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To cite this version:
Stavros Vakeroudis, Marc Yor. A scaling proof for Walsh’s Brownian motion extended arc-sine law. 2012. hal-00708858v3

HAL Id: hal-00708858
https://hal.science/hal-00708858v3
Preprint submitted on 29 Dec 2012
A scaling proof for Walsh’s Brownian motion extended arc-sine law

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December 29, 2012

Abstract

We present a new proof of the extended arc-sine law related to Walsh’s Brownian motion, known also as Brownian spider. The main argument mimics the scaling property used previously, in particular by D. Williams [12], in the 1-dimensional Brownian case, which can be generalized to the multivariate case. A discussion concerning the time spent positive by a skew Bessel process is also presented.

AMS 2010 subject classification: Primary: 60J60, 60J65; secondary: 60J70, 60G52.

Key words: Arc-sine law, Brownian spider, Skew Bessel process, Stable variables, Subordinators, Walsh Brownian motion.

1 Introduction

Recently, some renewed interest has been shown (see e.g. [9]) in the study of the law of the vector

\[ \overrightarrow{A}_i = \left( \int_0^1 1_{\{W_s \in I_i\}} ds; \ i = 1, 2, \ldots, n \right), \]

where \((W_s)\) denotes a Walsh Brownian motion, also called Brownian spider (see [10] for Walsh’s lyrical description) living on \(I = \bigcup_{i=1}^n I_i\), the union of \(n\) half-lines of the plane, meeting at 0.

For the sake of simplicity, we assume \(p_1 = p_2 = \ldots = p_n = 1/n\), i.e.: when returning to 0, Walsh’s Brownian motion chooses, loosely speaking, its "new" ray in a uniform way.

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In fact, excursion theory and/or the computation of the semi-group of Walsh’s Brownian motion (see [1]) allow to define the process rigorously.

Since \((d(0, W_s); s \geq 0)\), for \(d\) the Euclidian distance, is a reflecting Brownian motion, we denote by \((L_t, t \geq 0)\) the unique continuous increasing process such that:
\[(d(0, W_s) - L_s; s \geq 0)\] is a \(W_s = \sigma \{W_u, u \leq s\}\) Brownian motion.

Let
\[
\vec{A}_t = (A_t^{(1)}, A_t^{(2)}, \ldots, A_t^{(n)})
\]
denote the random vector of the times spent in the different rays. In Section 2 we will state and prove our main Theorem concerning the distribution of \(\vec{A}_t\) for a fixed time. Section 3 deals with the general case of stable variables. First, we recall some known results and then we state and prove our main Theorem. Finally, Section 4 is devoted to some remarks and comments.

b) **Reminder on the arc-sine law:**
A random variable \(A\) follows the arc-sine law if it admits the density:
\[
\frac{1}{\pi \sqrt{x(1-x)}} 1_{[0,1)}(x). \tag{1}
\]

Some well known representations of an arc-sine variable are the following:
\[
A \overset{(law)}{=} \frac{N^2}{N^2 + \tilde{N}^2} \overset{(law)}{=} \cos^2(U) \overset{(law)}{=} \frac{T}{T + \tilde{T}} \overset{(law)}{=} \frac{1}{1 + C^2}, \tag{2}
\]
where \(N, \tilde{N} \sim \mathcal{N}(0,1)\) and are independent, \(U\) is uniform on \([0, 2\pi]\), \(T\) and \(\tilde{T}\) stand for two iid stable \((1/2)\) unilateral variables, and \(C\) is a standard Cauchy variable.

With \((B_t, t \geq 0)\) denoting a real Brownian motion, two well known examples of arc-sine distributed variables are:
\[
g_t = \sup\{t < 1 : B_t = 0\}, \quad \text{and} \quad A_t^+ = \int_0^1 ds 1_{(B_s > 0)} ,
\]
a result that is due to Paul Lévy (see e.g. [6, 7, 13]).

c) This point gives some motivation for Section 3. From (2), one could think that more general studies of the time spent positive by diffusions may bring 2 independent gamma variables (this because \(N^2\) and \(\tilde{N}^2\) are distributed like two independent gamma variables of parameter \(1/2\)), or 2 independent stable \((\mu)\) variables. It turns out that it is the second case which seems to occur more naturally. We devote Section 3 to this case.

### 2 Main result

Our aim is to prove the following:

**Theorem 2.1.** The random vectors \(\vec{A}_T/T\) for:

(i) \(T = t\); (ii) \(T = \alpha_s^{(j)} = \inf\{t : A_t^{(j)} > s\}\); (iii) \(T = \tau_l, the inverse local times,\)
have the same distribution. In particular, it is specified by the iid stable \((1/2)\) subordinators:

\[
\left( (A_{\alpha_j}^{(j)}, l \geq 0) ; 1 \leq j \leq n \right).
\]

Hence:

\[
\overrightarrow{A}_1 \overset{(law)}{=} \frac{\overrightarrow{A}_{\tau_1}}{\tau_1},
\]

which yields that:

\[
\overrightarrow{A}_1 \overset{(law)}{=} \left( \frac{T_j}{\sum_{i=1}^{n} T_i} ; j \leq n \right),
\]

where \(T_j\) are iid, stable \((1/2)\) variables.

The law of the right-hand side of (3) is easily computed, and consequently so is its left-hand side. We refer the reader to [2] for explicit expressions of this law, which for \(n = 2\) reduces to the classical arc-sine law.

**Proof of Theorem 2.1.**

a) Clearly, (ii) plays a kind of "bridge" between (i) and (iii).

b) We shall work with \(\left( \alpha_{\alpha_1}^{(1)}, s \geq 0 \right)\), the inverse of \(\left( A_{\alpha_1}^{(1)}, t \geq 0 \right)\). It is more convenient to use the notation \(\left( \alpha_{\alpha_1}^{(2)}, s \geq 0 \right)\) for \(\left( \alpha_{\alpha_1}^{(1)}, s \geq 0 \right)\). We then follow the main steps of [13] (Section 3.4, p. 42), which themselves are inspired by Williams [12]; see also Watanabe (Proposition 1 in [11]) and Mc Kean [8].

\(\left( A_{\alpha_1}^{(j)} \right)\) denotes the time spent in \(I_j\), for any \(j \neq 1\). Since

\[
\begin{align*}
A_{\alpha_1}^{(j)} &\overset{\text{(law)}}{=} A_{\alpha_1^{(2)}}^{(j)} = \left( L_{\alpha_1^{(2)}}^{(j)} \right)^2 A_{\tau_1}^{(j)}, \\
\alpha_{\alpha_1}^{(2)} &\overset{\text{(law)}}{=} 1 + \sum_j A_{\alpha_1^{(2)}}^{(j)}, \\
\text{and} \\
\text{for every } u, t \geq 0, \quad \left( L_{\alpha_1^{(2)}}^{(j)} < t \right) = \left( u < A_{\alpha_1^{(2)}}^{(j)} \right),
\end{align*}
\]

and invoking the scaling property, we can write jointly for all \(j\)'s:

\[
\left( A_{\alpha_1^{(2)}}^{(j)}, L_{\alpha_1^{(2)}}^{(j)} \right) \overset{(law)}{=} \left( L_{\alpha_1^{(2)}}^{(j)} A_{\alpha_1^{(2)}}^{(j)} , 1 + \sum_j L_{\alpha_1^{(2)}}^{(j)} A_{\alpha_1^{(2)}}^{(j)} \right) \overset{(law)}{=} \left( \frac{A_{\alpha_1^{(2)}}^{(2j)}}{A_{\alpha_1^{(2)}}^{(2j)}}, \frac{\tau_1}{\tau_1} \right).
\]

Dividing now both sides by \(\alpha_{\alpha_1}^{(2)}\) and remarking that: \(\alpha_{\alpha_1}^{(2)} A_{\alpha_1^{(2)}}^{(1)} = \tau_1\), we deduce:

\[
\frac{1}{\alpha_{\alpha_1}^{(2)}} \left( A_{\alpha_1^{(2)}}^{(j)}, L_{\alpha_1^{(2)}}^{(j)} \right) \overset{(law)}{=} \frac{1}{\tau_1} \left( A_{\alpha_1^{(2)}}^{(j)}, 1 \right).
\]
With the help of the scaling Lemma below, we obtain:

\[
E \left[ 1_{(W_1 \in I_1)} f(\overrightarrow{A_1}, L_{11}) \right] = E \left[ \frac{1}{\alpha_1^{(+)}} f \left( \frac{A_{\alpha_1^{(+)}}}{\alpha_1^{(+)}} ; L_{\alpha_1^{(+)}} \right) \right]_{\text{from (5)}} = E \left[ \frac{A_{\tau_1}^{(i)}}{\tau_1} f \left( \frac{A_{\tau_1}^{(i)}}{\tau_1}, \frac{1}{\tau_1} \right) \right].
\]

(7)

\(I_1\) may be replaced by \(I_m\), for any \(m \in \{2, \ldots, n\}\). Adding the \(m\) quantities found in (7) and remarking that:

\[
\tau_1 = \sum_{i=1}^{n} A_{\tau_1}^{(i)},
\]

we get:

\[
E \left[ f(\overrightarrow{A_1}, L_{11}^2) \right] = E \left[ f \left( \frac{A_{\tau_1}^{(i)}}{\tau_1}, \frac{1}{\tau_1} \right) \right].
\]

which proves (3). Note that from (6), the latter also equals:

\[
E \left[ f \left( \frac{A_{\alpha_1^{(+)}}}{\alpha_1^{(+)}} ; L_{\alpha_1^{(+)}} \right) \right].
\]

Equality in law (4) follows now easily. Indeed, we denote by \(\nu\) the Itô measure of the Brownian spider, and we have:

\[
\nu = \frac{1}{n} \sum_{j=1}^{n} \nu_j,
\]

where \(\nu_j\) is the canonical image of \(n\), the standard Itô measure of the space of the excursions of the standard Brownian motion, on the space of the excursions on \(I_j\). Hence, with \(\lambda_j\), \(j = 1, \ldots, n\) denoting positive constants:

\[
E \left[ \exp \left( -\sum_{j=1}^{n} \lambda_j A_{\tau_1}^{(j)} \right) \right] = \exp \left( -\frac{1}{n} \sum_{j=1}^{n} \int \nu_j(d\varepsilon_j)(1 - e^{-\lambda_j \nu_j}) \right) = \exp \left( -\frac{1}{n} \sum_{j=1}^{n} \sqrt{2\lambda_j} \right),
\]

thus:

\[
\overrightarrow{A_{\tau_1}} = (A_{\tau_1}^{(j)} ; j \leq n) \overset{\text{law}}{=} \left( \frac{1}{n^2} T_j ; j \leq n \right).
\]

The latter, using (8) yields:

\[
\overrightarrow{A_1} = \frac{\overrightarrow{A_{\tau_1}}}{\tau_1} = \frac{\overrightarrow{A_{\tau_1}}}{\sum_{i=1}^{n} A_{\tau_1}^{(i)}} \overset{\text{(law)}}{=} \left( \frac{T_j}{n^2 \sum_{i=1}^{n} n^{-2} T_i} ; j \leq n \right),
\]

4
which finishes the proof.

It now remains to state the scaling Lemma which played a role in (7), and which we lift from [13] (Corollary 1, p. 40) in a "reduced" form.

**Lemma 2.2. (Scaling Lemma)** Let \( U_t = \int_0^t ds \theta_s \), with the pair \((W, \theta)\) satisfying:

\[
(W_t, \theta_t; t \geq 0) \overset{(law)}{=} (\sqrt{c}W_t, \theta_t; t \geq 0).
\]

Then,

\[
E[F(W_u, u \leq 1) \theta_1] = E\left[\frac{1}{\alpha_t} F\left(\frac{1}{\sqrt{\alpha_t}} W_{\alpha_t}, v \leq 1\right)\right],
\]

where \(\alpha_t = \inf\{s : U_s > t\}\).

### 3 Stable subordinators

#### 3.1 Reminder and preliminaries on stable variables

In this Section, we consider \( S_{\mu} \) and \( S_{\mu}' \) two independent stable variables with exponent \( \mu \in (0,1) \), i.e. for every \( \lambda \geq 0 \), the Laplace transform of \( S_{\mu} \) is given by:

\[
E[\exp(-\lambda S_{\mu})] = \exp(-\lambda^{\mu}).
\]

Concerning the law of \( S_{\mu} \), there is no simple expression for its density (except for the case \( \mu = 1/2 \); see e.g. Exercise 4.20 in [3]). However, we have that, for every \( s < 1 \) (see e.g. [15] or Exercise 4.19 in [3]):

\[
E[(S_{\mu})^{\mu s}] = \frac{\Gamma(1-s)}{\Gamma(1-\mu s)}.
\]

We consider now the random variable of the ratio of two \( \mu \)-stable variables:

\[
X = \frac{S_{\mu}}{S_{\mu}'}.
\]

Following e.g. Exercise 4.23 in [3], we have respectively the following formulas for the Stieltjes and the Mellin transforms of \( X \):

\[
E\left[\frac{1}{1 + sX}\right] = \frac{1}{1 + s^{\mu}}, \quad s \geq 0,
\]

\[
E[X^s] = \frac{\sin(\pi s)}{\mu \sin(\frac{\pi s}{\mu})}, \quad 0 < s < \mu.
\]

Moreover, the density of the random variable \( X^{\mu} \) is given by (see e.g. [14, 5] or Exercise 4.23 in [3]):

\[
P(X^{\mu} \in dy) = \frac{\sin(\pi \mu)}{\pi \mu} \frac{dy}{y^2 + 2y \cos(\pi \mu) + 1}, \quad y \geq 0,
\]
or equivalently:

$$\left( \frac{S_\mu}{S'_\mu} \right)^\mu = (C_\mu | C_\mu > 0),$$  

(18)

where, with $C$ denoting a standard Cauchy variable and $U$ a uniform variable in $[0, 2\pi)$,

$$C_\mu = \sin(\pi \mu) C - \cos(\pi \mu) \overset{\text{law}}{=} \frac{\sin(\pi \mu - U)}{U}. \quad (\text{law})$$

### 3.2 The case of 2 stable variables

We turn now our study to the random variable:

$$A = \frac{S'_\mu}{S'_\mu + S_\mu} = \frac{1}{1 + X},$$  

(19)

**Theorem 3.1.** The density function of the random variable $A$ is given by:

$$P(A \in dz) = \frac{\sin(\pi \mu)}{\pi} \frac{dz}{z(1-z) \left[ \left( \frac{1-z}{z} \right)^\mu + \left( \frac{z}{1-z} \right)^\mu + 2 \cos(\pi \mu) \right]}, \quad z \in [0, 1].$$  

(20)

**Proof of Theorem 3.1.**

Identity (19) is equivalent to:

$$X = \frac{1}{1 + Y} - 1.$$  

Hence, (15) yields:

$$E \left[ \frac{1}{1 + sX} \right] = E \left[ \frac{A}{(1-s)A+s} \right] = \frac{1}{1+s\mu}.$$  

We consider now a test function $f$ and invoking the density (17) we have ($\nu = \frac{1}{\mu} > 1$):

$$E \left[ f \left( \frac{1}{1 + X} \right) \right] = \frac{\sin(\pi \mu)}{\pi \mu} \int_0^\infty dy \frac{dy}{y^\nu + 2y \cos(\pi \mu) + 1} f \left( \frac{1}{1 + y^\nu} \right).$$

Changing the variables $z = \frac{1}{1+y^\nu}$, we deduce:

$$E \left[ f (A) \right] = \frac{\sin(\pi \mu)}{\pi} \int_0^1 dz (1-z)^{\mu-1} f (z) \Delta(z),$$

where:

$$\Delta(z) = \frac{1}{(z^{-1} - 1)^{2\mu} + 2(z^{-1} - 1)^{\mu} \cos(\pi \mu) + 1} \frac{z^{2\mu}}{z^{-2\mu}}.$$  

and (20) follows easily.

In Figure 1, we have plotted the density function $g$ of $A$, for several values of $\mu$.

**Remark 3.2.** Similar discussions have been made in [4] in the framework of a skew Bessel process with dimension $2 - 2\alpha$ and skewness parameter $p$. Formula (20) is a particular case of formula in [4] for the density of the time spent positive (called $f_{p,\alpha}$ in [4]).
3.3 The case of many stable (1/2) variables

In this Subsection, we consider again \( n \) iid stable (1/2) variables, i.e.: \( T_1, \ldots, T_n \), and we will study the distribution of:

\[
A^{(1)}_1 = \frac{T_1}{T_1 + \ldots + T_n}.
\] (21)

The following Theorem answers to an open question (and even in a more general sense) stated at the end of [9].

**Theorem 3.3.** The density function of the random variable \( A^{(1)}_1 \) is given by:

\[
P \left( A^{(1)}_1 \in dz \right) = \frac{1}{\pi} \sqrt{z} \sqrt{1-z} \left[ (n-1)z + \frac{1}{n-1}(1-z) \right], \quad z \in [0,1].
\] (22)

**Proof of Theorem 3.3.**

We first remark that, with \( C \) denoting a standard Cauchy variable, using e.g. (2):

\[
A^{(1)}_1 \overset{\text{(law)}}{=} \frac{T_1}{T_1 + (n-1)^2 T_2} \overset{\text{(law)}}{=} \frac{1}{1 + (n-1)^2 C^2}.
\] (23)

Hence, with \( f \) standing again for a test function, and invoking the density of a standard
Figure 2: The density function $h$ of $A_1^{(1)}$, for several values of $n$.

Cauchy variable, that is: for every $x \in \mathbb{R}$, $g(x) = \frac{1}{\pi (1 + x^2)}$ we have:

$$E \left[ f \left( A_1^{(1)} \right) \right] = E \left[ f \left( \frac{1}{1 + (n-1)^2 C^2} \right) \right]$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} f \left( \frac{1}{1 + (n-1)^2 x^2} \right)$$

Changing the variables $z = \frac{1}{1 + (n-1)^2 y}$, we deduce:

$$E \left[ f \left( A_1^{(1)} \right) \right] = \frac{2}{\pi} \int_{0}^{\infty} \frac{dy}{2 \sqrt{y}(1 + y)} f \left( \frac{1}{1 + (n-1)^2 y} \right)$$

and (22) follows easily.

Figure 2 presents the plot of the density function $h$ of $A_1^{(1)}$, for several values of $n$.

**Corollary 3.4.** The following convergence in law holds:

$$n^2 A_1^{(1)}(n) \xrightarrow{\text{law}} C^2 .$$  (24)
Proof of Corollary 3.4.

It follows from Theorem 3.3 by simply remarking that $C \overset{(\text{law})}{=} C^{-1}$. Hence:

\[
\frac{n^2 A_1^{(1)}(n)}{1 + (n - 1)^2 C^2} \overset{\text{law}}{\to} \frac{1}{\frac{1}{n} + (\frac{n-1}{n})^2 C^2} \quad \text{as} \quad n \to \infty \quad \overset{(\text{law})}{=} \quad C^2.
\]

\[
\square
\]

4 Conclusion and comments

We end up this article with some comments: usually, a scaling argument is "one-dimensional", as it involves a time-change. Exceptionally (or so it seems to the authors), here we could apply a scaling argument in a multivariate framework. We insist that the scaling Lemma plays a key role in our proof. The curious reader should also look at the totally different proof of this Theorem in [2], which mixes excursion theory and the Feynman-Kac method.

Acknowledgements

The author S. Vakeroudis is very grateful to Professor R.A. Doney for the invitation at the University of Manchester as a Post Doc fellow where he prepared a part of this work.

References


