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# A scaling proof for Walsh's Brownian motion extended arc-sine law

S. Vakeroudis <sup>\*†</sup> M. Yor <sup>\*‡</sup>

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## Abstract

We present a new proof of the extended arc-sine law related to Walsh's Brownian motion, known also as Brownian spider. The main argument mimics the scaling property used previously, in particular by D. Williams [12], in the 1-dimensional Brownian case, which can be generalized to the multivariate case. A discussion concerning the time spent positive by a skew Bessel process is also presented.

**AMS 2010 subject classification:** Primary: 60J60, 60J65;  
secondary: 60J70, 60G52.

**Key words:** Arc-sine law, Brownian spider, Skew Bessel process, Stable variables, Subordinators, Walsh Brownian motion.

## 1 Introduction

*a)* Recently, some renewed interest has been shown (see e.g. [9]) in the study of the law of the vector

$$\vec{A}_1 = \left( \int_0^1 1_{(W_s \in I_i)} ds; i = 1, 2, \dots, n \right),$$

where  $(W_s)$  denotes a Walsh Brownian motion, also called Brownian spider (see [10] for Walsh's lyrical description) living on  $I = \bigcup_{i=1}^n I_i$ , the union of  $n$  half-lines of the plane, meeting at 0.

To keep the discussion as simple as possible, we assume  $p_1 = p_2 = \dots = p_n = 1/n$ , i.e.: when returning to 0, Walsh's Brownian motion chooses, so to speak, its "new" ray

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in a uniform way. In fact, excursion theory and/or the computation of the semi-group of Walsh's Brownian motion (see [1]) allow to define the process rigorously.

Since  $(d(0, W_s); s \geq 0)$ , for  $d$  the Euclidian distance, is a reflecting Brownian motion, we denote by  $(L_t, t \geq 0)$  the unique continuous increasing process such that:  $(d(0, W_s) - L_s; s \geq 0)$  is a  $\mathcal{W}_s = \sigma\{W_u, u \leq s\}$  Brownian motion.

Let

$$\vec{A}_t = (A_t^{(1)}, A_t^{(2)}, \dots, A_t^{(n)})$$

denote the random vector of the times spent in the different rays. In Section 2 we will state and prove our main Theorem concerning the distribution of  $\vec{A}_t$  for a fixed time. Section 3 deals with the general case of stable variables, First, we recall some known results and then we state and prove our main Theorem. Finally, Section 4 is devoted to some remarks and comments.

### b) Reminder on the arc-sine law:

A random variable  $A$  follows the arc-sine law if it admits the density:

$$\frac{1}{\pi \sqrt{x(1-x)}} 1_{[0,1)}(x). \quad (1)$$

Some well known representations of an arc-sine variable are the following:

$$A \stackrel{(law)}{=} \frac{N^2}{N^2 + \hat{N}^2} \stackrel{(law)}{=} \cos^2(U) \stackrel{(law)}{=} \frac{T}{T + \hat{T}} \stackrel{(law)}{=} \frac{1}{1 + C^2}, \quad (2)$$

where  $N, \hat{N} \sim \mathcal{N}(0, 1)$  and are independent,  $U$  is uniform on  $[0, 2\pi]$ ,  $T$  and  $\hat{T}$  stand for two iid stable  $(1/2)$  unilateral variables, and  $C$  is a standard Cauchy variable.

With  $(B_t, t \geq 0)$  denoting a real Brownian motion, two well known examples of arc-sine distributed variables are:

$$g_1 = \sup\{t < 1 : B_t = 0\}, \quad \text{and} \quad A_1^+ = \int_0^1 ds 1_{(B_s > 0)},$$

a result that is due to Paul Lévy (see e.g. [6, 7, 13]).

c) This point may help to motivate Section 3. From (2), one may think that more general studies of the time spent positive by diffusions may bring 2 independent gamma variables (as  $N^2$  and  $\hat{N}^2$  are distributed like two independent gamma variables of parameter  $1/2$ ), or 2 independent stable  $(\mu)$  variables. It turns out that it is the second case which seems to occur most naturally. We devote Section 3 to this case.

## 2 Main result

Our aim is to prove the following:

**Theorem 2.1.** *The random vectors  $\vec{A}_T/T$  for:*

(i)  $T = t$ ; (ii)  $T = \alpha_s^{(j)} = \inf\{t : A_t^{(j)} > s\}$ ; (iii)  $T = \tau_l$ , the inverse local times,

have the same distribution. In particular, it is specified by the iid stable (1/2) subordinators:

$$((A_{\tau_l}^{(j)}, l \geq 0); 1 \leq j \leq n).$$

Hence:

$$\overrightarrow{A_1} \stackrel{(law)}{=} \frac{\overrightarrow{A_{\tau_1}}}{\tau_1}, \quad (3)$$

and consequently:

$$\overrightarrow{A_1} \stackrel{(law)}{=} \left( \frac{T_j}{\sum_{i=1}^n T_i}; j \leq n \right), \quad (4)$$

where  $T_j$  are iid, stable (1/2) variables.

Consequently, the law of the right-hand side of (3) is easily computed, and so is its left-hand side. We refer the reader to [2] for explicit expressions of this law, which for  $n = 2$  reduces to the classical arc-sine law.

### Proof of Theorem 2.1.

a) Clearly, (ii) plays a kind of "bridge" between (i) and (iii).

b) We shall work with  $(\alpha_s^{(1)}, s \geq 0)$ , the inverse of  $(A_t^{(1)}, t \geq 0)$ . It is more convenient to use the notation  $(\alpha_s^{(+)}, s \geq 0)$  for  $(\alpha_s^{(1)}, s \geq 0)$ . We then follow the main steps of [13] (Section 3.4, p. 42), which themselves are inspired by Williams [12]; see also Watanabe (Proposition 1 in [11]) and Mc Kean [8].

$(A_t^{(j)})$  denotes the time spent in  $I_j$ , for any  $j \neq 1$ . Since

$$\begin{cases} A_{\alpha_1^{(+)}}^{(j)} = A_{\tau(L_{\alpha_1^{(+)}})}^{(j)} \stackrel{(law)}{=} (L_{\alpha_1^{(+)}})^2 A_{\tau_1}^{(j)}, \\ \alpha_1^{(+)} = 1 + \sum_j A_{\alpha_1^{(+)}}^{(j)}, \\ \text{and} \\ \text{for every } u, t \geq 0, \quad (L_{\alpha_u^{(+)}}^2 < t) = (u < A_{\tau_{\sqrt{t}}}^{(1)}), \end{cases}$$

and invoking the scaling property, we can write jointly for all  $j$ 's:

$$\begin{aligned} \left( A_{\alpha_1^{(+)}}^{(j)}, L_{\alpha_1^{(+)}}^2, \alpha_1^{(+)} \right) &\stackrel{(law)}{=} \left( L_{\alpha_1^{(+)}}^2 A_{\tau_1}^{(j)}, L_{\alpha_1^{(+)}}^2, 1 + \sum_j L_{\alpha_1^{(+)}}^2 A_{\tau_1}^{(j)} \right) \\ &\stackrel{(law)}{=} \left( \frac{A_{\tau_1}^{(j)}}{A_{\tau_1}^{(1)}}, \frac{1}{A_{\tau_1}^{(1)}}, \frac{\tau_1}{A_{\tau_1}^{(1)}} \right), \end{aligned} \quad (5)$$

and now we deduce:

$$\frac{1}{\alpha_1^{(+)}} \left( A_{\alpha_1^{(+)}}^{(j)}, L_{\alpha_1^{(+)}}^2 \right) \stackrel{(law)}{=} \frac{1}{\tau_1} (A_{\tau_1}^{(j)}, 1). \quad (6)$$

With the help of the scaling Lemma below, we obtain:

$$\begin{aligned} E \left[ 1_{(W_1 \in I_1)} f(\vec{A}_1, L_1^2) \right] &= E \left[ \frac{1}{\alpha_1^{(+)} } f \left( \frac{\vec{A}_{\alpha_1^{(+)}}}{\alpha_1^{(+)}}, \frac{L_{\alpha_1^{(+)}}^2}{\alpha_1^{(+)}} \right) \right] \\ &\stackrel{\text{from (5)}}{=} E \left[ \frac{A_{\tau_1}^{(1)}}{\tau_1} f \left( \frac{\vec{A}_{\tau_1}}{\tau_1}, \frac{1}{\tau_1} \right) \right]. \end{aligned} \quad (7)$$

Replacing  $I_1$  by  $I_m$ , for any  $m \in \{2, \dots, n\}$ , and adding the  $m$  quantities found in (7), we obtain:

$$E \left[ f(\vec{A}_1, L_1^2) \right] = E \left[ f \left( \frac{\vec{A}_{\tau_1}}{\tau_1}, \frac{1}{\tau_1} \right) \right],$$

and from (6) this is also:

$$E \left[ f \left( \frac{\vec{A}_{\alpha_1^{(+)}}}{\alpha_1^{(+)}}, \frac{L_{\alpha_1^{(+)}}^2}{\alpha_1^{(+)}} \right) \right],$$

which proves (3).

Equality in law (4) follows now easily. Indeed, we denote by  $\nu$  the Itô measure of the Brownian spider, and we have:

$$\nu = \frac{1}{n} \sum_{j=1}^n \nu_j, \quad (8)$$

where  $\nu_j$  is the canonical image of  $\mathbf{n}$ , the standard Itô measure of the space of the excursions of the Brownian spider, on the space of the excursions on  $I_j$ . Hence, with  $\lambda_j$ ,  $j = 1, \dots, n$  denoting positive constants:

$$\begin{aligned} E \left[ \exp \left( - \sum_{j=1}^n \lambda_j A_{\tau_1}^{(j)} \right) \right] &= \exp \left( - \frac{1}{n} \sum_{j=1}^n \int \nu_j(d\varepsilon_j) (1 - e^{-\lambda_j \nu_j}) \right) \\ &= \exp \left( - \frac{1}{n} \sum_{j=1}^n \sqrt{2\lambda_j} \right), \end{aligned}$$

thus:

$$\vec{A}_{\tau_1} \stackrel{(law)}{=} \left( \frac{1}{n^2} T_j ; j \leq n \right),$$

which finishes the proof. ■

It now remains to state the scaling Lemma which played a role in (7), and which we lift from [13] (Corollary 1, p. 40) in a "reduced" form.

**Lemma 2.2. (Scaling Lemma)** Let  $U_t = \int_0^t ds \theta_s$ , with the pair  $(W, \theta)$  satisfying:

$$(W_{ct}, \theta_{ct}; t \geq 0) \stackrel{(law)}{=} (\sqrt{c}W_t, \theta_t; t \geq 0). \quad (9)$$

Then,

$$E[F(W_u, u \leq 1) \theta_1] = E\left[\frac{1}{\alpha_1} F\left(\frac{1}{\sqrt{\alpha_1}} W_{v\alpha_1}, v \leq 1\right)\right], \quad (10)$$

where  $\alpha_t = \inf\{s : U_s > t\}$ .

### 3 Stable subordinators

#### 3.1 Reminder and preliminaries on stable variables

In this Section, we consider  $S_\mu$  and  $S'_\mu$  two independent stable variables with exponent  $\mu \in (0, 1)$ , i.e. for every  $\lambda \geq 0$ , the Laplace transform of  $S_\mu$  is given by:

$$E[\exp(-\lambda S_\mu)] = \exp(-\lambda^\mu). \quad (11)$$

Concerning the law of  $S_\mu$ , there is no simple expression for its density (except for the case  $\mu = 1/2$ ; see e.g. Exercise 4.20 in [3]). However, we have that, for every  $s < 1$  (see e.g. [15] or Exercise 4.19 in [3]):

$$E[(S_\mu)^{\mu s}] = \frac{\Gamma(1-s)}{\Gamma(1-\mu s)}. \quad (12)$$

We consider now the random variable of the ratio of two  $\mu$ -stable variables:

$$X = \frac{S_\mu}{S'_\mu}. \quad (13)$$

Following e.g. Exercise 4.23 in [3], we have respectively the following formulas for the Stieltjes and the Mellin transforms of  $X$ :

$$E\left[\frac{1}{1+sX}\right] = \frac{1}{1+s^\mu}, \quad s \geq 0, \quad (14)$$

$$E[X^s] = \frac{\sin(\pi s)}{\mu \sin(\frac{\pi s}{\mu})}, \quad 0 < s < \mu. \quad (15)$$

Moreover, the density of the random variable  $X^\mu$  is given by (see e.g. [14, 5] or Exercise 4.23 in [3]):

$$P(X^\mu \in dy) = \frac{\sin(\pi\mu)}{\pi\mu} \frac{dy}{y^2 + 2y \cos(\pi\mu) + 1}, \quad y \geq 0, \quad (16)$$

or equivalently:

$$\left(\frac{S_\mu}{S'_\mu}\right)^\mu = (C_\mu | C_\mu > 0), \quad (17)$$

where, with  $C$  denoting a standard Cauchy variable and  $U$  a uniform variable in  $[0, 2\pi]$ ,

$$C_\mu = \sin(\pi\mu)C - \cos(\pi\mu) \stackrel{(law)}{=} \frac{\sin(\pi\mu - U)}{U}.$$

### 3.2 The case of 2 stable variables

We turn now our study to the random variable:

$$A = \frac{S'_\mu}{S'_\mu + S_\mu} = \frac{1}{1 + X}, \quad (18)$$

**Theorem 3.1.** *The density function of the random variable  $A$  is given by:*

$$P(A \in dz) = \frac{\sin(\pi\mu)}{\pi} \frac{dz}{z(1-z) \left[ \left(\frac{1-z}{z}\right)^\mu + \left(\frac{z}{1-z}\right)^\mu + 2\cos(\pi\mu) \right]}, \quad z \in [0, 1]. \quad (19)$$

**Proof of Theorem 3.1.**

Identity (18) is equivalent to:

$$X = \frac{1}{A} - 1.$$

Hence, (14) yields:

$$E \left[ \frac{1}{1 + sX} \right] = E \left[ \frac{A}{(1-s)A + s} \right] = \frac{1}{1 + s^\mu}.$$

We consider now a test function  $f$  and invoking the density (16) we have ( $\nu = \frac{1}{\mu} > 1$ ):

$$E \left[ f \left( \frac{1}{1 + X} \right) \right] = \frac{\sin(\pi\mu)}{\pi\mu} \int_0^\infty \frac{dy}{y^2 + 2y\cos(\pi\mu) + 1} f \left( \frac{1}{1 + y^\nu} \right).$$

Changing the variables  $z = \frac{1}{1+y^\nu}$ , we deduce:

$$E[f(A)] = \frac{\sin(\pi\mu)}{\pi} \int_0^1 \frac{dz(1-z)^{\mu-1}}{z^{\mu+1}} f(z) \Delta(z),$$

where:

$$\begin{aligned} \Delta(z) &= \frac{1}{(z^{-1} - 1)^{2\mu} + 2(z^{-1} - 1)^\mu \cos(\pi\mu) + 1} \\ &= \frac{z^{2\mu}}{(1 - z)^{2\mu} + 2(1 - z)^\mu z^\mu \cos(\pi\mu) + z^{2\mu}}, \end{aligned}$$

and (19) follows easily. ■

In Figure 1, we have plotted the density function  $g$  of  $A$ , for several values of  $\mu$ .

**Remark 3.2.** *Similar discussions have been made in [4] in the framework of a skew Bessel process with dimension  $2 - 2\alpha$  and skewness parameter  $p$ . Formula (19) is a particular case of the formula in [4] for the density of the time spent positive (called  $f_{p,\alpha}$  in [4]).*

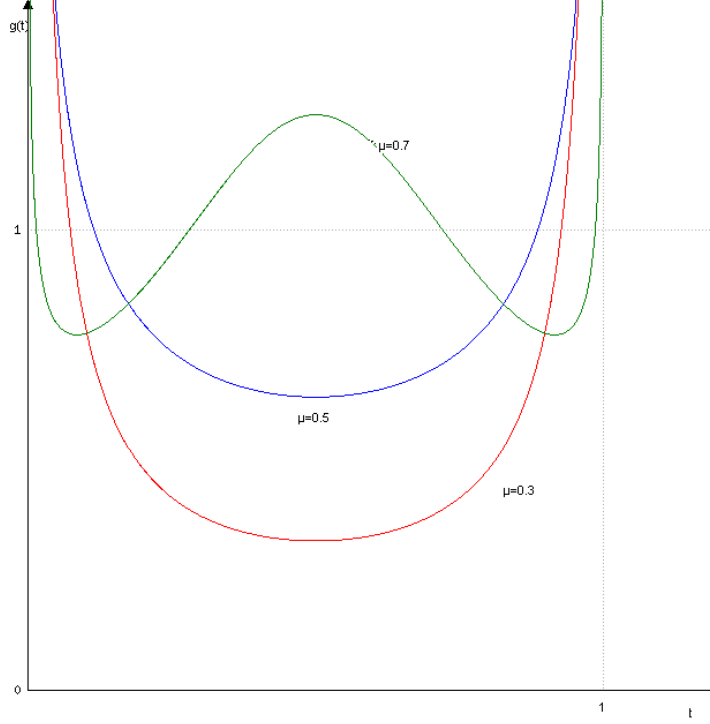


Figure 1: The density function  $g$  of  $A$ , for several values of  $\mu$ .

### 3.3 The case of many stable (1/2) variables

In this Subsection, we consider again  $n$  iid stable (1/2) variables, i.e.:  $T_1, \dots, T_n$ , and we will study the distribution of:

$$A_1^{(1)} = \frac{T_1}{T_1 + \dots + T_n} . \quad (20)$$

The following Theorem answers to an open question (and even in a more general sense) stated at the end of [9].

**Theorem 3.3.** *The density function of the random variable  $A_1^{(1)}$  is given by:*

$$P\left(A_1^{(1)} \in dz\right) = \frac{1}{\pi} \frac{dz}{\sqrt{z}\sqrt{1-z} \left[(n-1)z + \frac{1}{n-1}(1-z)\right]} , \quad z \in [0, 1]. \quad (21)$$

#### Proof of Theorem 3.3.

We first remark that, with  $C$  denoting a standard Cauchy variable, using e.g. (2):

$$A_1^{(1)} \stackrel{(law)}{=} \frac{T_1}{T_1 + (n-1)^2 T_2} \stackrel{(law)}{=} \frac{1}{1 + (n-1)^2 C^2} . \quad (22)$$

Hence, with  $f$  standing again for a test function, and invoking the density of a standard



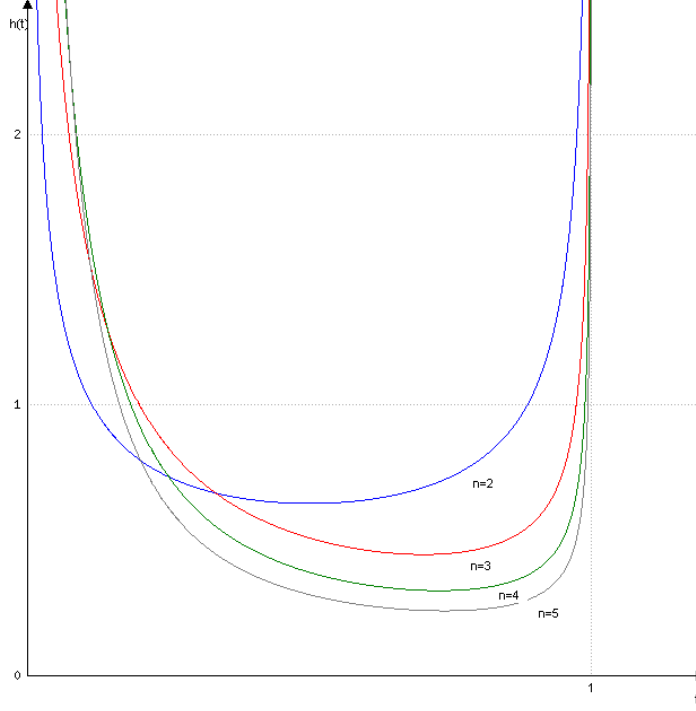


Figure 2: The density function  $h$  of  $A_1^{(1)}$ , for several values of  $n$ .

Cauchy variable, that is: for every  $x \in \mathbb{R}$ ,  $g(x) = \frac{1}{\pi(1+x^2)}$  we have:

$$\begin{aligned}
 E \left[ f \left( A_1^{(1)} \right) \right] &= E \left[ f \left( \frac{1}{1 + (n-1)^2 C^2} \right) \right] \\
 &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} f \left( \frac{1}{1 + (n-1)^2 x^2} \right) \\
 &\stackrel{x^2=y}{=} \frac{2}{\pi} \int_0^{\infty} \frac{dy}{2\sqrt{y}(1+y)} f \left( \frac{1}{1 + (n-1)^2 y} \right)
 \end{aligned}$$

Changing the variables  $z = \frac{1}{1+(n-1)^2 y}$ , we deduce:

$$E \left[ f \left( A_1^{(1)} \right) \right] = \frac{1}{\pi} \int_0^1 \frac{dz}{(n-1)^2 z^2} \frac{(n-1)\sqrt{z}}{\sqrt{z-1} \left( 1 + \frac{1}{(n-1)^2} \left( \frac{1}{z} - 1 \right) \right)} f(z),$$

and (21) follows easily. ■

Figure 2 presents the plot of the density function  $h$  of  $A_1^{(1)}$ , for several values of  $n$ .

**Corollary 3.4.** *The following convergence in law holds:*

$$n^2 A_1^{(1)}(n) \xrightarrow[n \rightarrow \infty]{(law)} C^2. \quad (23)$$

### Proof of Corollary 3.4.

It follows from Theorem 3.3 by simply remarking that  $C \stackrel{(law)}{=} C^{-1}$ . Hence:

$$n^2 A_1^{(1)}(n) = \frac{n^2}{1 + (n-1)^2 C^2} = \frac{1}{\frac{1}{n^2} + \left(\frac{n-1}{n}\right)^2 C^2} \xrightarrow{n \rightarrow \infty} \frac{1}{C^2} \stackrel{(law)}{=} C^2.$$

■

## 4 Conclusion and comments

We end up this article with some comments: usually, a scaling argument is "one-dimensional", as it involves a time-change. Exceptionally (or so it seems to the authors), here we could apply a scaling argument in a multivariate framework. We insist that the scaling Lemma plays a key role in our proof. The curious reader should also look at the totally different proof of this Theorem in [2], which mixes excursion theory and the Feynman-Kac method.

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