



HAL
open science

A scaling proof for Walsh's Brownian motion extended arc-sine law

Stavros Vakeroudis, Marc Yor

► **To cite this version:**

Stavros Vakeroudis, Marc Yor. A scaling proof for Walsh's Brownian motion extended arc-sine law. 2012. hal-00708858v1

HAL Id: hal-00708858

<https://hal.science/hal-00708858v1>

Preprint submitted on 16 Jun 2012 (v1), last revised 29 Dec 2012 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A scaling proof for Walsh's Brownian motion extended arc-sine law

S. Vakeroudis ^{*†} M. Yor ^{*‡}

June 16, 2012

Abstract

We present a new proof of the extended arc-sine law related to Walsh's Brownian motion, known also as Brownian spider. The main argument mimics the scaling property used previously, in particular by D. Williams [12], in the 1-dimensional Brownian case, which can be generalized to the multivariate case. A discussion concerning the time spent positive by a skew Bessel process is also presented.

AMS 2010 subject classification: Primary: 60J60, 60J65;
secondary: 60J70, 60G52.

Key words: Arc-sine law, Brownian spider, Skew Bessel process, Stable variables, Subordinators, Walsh Brownian motion.

1 Introduction

a) Recently, some renewed interest has been shown (see e.g. [9]) in the study of the law of the vector

$$\vec{A}_1 = \left(\int_0^1 1_{(W_s \in I_i)} ds; i = 1, 2, \dots, n \right),$$

where (W_s) denotes a Walsh Brownian motion, also called Brownian spider (see [10] for Walsh's lyrical description) living on $I = \bigcup_{i=1}^n I_i$, the union of n half-lines of the plane, meeting at 0.

To keep the discussion as simple as possible, we assume $p_1 = p_2 = \dots = p_n = 1/n$, i.e.: when returning to 0, Walsh's Brownian motion chooses, so to speak, its "new" ray

*Laboratoire de Probabilités et Modèles Aléatoires (LPMA) CNRS : UMR7599, Université Pierre et Marie Curie - Paris VI, Université Paris-Diderot - Paris VII, 4 Place Jussieu, 75252 Paris Cedex 05, France. E-mail: stavros.vakeroudis@upmc.fr

†Probability and Statistics Group, School of Mathematics, University of Manchester, Alan Turing Building, Oxford Road, Manchester M13 9PL, United Kingdom.

‡Institut Universitaire de France, Paris, France. E-mail: yormarc@aol.com

in a uniform way. In fact, excursion theory and/or the computation of the semi-group of Walsh's Brownian motion (see [1]) allow to define the process rigorously.

Since $(d(0, W_s); s \geq 0)$, for d the Euclidian distance, is a reflecting Brownian motion, we denote by $(L_t, t \geq 0)$ the unique continuous increasing process such that: $(d(0, W_s) - L_s; s \geq 0)$ is a $\mathcal{W}_s = \sigma\{W_u, u \leq s\}$ Brownian motion.

Let

$$\vec{A}_t = \left(A_t^{(1)}, A_t^{(2)}, \dots, A_t^{(n)} \right)$$

denote the random vector of the times spent in the different rays. In Section 2 we will state and prove our main Theorem concerning the distribution of \vec{A}_t for a fixed time. Section 3 deals with the general case of stable variables, First, we recall some known results and then we state and prove our main Theorem. Finally, Section 4 is devoted to some remarks and comments.

b) **Reminder on the arc-sine law:**

A random variable A follows the arc-sine law if it admits the density:

$$\frac{1}{\pi \sqrt{x(1-x)}} 1_{[0,1]}(x). \quad (1)$$

Some well known representations of an arc-sine variable are the following:

$$A \stackrel{(law)}{=} \frac{N^2}{N^2 + \hat{N}^2} \stackrel{(law)}{=} \cos^2(U) \stackrel{(law)}{=} \frac{T}{T + \hat{T}} \stackrel{(law)}{=} \frac{1}{1 + C^2}, \quad (2)$$

where $N, \hat{N} \sim \mathcal{N}(0, 1)$ and are independent, U is uniform on $[0, 2\pi]$, T and \hat{T} stand for two iid stable (1/2) unilateral variables, and C is a standard Cauchy variable.

With $(B_t, t \geq 0)$ denoting a real Brownian motion, two well known examples of arc-sine distributed variables are:

$$g_1 = \sup\{t < 1 : B_t = 0\}, \quad \text{and} \quad A_1^+ = \int_0^1 ds 1_{(B_s > 0)},$$

a result that is due to Paul Lévy (see e.g. [6, 7, 13]).

c) This point may help to motivate Section 3. From (2), one may think that more general studies of the time spent positive by diffusions may bring 2 independent gamma variables (as N and \hat{N} are distributed like two independent gamma variables of parameter 1/2), or 2 independent stable (μ) variables. It turns out that it is the second case which seems to occur most naturally. We devote Section 3 to this case.

2 Main result

Our aim is to prove the following:

Theorem 2.1. *The random vectors \vec{A}_T/T for:*

(i) $T = t$; (ii) $T = \alpha_s^{(j)} = \inf\{t : A_t^{(j)} > s\}$; (iii) $T = \tau_l$, the inverse local times,

have the same distribution. In particular, it is specified by the iid stable (1/2) subordinators:

$$\left((A_{\tau_l}^{(j)}, l \geq 0); 1 \leq j \leq n \right).$$

Hence:

$$\overrightarrow{A}_1 \stackrel{(law)}{=} \frac{\overrightarrow{A}_{\tau_1}}{\tau_1}, \quad (3)$$

and consequently:

$$\overrightarrow{A}_1 \stackrel{(law)}{=} \left(\frac{T_j}{\sum_{i=1}^n T_i}; j \leq n \right), \quad (4)$$

where T_j are iid, stable (1/2) variables.

Consequently, the law of the right-hand side of (3) is easily computed, and so is its left-hand side. We refer the reader to [2] for explicit expressions of this law, which for $n = 2$ reduces to the classical arc-sine law.

Proof of Theorem 2.1.

a) Clearly, (ii) plays a kind of "bridge" between (i) and (iii).

b) We shall work with $(\alpha_s^{(1)}, s \geq 0)$, the inverse of $(A_t^{(1)}, t \geq 0)$. It is more convenient to use the notation $(\alpha_s^{(+)}, s \geq 0)$ for $(\alpha_s^{(1)}, s \geq 0)$. We then follow the main steps of [13] (Section 3.4, p. 42), which themselves are inspired by Williams [12]; see also Watanabe (Proposition 1 in [11]) and Mc Kean [8].

$(A_t^{(j)})$ denotes the time spent in I_j , for any $j \neq 1$. Since

$$\begin{cases} A_{\alpha_1^{(+)}}^{(j)} = A_{\tau(L_{\alpha_1^{(+)}})}^{(j)} \stackrel{(law)}{=} (L_{\alpha_1^{(+)}})^2 A_{\tau_1}^{(j)}, \\ \alpha_1^{(+)} = 1 + \sum_j A_{\alpha_1^{(+)}}^{(j)}, \\ \text{and} \\ \text{for every } u, t \geq 0, \quad (L_{\alpha_u^{(+)}}^2 < t) = (u < A_{\tau\sqrt{t}}^{(1)}), \end{cases}$$

and invoking the scaling property, we can write jointly for all j 's:

$$\begin{aligned} \left(A_{\alpha_1^{(+)}}^{(j)}, L_{\alpha_1^{(+)}}^2, \alpha_1^{(+)} \right) &\stackrel{(law)}{=} \left(L_{\alpha_1^{(+)}}^2 A_{\tau_1}^{(j)}, L_{\alpha_1^{(+)}}^2, 1 + \sum_j L_{\alpha_1^{(+)}}^2 A_{\tau_1}^{(j)} \right) \\ &\stackrel{(law)}{=} \left(\frac{A_{\tau_1}^{(j)}}{A_{\tau_1}^{(1)}}, \frac{1}{A_{\tau_1}^{(1)}}, \frac{\tau_1}{A_{\tau_1}^{(1)}} \right), \end{aligned} \quad (5)$$

and now we deduce:

$$\frac{1}{\alpha_1^{(+)}} \left(A_{\alpha_1^{(+)}}^{(j)}, L_{\alpha_1^{(+)}}^2 \right) \stackrel{(law)}{=} \frac{1}{\tau_1} (A_{\tau_1}^{(j)}, 1). \quad (6)$$

With the help of the scaling Lemma below, we obtain:

$$\begin{aligned}
E \left[1_{(W_1 \in I_1)} f(\vec{A}_1, L_1^2) \right] &= E \left[\frac{1}{\alpha_1^{(+)} } f \left(\frac{\vec{A}_{\alpha_1^{(+)}}}{\alpha_1^{(+)}} , \frac{L_{\alpha_1^{(+)}}^2}{\alpha_1^{(+)}} \right) \right] \\
&\stackrel{\text{from(5)}}{=} E \left[\frac{A_{\tau_1}^{(1)}}{\tau_1} f \left(\frac{\vec{A}_{\tau_1}}{\tau_1} , \frac{1}{\tau_1} \right) \right].
\end{aligned} \tag{7}$$

Replacing I_1 by I_m , for any $m \in \{2, \dots, n\}$, and adding the m quantities found in (7), we obtain:

$$E \left[f(\vec{A}_1, L_1^2) \right] = E \left[f \left(\frac{\vec{A}_{\tau_1}}{\tau_1} , \frac{1}{\tau_1} \right) \right],$$

and from (6) this is also:

$$E \left[f \left(\frac{\vec{A}_{\alpha_1^{(+)}}}{\alpha_1^{(+)}} , \frac{L_{\alpha_1^{(+)}}^2}{\alpha_1^{(+)}} \right) \right],$$

which proves (3).

Equality in law (4) follows now easily. Indeed, we denote by ν the Itô measure of the Brownian spider, and we have:

$$\nu = \frac{1}{n} \sum_{j=1}^n \nu_j, \tag{8}$$

where ν_j is the canonical image of \mathbf{n} , the standard Itô measure of the space of the excursions of the Brownian spider, on the space of the excursions on I_j . Hence, with λ_j , $j = 1, \dots, n$ denoting positive constants:

$$\begin{aligned}
E \left[\exp \left(- \sum_{j=1}^n \lambda_j A_{\tau_1}^{(j)} \right) \right] &= \exp \left(- \frac{1}{n} \sum_{j=1}^n \int \nu_j(d\varepsilon_j) (1 - e^{-\lambda_j \nu_j}) \right) \\
&= \exp \left(- \frac{1}{n} \sum_{j=1}^n \sqrt{2\lambda_j} \right),
\end{aligned}$$

thus:

$$\vec{A}_1 \stackrel{(law)}{=} \left(\frac{1}{n^2} T_j ; j \leq n \right),$$

which finishes the proof. ■

It now remains to state the scaling Lemma which played a role in (7), and which we lift from [13] (Corollary 1, p. 40) in a "reduced" form.

Lemma 2.2. (Scaling Lemma) Let $U_t = \int_0^t ds \theta_s$, with the pair (W, θ) satisfying:

$$(W_{ct}, \theta_{ct}; t \geq 0) \stackrel{(law)}{=} (\sqrt{c}W_t, \theta_t; t \geq 0). \quad (9)$$

Then,

$$E[F(W_u, u \leq 1) \theta_1] = E \left[\frac{1}{\alpha_1} F \left(\frac{1}{\sqrt{\alpha_1}} W_{v\alpha_1}, v \leq 1 \right) \right], \quad (10)$$

where $\alpha_t = \inf\{s : U_s > t\}$.

3 Stable subordinators

3.1 Reminder and preliminaries on stable variables

In this Section, we consider S_μ and S'_μ two independent stable variables with exponent $\mu \in (0, 1)$, i.e. for every $\lambda \geq 0$, the Laplace transform of S_μ is given by:

$$E[\exp(-\lambda S_\mu)] = \exp(-\lambda^\mu). \quad (11)$$

Concerning the law of S_μ , there is no simple expression for its density (except for the case $\mu = 1/2$; see e.g. Exercise 4.20 in [3]). However, we have that, for every $s < 1$ (see e.g. [15] or Exercise 4.19 in [3]):

$$E[(S_\mu)^{\mu s}] = \frac{\Gamma(1-s)}{\Gamma(1-\mu s)}. \quad (12)$$

We consider now the random variable of the ratio of two μ -stable variables:

$$X = \frac{S_\mu}{S'_\mu}. \quad (13)$$

Following e.g. Exercise 4.23 in [3], we have respectively the following formulas for the Stieltjes and the Mellin transforms of X :

$$E \left[\frac{1}{1+sX} \right] = \frac{1}{1+s^\mu}, \quad s \geq 0, \quad (14)$$

$$E[X^s] = \frac{\sin(\pi s)}{\mu \sin(\frac{\pi s}{\mu})}, \quad 0 < s < \mu. \quad (15)$$

Moreover, the density of the random variable X^μ is given by (see e.g. [14, 5] or Exercise 4.23 in [3]):

$$P(X^\mu \in dy) = \frac{\sin(\pi\mu)}{\pi\mu} \frac{dy}{y^2 + 2y \cos(\pi\mu) + 1}, \quad y \geq 0, \quad (16)$$

or equivalently:

$$\left(\frac{S_\mu}{S'_\mu} \right)^\mu = (C_\mu | C_\mu > 0), \quad (17)$$

where, with C denoting a standard Cauchy variable and U a uniform variable in $[0, 2\pi)$,

$$C_\mu = \sin(\pi\mu)C - \cos(\pi\mu) \stackrel{(law)}{=} \frac{\sin(\pi\mu - U)}{U}.$$

3.2 The case of 2 stable variables

We turn now our study to the random variable:

$$A = \frac{S'_\mu}{S'_\mu + S_\mu} = \frac{1}{1 + X}, \quad (18)$$

Theorem 3.1. *The density function of the random variable A is given by:*

$$P(A \in dz) = \frac{\sin(\pi\mu)}{\pi} \frac{dz}{z(1-z) \left[\left(\frac{1-z}{z}\right)^\mu + \left(\frac{z}{1-z}\right)^\mu + 2\cos(\pi\mu) \right]}, \quad z \in [0, 1]. \quad (19)$$

Proof of Theorem 3.1.

Identity (18) is equivalent to:

$$X = \frac{1}{A} - 1.$$

Hence, (14) yields:

$$E \left[\frac{1}{1 + sX} \right] = E \left[\frac{A}{(1-s)A + s} \right] = \frac{1}{1 + s^\mu}.$$

We consider now a test function f and invoking the density (16) we have ($\nu = \frac{1}{\mu} > 1$):

$$E \left[f \left(\frac{1}{1 + X} \right) \right] = \frac{\sin(\pi\mu)}{\pi\mu} \int_0^\infty \frac{dy}{y^2 + 2y\cos(\pi\mu) + 1} f \left(\frac{1}{1 + y^\nu} \right).$$

Changing the variables $z = \frac{1}{1+y^\nu}$, we deduce:

$$E[f(A)] = \frac{\sin(\pi\mu)}{\pi} \int_0^1 \frac{dz(1-z)^{\mu-1}}{z^{\mu+1}} f(z) \Delta(z),$$

where:

$$\begin{aligned} \Delta(z) &= \frac{1}{(z^{-1} - 1)^{2\mu} + 2(z^{-1} - 1)^\mu \cos(\pi\mu) + 1} \\ &= \frac{z^{2\mu}}{(1-z)^{2\mu} + 2(1-z)^\mu z^\mu \cos(\pi\mu) + z^{2\mu}}, \end{aligned}$$

and (19) follows easily. ■

In Figure 1, we have plotted the density function g of A , for several values of μ .

Remark 3.2. *Similar discussions have been made in [4] in the framework of a skew Bessel process with dimension $2 - 2\alpha$ and skewness parameter p . Formula (19) is a particular case of the formula in [4] for the density of the time spent positive (called $f_{p,\alpha}$ in [4]).*

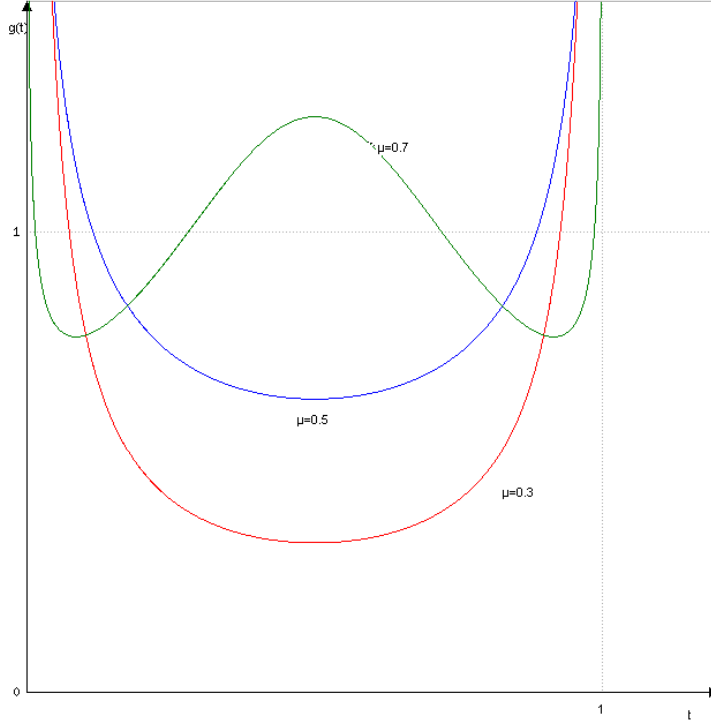


Figure 1: The density function g of A , for several values of μ .

3.3 The case of many stable (1/2) variables

In this Subsection, we consider again n iid stable (1/2) variables, i.e.: T_1, \dots, T_n , and we will study the distribution of:

$$A_1^{(1)} = \frac{T_1}{T_1 + \dots + T_n}. \quad (20)$$

The following Theorem answers to an open question (and even in a more general sense) stated at the end of [9].

Theorem 3.3. *The density function of the random variable $A_1^{(1)}$ is given by:*

$$P\left(A_1^{(1)} \in dz\right) = \frac{1}{\pi} \frac{dz}{\sqrt{z}\sqrt{1-z} \left[(n-1)z + \frac{1}{n-1}(1-z)\right]}, \quad z \in [0, 1]. \quad (21)$$

Proof of Theorem 3.3.

We first remark that, with C denoting a standard Cauchy variable, using e.g. (2):

$$A_1^{(1)} \stackrel{(law)}{=} \frac{T_1}{T_1 + (n-1)^2 T_2} \stackrel{(law)}{=} \frac{1}{1 + (n-1)^2 C^2}. \quad (22)$$

Hence, with f standing again for a test function, and invoking the density of a standard

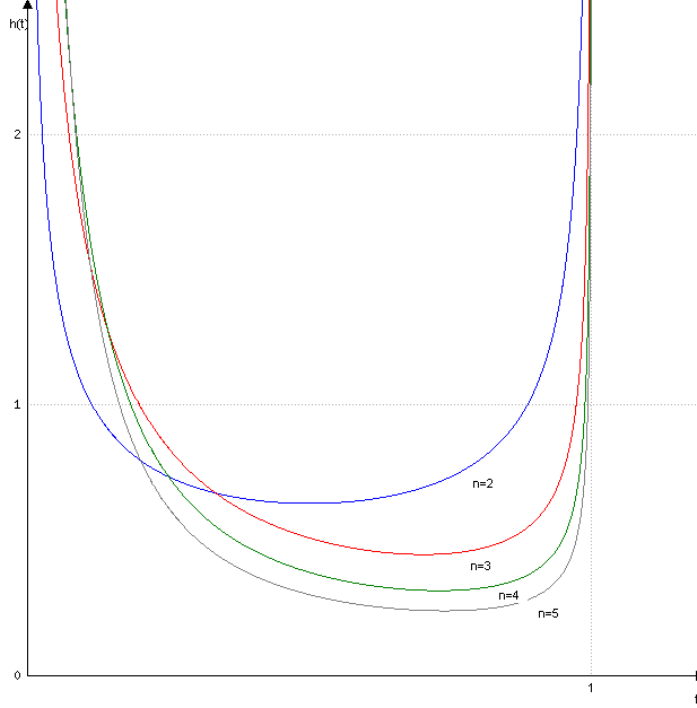


Figure 2: The density function h of $A_1^{(1)}$, for several values of n .

Cauchy variable, that is: for every $x \in \mathbb{R}$, $g(x) = \frac{1}{\pi(1+x^2)}$ we have:

$$\begin{aligned}
 E \left[f \left(A_1^{(1)} \right) \right] &= E \left[f \left(\frac{1}{1 + (n-1)^2 C^2} \right) \right] \\
 &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} f \left(\frac{1}{1 + (n-1)^2 x^2} \right) \\
 &\stackrel{x^2=y}{=} \frac{2}{\pi} \int_0^{\infty} \frac{dy}{2\sqrt{y}(1+y)} f \left(\frac{1}{1 + (n-1)^2 y} \right)
 \end{aligned}$$

Changing the variables $z = \frac{1}{1+(n-1)^2 y}$, we deduce:

$$E \left[f \left(A_1^{(1)} \right) \right] = \frac{1}{\pi} \int_0^1 \frac{dz}{(n-1)^2 z^2} \frac{(n-1)\sqrt{z}}{\sqrt{z-1} \left(1 + \frac{1}{(n-1)^2} \left(\frac{1}{z} - 1 \right) \right)} f(z),$$

and (21) follows easily. ■

Figure 2 presents the plot of the density function h of $A_1^{(1)}$, for several values of n .

Corollary 3.4. *The following convergence in law holds:*

$$n^2 A_1^{(1)}(n) \xrightarrow[n \rightarrow \infty]{(law)} C^2. \quad (23)$$

Proof of Corollary 3.4.

It follows from Theorem 3.3 by simply remarking that $C \stackrel{(law)}{=} C^{-1}$. Hence:

$$n^2 A_1^{(1)}(n) = \frac{n^2}{1 + (n-1)^2 C^2} = \frac{1}{\frac{1}{n^2} + \left(\frac{n-1}{n}\right)^2 C^2} \xrightarrow{n \rightarrow \infty} \frac{1}{C^2} \stackrel{(law)}{=} C^2.$$

■

4 Conclusion and comments

We end up this article with some comments: usually, a scaling argument is "one-dimensional", as it involves a time-change. Exceptionally (or so it seems to the authors), here we could apply a scaling argument in a multivariate framework. We insist that the scaling Lemma plays a key role in our proof. The curious reader should also look at the totally different proof of this Theorem in [2], which mixes excursion theory and the Feynman-Kac method.

Acknowledgements

The author S. Vakeroudis is very grateful to Professor R.A. Doney for the invitation at the University of Manchester as a Post Doc fellow where he prepared a part of this work.

References

- [1] M.T. Barlow, J.W. Pitman and M. Yor (1989). On Walsh's Brownian motion. *Sém. Prob. XXIII, Lect. Notes in Math.*, **1372**, Springer, Berlin Heidelberg New York, pp. 275-293.
- [2] M.T. Barlow, J.W. Pitman and M. Yor (1989). Une extension multidimensionnelle de la loi de l'arc sinus. *Sém. Prob. XXIII, Lect. Notes in Math.*, **1372**, Springer, Berlin Heidelberg New York, pp. 294-314.
- [3] L. Chaumont and M. Yor (2012). Exercises in Probability: A Guided Tour from Measure Theory to Random Processes, via Conditioning. Cambridge University Press, 2nd Edition.
- [4] Y. Kasahara and Y. Yano (2005). On a generalized arc-sine law for onedimensional diffusion processes. *Osaka J. Math.*, **42**, pp. 1-10.
- [5] J. Lamperti. An occupation time theorem for a class of stochastic processes. *Trans. Amer. Math. Soc.*, **88**, pp. 380-387.
- [6] P. Lévy (1939). Sur un problème de M. Marcinkiewicz. *C.R.A.S.*, **208**, pp. 318-321. Errata p. 776.
- [7] P. Lévy (1939). Sur certains processus stochastiques homogènes. *Compositio Math.*, t. **7**, pp. 283-339. Errata p. 776.

- [8] H.P. McKean (1975). Brownian local time. *Adv. Math.*, **16**, pp. 91-111.
- [9] V.G. Papanicolaou, E.G. Papageorgiou and D.C. Lepipas (2012). Random Motion on Simple Graphs. *Methodol. Comput. Appl. Probab.*, **14**, pp. 285-297.
- [10] J.B. Walsh (1978). A diffusion with discontinuous local time. *Astérisque*, **52-53**, pp. 37-45.
- [11] S. Watanabe (1995). Generalized arc-sine laws for one-dimensional diffusion processes and random walks. *Proc. Sympos. Pure Math.*, **57**, Stoch. Analysis, Cornell University (1993), Amer. Math. Soc., pp. 157-172.
- [12] D. Williams (1969). Markov properties of Brownian local time. *Bul. Am. Math. Soc.*, **76**, pp. 1035-1036.
- [13] M. Yor (1995). Local times and Excursions for Brownian motion: a concise introduction. Lecciones en Matemáticas Universidad Central de Venezuela, Caracas.
- [14] V.M. Zolotarev (1957). Mellin-Stieltjes transforms in probability theory. *Teor. Veroyatnost. i Primenen.*, **2**, pp. 444-469.
- [15] V.M. Zolotarev (1994). On the representation of the densities of stable laws by special functions. (In Russian.) *Teor. Veroyatnost. i Primenen.*, **39**, no. 2, pp. 429-437; translation in *Theory Probab. Appl.*, (1995) **39**, no. 2, pp. 354-362.