A . Let $\mu \in \mathfrak{M}(\partial D_A)$ be a positive measure with compact support contained in d_A (= the edge of D_A). Assume that μ is q -good relative to D_A . Let $R > 1$ be large enough so that $\text{supp } \mu \subset B_R^{N-k}(0)$ and let u be the solution of (1.1) in D_A^R with trace μ on d_A^R and trace zero on $\partial D_A^R \setminus d_A^R$. Then:*

(i) For every non-negative $\eta \in C_0^\infty(B_{3R/4}^{N-k}(0))$,

$$(4.10) \quad \left(\int_{d_A^R} \eta^{q'} d\mu \right) \leq cM^{q'} \int_{D_A^R} u^q \rho dx + cM^{q'} \left(\int_{D_A^R} u^q \rho dx \right)^{\frac{1}{q}} \left(1 + M^{-1} \|\eta\|_{L^{q'}(d_A^R)} \right).$$

where $M = \|\eta\|_{L^\infty}$ and ρ is the first eigenfunction of $-\Delta$ in D_A^R normalized by $\rho(x_0) = 1$ at some point $x_0 \in D_A^R$. The constant c depends only on $N, q, k, x_0, \lambda_1, R$ where λ_1 is the first eigenvalue.

(ii) For any compact set $E \subset d_A$,

$$(4.11) \quad C_{s,q}^{N-k}(E) = 0 \implies \mu(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'},$$

where $C_{s,q}^{N-k}$ denotes the Bessel capacity with the indicated indices in \mathbb{R}^{N-k} .

Remark. If we replace D_A^R by $D_A \cap B_{\tilde{R}}^k(0) \cap B_{\tilde{R}}^{N-k}(0)$, $\tilde{R} > 1$, then the constant c in (i) depends on \tilde{R} but *not on* R .

Proof. We identify d_A with \mathbb{R}^{N-k} and use the notation

$$x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \quad y = |x'|.$$

Let $\eta \in C_0^\infty(\mathbb{R}^{N-k})$ and let R be large enough so that $\text{supp } \eta \subset B_{R/2}^{N-k}(0)$. Let $w = w_R(t, x'')$ be the solution of the following problem in $\mathbb{R}_+ \times B_R^{N-k}(0)$:

$$(4.12) \quad \begin{aligned} \partial_t w - \Delta_{x''} w &= 0 && \text{in } \mathbb{R}_+ \times B_R^{N-k}(0), \\ w(0, x'') &= \eta(x'') && \text{in } B_R^{N-k}, \\ w(t, x'') &= 0 && \text{on } \partial B_R^{N-k}(0). \end{aligned}$$

Thus $w_R(t, \cdot) = S_R(t)[\eta]$ where $S_R(t)$ is the semi-group operator corresponding to the above problem. Denote,

$$(4.13) \quad H_R[\eta](x', x'') = w_R(|x'|^2, x'') = S_R(y^2)[\eta](x''), \quad y := |x'|.$$

We assume, as we may, that $R > 1$. Let ρ^R be the first eigenfunction of $-\Delta_{x''}$ in the ball $B_R^{N-k}(0)$ normalized by $\rho^R(0) = 1$ and let ρ_A be the first eigenfunction of $-\Delta_{x'}$ in C_A (where C_A denotes the cone with opening A in \mathbb{R}^k) normalized so that $\rho_A(x'_0) = 1$ at some point $x'_0 \in S_A$. Then $\rho^R \rho_A$ is the first eigenfunction of $-\Delta$ in $\{x \in D_A : |x''| < R\}$. Note that $\rho^R \leq 1$ and $\rho^R \rightarrow 1$ as $R \rightarrow \infty$ in $C^2(I)$ for any bounded set $I \subset \mathbb{R}^{N-k}$.

Let $h \in C^\infty(\mathbb{R})$ be a monotone decreasing function such that $h(t) = 1$ for $t < 1/2$ and $h(t) = 0$ for $t > 3/4$. Put

$$\psi_R(x') = h(|x'|/R)$$

and

$$(4.14) \quad \zeta_R := \rho_A \psi_R H_R[\eta]^{q'}.$$

If ρ_A^R is the first eigenfunction (normalized at x_0) of $D_A^R := D_A \cap \Gamma_R$ (Γ_R as in (2.25)) then

$$(4.15) \quad \rho_A \psi_R \leq c \rho_A^R$$

and $\rho^R \rho_A^R$ is the first eigenfunction in D_A^R .

Hereafter we shall drop the index R in ζ_R, H_R, w_R but keep it in the other notations in order to avoid confusion.

We shall verify that $\zeta \in D_A^R$. To this purpose we compute,

$$(4.16) \quad \begin{aligned} \Delta \zeta &= -\lambda_1 (\rho_A \psi_R) H[\eta]^{q'} + (\rho_A \psi_R) \Delta H[\eta]^{q'} + 2\nabla(\rho_A \psi_R) \cdot \nabla H[\eta]^{q'} \\ &= -\lambda_1 \zeta + q' (\rho_A \psi_R) (H[\eta])^{q'-1} \Delta H[\eta] \\ &\quad + q'(q' - 1) (\rho_A \psi_R) (H[\eta])^{q'-2} |\nabla H[\eta]|^2 \\ &\quad + 2q' (H[\eta])^{q'-1} \nabla(\rho_A \psi_R) \cdot \nabla H[\eta]. \end{aligned}$$

In addition,

$$\begin{aligned}\nabla H[\eta] &= \nabla_{x'} H[\eta] + \nabla_{x''} H[\eta] = \partial_y H[\eta] \frac{x'}{y} + \nabla_{x''} H[\eta] \\ &= 2y \partial_t w(y^2, x'') \frac{x'}{y} + \nabla_{x''} H[\eta](x', x'')\end{aligned}$$

and consequently (recall that y stands for $|x'|$),

$$\begin{aligned}\nabla H[\eta] \cdot \nabla(\rho_A \psi_R) &= 2\partial_t w(y^2, x'') x' \cdot \left(\psi_R(|x'|^{\kappa_+ - 1} (\kappa_+ \frac{x'}{y} \omega_k(x'/y) + |x'| \nabla \omega_k(x'/y))) + \rho_A \nabla \psi_R \right) \\ &= 2\kappa_+ \partial_t w(y^2, x'') |x'|^{\kappa_+} \omega_k(x'/y) = 2\partial_t w(y^2, x'') (\kappa_+ \rho_A \psi_R + \rho_A x' \cdot \nabla \psi_R).\end{aligned}$$

Since $w = w_R$ vanishes for $|x''| = R$ and $\eta = 0$ in a neighborhood of this sphere, $|\partial_t w(y^2, x'')| \leq c\rho^R$. As ψ_R vanishes for $|x'| > 3R/4$ we have $\rho_A \nabla \psi_R \leq c\rho_A^R$. Therefore

$$|\nabla H[\eta] \cdot \nabla \rho_A| \leq c\rho^R \rho_A^R$$

and, in view of (4.16),

$$(4.17) \quad |\Delta \zeta| \leq c\rho^R \rho_A^R.$$

Thus $\zeta \in X(D_A^R)$ and consequently

$$(4.18) \quad \int_{D_A^R} (-u\Delta\zeta + u^q\zeta) dx = - \int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta dx.$$

Since $q(q' - 1)\rho_A(H[\eta])^{q' - 2} |\nabla H[\eta]|^2 \geq 0$, we have

$$\begin{aligned}(4.19) \quad & \left| \int_{D_A^R} u\Delta\zeta dx \right| \\ & \leq \int_{D_A^R} u \left(\lambda_1 \zeta + q'(H[\eta])^{q' - 1} (\rho|\Delta H[\eta]| + 2|\nabla \rho \cdot \nabla H[\eta]|) \right) dx \\ & \leq \int_{D_A^R} u \left(\lambda_1 \zeta + q'\zeta^{1/q} \left(\rho^{1/q'} |\Delta H[\eta]| + 2\rho^{-1/q'} |\nabla \rho \cdot \nabla H[\eta]| \right) \right) dx \\ & \leq \left(\int_{D_A^R} u^q \zeta dx \right)^{\frac{1}{q}} \left(\lambda_1 \left(\int_{D_A^R} \zeta dx \right)^{\frac{1}{q'}} + q' \|L[\eta]\|_{L^{q'}(D_A^R)} \right)\end{aligned}$$

where

$$(4.20) \quad L[\eta] = \rho^{1/q'} |\Delta H[\eta]| + 2\rho^{-1/q'} |\nabla \rho \cdot \nabla H[\eta]|.$$

By Proposition 4.2

$$(4.21) \quad - \int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta dx = \int_{d_A^R} \eta^{q'} d\mu.$$

Therefore

$$(4.22) \quad \left(\int_{D_A^R} \eta^{q'} d\mu \right) \leq \int_{D_A^R} u^q \zeta dx + \left(\int_{D_A^R} u^q \zeta dx \right)^{\frac{1}{q}} \left(\lambda_1 \left(\int_{D_A^R} \zeta dx \right)^{\frac{1}{q'}} + q' \|L[\eta]\|_{L^{q'}(D_A^R)} \right).$$

Next we prove that

$$(4.23) \quad \|L[\eta]\|_{L^{q'}(D_A^R)} \leq C \|\eta\|_{W^{s,q'}(\mathbb{R}^{N-k})}$$

starting with the estimate of the first term on the right hand side of (4.20).

$$\begin{aligned} \Delta H[\eta] &= \Delta_{x'} H[\eta] + \Delta_{x''} H[\eta] = \partial_y^2 H[\eta] + \frac{k-1}{y} \partial_y H[\eta] + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w(y^2, x'') + k \partial_t w(y^2, x'') + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w(y^2, x'') + (k+1) \partial_t w(y^2, x''). \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^N} \rho |\Delta H[\eta]|^{q'} dx &\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_{tt} w(y^2, x'')|^{q'} dx'' y^{\kappa_+ + 2q' + k - 1} dy \\ &\quad + c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(y^2, x'')|^{q'} dx'' y^{\kappa_+ + k - 1} dy \\ &\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_{tt} w(t, x'')|^{q'} dx'' t^{(\kappa_+ + k)/2 + q'} \frac{dt}{t} \\ &\quad + c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(t, x'')|^{q'} dx'' t^{(\kappa_+ + k)/2} \frac{dt}{t} \\ &\leq c \int_0^1 \left\| t^{2 - (1 - \frac{\kappa_+ + k}{2q'})} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ &\quad + c \int_0^1 \left\| t^{1 - (1 - \frac{\kappa_+ + k}{2q'})} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t}. \end{aligned}$$

Put $\beta = \frac{\kappa_+ + k}{2q'}$ and note that $0 < \beta = \frac{1}{2}(2-s) < 1$. By standard interpolation theory,

$$\begin{aligned} &\int_0^1 \left\| t^{1 - (1 - \beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ &\approx \|\eta\|_{[W^{2,q'}, L^{q'}]_{1-\beta, q'}}^{q'} \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}, \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \left\| t^{2-(1-\beta)} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ & \approx \|\eta\|_{[W^{4,q'}, L^{q'}]_{\frac{1}{2}(1-\beta), q'}}^{q'} \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}. \end{aligned}$$

The second term on the right hand side of (4.20) is estimated in a similar way:

$$\begin{aligned} & \int_{\mathbb{R}^N} \rho^{-q'/q} |\nabla H[\eta] \cdot \nabla \rho|^{q'} dx \leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(y^2, x'')|^{q'} dx' y^{\kappa_+ + k - 1} dy \\ & \leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(t, x'')|^{q'} dx' t^{\frac{\kappa_+ + k}{2}} \frac{dt}{t} \\ & \leq c \int_0^1 \left\| t^{1-(\frac{1}{2}-\beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ & \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}. \end{aligned}$$

This proves (4.23). Further, (4.22) and (4.23) imply (4.10).

We turn to the proof of part (ii). Let E be a closed subset of $B_{R/2}^{N-k}(0)$ such that $C_{s, q'}^{N-k}(E) = 0$. Then there exists a sequence $\{\eta_n\}$ in $C_0^\infty(d_A)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of E (which may depend on n), $\text{supp } \eta_n \subset B_{3R/4}^{N-k}(0)$ and $\|\eta_n\|_{W^{s, q'}} \rightarrow 0$. Then, by (4.23),

$$\|L[\eta_n]\|_{L^{q'}(D_A^R)} \rightarrow 0.$$

Furthermore

$$\|w\|_{L^{q'}((0, R) \times B_R^{N-k}(0))} \leq c \|\eta_n\|_{L^{q'}(B_R^{N-k}(0))}$$

and consequently

$$H[\eta_n] \rightarrow 0 \text{ in } L^{q'}(D_A^R).$$

(Here we use the fact that $k \geq 2$.) In addition

$$0 \leq H[\eta_n] \leq 1, \quad H[\eta_n] \leq c(R - |x'|)$$

with a constant c independent of n . Hence (see (4.15))

$$\zeta_{n, R} := \rho_A \psi_R H[\eta_n]^{q'} \leq \rho^R \rho_A \psi_R H[\eta_n]^{q'-1} \leq \rho^R \rho_A^R H[\eta_n]^{q'-1}.$$

As $u^q \rho^R \rho_A^R \in L^1(D_A^R)$ we obtain,

$$\lim_{n \rightarrow \infty} \int_{D_A} u^q \zeta_n dx = 0.$$

This fact and (4.22) imply that

$$\int_{d_A^R} \eta_n^{q'} d\mu \rightarrow 0.$$

As $\eta_n = 1$ on a neighborhood of E in \mathbb{R}^{N-k} it follows that $\mu(E) = 0$. \square

Proposition 4.5. *Let D_A be a k -dihedron, $1 \leq k < N$. Let k_+ be as in (2.11) and let q_c^* and q_c be as in Proposition 2.1 and Proposition 2.2 respectively. Assume that $q_c \leq q < q_c^*$. A measure $\mu \in \mathfrak{M}(\partial D_A)$, with compact support contained in d_A , is q -good relative to D_A if and only if μ vanishes on every Borel set $E \subset d_A$ such that $C_{s,q'}(E) = 0$, where $s = 2 - \frac{k+\kappa_+}{q'}$.*

Remark. We shall use the notation $\mu \prec C_{s,q'}$ to say that μ vanishes on any Borel set $E \subset (d_A)$ such that $C_{s,q'}(E) = 0$.

In the case $k = N$: $D_A = C_A$ (= the cone with vertex 0 and opening A in \mathbb{R}^k) and $q_c = q_c^*$. By [23] (specifically the results quoted in subsection 2.2) $q_c = 1 - \frac{2}{\kappa_-} = \frac{N+\kappa_+}{N+\kappa_+-2}$ and if $1 < q < q_c$ then there exist solutions for every measure $\mu = k\delta_P$, $P \in d_A$.

In the case $k = 1$, $q_c^* = \infty$, $\kappa_+ = 1$ and $q_c = \frac{N+1}{N-1}$. Thus $s = 2/q$ and the statement of the theorem is well known (see [20]).

Proof. In view of the last remark, it remains to deal only with $2 \leq k \leq N-1$. We shall identify d_A with \mathbb{R}^{N-k} .

It is sufficient to prove the result for positive measures because $\mu \prec C_{s,q'}$ if and only if $|\mu| \prec C_{s,q'}$. In addition, if $|\mu|$ is a q -good measure then μ is a q -good measure.

First we show that if μ is non-negative and q -good then $\mu \prec C_{s,q'}$. If E is a Borel subset of $\partial\Omega$ then $\mu\chi_E$ is q -good. If E is compact and $C_{s,q'}(E) = 0$ then, by Proposition 4.4, E is a removable set. This means that the only positive solution of (1.1) in D_A such that $\mu(\partial\Omega \setminus E) = 0$ is the zero solution. This implies that $\mu\chi_E = 0$, i.e., $\mu(E) = 0$. If $C_{s,q'}(E) = 0$ but E is not compact then $\mu(E') = 0$ for every compact set $E' \subset E$. Therefore, we conclude again that $\mu(E) = 0$.

Next, assume that μ is a positive measure in $\mathfrak{M}(\partial D_A)$ supported in a compact subset of \mathbb{R}^{N-k} .

If $\mu \in B^{-s,q}(\mathbb{R}^{N-k})$ then, by Theorem 3.1, μ is admissible relative to $D_A \cap \Gamma_{k,R}$, for every $R > 0$. (As before $\Gamma_{k,R}$ is the cylinder with radius R around the 'axis' \mathbb{R}^{N-k} .) This implies that μ is q -good relative to D_A .

If $\mu \prec C_{s,q'}$ then, by a theorem of Feyel and de la Pradelle [10] (see also [3]), there exists a sequence $\{\mu_n\} \subset (B^{-s,q}(\mathbb{R}^{N-k}))_+$ such that $\mu_n \uparrow \mu$. As μ_n is q -good, it follows that μ is q -good. \square

Theorem 4.6. *Let P be an N -dimensional polyhedron as described in Proposition 4.3. Let μ be a bounded measure on ∂P , (may be a signed measure). Let $k = 1, \dots, N$, $j = 1, \dots, n_k$, and let $L_{k,j}$ and $A_{k,j}$ be defined as at the beginning of this section. Further, put*

$$(4.24) \quad s(k, j) = 2 - \frac{k + (\kappa_+)_{k,j}}{q'},$$

where $(\kappa_+)_{k,j}$ is defined as in (2.11) with $A = A_{k,j}$. Then $\mu \in \mathfrak{M}_q(\partial P)$, i.e., μ is a good measure for (1.1) relative to P , if and only if, for every pair

(k, j) as above and every Borel set $E \subset L_{k,j}$:

If $1 \leq k < N$ then

$$(4.25) \quad \begin{aligned} (q_c)_{k,j} \leq q < (q_c^*)_{k,j}, \quad C_{s(k,j),q'}^{N-k}(E) = 0 &\implies \mu(E) = 0 \\ q \geq (q_c^*)_{k,j} &\implies \mu(L_{N,j}) = 0 \end{aligned}$$

and if $k = N$, i.e., L is a vertex,

$$(4.26) \quad q \geq (q_c)_{k,j} = \frac{N+2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N-2 + \sqrt{(N-2)^2 + 4\lambda_A}} \implies \mu(L) = 0.$$

Here $(q_c^*)_{k,j}$ and $(q_c)_{k,j}$ are defined as in (2.32) and (2.36) respectively, with $A = A_{k,j}$.

If $1 < q < (q_c)_{k,j}$ then there is no restriction on $\mu \chi_{L_{k,j}}$.

Proof. This is an immediate consequence of Proposition 4.3 and Proposition 4.5 (see also the Remark following it). In the case $k = N$, $L_{N,j}$ is a vertex and the condition says merely that for $q \geq q_c(L_{N,j})$, μ does not charge the vertex. \square

4.3. Removable singular sets II..

Proposition 4.7. *Let A be a Lipschitz domain on S^{k-1} , $2 \leq k \leq N-1$, and let D_A be the k -dihedron with opening A . Let u be a positive solution of (1.1) in D_A^R , for some $R > 0$. Suppose that $F = \mathcal{S}(u) \subset d_A^R$ and let Q be an open neighborhood of F such that $\bar{Q} \subset d_A^R$. (Recall that $d_A^R = d_A \cap B_R^{N-k}(0)$ is an open subset of d_A .) Let μ be the trace of u on $\mathcal{R}(u)$.*

Let $\eta \in W_0^{s,q'}(d_A^R)$ such that

$$(4.27) \quad 0 \leq \eta \leq 1, \quad \eta = 0 \quad \text{on } Q.$$

Employing the notation in the proof of Proposition 4.4, put

$$(4.28) \quad \zeta := \rho_A \psi_R H_R[\eta]^{q'}.$$

Then

$$(4.29) \quad \int_{D_A^R} u^q \zeta \, dx \leq c(1 + \|\eta\|_{W^{s,q'}(d_A)})^{q'} + \mu(d_A^R \setminus Q)^q,$$

c independent of u and η .

Proof. First we prove (4.29) for $\eta \in C_0^\infty(d_A^R)$. Let σ_0 be a point in A and let $\{A_n\}$ be a Lipschitz exhaustion of A . If $0 < \epsilon < \text{dist}(\partial A, \partial A_n) = \bar{\epsilon}_n$ then

$$\epsilon \sigma_0 + C_{A_n} \subset C_A.$$

Denote

$$D_A^{R',R''} = D_A \cap [|x'| < R'] \cap [|x''| < R''].$$

Pick a sequence $\{\epsilon_n\}$ decreasing to zero such that $0 < \epsilon_n < \min(\bar{\epsilon}_n/2^n, R/8)$. Let u_n be the function given by

$$u_n(x'x'') = u(x' + \epsilon_n \sigma_0, x'') \quad \forall x \in D_{A_n}^{R_n, R}, \quad R_n = R - \epsilon_n.$$

Then u_n is a solution of (1.1) in $D_{A_n}^{R_n, R}$ belonging to $C^2(\bar{D}_{A_n}^{R_n, R})$ and we denote its boundary trace by h_n . Let

$$\zeta_n := \rho_{A_n} \psi_R H_R[\eta]^{q'},$$

with ψ_R and $H_R[\eta]$ as in the proof of Proposition 4.4. By Proposition 4.2

$$(4.30) \quad - \int_{D_{A_n}^{R_n, R}} \mathbb{P}[h_n] \Delta \zeta_n dx = \int_{B_R^{N-k}(0)} \eta^{q'} h_n d\omega_n$$

where ω_n is the harmonic measure on $d_{A_n}^R$ relative to $D_{A_n}^{R_n, R}$. (Note that $d_{A_n}^R = d_A^R$ and we may identify it with $B_R^{N-k}(0)$.) Hence

$$(4.31) \quad \int_{D_{A_n}^{R_n, R}} (-u_n \Delta \zeta_n + u_n^q \zeta_n) dx = - \int_{B_R^{N-k}(0)} \eta^{q'} h_n d\omega_n.$$

Further,

$$\int_{B_R^{N-k}(0)} \eta^{q'} h_n d\omega_n \rightarrow \int_{B_R^{N-k}(0)} \eta^{q'} d\mu \leq \mu(d_A^R \setminus Q),$$

because $\eta = 0$ in Q . By (4.19), (4.23) we obtain,

$$(4.32) \quad \left| \int_{D_{A_n}^{R_n, R}} u_n \Delta \zeta_n dx \right| \leq c \left(\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{\frac{1}{q}} \left(\left(\int_{D_{A_n}^{R_n, R}} \zeta_n dx \right)^{\frac{1}{q'}} + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))} \right).$$

From the definition of ζ_n it follows that

$$\int_{D_{A_n}^{R_n, R}} \zeta_n dx \leq \int_{D_{A_n}^{R_n, R}} \rho_n dx \rightarrow \int_{D_A^R} \rho dx,$$

where ρ (resp. ρ_n) is the first eigenfunction of $-\Delta$ in D_A^R (resp. $D_{A_n}^{R_n, R}$) normalized by 1 at some $x_0 \in D_{A_1}^{R_1, R}$. Therefore, by (4.31),

$$\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \leq c \left(\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{\frac{1}{q}} (1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))}) + \mu(d_A^R \setminus Q).$$

This implies

$$(4.33) \quad \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \leq c(1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})^{q'} + \mu(d_A^R \setminus Q)^q.$$

To verify this fact, put

$$m = \left(\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{1/q}, \quad b = \mu(d_A^R \setminus Q), \quad a = c(1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})$$

so that (4.33) becomes

$$m^q - am - b \leq 0.$$

If $b \leq m$ then

$$m^{q-1} - a - 1 \leq 0.$$

Therefore,

$$m \leq (a+1)^{\frac{1}{q-1}} + b$$

which implies (4.33). Finally, by the lemma of Fatou we obtain (4.29) for $\eta \in C_0^\infty$. By continuity we obtain the inequality for any $\eta \in W_0^{s,q'}$ satisfying (4.27). \square

Theorem 4.8. *Let A be a Lipschitz domain on S^{k-1} , $2 \leq k \leq N-1$, and let D_A be the k -dihedron with opening A . Let E be a compact subset of d_A^R and let u be a non-negative solution of (1.1) in D_A^R (for some $R > 0$) such that u vanishes on $\partial D_A^R \setminus E$. Then*

$$(4.34) \quad C_{s,q'}^{N-k}(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'} \implies u = 0,$$

where $C_{s,q'}^{N-k}$ denotes the Bessel capacity with the indicated indices in \mathbb{R}^{N-k} .

Proof. By Proposition 4.4, (4.34) holds under the additional assumption

$$(4.35) \quad \int_{D_A^R} u^q \rho_R \rho_A^R dx < \infty.$$

Indeed, by [23, Proposition 4.1], (4.35) implies that the solution u possesses a boundary trace μ on ∂D_A^R . By assumption, $\mu(\partial D_A^R \setminus E) = 0$. Therefore, by Proposition 4.5, the fact that $C_{s,q'}^{N-k}(E) = 0$ implies that $\mu(E) = 0$. Thus $\mu = 0$ and hence $u = 0$.

We show that, under the conditions of the theorem, if $C_{s,q'}^{N-k}(E) = 0$ then (4.35) holds.

By Proposition 4.7, for every $\eta \in W_0^{s,q'}(d_A^R)$ such that $0 \leq \eta \leq 1$ and $\eta = 0$ in a neighborhood of E ,

$$(4.36) \quad \int_{D_A^R} u^q \zeta dx \leq c(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))})^{q'},$$

for ζ as in (4.28). (Here we use the assumption that $u = 0$ on $\partial D_A^R \setminus E$.)

Let $a > 0$ be sufficiently small so that $E \subset B_{(1-4a)R}^{N-k}(0)$. Pick a sequence $\{\phi_n\}$ in $C_0^\infty(\mathbb{R}^{N-k})$ such that, for each n , there exists a neighborhood Q_n of E , $\bar{Q}_n \subset B_{(1-3a)R}^{N-k}(0)$ and

$$(4.37) \quad \begin{aligned} 0 &\leq \phi_n \leq 1 \text{ everywhere, } \phi_n = 1 \text{ in } Q_n, \\ \tilde{\phi}_n &:= \phi_n \chi_{[|x''| < (1-2a)R]} \in C_0^\infty(\mathbb{R}^{N-k}), \\ \|\tilde{\phi}_n\|_{W^{s,q'}(\mathbb{R}^{N-k})} &\rightarrow 0 \text{ as } n \rightarrow \infty \\ \eta_n &:= (1 - \phi_n)|_{[|x''| < R]} \in C_0^\infty(d_A^R), \\ \eta_n &= 0 \text{ in } [(1-a)R < |x''| < R]. \end{aligned}$$

Such a sequence exists because $C_{s,q'}^{N-k}(E) = 0$. Applying (4.36) to η_n we obtain,

$$(4.38) \quad \sup \int_{D_A^R} u^q \zeta_n dx \leq c < \infty,$$

where $\zeta_n = \rho_A \psi_R H_R^{q'}[\eta_n]$ (see (4.28)). By taking a subsequence we may assume that $\{\eta_n\}$ converges (say to η) in $L^{q'}(B_R^{N-k}(0))$ and consequently $H[\eta_n] \rightarrow H[\eta]$ in the sense that

$$H_R[\eta_n](x', \cdot) = w_{n,R}(y^2, \cdot) \rightarrow w_R(y^2, \cdot) = H_R[\eta](x', \cdot) \text{ in } L^{q'}$$

uniformly with respect to $y = |x'|$. It follows that

$$(4.39) \quad \int_{D_A^R} u^q \zeta dx < \infty, \quad \zeta = \rho_A \psi_R H_R^{q'}[\eta].$$

As $\tilde{\phi}_n \rightarrow 0$ in $W^{s,q'}(\mathbb{R}^{N-k})$ it follows that $\phi_n \rightarrow 0$ and hence $\eta_n \rightarrow 1$ a.e. in $B_{(1-2a)R}^{N-k}(0)$. Thus $\eta = 1$ in this ball, $\eta = 0$ in $[(1-a)R < |x''| < R]$ and $0 \leq \eta \leq 1$ everywhere.

Consequently, given $\delta > 0$, there exists an N -dimensional neighborhood O of $d_A \cap B_{(1-2a)R}^{N-k}(0)$ such that

$$1 - \delta < H_R[\eta] < 1 \text{ and } 1 - \delta < \psi_R / \rho_A^R < 1 \text{ in } O.$$

Therefore (4.39) implies that

$$(4.40) \quad \int_{D_A^{(1-3a)R}} u^q \rho^R \rho_A^R dx \leq c < \infty.$$

Recall that the trace of u on $\partial D_A^R \setminus d_A^{(1-4a)R}$ is zero. Therefore u is bounded in $D_A^R \setminus D_A^{(1-3a)R}$. This fact and (4.40) imply (4.35). \square

Definition 4.9. Let Ω be a bounded Lipschitz domain. Denote by ρ the first eigenfunction of $-\Delta$ in Ω normalized by $\rho(x_0) = 1$ for a fixed point $x_0 \in \Omega$.

For every compact set $K \subset \partial\Omega$ we define

$$M_{\rho,q}(K) = \{\mu \in \mathfrak{M}(\partial\Omega) : \mu \geq 0, \mu(\partial\Omega \setminus K) = 0, \mathbb{K}[\mu] \in L^q(\Omega)\}$$

and

$$\tilde{C}_{\rho,q'}(K) = \sup\{\mu(K)^q : \mu \in M_{\rho,q}(K), \int_{\Omega} \mathbb{K}[\mu]^q \rho dx = 1\}.$$

Finally we denote by $C_{\rho,q'}$ the outer measure generated by the above functional.

The following statement is verified by standard arguments:

Lemma 4.10. For every compact $K \subset \partial\Omega$, $C_{\rho,q'}(K) = \tilde{C}_{\rho,q'}(K)$. Thus $C_{\rho,q'}$ is a capacity and,

$$(4.41) \quad C_{\rho,q'}(K) = 0 \iff M_{\rho,q}(K) = \{0\}.$$

Theorem 4.11. *Let Ω be a bounded polyhedron in \mathbb{R}^N . A compact set $K \subset \partial\Omega$ is removable if and only if*

$$(4.42) \quad C_{s(k,j),q'}(K \cap L_{k,j}) = 0,$$

for $k = 1, \dots, N$ $j = 1, \dots, n_k$, where $s(k,j)$ is defined as in (4.24). This condition is equivalent to

$$(4.43) \quad C_{\rho,q'}(K) = 0.$$

A measure $\mu \in \mathfrak{M}(\partial\Omega)$ is q -good if and only if it does not charge sets with $C_{\rho,q'}$ -capacity zero.

Proof. The first assertion is an immediate consequence of Proposition 4.3 and Theorem 4.8. The second assertion follows from the fact that

$$C_{\rho,q'}(K \cap L_{k,j}) = C_{s(k,j),q'}(K \cap L_{k,j}).$$

The third assertion follows from Theorem 4.6 and the previous statement. \square

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