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# BOUNDARY TRACE OF POSITIVE SOLUTIONS OF SUPERCRITICAL SEMILINEAR ELLIPTIC EQUATIONS IN DIHEDRAL DOMAINS

MOSHE MARCUS AND LAURENT VERON

ABSTRACT. We study the generalized boundary value problem for (E)  $-\Delta u + |u|^{q-1}u = 0$  in a dihedral domain  $\Omega$ , when  $q > 1$  is supercritical. The value of the critical exponent can take only a finite number of values depending on the geometry of  $\Omega$ . When  $\mu$  is a bounded Borel measure in a  $k$ -wedge, we give necessary and sufficient conditions in order it be the boundary value of a solution of (E). We also give conditions which ensure that a boundary compact subset is removable. These conditions are expressed in terms of Bessel capacities  $B_{s,q}$  in  $\mathbb{R}^{N-k}$  where  $s$  depends on the characteristics of the wedge. This allows us to describe the boundary trace of a positive solution of (E).

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*Key words.* Laplacian; Poisson potential; Borel measures; Besov spaces; harmonic lifting; Bessel capacities.

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## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $\rho$  the first eigenfunction of  $-\Delta$  in  $W_0^{1,2}(\Omega)$  with supremum 1 and  $\lambda$  the corresponding eigenvalue, and let  $q > 1$ . If  $\partial\Omega$  is  $C^2$ , a very long-term research on the equation

$$(1.1) \quad -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega,$$

has been carried out since twenty years by probabilistic or analytic methods. The associated Dirichlet problem

$$(1.2) \quad -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \quad u = \mu \quad \text{in } \partial\Omega$$

where  $\mu$  is a Radon measure on  $\partial\Omega$  has played a fundamental role in these studies. By a (weak) solution  $u := u_\mu$  to the Dirichlet problem (1.2) we mean a function  $u \in L_\rho^q(\Omega)$  such that, for any  $\eta \in X(\Omega)$ , there holds

$$(1.3) \quad \int_{\Omega} (-u\Delta\eta + |u|^{q-1}u) dx = - \int_{\Omega} \mathbb{K}[\mu]\Delta\eta dx,$$

where  $X(\Omega)$  is the space of test functions

$$(1.4) \quad X(\Omega) = \left\{ \eta \in W_0^{1,2}(\Omega) : \rho^{-1}\Delta\eta \in L^\infty(\Omega) \right\}$$

and  $\mathbb{K}[\mu]$  the harmonic function in  $\Omega$  with boundary trace  $\mu$ . The following two observations are fundamental in this regard:

1- For any *admissible* bounded Borel measure  $\mu$  on  $\partial\Omega$ , i.e. which satisfies

$$(1.5) \quad \int_{\Omega} (\mathbb{K}[|\mu|])^q \rho dx < \infty,$$

there exists a (unique) solution to (1.2).

2- Any positive solution has a *boundary trace*  $\nu$  in the class of outer regular Borel measures on  $\partial\Omega$ , not necessarily locally bounded [13], [16].

The complementary problem associated to 1 is to find condition on a bounded Borel measure  $\mu$  so that (1.5) is satisfied. It leads naturally to existence of a threshold of criticality corresponding to a particular exponent  $q_c = \frac{N+1}{N-1}$ . When  $1 < q < q_c$ , i.e. *in the subcritical case*, every bounded Borel measure is admissible, while this is no longer valid in the *supercritical case*, that is when  $q \geq q_c$ . It is shown in that case that the Dirichlet problem (1.5) with  $\mu \geq 0$  can be solved if and only if  $\mu(E) = 0$  for any Borel subset  $E \subset \partial\Omega$  such that  $C_{\frac{2}{q},q'}(E) = 0$  where  $q' = \frac{q}{q-1}$  and  $C_{\frac{2}{q},q'}$  denotes the Bessel capacity in  $\mathbb{R}^{N-1}$ . The complementary problem to 2 is, given an arbitrary outer regular Borel measure  $\nu$  on  $\partial\Omega$ , to find if there exists a positive solution  $u$  of (1.1) with boundary trace  $\nu$ . In the subcritical case  $1 < q < q_c$  it was shown by LeGall [13], by probabilistic methods in the case  $N = q = 2$  and Marcus and Véron ( $N \geq 2$ ,  $1 < q < q_c$ ) [16] that the boundary trace establishes a one to one correspondence between the positive solutions of (1.1) and the set of outer regular Borel measures. When  $q \geq q_c$  a compact subset  $E$  of  $\partial\Omega$  is removable if and only if  $C_{\frac{2}{q},q'}(E) = 0$ ; it means that any

positive solution  $u \in C(\overline{\Omega} \setminus K)$  of (1.1) which vanishes on  $\partial\Omega \setminus K$  is identically zero [13], [6], [16]. Actually, Marcus and Véron obtained necessary and sufficient conditions of compatibility between the locally bounded part of  $\nu$  and its singular part in order there exists a maximal solution of (1.1) with boundary trace  $\nu$ . Moreover uniqueness in general no longer holds, which leads to the introduction of new tools for describing the boundary trace. The culmination of the study of equation (1.1) has been carried out by Mselati [23] when  $q = 2$  and Marcus [15] when  $q \geq q_c$  who obtain a *complete description* of the set of positive solutions via the notion of fine trace or precise trace.

When  $\partial\Omega$  is no longer smooth it is still possible to define a boundary trace with the help of the harmonic measure however the situation becomes much more complicated since the critical exponent  $q_{c,y}$  which depends of the opening of the domain at each boundary point  $y$ . The subcritical case of equation (1.1) which corresponds to the fact that the exponent  $q$  is smaller than any  $q_{c,y}$  is completely described in [22], without the use of any Bessel capacity.

We recall here some elements of local analysis when  $\Omega = C_S \cap B_1$  where  $C_S$  is a cone with vertex 0 and the opening  $S$  is a Lipschitz domain of  $S^{N-1}$ . Denote by  $\lambda_S$  the first eigenvalue and by  $\phi_S$  the first eigenfunction of  $-\Delta'$  in  $W_0^{1,2}(S)$  (normalized by  $\max \phi_S = 1$ ). Put

$$(1.6) \quad \alpha_S = \frac{1}{2}(N - 2 + \sqrt{(N - 2)^2 + 4\lambda_S})$$

and

$$\Phi_1(x) = \frac{1}{\gamma} x^{-\alpha_S} \phi_S(x/|x|)$$

where  $\gamma$  is a positive number;  $\Phi_1$  is a harmonic function in  $C_S$  vanishing on  $\partial C_S \setminus \{0\}$  and  $\gamma = \gamma_S$  is chosen so that the boundary trace of  $\Phi_1$  is  $\delta_0$  (=Dirac measure on  $\partial C_S$  with mass 1 at the origin).

(i) If  $q \geq 1 + \frac{2}{\alpha_S}$ , it was shown in [8] that, there is no solution of (1.1) in  $\Omega_S$  with isolated singularity at 0.

(ii) If  $1 < q < 1 + \frac{2}{\alpha_S}$ , then for any  $k > 0$  there exists a unique solution  $u := u_k$  to problem (1.2) with  $\mu = k\delta_0$ . Furthermore [22, Theorem 5.7]

$$(1.7) \quad u_k(x) = k\Phi_1(x)(1 + o(1)) \quad \text{as } x \rightarrow 0.$$

The function  $u_\infty = \lim_{k \rightarrow \infty} u_k$  is a positive solution of (1.1) in  $\Omega_S$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies

$$(1.8) \quad u_\infty(x) = |x|^{-\frac{2}{q-1}} \omega_S(x/|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0$$

where  $\omega$  is the (unique) positive solution of

$$(1.9) \quad -\Delta' \omega - \lambda_{N,q} \omega + |\omega|^{q-1} \omega = 0$$

on  $S^{N-1}$  with

$$(1.10) \quad \lambda_{N,q} = \frac{2}{q-1} \left( \frac{2q}{q-1} - N \right).$$

Finally the following classification holds [22, Theorem 5.9]: if  $u \in C(\bar{\Omega}_S \setminus \{0\})$  is a positive solution of (1.1) vanishing on  $(\partial C_S \cap B_{r_0}(0)) \setminus \{0\}$ , then it satisfies either (1.7) for some  $k \geq 0$  or (1.8).

If  $\Omega$  is Lipschitz and  $\xi \in \partial\Omega$ ,  $q_\xi$  is the critical value for (1.1) at  $\xi$  if, for  $1 < q < q_\xi$ , problem (1.2) admits a solution with  $\mu = \delta_\xi$  (and  $k\delta_\xi$  for any  $k > 0$ ) while, for  $q > q_\xi$  no such solution exists. The secondary critical value  $q_\xi^\sharp$  at  $\xi$  is defined by the fact that for  $1 < q < q_\xi^\sharp$  there exists a non-trivial solution of (1.1) which vanishes on  $\partial\Omega \setminus \{\xi\}$  but for  $q > q_\xi^\sharp$  no such solution exists.

In the case of smooth domains,  $q_\xi = q_\xi^\sharp = (N+1)/(N-1)$  for every boundary point  $\xi$ . If  $\Omega$  is a polyhedron,  $q_\xi = q_\xi^\sharp$  at every point and the function  $\xi \rightarrow q_\xi$  obtains only a finite number of values. In fact it is constant on each open face and each open edge, of any dimension. In addition, if  $q = q_\xi$ , an isolated singularity at  $\xi$  is removable.

**A. On the action of Poisson type kernels with fractional dimension.**

In preparation for the study of supercritical boundary value problems we establish an harmonic analytic result, extending a well known result on the action of Poisson kernels on Besov spaces with negative index (see [26, 1.14.4.] and [20]). We first quote the classical result for comparison purposes.

**Proposition 1.1.** *Let  $1 < q < \infty$  and  $s > 0$ . Then, for any bounded Borel measure  $\mu$  in  $\mathbb{R}^{n-1}$ ,*

$$(1.11) \quad I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|_{B^{-s,q}(\mathbb{R}^{n-1})}^q.$$

Here  $\mathbb{K}_n[\mu]$  denotes the Poisson potential of  $\mu$  in  $\mathbb{R}^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ , namely,

$$(1.12) \quad \mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{n-1}} \frac{d\mu(z)}{(y_1^2 + |\zeta - z|^2)^{n/2}} \quad \forall y = (y_1, \zeta) \in \mathbb{R}_+^n$$

where  $\gamma_n$  is a constant depending only on  $n$ . The notation  $I \approx J$  means that  $c^{-1}I \leq J \leq cI$  for some  $c > 0$ .

In this paper we prove,

**Theorem 1.2.** *Let  $1 < q, m$  a positive integer and  $\nu \in \mathbb{R}$  such that  $m+1 \leq \nu$ . For every  $s \in (0, m/q')$  there exists a positive constant  $c$  such that, for every bounded positive measure  $\mu$  supported in  $\mathbb{R}^m \cap B_{R/2}(0)$ ,  $R > 1$ ,*

$$(1.13) \quad \begin{aligned} \frac{1}{c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q &\leq \int_0^R \left( \int_{|\zeta| < R} |\mathbb{K}_{\nu,m}[\mu](\tau, \zeta)|^q d\zeta \right) \tau^{sq-1} d\tau \\ &\leq cR^{(s+\nu-m)q+1} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q. \end{aligned}$$

Here

$$(1.14) \quad \mathbb{K}_{\nu,m}[\mu](\tau, \zeta) = \int_{\mathbb{R}^m} \frac{\tau^{\nu-m} d\mu(z)}{(\tau^2 + |\zeta - z|^2)^{\nu/2}} \quad \forall \tau \in [0, \infty).$$

This also holds when  $s = m/q'$ , provided that the diameter of  $\text{supp } \mu$  is sufficiently small.

This is proved in Section 2 (see Theorem 3.8) using a slightly different notation. Note that

$$\mathbb{K}_n[\mu] = \gamma_n \mathbb{K}_{n,n-1}[\mu].$$

**B.** *The admissibility condition and the critical value in a  $k$ -wedge.*

The next step towards the study of boundary value problems in a polyhedron is the treatment of such problems in a  $k$ -wedge (or  $k$ -dihedron) i.e., the domain defined by the intersection of  $k$  hyperplanes in  $\mathbb{R}^N$ ,  $1 < k < N$ . The edge is an  $(N - k)$  dimensional space. We note that the case  $k = N$  corresponds to a cone while the case  $N = 1$  (i.e. a half space) corresponds to the smooth case. Both cases have already been treated.

We denote by  $D_A$  a  $k$ -wedge such that, its edge  $d_A$  is identified with  $\mathbb{R}^{N-k}$  and the 'opening' of the wedge is  $A = D_A \cap S^{k-1}$ . If  $S_A$  denotes the spherical domain

$$(1.15) \quad S_A = \{x \in \mathbb{R}^N : |x| = 1, x \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}$$

then

$$D_A = \{x = (r, \sigma) : r > 0, \sigma \in S_A\}, \quad D_{A,R} = D_A \cap \Gamma_R$$

where

$$\Gamma_R = \{x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k} : |x'| < r, |x''| < R\}.$$

Let  $\lambda_A$  be the first eigenvalue of  $-\Delta_{S^{N-1}}$  in  $W_0^{1,2}(S_A)$  and denote

$$(1.16) \quad \begin{aligned} \kappa_+ &= \frac{1}{2} \left( 2 - N + \sqrt{(N-2)^2 + 4\lambda_A} \right) \\ \kappa_- &= \frac{1}{2} \left( 2 - N - \sqrt{(N-2)^2 + 4\lambda_A} \right). \end{aligned}$$

One can show that the Martin kernel  $K_A$  in  $D_A$  relative to points  $z \in d_A$  is given by

$$(1.17) \quad K_A(x, z) = c_A \frac{|x'|^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})}{|x - z|^{(N-2+2\kappa_+)}}$$

where  $\omega^{\{N-k+1\}}$  is a related eigenfunction in  $A$  and  $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ . Using this formula we obtain the admissibility condition for a measure  $\mu \in \mathfrak{M}(d_A)$  such that  $\text{supp } \mu \subset B_R(0)$ :

$$(1.18) \quad \int_{\Gamma_R} \left( \int_{\mathbb{R}^{N-k}} \frac{|x'|^{\kappa_+} d|\mu|(z)}{(|x'|^2 + |x''|^2)^{(N-2+2\kappa_+)/2}} \right)^q |x'|^{\kappa_+} dx < \infty$$

where  $\Gamma_R := \{x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k} : |x'| < R, |x''| < R\}$ .

Using this expression we show that the condition

$$(1.19) \quad 1 < q < q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}.$$

is necessary and sufficient in order that the Dirac measure  $\mu = \delta_P$ , supported at a point  $P \in d_A$ , be admissible.

In addition we show that the condition

$$(1.20) \quad 1 < q < q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}$$

is necessary and sufficient for the existence of a non-trivial solution  $u$  of (1.1) in  $D_A$  which vanishes on  $\partial D_A \setminus d_A$ . Furthermore, when this condition holds, there exist admissible non-trivial positive *bounded measures*  $\mu$  on  $d_A$ , i.e., measures such that  $\mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A)$ .

Finally we have the following removability result:

**Theorem 1.3.** *Assume that  $q_c \leq q < q_c^*$ . A measure  $\mu \in \mathfrak{M}(\partial D_A)$ , with compact support contained in  $d_A$ , is good relative to (1.2) in  $D_A$  if and only if  $\mu$  vanishes on every Borel set  $E \subset d_A$  such that  $C_{s,q'}^{N-k}(E) = 0$ , where  $s = 2 - \frac{k+\kappa_+}{q'}$  and  $C_{s,q'}^{N-k}$  is the Bessel capacity with the indicated indices in  $\mathbb{R}^{N-k}$  (which we identify with the edge  $d_A$ ).*

Note that  $\mu$  is a *good measure* if the specified equation possesses a solution with boundary data  $\mu$ . The above result implies in particular that sets with  $C_{s,q'}^{N-k}$ -capacity zero are conditionally removable. However we obtain a much stronger result later on.

### C. Boundary value problems in a polyhedron: the supercritical case.

In the final part of the paper (Sections 4) we study boundary value problems in the supercritical case in polyhedrons, with trace given by bounded measures. For such domains  $\Omega$  we provide a complete characterization of 'good measures', i.e., measures  $\mu$  on  $\partial\Omega$  such that (1.1) possesses a (unique) solution with boundary trace  $\mu$ . We also provide a complete characterization of *removable sets*. These results, with rather obvious modifications, also apply to piecewise  $C^2$  domains. The case of general Lipschitz domains and boundary trace given by unbounded Borel measures will be treated in a subsequent paper.

**Theorem 1.4.** *Let  $\Omega$  be an  $N$ -dimensional polyhedron. Let  $L$  denote one of the faces, or edges, or vertices of  $\Omega$  and let  $Q_L$  denote the half space with boundary  $L$ , or the wedge with edge  $L$ , or the cone with vertex  $L$  such that  $\Omega \subset D_L$  and  $\partial Q_L$  is determined by the faces of  $\Omega$  adjacent to  $L$ . Thus  $\partial\Omega$  is the union of the sets  $\partial\Omega \cap \partial Q_L$ .*

Denote by  $A_L$  the opening of  $Q_L$  so that, in the notation of  $\mathbf{G}$ ,  $Q_L = D_{A_L}$  and denote by  $\kappa_+(L)$ ,  $q_c(L)$  etc. the various notations introduced in  $\mathbf{G}$

relative to  $A_L$ . In particular let  $k(L)$  denote the co-dimension of the linear space spanned by  $L$  and put

$$s(L) = 2 - \frac{k(L) + \kappa_+(L)}{q'}.$$

Let  $\mu$  be a bounded measure on  $\partial\Omega$ , (possibly a signed measure). Then  $\mu$  is a good measure relative to (1.2) in  $\Omega$ , if and only if, for every  $L$  as above and every Borel set  $E \subset L$  the following condition holds.

If  $1 \leq k = \text{codim}L < N$  then

$$(1.21) \quad \begin{aligned} C_{s(L),q'}^{N-k}(E) = 0 &\implies \mu(E) = 0 && \text{if } q_c(L) \leq q < q_c^*(L) \\ q \geq q_c^*(L) &\implies \mu(L) = 0 && \text{if } q \geq q_c^*(L) \end{aligned}$$

and if  $k = N$  (i.e.,  $L$  is a vertex)

$$(1.22) \quad q \geq q_c(L) = \frac{N+2 + \sqrt{(N-2)^2 + 4\lambda_{A_L}}}{N-2 + \sqrt{(N-2)^2 + 4\lambda_{A_L}}} \implies \mu(L) = 0.$$

In all cases, if  $1 < q < q_c(L)$  then there is no restriction on  $\mu\chi_L$ .

#### D. Characterization of removable sets.

Let  $\Omega$  be an  $N$ -dimensional polyhedron. Theorem 1.4 provides a necessary and sufficient condition for the removability of a singular set  $E$  relative to the family of solutions  $u$  such that

$$\int_{\Omega} |u|^q \rho \, dx < \infty.$$

The next result provides a necessary and sufficient condition for *removability* in the sense that the only non-negative solution  $u \in C(\bar{\Omega} \setminus E)$  which vanishes on  $\bar{\Omega} \setminus E$  is the trivial solution  $u = 0$ .

**Theorem 1.5.** *Let  $\Omega$  be an  $N$ -dimensional polyhedron and let  $E$  be a compact subset of  $\partial\Omega$ . A nonempty compact set  $E \subset \partial\Omega$  is removable if and only if, for every  $L$  as in **G** such that  $1 \leq k = \text{codim}L < N$  the following condition holds:*

*either*

$$q_c(L) \leq q < q_c^*(L) \quad \text{and} \quad C_{s(L),q'}^{N-k}(E) = 0$$

*or  $q \geq q_c^*(L)$ . In the case  $k = N$  the condition is  $q \geq q_c^*(L) = q_c(L)$ .*

## 2. THE MARTIN KERNEL AND CRITICAL VALUES FOR A CONE.

**2.1. The geometric framework.** An  $N$ -dim polyhedra  $P$  is the bounded domain bordered by a finite number of hyperplanes. Thus characteristic elements of the boundary of  $P$  are the faces (subsets of an hyperplane), the vertex (intersection of  $N$  hyperplanes) and a wide variety of  $N-k$  dimensional edges, where  $k$  ranges from 2 to  $N$ . An  $N-k$  dimensional edge, i.e. an



intersection of  $k$  hyperplanes, will be described by the characteristic angles of these hyperplanes.

We recall that the spherical coordinates in  $\mathbb{R}^N = \{x = (x_1, \dots, x_N)\}$  are expressed by

$$(2.1) \quad \begin{cases} x_1 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \sin \theta_1 \\ x_2 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \cos \theta_1 \\ x_3 = r \sin \theta_{N-1} \sin \theta_{N-2} \dots \cos \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{N-1} = r \sin \theta_{N-1} \cos \theta_{N-2}, \\ x_N = r \cos \theta_{N-1} \end{cases}$$

where  $\theta_1 \in [0, 2\pi]$  and  $\theta_\ell \in [0, \pi]$  for  $\ell = 2, 3, \dots, N-1$  (the  $\theta_j$  are the Euler angles). Thus the "angular" component  $\sigma \in S^{N-1}$  of the spherical coordinates  $(r, \sigma)$  of  $x \in \mathbb{R}^N$  is denoted by  $\sigma = (\theta_1, \dots, \theta_{N-1})$ .

We consider an unbounded *non-degenerate k-dihedron* defined as follows. Let  $k \in [2, N] \cap \mathbb{N}$  and let  $A$  be given by

$$A = \begin{cases} (0, \alpha_1) \times \prod_{j=2}^{k-1} (\alpha_j, \alpha'_j) & \text{if } k > 2 \\ (0, \alpha_1) & \text{if } k = 2 \end{cases}$$

where

$$0 < \alpha_1 < 2\pi, \quad 0 < \alpha_j < \alpha'_j < \pi \quad j = 2, \dots, k-1.$$

We denote by  $S_A$  the spherical domain

$$(2.2) \quad S_A = \{x \in \mathbb{R}^N : |x| = 1, \sigma \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}$$

and by  $D_A$  the corresponding  $k$ -dihedron,

$$D_A = \{x = (r, \sigma) : r > 0, \sigma \in S_A\}.$$

The *edge* of  $D_A$  is the  $(N-k)$ -dimensional space

$$(2.3) \quad d_A = \{x : x_1 = x_2 = \dots = x_k = 0\}.$$

**2.2. Separable harmonic functions and the Martin kernel in a  $k$ -dihedron.** In the system of spherical coordinates, the Laplacian takes the form

$$\Delta u = \partial_{rr} u + \frac{N-1}{r} \partial_r u + \frac{1}{r^2} \Delta_{S^{N-1}} u$$

where the Laplace-Beltrami operator  $\Delta_{S^{N-1}}$  is expressed by induction by

$$(2.4) \quad \begin{aligned} \Delta_{S^{N-1}} u &= \frac{1}{(\sin \theta_{N-1})^{N-2}} \frac{\partial}{\partial \theta_{N-1}} \left( (\sin \theta_{N-1})^{N-2} \frac{\partial u}{\partial \theta_{N-1}} \right) \\ &+ \frac{1}{(\sin \theta_{N-1})^2} \Delta_{S^{N-2}} u. \end{aligned}$$

and

$$(2.5) \quad \Delta_{S^1} u = \partial_{\theta_1 \theta_1} u$$

If we compute the positive harmonic functions in the k-dihedron  $D_A$  of the form

$$v(x) = v(r, \sigma) = r^\kappa \omega(\sigma) \quad \text{in } D_A, \quad v = 0 \quad \text{in } \partial D_A \setminus \{0\}.$$

we find that  $\kappa$  satisfies the algebraic equation

$$(2.6) \quad \kappa^2 + (N-2)\kappa - \lambda_A = 0$$

where  $\lambda_A$  is the first eigenvalue of  $-\Delta_{S^{N-1}}$  in  $W_0^{1,2}(S_A)$  and  $\omega$  is the corresponding normalized eigenfunction:

$$(2.7) \quad \begin{cases} \Delta_{S^{N-1}} \omega + \lambda_A \omega = 0 & \text{in } S_A \\ \omega = 0 & \text{on } \partial S_A. \end{cases}$$

Thus

$$(2.8) \quad \begin{aligned} \kappa_+ &= \frac{1}{2} \left( 2 - N + \sqrt{(N-2)^2 + 4\lambda_A} \right) \\ \kappa_- &= \frac{1}{2} \left( 2 - N - \sqrt{(N-2)^2 + 4\lambda_A} \right). \end{aligned}$$

Since

$$(2.9) \quad S^{N-1} = \{ \sigma \in \mathbb{R}^{N-1} \times \mathbb{R} : \sigma = (\sigma_2 \sin \theta_{N-1}, \cos \theta_{N-1}), \sigma_2 \in S^{N-2} \},$$

we look for  $\omega := \omega^{\{1\}}$  of the form

$$\omega^{\{1\}}(\sigma) = (\sin \theta_{N-1})^{\kappa_+} \omega^{\{2\}}(\sigma_2), \quad \theta_{N-1} \in (0, \pi), \quad \sigma_2 \in S^{N-2}.$$

Here  $S^{N-2} = S^{N-1} \cap \{x_N = 0\}$  and we denote

$$S_A^{\{N-2\}} = S_A \cap \{x_N = 0\}, \quad D_A^{\{N-2\}} := D_A \cap \{x_N = 0\} \subset \mathbb{R}^{N-1}.$$

Then (2.8) jointly with relation (2.4) implies

$$(2.10) \quad \begin{cases} \Delta_{S^{N-2}} \omega^{\{2\}} + (\lambda_A - \kappa_+) \omega^{\{2\}} = 0 & \text{on } S_A^{\{N-2\}} \\ \omega^{\{2\}} = 0 & \text{on } \partial S_A^{\{N-2\}}. \end{cases}$$

Since we are interested in  $\omega^{\{2\}}$  positive,  $\lambda_A^{\{2\}} := \lambda_A - \kappa_+$  must be the first eigenvalue of  $-\Delta_{S^{N-2}}$  in  $W_0^{1,2}(S_A^{\{N-2\}})$ .

Next we look for positive harmonic functions  $\tilde{u}$  in  $D_A^{\{N-2\}}$  such that

$$\tilde{u}(x_1, \dots, x_{N-1}) = r^{\kappa'} \omega(\sigma_2), \quad \tilde{u} = 0 \quad \text{on } \partial D_A^{\{N-2\}}$$

The algebraic equation which gives the exponents is

$$(\kappa')^2 + (N-3)\kappa' - \lambda_A^{\{2\}} = 0.$$

Denote by  $\kappa'_+$  the positive root of this equation. By the definition of  $\lambda_A^{\{2\}}$ ,

$$\kappa'_+{}^2 + (N-3)\kappa'_+ - \lambda_A^{\{2\}} = \kappa'_+{}^2 + (N-2)\kappa'_+ - \lambda_A = 0.$$

Therefore  $\kappa'_+ = \kappa_+$ . Accordingly, if  $k \geq 3$ , we set

$$\omega^{\{2\}}(\sigma_2) = (\sin \theta_{N-2})^{\kappa_+} \omega^{\{3\}}(\sigma_3),$$

and find that  $\omega^{\{3\}}$  satisfies

$$(2.11) \quad \begin{cases} \Delta_{S^{N-3}} \omega^{\{3\}} + (\lambda_A - 2\kappa_+) \omega^{\{3\}} = 0 & \text{in } S_A^{\{N-3\}} \\ \omega^{\{3\}} = 0 & \text{on } \partial S_A^{\{N-3\}}, \end{cases}$$

where

$$S_A^{\{N-3\}} = S_A \cap \{x_N = x_{N-1} = 0\}.$$

Performing this reduction process  $(N-k)$  times, we obtain the following results.

(i) If  $k > 2$  then

$$(2.12) \quad \omega(\sigma) = (\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k)^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})$$

where

$$\sigma_{N-k+1} \in S^{k-1} = S^{N-1} \cap \{x_N = x_{N-1} = \dots = x_{k+1} = 0\},$$

and  $\omega' := \omega^{\{N-k+1\}}$  satisfies

$$(2.13) \quad \begin{cases} \Delta_{S^{k-1}} \omega' + (\lambda_A - (N-k)\kappa_+) \omega' = 0, & \text{in } S_A^{\{k-1\}} \\ \omega' = 0, & \text{on } \partial S_A^{\{k-1\}}, \end{cases}$$

$$S_A^{\{k-1\}} = S_A \cap \{x_N = x_{N-1} = \dots = x_{k+1} = 0\} \approx A$$

and  $\lambda_A - (N-k)\kappa_+$  is the first eigenvalue of the problem. In such a case, it is usually impossible to determine more explicitly  $\omega^{\{N-k+1\}}$  and  $\lambda_A - (N-k)\kappa_+$ , except for some very specific values of  $\alpha_j$  and  $\alpha'_j$ , associated to consecutive zeros of generalized Legendre functions.

(ii) If  $k = 2$  then

$$(2.14) \quad \omega(\sigma) = (\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2)^{\kappa_+} \omega^{\{N-1\}}(\theta_1)$$

where  $\sigma_{N-1} \in S^1 \approx \theta_1 \in (0, 2\pi)$ , and  $\omega^{\{N-1\}}$  satisfies

$$(2.15) \quad \begin{cases} \Delta_{S^1} \omega^{\{N-1\}} + (\lambda_A - (N-2)\kappa_+) \omega^{\{N-1\}} = 0 & \text{on } S_A^{\{1\}} \\ \omega^{\{N-1\}} = 0 & \text{on } \partial S_A^{\{1\}}, \end{cases}$$

with  $\partial S_A^{\{1\}} \approx (0, \alpha)$ . In this case

$$(2.16) \quad \kappa_+ = \frac{\pi}{\alpha}, \quad \omega^{\{N-1\}}(\theta_1) = \sin(\pi\theta_1/\alpha),$$

and

$$(2.17) \quad \lambda_A - (N-2)\kappa_+ = \frac{\pi^2}{\alpha^2} \implies \lambda_A = \frac{\pi^2}{\alpha^2} + (N-2)\frac{\pi}{\alpha}.$$

Observe that  $\frac{1}{2} \leq \kappa_+$  with equality holding only in the degenerate case  $\alpha = 2\pi$  (which we exclude).

In either case, we find a positive harmonic function  $v_A$  in  $D_A$ , vanishing on  $\partial D_A$ , of the form

$$v_A(x) = |x|^{\kappa_+} \omega(x/|x|)$$

with  $\omega$  as in (2.12) (for  $k > 2$ ) or (2.16) (for  $k=2$ ).

Similarly we find a positive harmonic function in  $D_A$  vanishing on  $\partial D_A \setminus \{0\}$ , singular at the origin, of the form

$$K'_A(x) = |x|^{\kappa_-} \omega(x/|x|), \quad \kappa_- = 2 - N - \kappa_+.$$

Because of the uniqueness of the kernel function (see e.g. [3]),  $K'_A(x)$  is, up to a multiplicative constant  $c_A$ , the Martin kernel of the Laplacian in  $D_A$ , with singularity at 0. The Martin kernel, with singularity at a point  $z \in d_A$ , is given by

$$(2.18) \quad K_A(x, z) = c_A \frac{(\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k)^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})}{|x-z|^{N-2+\kappa_+}}$$

for every  $x \in D_A$ . From (2.1)

$$\sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_k = |x-z|^{-1} \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}.$$

Therefore, if we write  $x \in \mathbb{R}^N$  in the form  $x = (x', x'')$ ,  $x' = (x_1, \dots, x_k)$ ,  $x'' = (x_{k+1}, \dots, x_N)$ , we obtain the formula,

$$(2.19) \quad \begin{aligned} K_A(x, z) &= c_A \frac{|x'|^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})}{|x-z|^{(N-2+2\kappa_+)}} \\ &= c_A \frac{|x'|^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})}{(|x'|^2 + |x''-z|^2)^{(N-2+2\kappa_+)/2}}. \end{aligned}$$

Therefore, the Poisson potential of a measure  $\mu \in \mathfrak{M}(d_A)$  is expressed by

$$(2.20) \quad \begin{aligned} \mathbb{K}[\mu](x) &= c_A |x'|^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1}) \\ &\quad \times \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(|x'|^2 + |x''-z|^2)^{(N-2+2\kappa_+)/2}}. \end{aligned}$$

**2.3. The admissibility condition.** Consider the boundary value problem

$$(2.21) \quad \begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } D_A \\ u = \mu \in \mathfrak{M}(\partial D_A). \end{cases}$$

Let  $r' = |x'|$ ,  $r'' = |x''|$ ,

$$(2.22) \quad \Gamma_R = \{x = (x', x'') : r' \leq R, r'' \leq R\}$$

and let  $\rho_R$  denote the first (positive) eigenfunction in  $D_{A,R} := D_A \cap \Gamma_R$ .

By (1.5), the admissibility condition for a measure  $\mu \in \mathfrak{M}(d_A \cap \Gamma_R)$  relative to (2.21) in  $D_{A,R}$  is

$$(2.23) \quad \int_{D_{A,R}} \mathbb{K}^R[|\mu|](x)^q \rho_R(x) dx < \infty.$$

where  $\mathbb{K}^R$  is the Martin kernel of  $-\Delta$  in  $D_{A,R}$ . Near  $d_A$  this kernel behaves like the Martin kernel of the dihedron  $D_A$ . Furthermore, the first eigenfunction  $\rho_R$  of  $-\Delta$  in  $W_0^{1,2}(D_{A,R})$  behaves like the regular harmonic function  $v_A$ . Therefore

$$(2.24) \quad \rho_R(x) \approx (r')^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})$$

and the admissibility condition for a measure  $\mu \in \mathfrak{M}(d_A)$  is

$$(2.25) \quad \int_{\Gamma_R \cap D_A} \mathbb{K}[|\mu|](x)^q \rho(x) dx < \infty \quad \forall R > 0,$$

with  $\Gamma_R$  as in (2.22). By (2.20),

$$(2.26) \quad \mathbb{K}[|\mu|](x) \leq c_A (r')^{\kappa_+} \int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z)$$

where

$$(2.27) \quad j(x) = |x|^{-N+2-2\kappa_+} \quad \forall x \in \mathbb{R}^N.$$

Therefore, using (2.24), condition (2.25) becomes

$$(2.28) \quad \int_0^R \int_{|x''| < R} \left( \int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z) \right)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty$$

for every  $R > 0$ .

#### 2.4. The critical values. Relative to the equation

$$(2.29) \quad -\Delta u + |u|^{q-1} u = 0$$

there exist two thresholds of criticality associated with the edge  $d_A$ .

The first is the value  $q_c^*$  such that, for  $q_c^* \leq q$  the whole edge  $d_A$  is removable relative to this equation, but for  $1 < q < q_c^*$  there exist non-trivial solutions in  $D_A$  which vanish on  $\partial D_A \setminus d_A$ . The second  $q_c < q_c^*$  corresponds to the removability of points on  $d_A$ . For  $q \geq q_c$  points on  $d_A$  are removable while for  $1 < q < q_c$  there exist solutions with isolated point singularities on  $d_A$ . In the next two propositions we determine these critical values.

**Proposition 2.1.** *Assume  $q > 1$ ,  $1 \leq k < N$ . Then the condition*

$$(2.30) \quad q < q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}$$

*is necessary and sufficient for the existence of a non-trivial solution  $u$  of (2.29) in  $D_A$  which vanishes on  $\partial D_A \setminus d_A$ . Furthermore, when this condition holds, there exist non-trivial positive bounded measures  $\mu$  on  $d_A$  such that  $\mathbb{K}[\mu] \in L_\rho^q(\Gamma_R \cap D_A)$ .*

*Remark.* The statement remains true in the case  $k = N$ , which is the case of the cone. In this case  $q_c = q_c^* = 1 + (2/\alpha_S)$  (defined in (1.6)). However, in the present notation,  $\alpha_S = -\kappa_-$  and a straightforward computation yields:

$$(2.31) \quad q_c = \frac{N + 2 + \sqrt{(N - 2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N - 2)^2 + 4\lambda_A}}.$$

*Proof.* Recall that  $\lambda_A - (N - k)\kappa_+$  is the first eigenvalue in  $S_A^{\{k-1\}}$  (see (2.13) and the remarks following it). Let  $\kappa'_+, \kappa'_-$  be the two roots of the equation

$$X^2 + (k - 2)X - (\lambda_A - (N - k)\kappa_+) = 0,$$

i.e.

$$\kappa'_\pm = \frac{1}{2}(2 - k \pm \sqrt{(k - 2)^2 + 4(\lambda_A - (N - k)\kappa_+)}).$$

Then, by (1.7) and (1.8), if  $1 < q < 1 - (2/\kappa'_-)$  (note that because of a change in notation the entity denoted by  $\alpha_S$  in (1.6) is the same as  $-\kappa'_-$  in the present section) there exists a unique solution of (2.29) in the cone  $C_{S_A^{k-1}}$  i.e. the cone with opening  $S_A^{k-1} \subset S^{k-1} \subset \mathbb{R}^k$  with trace  $a\delta_0$  (where  $\delta_0$  denotes the Dirac measure at the vertex of the cone and  $a > 0$ ). By (1.8) this solution satisfies

$$(2.32) \quad u_a(x) = a|x|^{-\alpha} \phi(x/|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0,$$

where  $\phi$  is the first positive eigenfunction of  $-\Delta'$  in  $W_0^{1,2}(S_A^{k-1})$  normalized so that  $u_1$  possesses trace  $\delta_0$ .

The function  $u$  given by

$$\tilde{u}_a(x', x'') = u_a(x') \quad \forall (x', x'') \in D_A = C_{S_A^{k-1}} \times \mathbb{R}^{N-k},$$

is a nonzero solution of (2.29) in  $D_A$  which vanishes on  $\partial D_A \setminus d_A$  and has bounded trace on  $d_A$ .

A simple calculation shows that  $1 - (2/\kappa_-)$  equals  $q_c^*$  as given in (2.30).

Next, assume that  $q \geq q_c^*$  and let  $u$  be a solution of (2.29) in  $D_A$  which vanishes on  $\partial D_A \setminus d_A$ .

Given  $\epsilon > 0$  let  $v_\epsilon$  be the solution of (2.29) in  $D_A^{\{N-k-1\}} \setminus \{x' \in \mathbb{R}^k : |x'| \leq \epsilon\}$  such that

$$v_\epsilon(x') = \begin{cases} 0, & \text{if } x' \in \partial D_A^{\{N-k-1\}} \\ \infty, & \text{if } |x'| = \epsilon. \end{cases}$$

Given  $R > 0$  let  $w_R$  be the maximal solution in  $\{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$ .

Then the function  $u^*$  given by

$$u^*(x', x'') = v_\epsilon(x') + w_R(x'')$$

is a supersolution of (2.29) in  $D_A \setminus \{(x', x'') : |x'| > \epsilon, |x''| < R\}$  and it dominates  $u$  in this domain. But  $w_R(x'') \rightarrow 0$  as  $R \rightarrow \infty$  and, by [8],  $v_\epsilon(x') \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Therefore  $u_+ = 0$  and, by the same token,  $u_- = 0$ .  $\square$

**Proposition 2.2.** *Let  $A$  be defined as before. Then*

$$(2.33) \quad \mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A) \quad \forall \mu \in \mathfrak{M}(d_A), \quad \forall R > 0$$

*if and only if*

$$(2.34) \quad 1 < q < q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}.$$

*This statement is equivalent to the following:*

*Condition (2.34) is necessary and sufficient in order that the Dirac measure  $\mu = \delta_P$ , supported at a point  $P \in d_A$ , satisfy (2.33).*

*Proof.* It is sufficient to prove the result relative to the family of measures  $\mu$  such that  $\mu$  is positive, has compact support and  $\mu(d_A) = 1$ . Let  $R > 1$  be sufficiently large so that the support of  $\mu$  is contained in  $\Gamma_{R/2}$ . The measure  $\mu$  can be approximated (in the sense of weak convergence of measures) by a sequence  $\{\mu_n\}$  of convex combinations of Dirac measures supported in  $d_A \cap \Gamma_{R/2}$ . For such a sequence  $\mathbb{K}[\mu_n] \rightarrow \mathbb{K}[\mu]$  pointwise and  $\{\mathbb{K}[\mu_n]\}$  is uniformly bounded in  $D_A \setminus \Gamma_{3R/4}$ . Therefore it is sufficient to prove the result when  $\mu = \delta_0$ . In this case the admissibility condition (1.5)) is

$$\int_0^R \int_{|x''| < R} j(x)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty,$$

i.e.,

$$\int_0^R \int_0^R |x|^{q(2-N-2\kappa_+)} (r')^{(q+1)\kappa_+ + k - 1} (r'')^{N-k-1} dr'' dr' < \infty.$$

Substituting  $\tau := r''/r'$  the condition becomes

$$\int_0^R \int_0^{R/r'} (1 + \tau^2)^{\frac{q}{2}(2-N-2\kappa_+)} (r')^{q(2-N-\kappa_+) + \kappa_+ + N - 1} \tau^{N-k-1} d\tau dr' < \infty.$$

This holds if and only if  $q < (\kappa_+ + N)/(\kappa_+ + N - 2)$ .  $\square$

*Remark.* It is interesting to notice that  $k$  does not appear explicitly in (2.34). Furthermore, we observe that

$$(2.35) \quad \frac{2}{q_c - 1} \left( \frac{2q_c}{q_c - 1} - N \right) = \lambda_A \iff \kappa_+ (\kappa_+ + N - 2) = \lambda_A,$$

which follows from (2.8). This implies that there does not exist a nontrivial solution of the nonlinear eigenvalue problem

$$(2.36) \quad \begin{aligned} -\Delta_{S^{N-1}} \psi - \frac{2}{q-1} \left( \frac{2q}{q-1} - N \right) \psi + |\psi|^{q-1} \psi &= 0 \quad \text{in } S_{D_A} \\ \psi &= 0 \quad \text{in } \partial S_{D_A} \end{aligned}$$

which, in turn, implies that there does not exist a nontrivial solution of (2.29) of the form  $u(x) = u(r, \sigma) = |x|^{-2/(q-1)} \psi(\sigma)$ , and also no solution of this equation in  $D_A$  which vanishes on  $\partial D_A \setminus \{0\}$ . This is the classical ansatz for the removability of isolated singularities in  $d_A$ .

### 3. THE HARMONIC LIFTING OF A BESOV SPACE $B^{-s,p}(d_A)$ .

Denote by  $W^{\sigma,p}(\mathbb{R}^\ell)$  ( $\sigma > 0$ ,  $1 \leq p \leq \infty$ ) the Sobolev spaces over  $\mathbb{R}^\ell$ . In order to use interpolation, it is useful to introduce the Besov space  $B^{\sigma,p}(\mathbb{R}^\ell)$  ( $\sigma > 0$ ). If  $\sigma$  is not an integer then

$$(3.1) \quad B^{\sigma,p}(\mathbb{R}^\ell) = W^{\sigma,p}(\mathbb{R}^\ell).$$

If  $\sigma$  is an integer the space is defined as follows. Put

$$\Delta_{x,y}f = f(x+y) + f(x-y) - 2f(x).$$

Then

$$(3.2) \quad B^{1,p}(\mathbb{R}^\ell) = \left\{ f \in L^p(\mathbb{R}^\ell) : \frac{\Delta_{x,y}f}{|y|^{1+\ell/p}} \in L^p(\mathbb{R}^\ell \times \mathbb{R}^\ell) \right\},$$

with norm

$$(3.3) \quad \|f\|_{B^{1,p}} = \|f\|_{L^p} + \left( \iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|\Delta_{x,y}f|^p}{|y|^{\ell+p}} dx dy \right)^{1/p},$$

(with standard modification if  $p = \infty$ ) and

$$(3.4) \quad B^{m,p}(\mathbb{R}^\ell) = \left\{ f \in W^{m-1,p}(\mathbb{R}^\ell) : \right. \\ \left. D_x^\alpha f \in B^{1,p}(\mathbb{R}^\ell) \forall \alpha \in \mathbb{N}^\ell, |\alpha| = m-1 \right\}$$

with norm

$$(3.5) \quad \|f\|_{B^{m,p}} = \|f\|_{W^{m-1,p}} + \left( \sum_{|\alpha|=m-1} \iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|D_x^\alpha \Delta_{x,y}f|^p}{|y|^{\ell+p}} dx dy \right)^{1/p}.$$

We recall that the following inclusions hold ([25, p 155])

$$(3.6) \quad \begin{aligned} W^{m,p}(\mathbb{R}^\ell) &\subset B^{m,p}(\mathbb{R}^\ell) && \text{if } p \geq 2 \\ B^{m,p}(\mathbb{R}^\ell) &\subset W^{m,p}(\mathbb{R}^\ell) && \text{if } 1 \leq p \leq 2. \end{aligned}$$

When  $1 < p < \infty$ , the dual spaces of  $W^{s,p}$  and  $B^{m,p}$  are respectively denoted by  $W^{-s,p'}$  and  $B^{-m,p'}$ .

The following is the main result of this section.

**Theorem 3.1.** *Suppose that  $q_c < q < q_c^*$  and let  $A$  be defined as in subsection 2.1. Then there exist positive constants  $c_1, c_2$ , depending on  $q, N, k, \kappa_+$ , such that for any  $R > 1$  and any  $\mu \in \mathfrak{M}_+(d_A)$  with support in  $B_{R/2}$ :*

$$(3.7) \quad \begin{aligned} &c_1 \|\mu\|_{B^{-s,q}(\mathbb{R}^{N-k})}^q \\ &\leq \int_{D_{A,R}} \mathbb{K}[|\mu|^q(x)] \rho(x) dx \leq c_2 (1+R)^\beta \|\mu\|_{B^{-s,q}(\mathbb{R}^{N-k})}^q, \end{aligned}$$

where  $s = 2 - \frac{\kappa_+ + k}{q}$ ,  $\beta = (q+1)\kappa_+ + k - 1$  and  $D_{A,R} = D_A \cap \Gamma_R$ . If  $q = q_c$  the estimate remains valid for measures  $\mu$  such that the diameter of  $\text{supp } \mu$  is sufficiently small (depending on the parameters mentioned before).



*Remark.* When  $q \geq 2$  the norms in the Besov space may be replaced by the norms in the corresponding Sobolev spaces.

Recall the admissibility condition for a measure  $\mu \in \mathfrak{M}_+(d_A)$ :

$$\int_{D_{A,R}} \mathbb{K}[\mu]^q(x) \rho(x) dx < \infty \quad \forall R > 0$$

and the equivalence (see (2.25)–(2.28))

$$(3.8) \quad \int_{D_{A,R}} \mathbb{K}[\mu]^q(x) \rho(x) dx \approx J^{A,R}(\mu) := \int_0^R \int_{B_R''} \left( \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(\tau^2 + |x'' - z|^2)^{(N-2+2\kappa_+)/2}} \right)^q \tau^{(q+1)\kappa_+ + k - 1} dx'' d\tau,$$

where  $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ ,  $\tau = |x'|$  and  $B_R'' = \{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$ . We denote,

$$(3.9) \quad \nu = N - 2 + 2\kappa_+.$$

If  $2\kappa_+$  is an integer, it is natural to relate (3.8) to the Poisson potential of  $\mu$  in  $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}_{n-1}$  where  $n = N - 2 + 2\kappa_+$ . We clarify this statement below.

Assuming that  $2 \leq n + k - N$ , denote

$$y = (y_1, \tilde{y}, y'') \in \mathbb{R}^n, \quad \tilde{y} = (y_2, \dots, y_{n+k-N}), \quad y'' = (y_{n+k-N+1}, \dots, y_n).$$

The Poisson kernel in  $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}_{n-1}$  is given by

$$(3.10) \quad P_n(y) = \gamma_n y_1 |y|^{-n} \quad y_1 > 0,$$

for some  $\gamma_n > 0$ , and the Poisson potential of a bounded Borel measure  $\mu$  with support in

$$\mathbf{d} := \{y = (0, y'') \in \mathbb{R}^n : y'' \in \mathbb{R}_{N-k}\}$$

is

$$(3.11) \quad \mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2)^{n/2}} \quad \forall y \in \mathbb{R}_+^n.$$

In particular, for  $\tilde{y} = 0$ ,

$$(3.12) \quad \mathbb{K}_n[\mu](y_1, 0, y'') = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(y_1^2 + |y'' - z|^2)^{n/2}}.$$

The integral in (3.12) is precisely the same as the inner integral in (3.8).

In fact, it will be shown that, if we set

$$(3.13) \quad n := \{\nu\} = \inf\{m \in \mathbb{N} : m \geq \nu\},$$

this approach also works when  $2\kappa_+$  is not an integer. We note that, for  $n$  given by (3.13),

$$(3.14) \quad n - N + k \geq 2,$$

with equality only if  $k = 3$  and  $\kappa_+ \leq 1/2$  or  $k = 2$  and  $\kappa_+ \in (1/2, 1]$ . Indeed,

$$n - N + k = k + \{2\kappa_+\} - 2$$

and (as  $\kappa_+ > 0$ )  $\{2\kappa_+\} \geq 1$ . If  $k = 2$  then  $\kappa_+ > 1/2$  and consequently  $\{2\kappa_+\} \geq 2$ . These facts imply our assertion.

We also note that  $\kappa_+$  is strictly increasing relative to  $\lambda_A$  and

$$(3.15) \quad \kappa_+ \begin{cases} = 1, & \text{if } D_A = \mathbb{R}_+^N, \\ < 1, & \text{if } D_A \subsetneq \mathbb{R}_+^N, \\ > 1, & \text{if } D_A \supsetneq \mathbb{R}_+^N. \end{cases}$$

Finally we observe that  $\gamma := \lambda_A - (N - k)\kappa_+ > 0$  (see (2.13)) and, by (2.8) and (2.30):

$$(3.16) \quad \gamma = \kappa_+^2 + (k - 2)\kappa_+, \quad q_c^* = 1 + \frac{-(k - 2) + \sqrt{(k - 2)^2 + 4\gamma}}{\gamma}.$$

Therefore  $q_c^*$  is strictly decreasing relative to  $\gamma$  and consequently also relative to  $\kappa_+$ .

The proof of the theorem is based on the following important result proved in [26, 1.14.4.]

**Proposition 3.2.** *Let  $1 < q < \infty$  and  $s > 0$ . Then for any bounded Borel measure  $\mu$  in  $\mathbb{R}^{n-1}$  there holds*

$$(3.17) \quad I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|_{B^{-s,q}(\mathbb{R}^{n-1})}^q.$$

In the first part of the proof we derive inequalities comparing  $I(\mu)$  and  $J^{A,R}(\mu)$ . Actually, it is useful to consider a slightly more general expression than  $I(\mu)$ , namely:

$$(3.18) \quad I_{\nu,\sigma}^{m,j}(\mu) := \int_{\mathbb{R}_+^{m+j}} \left| \int_{\mathbb{R}^m} \frac{y_1 d\mu(z)}{(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2)^{\nu/2}} \right|^q e^{-y_1} y_1^{\sigma q-1} dy,$$

where  $\nu$  is an arbitrary number such that  $\nu > m$ ,  $j \geq 1$  and  $\sigma > 0$ . A point  $y \in \mathbb{R}_+^{m+j}$  is written in the form  $y = (y_1, \tilde{y}, y'') \in \mathbb{R}_+ \times \mathbb{R}^{j-1} \times \mathbb{R}^m$ . We assume that  $\mu$  is supported in  $\mathbb{R}^m$ . Note that,

$$(3.19) \quad I(\mu) = \gamma_n^q I_{n,s}^{m,j} \quad \text{where } m = N - k, \quad j = n - m = n - N + k.$$

Put

$$(3.20) \quad F_{\nu,m}[\mu](\tau) := \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty).$$

With this notation, if  $j \geq 2$  then

$$(3.21) \quad I_{\nu,\sigma}^{m,j}(\mu) := \int_0^\infty \int_{\mathbb{R}^{j-1}} F_{\nu,m}[\mu](\sqrt{y_1^2 + |\tilde{y}|^2}) e^{-y_1} y_1^{(\sigma+1)q-1} d\tilde{y} dy_1$$

and if  $j = 1$

$$(3.22) \quad I_{\nu,\sigma}^{m,1}(\mu) := \int_0^\infty F_{\nu,m}[\mu](y_1) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1$$

**Lemma 3.3.** *Assume that  $m < \nu$ ,  $0 < \sigma$ ,  $2 \leq j$  and  $1 < q < \infty$ . Then there exists a positive constant  $c$ , depending on  $m, j, \nu, \sigma, q$ , such that, for every bounded Borel measure  $\mu$  with support in  $\mathbb{R}^m$ :*

$$(3.23) \quad \frac{1}{c} \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau \leq I_{\nu,\sigma}^{m,j}(\mu) \leq c \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau,$$

where  $F_{\nu,m}$  is given by (3.20) and, for every  $\tau > 0$ ,

$$(3.24) \quad h_{\sigma,j}(\tau) = \begin{cases} \frac{\tau^{(\sigma+1)q+j-2}}{(1+\tau)^{(\sigma+1)q}}, & \text{if } j \geq 2, \\ e^{-\tau} \tau^{(\sigma+1)q-1}, & \text{if } j = 1. \end{cases}$$

*Proof.* There is nothing to prove in the case  $j = 1$ . Therefore we assume that  $j \geq 2$ .

We use the notation  $y = (y_1, \tilde{y}, y'') \in \mathbb{R} \times \mathbb{R}^{j-1} \times \mathbb{R}^m$ . The integrand in (3.21) depends only on  $y_1$  and  $\rho := |\tilde{y}|$ . Therefore,  $I_{\nu,\sigma}^{m,j}$  can be written in the form

$$I_{\nu,\sigma}^{m,j}(\mu) = c_{m,j} \int_0^\infty \int_0^\infty F_{\nu,m}[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1 \rho^{j-2} d\rho.$$

We substitute  $y_1 = (\tau^2 - \rho^2)^{1/2}$ , then change the order of integration and finally substitute  $\rho = r\tau$ . This yields,

$$\begin{aligned} & c_{m,j}^{-1} I_{\nu,\sigma}^{m,j}(\mu) \\ &= \int_0^\infty \int_\rho^\infty F_{\nu,m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2 - \rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau d\tau d\rho \\ &= \int_0^\infty \int_0^\tau F_{\nu,m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2 - \rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau d\rho d\tau \\ &= \int_0^\infty \int_0^1 F_{\nu,m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr d\tau, \end{aligned}$$

where

$$f(r) = r^{j-2} (1-r^2)^{(\sigma+1)q/2-1}.$$

We denote

$$I_\sigma^j(\tau) = \int_0^1 e^{-\tau\sqrt{1-r^2}} f(r) dr,$$

so that

$$(3.25) \quad I_{\nu,\sigma}^{m,j}(\mu) = c_{m,j} \int_0^\infty F_{\nu,m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} I_\sigma^j(\tau) d\tau.$$

To complete the proof we estimate  $I_\sigma^j$ . Since  $j \geq 2$ ,  $f \in L^1(0, 1)$  and  $I_\sigma^j$  is continuous in  $[0, \infty)$  and positive everywhere. Hence, for every  $\alpha > 0$ , there exists a positive constant  $c_\alpha = c_\alpha(\sigma)$  such that

$$(3.26) \quad \frac{1}{c_\alpha} \leq I_\sigma^j \leq c_\alpha \quad \text{in } [0, \alpha].$$

Next we estimate  $I_\sigma^j$  for large  $\tau$ . Since  $j \geq 2$ ,

$$I_\sigma^j \leq 2^{(\sigma+1)q/2-1} \int_0^1 (1-r)^{(\sigma+1)q/2-1} e^{-\tau\sqrt{1-r}} dr.$$

Substituting  $r = 1 - t^2$  we obtain,

$$(3.27) \quad I_\sigma^j \leq 2^{(\sigma+1)q/2} \int_0^1 t^{(\sigma+1)q-1} e^{-t\tau} dt = c(\sigma, q) \tau^{-(\sigma+1)q}.$$

On the other hand, if  $\tau \geq 2$ ,

$$(3.28) \quad \begin{aligned} I_\sigma^j(\tau) &= \int_0^1 (1-t^2)^{(j-3)/2} t^{(\sigma+1)q-1} e^{-\tau t} dt \\ &= \tau^{-(\sigma+1)q} \int_0^\tau (1-(s/\tau)^2)^{(j-3)/2} s^{(\sigma+1)q-1} e^{-s} ds \\ &\geq \tau^{-(\sigma+1)q} 2^{-(j-3)} \int_0^1 s^{(\sigma+1)q-1} e^{-s} ds. \end{aligned}$$

Combining (3.25) with (3.26)–(3.28) we obtain (3.23).  $\square$

Next we derive an estimate in which integration over  $\mathbb{R}_+^n = \mathbb{R}_+^j \times \mathbb{R}^m$  is replaced by integration over a bounded domain, for measures supported in a fixed bounded subset of  $\mathbb{R}^m$ .

Let  $B_R^j(0)$  and  $B_R^m(0)$  denote the balls of radius  $R$  centered at the origin, in  $\mathbb{R}^j$  and  $\mathbb{R}^m$  respectively. Denote

$$(3.29) \quad F_{\nu, m}^R[\mu](\tau) = \int_{B_R^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty)$$

and, if  $j \geq 2$ ,

$$(3.30) \quad I_{\nu, \sigma}^{m, j}(\mu; R) = \int_{B_R^j \cap \{0 < y_1\}} F_{\nu, m}^R[\mu](\sqrt{y_1^2 + |\tilde{y}|^2}) e^{-y_1} y_1^{\sigma q - 1} d\tilde{y} dy_1.$$

where  $(y_1, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^{j-1}$ . If  $j = 1$  we denote,

$$(3.31) \quad I_{\nu, \sigma}^{m, 1}(\mu; R) = \int_0^R F_{\nu, m}^R[\mu](y_1) e^{-y_1} y_1^{\sigma q - 1} dy_1.$$

Similarly to Lemma 3.3 we obtain,

**Lemma 3.4.** *If  $j \geq 1$ , there exists a positive constant  $c$  such that, for any bounded Borel measure  $\mu$  with support in  $\mathbb{R}^m \cap B_R$*

(3.32)

$$c^{-1} \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \leq I_{\nu,\sigma}^{m,j}(\mu; R) \leq c \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau$$

with  $h_{\sigma,j}$  as in (3.24).

*Proof.* In the case  $j = 1$  there is nothing to prove. Therefore we assume that  $j \geq 2$ .

From (3.30) we obtain,

$$I_{\nu,\sigma}^{m,j}(\mu; R) = c_{m,j} \int_0^R \int_0^{\sqrt{R^2-\rho^2}} F_{\nu,m}^R[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1} dy_1 \rho^{j-2} d\rho.$$

Substituting  $y_1 = (\tau^2 - \rho^2)^{1/2}$ , then changing the order of integration and finally substituting  $\rho = r\tau$  we obtain,

$$c_{m,j}^{-1} I_{\nu,\sigma}^{m,j}(\mu; R) = \int_0^R \int_0^1 F_{\nu,\mu}^R[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr d\tau.$$

where

$$f(r) = r^{j-2}(1-r^2)^{(\sigma+1)q/2-1}.$$

The remaining part of the proof is the same as for Lemma 3.3.  $\square$

**Lemma 3.5.** *Let  $1 < q$ ,  $0 < \sigma$  and assume that  $m < \nu q$  and  $0 \leq j-1 < \nu$ . Then there exists a positive constant  $\bar{c}$ , depending on  $j, m, q, \sigma, \nu$ , such that, for every  $R \geq 1$  and every bounded Borel measure  $\mu$  with support in  $B_{R/2}(0) \cap \mathbb{R}^m$ ,*

$$(3.33) \quad \left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \bar{c} R^{(\sigma+1-\nu)q+m+j-1} \|\mu\|_{\mathfrak{M}}^q$$

with  $h_{\sigma,j}$  as in (3.24).

*Proof.* We estimate,

$$(3.34) \quad \left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F_{\nu,m}^R[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \int_R^\infty |F_{\nu,m}[\mu](\tau)| h_{\sigma,j}(\tau) d\tau + \int_0^R |F_{\nu,m}[\mu] - F_{\nu,m}^R[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau.$$

For every  $\tau > 0$ ,

$$(3.35) \quad |F_{\nu,m}[\mu]|(\tau) \leq \tau^{-\nu q} \|\mu\|_{\mathfrak{M}}^q.$$

Since  $j - 1 < \nu q$ , it follows that

$$\begin{aligned}
\int_R^\infty |F_{\nu,m}[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau &\leq \|\mu\|_{\mathfrak{M}}^q \int_R^\infty \tau^{-\nu q} h_{\sigma,j}(\tau) d\tau \\
(3.36) \qquad \qquad \qquad &\leq c(\sigma, q) \|\mu\|_{\mathfrak{M}}^q \int_R^\infty \frac{\tau^{(\sigma+1)q+j-2-\nu q}}{(1+\tau)^{(\sigma+1)q}} d\tau \\
&\leq \frac{c(\sigma, q)}{\nu q - j + 1} \|\mu\|_{\mathfrak{M}}^q R^{j-1-\nu q}.
\end{aligned}$$

Since, by assumption,  $\text{supp } \mu \subset B_{R/2}$ , we have

$$\begin{aligned}
&\int_0^R |F_{\nu,m}[\mu] - F_{\nu,m}^R[\mu]|(\tau) h_{\sigma,j}(\tau) d\tau \\
&\leq \int_0^R \int_{|y''|>R} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' h_{\sigma,j}(\tau) d\tau \\
&\leq \|\mu\|_{\mathfrak{M}}^q \int_0^R \int_{|\zeta|>R/2} (|\tau^2 + |\zeta|^2)^{-\nu q/2} d\zeta h_{\sigma,j} d\tau \\
(3.37) \qquad \qquad \qquad &\leq c(m, q) \|\mu\|_{\mathfrak{M}}^q \int_0^R \int_{R/2}^\infty (\tau^2 + \rho^2)^{-\nu q/2} \rho^{m-1} d\rho h_{\sigma,j} d\tau \\
&\leq c(m, q) \|\mu\|_{\mathfrak{M}}^q \int_0^R \tau^{m-\nu q} \int_{R/2\tau}^\infty (1 + \eta^2)^{-\nu q/2} \eta^{m-1} d\eta h_{\sigma,j} d\tau \\
&\leq \frac{c(m, q)}{\nu q - m} \|\mu\|_{\mathfrak{M}}^q R^{m-\nu q} \int_0^R \tau^{(\sigma+1)q+j-2} d\tau \\
&\leq \frac{c(m, q)}{(\nu q - m)(\sigma + 1)q + j - 1} \|\mu\|_{\mathfrak{M}}^q R^{(\sigma+1)q+j-1+m-\nu q}.
\end{aligned}$$

Combining (3.34)–(3.37) we obtain (3.33).  $\square$

**Corollary 3.6.** *For every  $R > 0$  put*

$$(3.38) \qquad J_{\nu,\sigma}^{m,j}(\mu; R) := \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(\sigma+1)q+j-2} d\tau.$$

Then

$$\begin{aligned}
(3.39) \qquad \frac{1}{c} I_{\nu,\sigma}^{m,j}(\mu) - \bar{c} R^\beta \|\mu\|_{\mathfrak{M}}^q &\leq J_{\nu,\sigma}^{m,j}(\mu; R) \leq c R^{(\sigma+1)q} I_{\nu,\sigma}^{m,j}(\mu), \\
\beta &= (\sigma + 1 - \nu)q + j + m - 1,
\end{aligned}$$

for every  $R > 1$  and every bounded Borel measure  $\mu$  with support in  $B_{R/2}^m(0) := B_{R/2}(0) \cap \mathbb{R}^m$ .

*Proof.* This is an immediate consequence of Lemma 3.5 and Lemma 3.3.  $\square$

**Lemma 3.7.** *Let  $m, j$  be positive integers such that  $j \geq 1$  and let  $1 < q$ ,  $0 < \sigma$ . Put  $n := m + j$ .*

Then there exist positive constants  $c, \bar{c}$ , depending on  $j, m, q, \sigma$ , such that, for every  $R > 1$  and every measure  $\mu \in \mathfrak{M}_+(B_{R/2}^m(0))$ ,

$$(3.40) \quad \begin{aligned} \frac{1}{c} \|\mu\|_{B^{-\sigma, q}(\mathbb{R}^{n-1})}^q - \bar{c} R^{q(\sigma - \frac{n-1}{q'})} \|\mu\|_{\mathfrak{M}}^q &\leq J_{n, \sigma}^{m, j}(\mu; R) \\ &\leq c R^{(\sigma+1)q} \|\mu\|_{B^{-\sigma, q}(\mathbb{R}^{n-1})}^q. \end{aligned}$$

If  $\sigma < \frac{n-1}{q'}$ , there exists  $R_0 > 1$  such that, for all  $R > R_0$

$$(3.41) \quad \frac{1}{2c} \|\mu\|_{B^{-\sigma, q}(\mathbb{R}^{n-1})}^q \leq J_{n, \sigma}^{m, j}(\mu; R).$$

If  $\sigma = \frac{n-1}{q'}$  then, there exists  $a > 0$  such that the inequality remains valid for measures  $\mu$  such that  $\text{diam}(\text{supp } \mu) \leq a$ .

If, in addition,  $\frac{j-1}{q'} < \sigma$  then

$$(3.42) \quad \frac{1}{2c} \|\mu\|_{B^{-s, q}(\mathbb{R}^m)}^q \leq J_{n, \sigma}^{m, j}(\mu; R) \leq c R^{(\sigma+1)q} \|\mu\|_{B^{-s, q}(\mathbb{R}^m)}^q,$$

where  $s := \sigma - \frac{j-1}{q'}$ .

*Remark.* Assume that  $\mu \geq 0$ . Then:

(i) If  $\mu \in B^{-\sigma, q}(\mathbb{R}^{n-1})$  and  $\frac{j-1}{q'} \geq \sigma$  then  $\mu(\mathbb{R}^m) = 0$ .

(ii) If  $\mu \in B^{-s, q}(\mathbb{R}^m)$  and  $\sigma > (n-1)/q'$  then  $s > m/q'$  and therefore  $B^{s, q'}(\mathbb{R}^m)$  can be embedded in  $C(\mathbb{R}^m)$ .

*Proof.* Inequality (3.40) follows from (3.39) and Proposition 3.2 (see also (3.19)).

For positive measures  $\mu$ ,

$$\|\mu\|_{\mathfrak{M}} = \mu(\mathbb{R}^{n-1}) \leq \|\mu\|_{B^{-\sigma, q}(\mathbb{R}^{n-1})}^q.$$

Therefore, if  $\sigma < \frac{n-1}{q'}$ , (3.40) implies that there exists  $R_0 > 1$  such that (3.41) holds for all  $R > R_0$ .

If  $\sigma = \frac{n-1}{q'}$  (3.40) implies that

$$\frac{1}{c} \|\mu\|_{B^{-\sigma, q}(\mathbb{R}^{n-1})}^q - \bar{c} \|\mu\|_{\mathfrak{M}}^q \leq J_{n, \sigma}^{m, j}(\mu; R).$$

But if  $\mu$  is a positive bounded measure such that  $\text{diam}(\text{supp } \mu) \leq a$  then

$$\|\mu\|_{\mathfrak{M}} / \|\mu\|_{B^{-\sigma, q}(\mathbb{R}^{n-1})}^q \rightarrow 0 \quad \text{as } a \rightarrow 0.$$

The last inequality follows from the imbedding theorem for Besov spaces according to which there exists a continuous trace operator  $T : B^{\sigma, q'}(\mathbb{R}^{n-1}) \mapsto B^{s, q'}(\mathbb{R}^m)$  and a continuous lifting  $T' : B^{s, q'}(\mathbb{R}^m) \mapsto B^{\sigma, q'}(\mathbb{R}^{n-1})$  where  $s = \sigma - \frac{n-m-1}{q'}$ .  $\square$

If  $\nu \in \mathbb{N}$  and  $\sigma = s + \frac{\nu-m-1}{q'}$ ,

$$\begin{aligned} J_{\nu,\sigma}^{m,\nu-m}(\mu; R) &= \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(\sigma+1)q+\nu-m-2} d\tau \\ &= \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(s+\nu-m)q-1} d\tau. \end{aligned}$$

However, if  $\mu$  is positive, the expression

$$(3.43) \quad M_{\nu,s}^m(\mu; R) := \int_0^R F_{\nu,m}^R[\mu](\tau) \tau^{(s+\nu-m)q-1} d\tau,$$

is meaningful for any real  $\nu > m$  and  $s > 0$ . Furthermore, as shown below, the results stated in Lemma 3.7 can be extended to this general case.

**Theorem 3.8.** *Let  $1 < q$ ,  $\nu \in \mathbb{R}$  and  $m$  a positive integer. Assume that  $1 \leq \nu - m$  and  $0 < s < m/q'$ . Then there exists a positive constant  $c$  such that, for every bounded positive measure  $\mu$  supported in  $\mathbb{R}^m \cap B_{R/2}(0)$ ,  $R > 1$ ,*

$$(3.44) \quad \frac{1}{c} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q \leq M_{\nu,s}^m(\mu; R) \leq cR^{(s+\nu-m)q+1} \|\mu\|_{B^{-s,q}(\mathbb{R}^m)}^q.$$

*This also holds when  $s = m/q'$ , provided that the diameter of  $\text{supp } \mu$  is sufficiently small.*

*Proof.* If  $\nu$  is an integer and  $j := \nu - m$  then this statement is part of Lemma 3.7. Indeed the condition  $s > 0$  means that  $\sigma = s + \frac{j-1}{q'} > \frac{j-1}{q'}$  and the condition  $s < m/q'$  means that  $\sigma < \frac{n-1}{q'}$ .

Therefore we assume that  $\nu \notin \mathbb{N}$ . Let  $n := \{\nu\}$  and  $\theta := n - \nu$  so that  $0 < \theta < 1$ . Our assumptions imply that  $1 \leq n - m - 1$  because (as  $\nu$  is not an integer)  $\nu - m > 1$  and consequently  $n - m \geq 2$ .

If  $a, b$  are positive numbers, put

$$A_\nu := \frac{a^{(s+\nu-m)q-1}}{(a^2 + b^2)^{\nu q/2}}.$$

Obviously  $A_\nu$  decreases as  $\nu$  increases. Therefore,  $A_n \leq A_\nu \leq A_{n-1}$  which in turn implies,

$$M_{n,s}^m \leq M_{\nu,s}^m \leq M_{n-1,s}^m.$$

By Lemma 3.7, the assertions of the theorem are valid in the case that  $\nu = n$  or  $\nu = n - 1$ . Therefore the previous inequality implies that the assertions hold for any real  $\nu$  subject to the conditions imposed.  $\square$

By (3.8),

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau) \tau^{(q+1)\kappa_+ + k-1} d\tau,$$

where  $m = N - k$  and  $\nu = N - 2 + 2\kappa_+$ . Consequently, by (3.38),

$$J^{A,R} = M_{\nu,s}^m$$



where  $s$  is determined by,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

It follows that

$$sq = -(k - 2 + 2\kappa_+)q + (q + 1)\kappa_+ + k = k(1 - q) + 2q - \kappa_+(q - 1)$$

and therefore

$$s = 2 - \frac{k + \kappa_+}{q'}.$$

*Proof of Theorem 3.1.*

Put

$$(3.45) \quad \nu := N - 2 + 2\kappa_+, \quad s := 2 - \frac{\kappa_+ + k}{q'}, \quad m := N - k.$$

Recall that in the case  $k = 2$  we have  $\kappa_+ > 1/2$ . Therefore

$$(3.46) \quad \nu - m - 1 = k - 3 + 2\kappa_+ > 0.$$

Furthermore,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

Thus

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau) \tau^{(q+1)\kappa_+ + k - 1} d\tau = M_{\nu,s}^m.$$

Next we show that  $0 < s \leq m/q'$ . More precisely we prove

$$(3.47) \quad 0 < s \leq m/q' \iff q_c \leq q < q_c^*.$$

Let  $\mu$  be a bounded non-negative Borel measure in  $B^{-s,q}(\mathbb{R}^m)$ . If  $s \leq 0$ ,  $B^{-s,q}(\mathbb{R}^m) \subset L^q(\mathbb{R}^m)$ . Therefore, in this case, every bounded Borel measure on  $\mathbb{R}^m$  is admissible i.e. satisfies (2.33). Consequently, by Proposition 2.2,  $q < q_c$ . As we assume  $q \geq q_c$  it follows that  $s > 0$ .

If,  $s > 0$  and  $sq' - m \geq 0$  then  $C_{s,q'}(K) = 0$  for every compact subset of  $\mathbb{R}^m$  and consequently  $\mu(K) = 0$  for any such set. Conversely, if  $sq' - m < 0$  then there exist non-trivial positive bounded measures in  $B^{-s,q}(\mathbb{R}^m)$ . Therefore, by Proposition 2.1,  $sq' < m$  if and only if  $q < q_c^*$ .

In conclusion,  $0 < s \leq m/q'$  and  $\nu - m \geq 1$ ; therefore Theorem 3.1 is a consequence of Theorem 3.8.  $\square$

*Remark.* Note that the critical exponent for the imbedding of  $B^{2 - \frac{\kappa_+ + k}{q'}, q'}(\mathbb{R}^{N-k})$  into  $C(\mathbb{R}^{N-k})$  is again

$$q = q_c = \frac{N + \kappa_+}{N + \kappa_+ - 2}.$$

## 4. SUPERCRITICAL EQUATIONS IN A POLYHEDRAL DOMAIN

In this section  $q$  is a real number larger than 1 and  $P$  an  $N$ -dim polyhedral domain as described in subsection 6.1. Denote by  $\{L_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$  the family of faces, edges and vertices of  $P$ . In this notation,  $L_{1,j}$  denotes one of the open faces of  $P$ ; for  $k = 2, \dots, N - 1$ ,  $L_{k,j}$  denotes a relatively open  $N - k$ -dimensional edge and  $L_{N,j}$  denotes a vertex. For  $1 \leq k < N$ , the  $(N - k)$  dimensional space which contains  $L_{k,j}$  is denoted by  $\mathbb{R}_j^{N-k}$ . If  $1 < k < N$ , the cylinder of radius  $r$  around the axis  $\mathbb{R}_j^{N-k}$  will be denoted by  $\Gamma_{k,j,r}^\infty$  and the subset  $A_{k,j}$  of  $S^{k-1}$  is defined by

$$\lim_{r \rightarrow 0} \frac{1}{r} (\partial \Gamma_{k,j,r}^\infty \cap P) = L_{k,j} \times A_{k,j}.$$

$A_{k,j}$  is the 'opening' of  $P$  at the edge  $L_{k,j}$ . For  $k = N$  we replace in this definition the cylinder  $\Gamma_{N,j,r}^\infty$  by the ball  $B_r(L_{N,j})$ . For  $1 < k \leq N$  and  $A = A_{k,j}$  we use  $d_A$  as an alternative notation for  $\mathbb{R}_j^{N-k}$  and denote by  $D_A$  the  $k$ -dihedron with edge  $d_A$  and opening  $A$  as in subsection 6.1 (with  $S_A$  defined as in (2.2)). For  $k = 1$ ,  $D_A$  stands for the half space  $\mathbb{R}_j^{N-1} \times (0, \infty)$ .

In what follows we denote by  $\mathfrak{M}_q^\Omega$  the set of bounded measures  $\mu$  on the boundary of a Lipschitz domain  $\Omega$  such that the boundary value problem

$$(4.1) \quad -\Delta u + u^q = 0 \text{ in } \Omega, \quad u = \mu \text{ on } \partial\Omega$$

possesses a solution. A measure  $\mu$  in this space is called a *q-good measure*. Furthermore, if  $u$  is a positive solution of (1.1) in  $\Omega$  such that

$$(4.2) \quad \int_{\Omega} u^q \rho dx < \infty,$$

it possesses a boundary trace which is a bounded Borel measure, and thus a  $q$ -good measure [22, Proposition 4.1].

The following statements can be proved in the same way as in the case of smooth domains. For the proof in that case see [17].

**I.**  $\mathfrak{M}_q^\Omega$  is a linear space and

$$\mu \in \mathfrak{M}_q^\Omega \iff |\mu| \in \mathfrak{M}_q^\Omega.$$

**II.** If  $\{\mu_n\}$  is an increasing sequence of measures in  $\mathfrak{M}_q^\Omega$  and  $\mu := \lim \mu_n$  is a finite measure then  $\mu \in \mathfrak{M}_q^\Omega$ .

**Proposition 4.1.** *Let  $\mu$  be a bounded measure on  $\partial P$ . ( $\mu$  may be a signed measure.) For  $i = 1, \dots, N$ ,  $j = 1, \dots, n_i$ , we define the measure  $\mu_{k,j}$  on  $d_{A_{k,j}}$  by,*

$$\mu_{k,j} = \mu \text{ on } L_{k,j}, \quad \mu_{k,j} = 0 \text{ on } d_{A_{k,j}} \setminus L_{k,j}.$$

*Then  $\mu \in \mathfrak{M}_q^P$ , i.e., problem*

$$(4.3) \quad -\Delta u + u^q = 0 \text{ in } P, \quad u = \mu \text{ on } \partial P$$

possesses a solution, if and only if,  $\mu_{k,j}$  is a  $q$ -good measure relative to  $D_{A_{k,j}}$  for all  $(k, j)$  as above.

*Proof.* In view of statement **I** above, it is sufficient to prove the proposition in the case that  $\mu$  is non-negative. This is assumed hereafter. If  $\mu \in \mathfrak{M}_q^P$  then any measure  $\nu$  on  $\partial P$  such that  $0 \leq \nu \leq \mu$  is a  $q$ -good measure relative to  $P$ . Therefore

$$\mu \in \mathfrak{M}_q^P \implies \mu'_{k,j} := \mu \chi_{L_{k,j}} \in \mathfrak{M}_q^P.$$

Assume that  $\mu \in \mathfrak{M}_q^P$  and let  $u_{k,j}$  be the solution of (4.3) when  $\mu$  is replaced by  $\mu'_{k,j}$ . Denote by  $u'_{k,j}$  the extension of  $u_{k,j}$  by zero to the  $k$ -dihedron  $D_{A_{k,j}}$ . Then  $u'_{k,j}$  is a subsolution of (1.1) in  $D_{A_{k,j}}$  with boundary data  $\mu_{k,j}$ . In the present case there always exists a supersolution, e.g. the maximal solution of (1.1) in  $D_{A_{k,j}}$  vanishing outside  $d_{A_{k,j}} \setminus \bar{L}_{k,j}$ . Therefore there exists a solution  $v_{k,j}$  of this equation in  $D_{A_{k,j}}$  with boundary data  $\mu_{k,j}$ , i.e.,  $\mu_{k,j}$  is  $q$ -good relative to  $D_{A_{k,j}}$ .

Next assume that  $\mu \in \mathfrak{M}(\partial P)$  and that  $\mu_{k,j}$  is  $q$ -good relative to  $D_{A_{k,j}}$  for every  $(k, j)$  as above. Let  $v_{k,j}$  be the solution of (1.1) in  $D_{A_{k,j}}$  with boundary data  $\mu_{k,j}$ . Then  $v_{k,j}$  is a supersolution of problem (4.3) with  $\mu$  replaced by  $\mu'_{k,j}$  and consequently there exists a solution  $u_{k,j}$  of this problem. It follows that

$$w := \max\{u_{k,j} : k = 1, \dots, N, j = 1, \dots, n_k\}$$

is a subsolution while

$$\bar{w} := \sum_{k=1, \dots, N, j=1, \dots, n_k} u_{k,j}$$

is a supersolution of (4.3). Consequently there exists a solution of this problem, i.e.,  $\mu \in \mathfrak{M}_q^P$ .  $\square$

#### 4.1. Removable singular sets and 'good measures', I.

**Proposition 4.2.** *Let  $A$  be a Lipschitz domain on  $S^{k-1}$ ,  $2 \leq k \leq N-1$ , and let  $D_A$  be the  $k$ -dihedron with opening  $A$ . Let  $\mu \in \mathfrak{M}(\partial D_A)$  be a positive measure with compact support contained in  $d_A$  (= the edge of  $D_A$ ). Assume that  $\mu$  is  $q$ -good relative to  $D_A$ . Let  $R > 1$  be large enough so that  $\text{supp } \mu \subset B_R^{N-k}(0)$  and let  $u$  be the solution of (1.1) in  $D_A^R$  with trace  $\mu$  on  $d_A^R$  and trace zero on  $\partial D_A^R \setminus d_A^R$ . Then:*

(i) *For every non-negative  $\eta \in C_0^\infty(B_{3R/4}^{N-k}(0))$ ,*

$$(4.4) \quad \left( \int_{d_A^R} \eta^{q'} d\mu \right) \leq cM^{q'} \int_{D_A^R} u^q \rho dx + cM^{q'} \left( \int_{D_A^R} u^q \rho dx \right)^{\frac{1}{q}} \left( 1 + M^{-1} \|\eta\|_{L^{q'}(d_A^R)} \right).$$

where  $M = \|\eta\|_{L^\infty}$  and  $\rho$  is the first eigenfunction of  $-\Delta$  in  $D_A^R$  normalized by  $\rho(x_0) = 1$  at some point  $x_0 \in D_A^R$ . The constant  $c$  depends only on  $N, q, k, x_0, \lambda_1, R$  where  $\lambda_1$  is the first eigenvalue.

(ii) For any compact set  $E \subset d_A$ ,

$$(4.5) \quad C_{s,q}^{N-k}(E) = 0 \implies \mu(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'},$$

where  $C_{s,q}^{N-k}$  denotes the Bessel capacity with the indicated indices in  $\mathbb{R}^{N-k}$ .

*Remark.* If we replace  $D_A^R$  by  $D_A^{\tilde{R},R} = D_A \cap B_{\tilde{R}}^k(0) \cap B_R^{N-k}(0)$ ,  $\tilde{R} > 1$ , then the constant  $c$  in (i) depends on  $\tilde{R}$  but not on  $R$ .

*Proof.* We identify  $d_A$  with  $\mathbb{R}^{N-k}$  and use the notation

$$x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \quad y = |x'|.$$

Let  $\eta \in C_0^\infty(\mathbb{R}^{N-k})$  and let  $R$  be large enough so that  $\text{supp } \eta \subset B_{R/2}^{N-k}(0)$ .

Let  $w = w_R(t, x'')$  be the solution of the following problem in  $\mathbb{R}_+ \times B_R^{N-k}(0)$ :

$$(4.6) \quad \begin{aligned} \partial_t w - \Delta_{x''} w &= 0 && \text{in } \mathbb{R}_+ \times B_R^{N-k}(0), \\ w(0, x'') &= \eta(x'') && \text{in } B_R^{N-k}, \\ w(t, x'') &= 0 && \text{on } \partial B_R^{N-k}(0). \end{aligned}$$

Thus  $w_R(t, \cdot) = S_R(t)[\eta]$  where  $S_R(t)$  is the semi-group operator corresponding to the above problem. Denote,

$$(4.7) \quad H_R[\eta](x', x'') = w_R(|x'|^2, x'') = S_R(y^2)[\eta](x''), \quad y := |x'|.$$

We assume, as we may, that  $R > 1$ . Let  $\rho^R$  be the first eigenfunction of  $-\Delta_{x''}$  in the ball  $B_R^{N-k}(0)$  normalized by  $\rho^R(0) = 1$  and let  $\rho_A$  be the first eigenfunction of  $-\Delta_{x'}$  in  $C_A$  (where  $C_A$  denotes the cone with opening  $A$  in  $\mathbb{R}^k$ ) normalized so that  $\rho_A(x'_0) = 1$  at some point  $x'_0 \in S_A$ . Then  $\rho^R \rho_A$  is the first eigenfunction of  $-\Delta$  in  $\{x \in D_A : |x''| < R\}$ . Note that  $\rho^R \leq 1$  and  $\rho^R \rightarrow 1$  as  $R \rightarrow \infty$  in  $C^2(I)$  for any bounded set  $I \subset \mathbb{R}^{N-k}$ .

Let  $h \in C^\infty(\mathbb{R})$  be a monotone decreasing function such that  $h(t) = 1$  for  $t < 1/2$  and  $h(t) = 0$  for  $t > 3/4$ . Put

$$\psi_R(x') = h(|x'|/R)$$

and

$$(4.8) \quad \zeta_R := \rho_A \psi_R H_R[\eta]^{q'}.$$

If  $\rho_A^R$  is the first eigenfunction (normalized at  $x_0$ ) of  $D_A^R := D_A \cap \Gamma_R$  ( $\Gamma_R$  as in (2.22)) then

$$(4.9) \quad \rho_A \psi_R \leq c \rho_A^R$$

and  $\rho^R \rho_A^R$  is the first eigenfunction in  $D_A^R$ .

Hereafter we shall drop the index  $R$  in  $\zeta_R, H_R, w_R$  but keep it in the other notations in order to avoid confusion.

We shall verify that  $\zeta \in D_A^R$ . To this purpose we compute,

$$\begin{aligned}
(4.10) \quad \Delta\zeta &= -\lambda_1(\rho_A\psi_R)H[\eta]^{q'} + (\rho_A\psi_R)\Delta H[\eta]^{q'} + 2\nabla(\rho_A\psi_R) \cdot \nabla H[\eta]^{q'} \\
&= -\lambda_1\zeta + q'(\rho_A\psi_R)(H[\eta])^{q'-1}\Delta H[\eta] \\
&\quad + q(q'-1)(\rho_A\psi_R)(H[\eta])^{q'-2}|\nabla H[\eta]|^2 \\
&\quad + 2q'(H[\eta])^{q'-1}\nabla(\rho_A\psi_R) \cdot \nabla H[\eta].
\end{aligned}$$

In addition,

$$\begin{aligned}
\nabla H[\eta] &= \nabla_{x'}H[\eta] + \nabla_{x''}H[\eta] = \partial_y H[\eta] \frac{x'}{y} + \nabla_{x''}H[\eta] \\
&= 2y\partial_t w(y^2, x'') \frac{x'}{y} + \nabla_{x''}H[\eta](x', x'')
\end{aligned}$$

and consequently (recall that  $y$  stands for  $|x'|$ ),

$$\begin{aligned}
&\nabla H[\eta] \cdot \nabla(\rho_A\psi_R) \\
&= 2\partial_t w(y^2, x'')x' \cdot \left( \psi_R(|x'|^{\kappa_+ - 1}(\kappa_+ \frac{x'}{y}\omega_k(x'/y) + |x'|\nabla\omega_k(x'/y))) + \rho_A\nabla\psi_R \right) \\
&= 2\kappa_+\partial_t w(y^2, x'')|x'|^{\kappa_+}\omega_k(x'/y) = 2\partial_t w(y^2, x'')(\kappa_+\rho_A\psi_R + \rho_Ax' \cdot \nabla\psi_R).
\end{aligned}$$

Since  $w = w_R$  vanishes for  $|x''| = R$  and  $\eta = 0$  in a neighborhood of this sphere,  $|\partial_t w(y^2, x'')| \leq c\rho^R$ . As  $\psi_R$  vanishes for  $|x'| > 3R/4$  we have  $\rho_A\nabla\psi_R \leq c\rho_A^R$ . Therefore

$$|\nabla H[\eta] \cdot \nabla\rho_A| \leq c\rho^R\rho_A^R$$

and, in view of (4.10),

$$(4.11) \quad |\Delta\zeta| \leq c\rho^R\rho_A^R.$$

Thus  $\zeta \in X(D_A^R)$  and consequently

$$(4.12) \quad \int_{D_A^R} (-u\Delta\zeta + u^q\zeta) dx = - \int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta dx.$$

Since  $q(q'-1)\rho_A(H[\eta])^{q'-2}|\nabla H[\eta]|^2 \geq 0$ , we have

$$\begin{aligned}
(4.13) \quad &\left| \int_{D_A^R} u\Delta\zeta dx \right| \\
&\leq \int_{D_A^R} u \left( \lambda_1\zeta + q'(H[\eta])^{q'-1}(\rho|\Delta H[\eta]| + 2|\nabla\rho \cdot \nabla H[\eta]|) \right) dx \\
&\leq \int_{D_A^R} u \left( \lambda_1\zeta + q'\zeta^{1/q} \left( \rho^{1/q'}|\Delta H[\eta]| + 2\rho^{-1/q'}|\nabla\rho \cdot \nabla H[\eta]| \right) \right) dx \\
&\leq \left( \int_{D_A^R} u^q\zeta dx \right)^{\frac{1}{q}} \left( \lambda_1 \left( \int_{D_A^R} \zeta dx \right)^{\frac{1}{q'}} + q' \|L[\eta]\|_{L^{q'}(D_A^R)} \right)
\end{aligned}$$

where

$$(4.14) \quad L[\eta] = \rho^{1/q'} |\Delta H[\eta]| + 2\rho^{-1/q} |\nabla \rho \cdot \nabla H[\eta]|.$$

By Proposition 5.2

$$(4.15) \quad - \int_{D_A^R} \mathbb{K}[\mu] \Delta \zeta dx = \int_{d_A^R} \eta^{q'} d\mu.$$

Therefore

$$(4.16) \quad \left( \int_{d_A^R} \eta^{q'} d\mu \right) \leq \int_{D_A^R} u^q \zeta dx + \left( \int_{D_A^R} u^q \zeta dx \right)^{\frac{1}{q}} \left( \lambda_1 \left( \int_{D_A^R} \zeta dx \right)^{\frac{1}{q'}} + q' \|L[\eta]\|_{L^{q'}(D_A^R)} \right).$$

Next we prove that

$$(4.17) \quad \|L[\eta]\|_{L^{q'}(D_A^R)} \leq C \|\eta\|_{W^{s,q'}(\mathbb{R}^{N-k})}$$

starting with the estimate of the first term on the right hand side of (4.14).

$$\begin{aligned} \Delta H[\eta] &= \Delta_{x'} H[\eta] + \Delta_{x''} H[\eta] = \partial_y^2 H[\eta] + \frac{k-1}{y} \partial_y H[\eta] + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w(y^2, x'') + k \partial_t w(y^2, x'') + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w(y^2, x'') + (k+1) \partial_t w(y^2, x''). \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^N} \rho |\Delta H[\eta]|^{q'} dx &\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_{tt} w(y^2, x'')|^{q'} dx'' y^{\kappa_+ + 2q' + k - 1} dy \\ &\quad + c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(y^2, x'')|^{q'} dx'' y^{\kappa_+ + k - 1} dy \\ &\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_{tt} w(t, x'')|^{q'} dx'' t^{(\kappa_+ + k)/2 + q'} \frac{dt}{t} \\ &\quad + c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(t, x'')|^{q'} dx'' t^{(\kappa_+ + k)/2} \frac{dt}{t} \\ &\leq c \int_0^1 \left\| t^{2 - (1 - \frac{\kappa_+ + k}{2q'})} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ &\quad + c \int_0^1 \left\| t^{1 - (1 - \frac{\kappa_+ + k}{2q'})} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t}. \end{aligned}$$

Put  $\beta = \frac{\kappa_+ + k}{2q'}$  and note that  $0 < \beta = \frac{1}{2}(2-s) < 1$ . By standard interpolation theory,

$$\begin{aligned} & \int_0^1 \left\| t^{1-(1-\beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ & \approx \|\eta\|_{[W^{2,q'}, L^{q'}]_{1-\beta, q'}}^{q'} \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}, \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \left\| t^{2-(1-\beta)} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ & \approx \|\eta\|_{[W^{4,q'}, L^{q'}]_{\frac{1}{2}(1-\beta), q'}}^{q'} \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}. \end{aligned}$$

The second term on the right hand side of (4.14) is estimated in a similar way:

$$\begin{aligned} & \int_{\mathbb{R}^N} \rho^{-q'/q} |\nabla H[\eta] \cdot \nabla \rho|^{q'} dx \leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(y^2, x'')|^{q'} dx' y^{\kappa_+ + k - 1} dy \\ & \leq c \int_0^1 \int_{\mathbb{R}^{N-k}} |\partial_t w(t, x'')|^{q'} dx' t^{\frac{\kappa_+ + k}{2}} \frac{dt}{t} \\ & \leq c \int_0^1 \left\| t^{1-(\frac{1}{2}-\beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ & \approx \|\eta\|_{W^{2(1-\beta), q'}(\mathbb{R}^{N-k})}^{q'}. \end{aligned}$$

This proves (4.17). Further, (4.16) and (4.17) imply (4.4).

We turn to the proof of part (ii). Let  $E$  be a closed subset of  $B_{R/2}^{N-k}(0)$  such that  $C_{s, q'}^{N-k}(E) = 0$ . Then there exists a sequence  $\{\eta_n\}$  in  $C_0^\infty(d_A)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood of  $E$  (which may depend on  $n$ ),  $\text{supp } \eta_n \subset B_{3R/4}^{N-k}(0)$  and  $\|\eta_n\|_{W^{s, q'}} \rightarrow 0$ . Then, by (4.17),

$$\|L[\eta_n]\|_{L^{q'}(D_A^R)} \rightarrow 0.$$

Furthermore

$$\|w\|_{L^{q'}((0, R) \times B_R^{N-k}(0))} \leq c \|\eta_n\|_{L^{q'}(B_R^{N-k}(0))}$$

and consequently

$$H[\eta_n] \rightarrow 0 \text{ in } L^{q'}(D_A^R).$$

(Here we use the fact that  $k \geq 2$ .) In addition

$$0 \leq H[\eta_n] \leq 1, \quad H[\eta_n] \leq c(R - |x'|)$$

with a constant  $c$  independent of  $n$ . Hence (see (4.9))

$$\zeta_{n, R} := \rho_A \psi_R H[\eta_n]^{q'} \leq \rho^R \rho_A \psi_R H[\eta_n]^{q'-1} \leq \rho^R \rho_A^R H[\eta_n]^{q'-1}.$$

As  $u^q \rho^R \rho_A^R \in L^1(D_A^R)$  we obtain,

$$\lim_{n \rightarrow \infty} \int_{D_A} u^q \zeta_n dx = 0.$$

This fact and (4.16) imply that

$$\int_{d_A^R} \eta_n^{q'} d\mu \rightarrow 0.$$

As  $\eta_n = 1$  on a neighborhood of  $E$  in  $\mathbb{R}^{N-k}$  it follows that  $\mu(E) = 0$ .  $\square$

**Proposition 4.3.** *Let  $D_A$  be a  $k$ -dihedron,  $1 \leq k < N$ . Let  $k_+$  be as in (2.8) and let  $q_c^*$  and  $q_c$  be as in Proposition 2.1 and Proposition 2.2 respectively. Assume that  $q_c \leq q < q_c^*$ . A measure  $\mu \in \mathfrak{M}(\partial D_A)$ , with compact support contained in  $d_A$ , is  $q$ -good relative to  $D_A$  if and only if  $\mu$  vanishes on every Borel set  $E \subset d_A$  such that  $C_{s,q'}(E) = 0$ , where  $s = 2 - \frac{k+\kappa_+}{q'}$ .*

*Remark.* We shall use the notation  $\mu \prec C_{s,q'}$  to say that  $\mu$  vanishes on any Borel set  $E \subset (d_A)$  such that  $C_{s,q'}(E) = 0$ .

In the case  $k = N$ :  $D_A = C_A$  (= the cone with vertex 0 and opening  $A$  in  $\mathbb{R}^k$ ) and  $q_c = q_c^*$ . By (1.7),  $q_c = 1 - \frac{2}{\kappa_-} = \frac{N+\kappa_+}{N+\kappa_+-2}$ . (Note the difference in notation; the entity denoted by  $\kappa_-$  in section 6 and in the present section is denoted by  $-\alpha_S$  in (1.6) and (2.8).) If  $1 < q < q_c$  then, again by Theorem ??, there exist solutions for every measure  $\mu = k\delta_0$  on  $\partial C_A$ .

In the case  $k = 1$ ,  $q_c^* = \infty$ ,  $\kappa_+ = 1$  and  $q_c = \frac{N+1}{N-1}$ . Thus  $s = 2/q$  and the statement of the theorem is well known (see [18]).

*Proof.* In view of the last remark, it remains to deal only with  $2 \leq k \leq N-1$ . We shall identify  $d_A$  with  $\mathbb{R}^{N-k}$ .

It is sufficient to prove the result for positive measures because  $\mu \prec C_{s,q'}$  if and only if  $|\mu| \prec C_{s,q'}$ . In addition, if  $|\mu|$  is a  $q$ -good measure then  $\mu$  is a  $q$ -good measure.

First we show that if  $\mu$  is non-negative and  $q$ -good then  $\mu \prec C_{s,q'}$ . If  $E$  is a Borel subset of  $\partial\Omega$  then  $\mu\chi_E$  is  $q$ -good. If  $E$  is compact and  $C_{s,q'}(E) = 0$  then, by Proposition 4.2,  $E$  is a removable set. This means that the only solution of (4.1) such that  $\mu(\partial\Omega \setminus E) = 0$  is the zero solution. This implies that  $\mu\chi_E = 0$ , i.e.,  $\mu(E) = 0$ . If  $C_{s,q'}(E) = 0$  but  $E$  is not compact then  $\mu(E') = 0$  for every compact set  $E' \subset E$ . Therefore, we conclude again that  $\mu(E) = 0$ .

Next, assume that  $\mu$  is a positive measure in  $\mathfrak{M}(\partial D_A)$  supported in a compact subset of  $\mathbb{R}^{N-k}$ .

If  $\mu \in B^{-s,q}(\mathbb{R}^{N-k})$  then, by Theorem 3.1,  $\mu$  is admissible relative to  $D_A \cap \Gamma_{k,R}$ , for every  $R > 0$ . (As before  $\Gamma_{k,R}$  is the cylinder with radius  $R$  around the 'axis'  $\mathbb{R}^{N-k}$ .) This implies that  $\mu$  is  $q$ -good relative to  $D_A$ .

If  $\mu \prec C_{s,q'}$  then, by a theorem of Feyel and de la Pradelle [9] (see also [2]), there exists a sequence  $\{\mu_n\} \subset (B^{-s,q}(\mathbb{R}^{N-k}))_+$  such that  $\mu_n \uparrow \mu$ . As  $\mu_k$  is  $q$ -good, it follows that  $\mu$  is  $q$ -good.  $\square$

**Theorem 4.4.** *Let  $P$  be an  $N$ -dimensional polyhedron as described in Proposition 4.1. Let  $\mu$  be a bounded measure on  $\partial P$ , (may be a signed measure). Let  $k = 1, \dots, N$ ,  $j = 1, \dots, n_k$ , and let  $L_{k,j}$  and  $A_{k,j}$  be defined as at the*



beginning of section 8. Further, put

$$(4.18) \quad s(k, j) = 2 - \frac{k + (\kappa_+)_{k,j}}{q'},$$

where  $(\kappa_+)_{k,j}$  is defined as in (2.8) with  $A = A_{k,j}$ . Then  $\mu \in \mathfrak{M}_q^P$ , i.e.,  $\mu$  is a good measure for (1.1) relative to  $P$ , if and only if, for every pair  $(k, j)$  as above and every Borel set  $E \subset L_{k,j}$ :

If  $1 \leq k < N$  then

$$(4.19) \quad \begin{aligned} (q_c)_{k,j} \leq q < (q_c^*)_{k,j}, \quad C_{s(k,j),q'}^{N-k}(E) = 0 &\implies \mu(E) = 0 \\ q \geq (q_c^*)_{k,j} &\implies \mu(L_{N,j}) = 0 \end{aligned}$$

and if  $k = N$ , i.e.,  $L$  is a vertex,

$$(4.20) \quad q \geq (q_c)_{k,j} = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}} \implies \mu(L) = 0.$$

Here  $(q_c^*)_{k,j}$  and  $(q_c)_{k,j}$  are defined as in (2.30) and (2.34) respectively, with  $A = A_{k,j}$ .

If  $1 < q < (q_c)_{k,j}$  then there is no restriction on  $\mu_{\chi_{L_{k,j}}}$ .

*Proof.* This is an immediate consequence of Proposition 4.1 and Proposition 4.3 (see also the Remark following it). In the case  $k = N$ ,  $L_{N,j}$  is a vertex and the condition says merely that for  $q \geq q_c(L_{N,j})$ ,  $\mu$  does not charge the vertex.  $\square$

## 4.2. Removable singular sets II.

**Proposition 4.5.** *Let  $A$  be a Lipschitz domain on  $S^{k-1}$ ,  $2 \leq k \leq N-1$ , and let  $D_A$  be the  $k$ -dihedron with opening  $A$ . Let  $u$  be a positive solution of (1.1) in  $D_A^R$ , for some  $R > 0$ . Suppose that  $F = \mathcal{S}(u) \subset d_A^R$  and let  $Q$  be an open neighborhood of  $F$  such that  $\bar{Q} \subset d_A^R$ . (Recall that  $d_A^R = d_A \cap B_R^{N-k}(0)$  is an open subset of  $d_A$ .) Let  $\mu$  be the trace of  $u$  on  $\mathcal{R}(u)$ .*

Let  $\eta \in W_0^{s,q'}(d_A^R)$  such that

$$(4.21) \quad 0 \leq \eta \leq 1, \quad \eta = 0 \quad \text{on } Q.$$

Employing the notation in the proof of Proposition 4.2, put

$$(4.22) \quad \zeta := \rho_A \psi_R H_R[\eta]^{q'}.$$

Then

$$(4.23) \quad \int_{D_A^R} u^q \zeta \, dx \leq c(1 + \|\eta\|_{W^{s,q'}(d_A)})^{q'} + \mu(d_A^R \setminus Q)^q,$$

$c$  independent of  $u$  and  $\eta$ .

*Proof.* First we prove (4.23) for  $\eta \in C_0^\infty(d_A^R)$ . Let  $\sigma_0$  be a point in  $A$  and let  $\{A_n\}$  be a Lipschitz exhaustion of  $A$ . If  $0 < \epsilon < \text{dist}(\partial A, \partial A_n) = \bar{\epsilon}_n$  then

$$\epsilon \sigma_0 + C_{A_n} \subset C_A.$$

Denote

$$D_A^{R',R''} = D_A \cap [|x'| < R'] \cap [|x''| < R''].$$

Pick a sequence  $\{\epsilon_n\}$  decreasing to zero such that  $0 < \epsilon_n < \min(\bar{\epsilon}_n/2^n, R/8)$ . Let  $u_n$  be the function given by

$$u_n(x'x'') = u(x' + \epsilon_n \sigma_0, x'') \quad \forall x \in D_{A_n}^{R_n, R}, \quad R_n = R - \epsilon_n.$$

Then  $u_n$  is a solution of (1.1) in  $D_{A_n}^{R_n, R}$  belonging to  $C^2(\bar{D}_{A_n}^{R_n, R})$  and we denote its boundary trace by  $h_n$ . Let

$$\zeta_n := \rho_{A_n} \psi_R H_R[\eta]^{q'},$$

with  $\psi_R$  and  $H_R[\eta]$  as in the proof of Proposition 4.2. By Proposition 5.2

$$(4.24) \quad - \int_{D_{A_n}^{R_n, R}} \mathbb{P}[h_n] \Delta \zeta_n dx = \int_{B_R^{N-k}(0)} \eta^{q'} h_n d\omega_n$$

where  $\omega_n$  is the harmonic measure on  $d_{A_n}^R$  relative to  $D_{A_n}^{R_n, R}$ . (Note that  $d_{A_n}^R = d_A^R$  and we may identify it with  $B_R^{N-k}(0)$ .) Hence

$$(4.25) \quad \int_{D_{A_n}^{R_n, R}} (-u_n \Delta \zeta_n + u_n^q \zeta_n) dx = - \int_{B_R^{N-k}(0)} \eta^{q'} h_n d\omega_n.$$

Further,

$$\int_{B_R^{N-k}(0)} \eta^{q'} h_n d\omega_n \rightarrow \int_{B_R^{N-k}(0)} \eta^{q'} d\mu \leq \mu(d_A^R \setminus Q),$$

because  $\eta = 0$  in  $Q$ . By (4.13), (4.17) we obtain,

$$(4.26) \quad \left| \int_{D_{A_n}^{R_n, R}} u_n \Delta \zeta_n dx \right| \leq c \left( \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{\frac{1}{q}} \left( \left( \int_{D_{A_n}^{R_n, R}} \zeta_n dx \right)^{\frac{1}{q'}} + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))} \right).$$

From the definition of  $\zeta_n$  it follows that

$$\int_{D_{A_n}^{R_n, R}} \zeta_n dx \leq \int_{D_{A_n}^{R_n, R}} \rho_n dx \rightarrow \int_{D_A^R} \rho dx,$$

where  $\rho$  (resp.  $\rho_n$ ) is the first eigenfunction of  $-\Delta$  in  $D_A^R$  (resp.  $D_{A_n}^{R_n, R}$ ) normalized by 1 at some  $x_0 \in D_{A_1}^{R_1, R}$ . Therefore, by (4.25),

$$\int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \leq c \left( \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{\frac{1}{q}} (1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))}) + \mu(d_A^R \setminus Q).$$

This implies

$$(4.27) \quad \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \leq c (1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})^{q'} + \mu(d_A^R \setminus Q)^q.$$

To verify this fact, put

$$m = \left( \int_{D_{A_n}^{R_n, R}} u_n^q \zeta_n dx \right)^{1/q}, \quad b = \mu(d_A^R \setminus Q), \quad a = c(1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})$$

so that (4.27) becomes

$$m^q - am - b \leq 0.$$

If  $b \leq m$  then

$$m^{q-1} - a - 1 \leq 0.$$

Therefore,

$$m \leq (a + 1)^{\frac{1}{q-1}} + b$$

which implies (4.27). Finally, by the lemma of Fatou we obtain (4.23) for  $\eta \in C_0^\infty$ . By continuity we obtain the inequality for any  $\eta \in W_0^{s, q'}$  satisfying (4.21).  $\square$

**Theorem 4.6.** *Let  $A$  be a Lipschitz domain on  $S^{k-1}$ ,  $2 \leq k \leq N - 1$ , and let  $D_A$  be the  $k$ -dihedron with opening  $A$ . Let  $E$  be a compact subset of  $d_A^R$  and let  $u$  be a non-negative solution of (1.1) in  $D_A^R$  (for some  $R > 0$ ) such that  $u$  vanishes on  $\partial D_A^R \setminus E$ . Then*

$$(4.28) \quad C_{s, q'}^{N-k}(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'} \implies u = 0,$$

where  $C_{s, q'}^{N-k}$  denotes the Bessel capacity with the indicated indices in  $\mathbb{R}^{N-k}$ .

*Proof.* By Proposition 4.2, (4.28) holds under the additional assumption

$$(4.29) \quad \int_{D_A^R} u^q \rho_R \rho_A^R dx < \infty.$$

Indeed, by [22, Proposition 4.1], (4.29) implies that the solution  $u$  possesses a boundary trace  $\mu$  on  $\partial D_A^R$ . By assumption,  $\mu(\partial D_A^R \setminus E) = 0$ . Therefore, by Proposition 4.3, the fact that  $C_{s, q'}^{N-k}(E) = 0$  implies that  $\mu(E) = 0$ . Thus  $\mu = 0$  and hence  $u = 0$ .

We show that, under the conditions of the theorem, if  $C_{s, q'}^{N-k}(E) = 0$  then (4.29) holds.

By Proposition 4.5, for every  $\eta \in W_0^{s, q'}(d_A^R)$  such that  $0 \leq \eta \leq 1$  and  $\eta = 0$  in a neighborhood of  $E$ ,

$$(4.30) \quad \int_{D_A^R} u^q \zeta dx \leq c(1 + \|\eta\|_{W^{s, q'}(B_R^{N-k}(0))})^{q'},$$

for  $\zeta$  as in (4.22). (Here we use the assumption that  $u = 0$  on  $\partial D_A^R \setminus E$ .)

Let  $a > 0$  be sufficiently small so that  $E \subset B_{(1-4a)R}^{N-k}(0)$ . Pick a sequence  $\{\phi_n\}$  in  $C_0^\infty(\mathbb{R}^{N-k})$  such that, for each  $n$ , there exists a neighborhood  $Q_n$

of  $E$ ,  $\bar{Q}_n \subset B_{(1-3a)R}^{N-k}(0)$  and

$$(4.31) \quad \begin{aligned} 0 &\leq \phi_n \leq 1 \text{ everywhere, } \phi_n = 1 \text{ in } Q_n, \\ \tilde{\phi}_n &:= \phi_n \chi_{[|x''| < (1-2a)R]} \in C_0^\infty(\mathbb{R}^{N-k}), \\ \|\tilde{\phi}_n\|_{W^{s,q'}(\mathbb{R}^{N-k})} &\rightarrow 0 \text{ as } n \rightarrow \infty \\ \eta_n &:= (1 - \phi_n)|_{[|x''| < R]} \in C_0^\infty(d_A^R), \\ \eta_n &= 0 \text{ in } [(1-a)R < |x''| < R]. \end{aligned}$$

Such a sequence exists because  $C_{s,q'}^{N-k}(E) = 0$ . Applying (4.30) to  $\eta_n$  we obtain,

$$(4.32) \quad \sup \int_{D_A^R} u^q \zeta_n dx \leq c < \infty,$$

where  $\zeta_n = \rho_A \psi_R H_R^{q'}[\eta_n]$  (see (4.22)). By taking a subsequence we may assume that  $\{\eta_n\}$  converges (say to  $\eta$ ) in  $L^{q'}(B_{(1-2a)R}^{N-k}(0))$  and consequently  $H[\eta_n] \rightarrow H[\eta]$  in the sense that

$$H_R[\eta_n](x', \cdot) = w_{n,R}(y^2, \cdot) \rightarrow w_R(y^2, \cdot) = H_R[\eta](x', \cdot) \text{ in } L^q$$

uniformly with respect to  $y = |x'|$ . It follows that

$$(4.33) \quad \int_{D_A^R} u^q \zeta dx < \infty, \quad \zeta = \rho_A \psi_R H_R^{q'}[\eta].$$

As  $\tilde{\phi}_n \rightarrow 0$  in  $W^{s,q'}(\mathbb{R}^{N-k})$  it follows that  $\phi_n \rightarrow 0$  and hence  $\eta_n \rightarrow 1$  a.e. in  $B_{(1-2a)R}^{N-k}(0)$ . Thus  $\eta = 1$  in this ball,  $\eta = 0$  in  $[(1-a)R < |x''| < R]$  and  $0 \leq \eta \leq 1$  everywhere.

Consequently, given  $\delta > 0$ , there exists an  $N$ -dimensional neighborhood  $O$  of  $d_A \cap B_{(1-2a)R}^{N-k}(0)$  such that

$$1 - \delta < H_R[\eta] < 1 \text{ and } 1 - \delta < \psi_R / \rho_A^R < 1 \text{ in } O.$$

Therefore (4.33) implies that

$$(4.34) \quad \int_{D_A^{(1-3a)R}} u^q \rho^R \rho_A^R dx \leq c < \infty.$$

Recall that the trace of  $u$  on  $\partial D_A^R \setminus d_A^{(1-4a)R}$  is zero. Therefore  $u$  is bounded in  $D_A^R \setminus D_A^{(1-3a)R}$ . This fact and (4.34) imply (4.29).  $\square$

**Definition 4.7.** Let  $\Omega$  be a bounded Lipschitz domain. Denote by  $\rho$  the first eigenfunction of  $-\Delta$  in  $\Omega$  normalized by  $\rho(x_0) = 1$  for a fixed point  $x_0 \in \Omega$ .

For every compact set  $K \subset \partial\Omega$  we define

$$M_{\rho,q}(K) = \{\mu \in \mathfrak{M}(\partial\Omega) : \mu \geq 0, \mu(\partial\Omega \setminus K) = 0, \mathbb{K}[\mu] \in L_\rho^q(\Omega)\}$$

and

$$\tilde{C}_{\rho,q'}(K) = \sup\{\mu(K)^q : \mu \in M_{\rho,q}(K), \int_{\Omega} \mathbb{K}[\mu]^q \rho \, dx = 1\}.$$

Finally we denote by  $C_{\rho,q'}$  the outer measure generated by the above functional.

The following statement is verified by standard arguments:

**Lemma 4.8.** *For every compact  $K \subset \partial\Omega$ ,  $C_{\rho,q'}(K) = \tilde{C}_{\rho,q'}(K)$ . Thus  $C_{\rho,q'}$  is a capacity and,*

$$(4.35) \quad C_{\rho,q'}(K) = 0 \iff M_{\rho,q}(K) = \{0\}.$$

**Theorem 4.9.** *Let  $\Omega$  be a bounded polyhedron in  $\mathbb{R}^N$ . A compact set  $K \subset \partial\Omega$  is removable if and only if*

$$(4.36) \quad C_{s(k,j),q'}(K \cap L_{k,j}) = 0,$$

for  $k = 1, \dots, N$   $j = 1, \dots, n_k$ , where  $s(k, j)$  is defined as in (4.18). This condition is equivalent to

$$(4.37) \quad C_{\rho,q'}(K) = 0.$$

A measure  $\mu \in \mathfrak{M}(\partial\Omega)$  is  $q$ -good if and only if it does not charge sets with  $C_{\rho,q'}$ -capacity zero.

*Proof.* The first assertion is an immediate consequence of Proposition 4.1 and Theorem 4.6. The second assertion follows from the fact that

$$C_{\rho,q'}(K \cap L_{k,j}) = C_{s(k,j),q'}(K \cap L_{k,j}).$$

The third assertion follows from Theorem 4.4 and the previous statement.  $\square$

## 5. APPENDIX

If  $\Omega$  is a bounded Lipschitz domain we say that  $\{\Omega_n\}$  is *Lipschitz exhaustion* of  $\Omega$  if, for every  $n$ ,  $\Omega_n$  is Lipschitz and

$$(5.1) \quad \Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}, \quad \Omega = \cup \Omega_n, \quad \mathbb{H}_{N-1}(\partial\Omega_n) \rightarrow \mathbb{H}_{N-1}(\partial\Omega).$$

If  $\omega_n$  (respectively  $\omega$ ) is the harmonic measure in  $\Omega_n$  (respectively  $\Omega$ ) relative to  $x_0 \in \Omega_1$ , then, for every  $Z \in C(\bar{\Omega})$ ,

$$(5.2) \quad \lim_{n \rightarrow \infty} \int_{\partial\Omega_n} Z \, d\omega_n = \int_{\partial\Omega} Z \, d\omega.$$

[22, Lemma 2.1]. Furthermore, if  $\mu$  is a bounded Borel measure on  $\partial\Omega$  and  $v := \mathbb{K}^{\Omega}[\mu]$ , there holds

$$(5.3) \quad \lim_{n \rightarrow \infty} \int_{\partial\Omega_n} Z v \, d\omega_n = \int_{\partial\Omega} Z \, d\mu,$$

by [22, Lemma 2.2]. The following estimates are proved in [22, Lemma 2.3]

**Proposition 5.1.** *Let  $\mu$  be bounded Borel measures on  $\partial\Omega$ . Then  $\mathbb{K}[\mu] \in L^1_\rho(\Omega)$  and there exists a constant  $C = C(\Omega)$  such that*

$$(5.4) \quad \|\mathbb{K}[\mu]\|_{L^1_\rho(\Omega)} \leq C \|\mu\|_{\mathfrak{M}(\partial\Omega)}.$$

*In particular if  $h \in L^1(\partial\Omega; \omega)$  then*

$$(5.5) \quad \|\mathbb{P}[h]\|_{L^1_\rho(\Omega)} \leq C \|h\|_{L^1(\partial\Omega; \omega)}.$$

The next result is useful in a  $k$ -dimensional dihedron in the case where  $\mu$  is concentrated on the edge. In such a case one can find, for every smooth function on the edge, a lifting  $Z$  such that condition (5.6) holds.

**Proposition 5.2.** *We denote by  $G^{\Omega_n}$  (respectively  $G^\Omega$ ) the Green function in  $\Omega_n$  (respectively  $\Omega$ ). Let  $v$  be a positive harmonic function in  $\Omega$  with boundary trace  $\mu$ . Let  $Z \in C^2(\bar{\Omega})$  and let  $\tilde{G} \in C^\infty(\Omega)$  be a function that coincides with  $x \mapsto G(x, x_0)$  in  $Q \cap \Omega$  for some neighborhood  $Q$  of  $\partial\Omega$  and some fixed  $x_0 \in \Omega$ . In addition assume that there exists a constant  $c > 0$  such that*

$$(5.6) \quad |\nabla Z \cdot \nabla \tilde{G}| \leq c\rho.$$

*Under these assumptions, if  $\zeta := Z\tilde{G}$  then*

$$(5.7) \quad -\int_{\Omega} v\Delta\zeta \, dx = \int_{\partial\Omega} Z d\mu.$$

*Proof.* Let  $\{\Omega_n\}$  be a  $C^1$  exhaustion of  $\Omega$ . We assume that  $\partial\Omega_n \subset Q$  for all  $n$  and  $x_0 \in \Omega_1$ . Let  $\tilde{G}_n(x)$  be a function in  $C^1(\Omega_n)$  such that  $\tilde{G}_n$  coincides with  $G^{\Omega_n}(\cdot, x_0)$  in  $Q \cap \Omega_n$ ,  $\tilde{G}_n(\cdot, x_0) \rightarrow \tilde{G}(\cdot, x_0)$  in  $C^2(\Omega \setminus Q)$  and  $\tilde{G}_n(\cdot, x_0) \rightarrow \tilde{G}(\cdot, x_0)$  in  $\text{Lip}(\Omega)$ . If  $\zeta_n = Z\tilde{G}_n$  we have,

$$\begin{aligned} -\int_{\Omega_n} v\Delta\zeta_n \, dx &= \int_{\partial\Omega_n} v\partial_{\mathbf{n}}\zeta \, dS = \int_{\partial\Omega_n} vZ\partial_{\mathbf{n}}\tilde{G}_n(\xi, x_0) \, dS \\ &= \int_{\partial\Omega_n} vZP^{\Omega_n}(x_0, \xi) \, dS = \int_{\partial\Omega_n} vZ \, d\omega_n. \end{aligned}$$

By (5.3),

$$\int_{\partial\Omega_n} vZ \, d\omega_n \rightarrow \int_{\partial\Omega} Z \, d\mu.$$

On the other hand, in view of (5.6), we have

$$\Delta\zeta_n = \tilde{G}_n\Delta Z + Z\Delta\tilde{G}_n + 2\nabla Z \cdot \nabla\tilde{G}_n \rightarrow \Delta Z$$

in  $L^1_\rho(\Omega)$ ; therefore,

$$-\int_{\Omega_n} v\Delta\zeta_n \, dx \rightarrow -\int_{\Omega} v\Delta\zeta \, dx.$$

□

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