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A scaling proof for Walsh’s Brownian motion extended arc-sine law

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Abstract

We present a new proof of the extended arc-sine law related to Walsh’s Brownian motion, known also as Brownian spider. The main argument mimics the scaling property used previously, in particular by D. Williams [12], in the 1-dimensional Brownian case, which can be generalized to the multivariate case. A discussion concerning the time spent positive by a skew Bessel process is also presented.

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1 Introduction

a) Recently, some renewed interest has been shown (see e.g. [9]) in the study of the law of the vector

$$\bar{A}_i = \left( \int_0^1 1_{\{W_s \in I_i\}} ds; \ i = 1, 2, \ldots, n \right),$$

where $W_s$ denotes a Walsh Brownian motion, also called Brownian spider (see [10] for Walsh’s lyrical description) living on $I = \bigcup_{i=1}^n I_i$, the union of $n$ half-lines of the plane, meeting at 0.

For the sake of simplicity, we assume $p_1 = p_2 = \ldots = p_n = 1/n$, i.e.: when returning to 0, Walsh’s Brownian motion chooses, loosely speaking, its "new" ray in a uniform way.

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In fact, excursion theory and/or the computation of the semi-group of Walsh’s Brownian motion (see [1]) allow to define the process rigorously.

Since $(d(0, W_s); s \geq 0)$, for $d$ the Euclidian distance, is a reflecting Brownian motion, we denote by $(L_t; t \geq 0)$ the unique continuous increasing process such that: 
$(d(0, W_s) - L_s; s \geq 0)$ is a $W_s = \sigma \{W_u, u \leq s\}$ Brownian motion. Let 
$\mathbf{A} = (A_t(1), A_t(2), \ldots, A_t(n))$
denote the random vector of the times spent in the different rays. In Section 2 we will state and prove our main Theorem concerning the distribution of $\mathbf{A}$ for a fixed time. Section 3 deals with the general case of stable variables, First, we recall some known results and then we state and prove our main Theorem. Finally, Section 4 is devoted to some remarks and comments.

b) Reminder on the arc-sine law:
A random variable $A$ follows the arc-sine law if it admits the density:
$$\frac{1}{\pi \sqrt{x(1-x)}} 1_{(0,1)}(x). \quad (1)$$

Some well known representations of an arc-sine variable are the following:
$$A \stackrel{\text{law}}{=} \frac{N^2}{N^2 + \hat{N}^2} \stackrel{\text{law}}{=} \cos^2(U) \stackrel{\text{law}}{=} \frac{T}{T + \hat{T}} \stackrel{\text{law}}{=} \frac{1}{1 + C^2}, \quad (2)$$

where $N, \hat{N} \sim \mathcal{N}(0, 1)$ and are independent, $U$ is uniform on $[0, 2\pi]$, $T$ and $\hat{T}$ stand for two iid stable $(1/2)$ unilateral variables, and $C$ is a standard Cauchy variable.

With $(B_t, t \geq 0)$ denoting a real Brownian motion, two well known examples of arc-sine distributed variables are:
$$g_1 = \sup \{t < 1 : B_t = 0\}, \quad \text{and} \quad A_1^+ = \int_0^1 ds \ 1_{(B_s > 0)},$$
a result that is due to Paul Lévy (see e.g. [6, 7, 13]).

c) This point gives some motivation for Section 3. From (2), one could think that more general studies of the time spent positive by diffusions may bring 2 independent gamma variables (this because $N^2$ and $\hat{N}^2$ are distributed like two independent gamma variables of parameter 1/2), or 2 independent stable ($\mu$) variables. It turns out that it is the second case which seems to occur more naturally. We devote Section 3 to this case.

2 Main result

Our aim is to prove the following:

**Theorem 2.1.** The random vectors $\mathbf{A}_T / T$ for:

(i) $T = t$;  
(ii) $T = \alpha_s^{(j)} = \inf \{t : A_t^{(j)} > s\}$;  
(iii) $T = \tau_l$, the inverse local times,
have the same distribution. In particular, it is specified by the iid stable (1/2) subordinators:

$$((A^{(j)}_\tau, \tau \geq 0) ; 1 \leq j \leq n).$$

Hence:

$$\bar{A}_1^{(\text{law})} = \frac{\bar{A}_n}{\bar{\tau}_1},$$

which yields that:

$$\bar{A}_1^{(\text{law})} = \left(\frac{T_j}{\sum_{i=1}^n T_i} ; j \leq n\right),$$

where $T_j$ are iid, stable (1/2) variables.

The law of the right-hand side of (3) is easily computed, and consequently so is its left-hand side. We refer the reader to [2] for explicit expressions of this law, which for $n = 2$ reduces to the classical arc-sine law.

**Proof of Theorem 2.1.**

a) Clearly, (ii) plays a kind of "bridge" between (i) and (iii).

b) We shall work with $\left(\alpha_s^{(1)}, s \geq 0\right)$, the inverse of $\left(A^{(1)}_t, t \geq 0\right)$. It is more convenient to use the notation $\left(\alpha_s^{(+)}, s \geq 0\right)$ for $\left(\alpha_s^{(1)}, s \geq 0\right)$. We then follow the main steps of [13] (Section 3.4, p. 42), which themselves are inspired by Williams [12]; see also Watanabe (Proposition 1 in [11]) and McKean [8].

$\left(A^{(j)}_t\right)$ denotes the time spent in $I_j$, for any $j \neq 1$. Since

$$\left\{\begin{align*}
A^{(j)}_{\alpha_1^{(+)}} &= A^{(j)}_{\tau(L^{(+)}/\alpha_1^{(+)})} \overset{\text{(law)}}{=} \left(L^{(+)}/\alpha_1^{(+)}\right)^2 A^{(j)}_{\tau_1}, \\
\alpha_1^{(+)} &= 1 + \sum_j A^{(j)}_{\alpha_1^{(+)}} , \\
\text{and} & \\
\text{for every} \ u, t \geq 0, \ \left(L^{(+)}/\alpha_u^{(+)}<t\right) &= \left(u < A^{(1)}_{\tau \sqrt{t}}\right),
\end{align*}\right.$$

and invoking the scaling property, we can write jointly for all $j$'s:

$$\left(A^{(j)}_{\alpha_1^{(+)}} , L^{2}_{\alpha_1^{(+)}} , \alpha_1^{(+)}\right) \overset{\text{(law)}}{=} \left(L^{2}_{\alpha_1^{(+)}} A^{(j)}_{\tau_1} , L^{2}_{\alpha_1^{(+)}} , 1 + \sum_j L^{2}_{\alpha_1^{(+)}} A^{(j)}_{\tau_1}\right)$$

$$\overset{\text{(law)}}{=} \left(A^{(j)}_{\tau_1} , \frac{1}{A^{(1)}_{\tau_1}}, \frac{\tau_1}{A^{(1)}_{\tau_1}}\right).$$

Dividing now both sides by $\alpha_1^{(+)}$ and remarking that: $\alpha_1^{(+)}/A^{(1)}_{\tau_1} = \tau_1$, we deduce:

$$\frac{1}{\alpha_1^{(+)}} \left(A^{(j)}_{\alpha_1^{(+)}} , L^{2}_{\alpha_1^{(+)}}\right) \overset{\text{(law)}}{=} \frac{1}{\tau_1} \left(A^{(j)}_{\tau_1} , 1\right).$$
With the help of the scaling Lemma below, we obtain:

\[
E \left[ 1_{\{W_1 \in I_1\}} f(\vec{A}_1, L_1^2) \right] = E \left[ \frac{1}{\alpha_1^{(+)}} f \left( \frac{A_{1_1}^{(+)}}{\alpha_1^{(+)}}, \frac{L_1^{2(+)}}{\alpha_1^{(+)}} \right) \right]
\]

from (5)

\[
= \frac{A_{1_1}^{(+)}}{\tau_1} f \left( \frac{A_{1_1}^{(-)}, 1}{\tau_1} \right)
\]

(7)

\(I_1\) may be replaced by \(I_m\), for any \(m \in \{2, \ldots, n\}\). Adding the \(m\) quantities found in (7) and remarking that:

\[
\tau_1 = \sum_{i=1}^{n} A_{1_1}^{(i)}
\]

we get:

\[
E \left[ f(\vec{A}_1, L_1^2) \right] = E \left[ f \left( \frac{A_{1_1}}{\tau_1}, \frac{1}{\tau_1} \right) \right]
\]

which proves (3). Note that from (6), the latter also equals:

\[
E \left[ f \left( \frac{A_{1_1}^{(+)}}{\alpha_1^{(+)}}, \frac{L_1^{2(+)}}{\alpha_1^{(+)}} \right) \right]
\]

Equality in law (4) follows now easily. Indeed, we denote by \(\nu\) the Itô measure of the Brownian spider, and we have:

\[
\nu = \frac{1}{n} \sum_{j=1}^{n} \nu_j
\]

where \(\nu_j\) is the canonical image of \(n\), the standard Itô measure of the space of the excursions of the standard Brownian motion, on the space of the excursions on \(I_j\). Hence, with \(\lambda_j, j = 1, \ldots, n\) denoting positive constants:

\[
E \left[ \exp \left( -\sum_{j=1}^{n} \lambda_j A_{1_1}^{(j)} \right) \right] = \exp \left( -\frac{1}{n} \sum_{j=1}^{n} \int \nu_j(\varepsilon_j)(1 - e^{-\lambda_j \nu_j}) \right)
\]

\[
= \exp \left( -\frac{1}{n} \sum_{j=1}^{n} \sqrt{2\lambda_j} \right)
\]

thus:

\[
\vec{A}_{1_1} = (A_{1_1}^{(j)}; j \leq n) \stackrel{(law)}{=} \left( \frac{1}{n^2} T_j; j \leq n \right).
\]

The latter, using (8) yields:

\[
\vec{A}_1 = \frac{\vec{A}_{1_1}}{\tau_1} = \frac{\vec{A}_{1_1}}{\sum_{i=1}^{n} A_{1_1}^{(i)}} \stackrel{(law)}{=} \left( \frac{T_j}{n^2 \sum_{i=1}^{n} n^{-2} T_i}; j \leq n \right).
\]
which finishes the proof.

It now remains to state the scaling Lemma which played a role in (7), and which we
lift from [13] (Corollary 1, p. 40) in a "reduced" form.

Lemma 2.2. (Scaling Lemma) Let $U_t = \int_0^t ds \theta_s$, with the pair $(W, \theta)$ satisfying:

$$(W_{ct}, \theta_{ct}; t \geq 0) \overset{(law)}{=} \left(\sqrt{c}W_t, \theta_t; t \geq 0\right).$$

Then,

$$E[F(W_u, u \leq 1) \theta_t] = E\left[\frac{1}{\alpha_1} F\left(\frac{1}{\sqrt{\alpha_1}} W_{\alpha_1}, v \leq 1\right)\right],$$

where $\alpha_t = \inf\{s : U_s > t\}$.

3 Stable subordinators

3.1 Reminder and preliminaries on stable variables

In this Section, we consider $S_\mu$ and $S'_\mu$ two independent stable variables with exponent
$\mu \in (0, 1)$, i.e. for every $\lambda \geq 0$, the Laplace transform of $S_\mu$ is given by:

$$E[\exp(-\lambda S_\mu)] = \exp(-\lambda^\mu).$$

Concerning the law of $S_\mu$, there is no simple expression for its density (except for the
case $\mu = 1/2$; see e.g. Exercise 4.20 in [3]). However, we have that, for every $s < 1$ (see
e.g. [15] or Exercise 4.19 in [3]):

$$E[(S_\mu)^{\mu s}] = \frac{\Gamma(1-s)}{\Gamma(1-\mu s)}.$$

We consider now the random variable of the ratio of two $\mu$-stable variables:

$$X = \frac{S_\mu}{S'_\mu}.$$

Following e.g. Exercise 4.23 in [3], we have respectively the following formulas for the
Stieltjes and the Mellin transforms of $X$:

$$E\left[\frac{1}{1+sX}\right] = \frac{1}{1+s^\mu}, \quad s \geq 0,$$

$$E[X^s] = \frac{\sin(\pi s)}{\mu \sin(\pi s)}, \quad 0 < s < \mu.$$

Moreover, the density of the random variable $X^\mu$ is given by (see e.g. [14, 5] or Exercise
4.23 in [3]):

$$P(X^\mu \in dy) = \frac{\sin(\pi \mu)}{\pi \mu} \frac{dy}{y^2 + 2y \cos(\pi \mu) + 1}, \quad y \geq 0,$$
or equivalently:

\[
\left( \frac{S_\mu}{S_\mu'} \right)^\mu = (C_\mu | C_\mu > 0),
\]

where, with \( C \) denoting a standard Cauchy variable and \( U \) a uniform variable in \([0, 2\pi)\),

\[
C_\mu = \sin(\pi \mu) C - \cos(\pi \mu) \overset{(law)}{=} \frac{\sin(\pi \mu - U)}{U}. \tag{18}
\]

### 3.2 The case of 2 stable variables

We turn now our study to the random variable:

\[
A = \frac{S_\mu'}{S_\mu' + S_\mu} = \frac{1}{1 + X}, \tag{19}
\]

**Theorem 3.1.** The density function of the random variable \( A \) is given by:

\[
P(A \in dz) = \frac{\sin(\pi \mu)}{\pi} \frac{dz}{z(1-z) \left[ \left( \frac{1-z}{z} \right)^\mu + \left( \frac{z}{1-z} \right)^\mu + 2 \cos(\pi \mu) \right]}, \quad z \in [0, 1]. \tag{20}
\]

**Proof of Theorem 3.1.**

Identity (19) is equivalent to:

\[
X = \frac{1}{A} - 1.
\]

Hence, (15) yields:

\[
E \left[ \frac{1}{1 + sX} \right] = E \left[ \frac{A}{(1-s)A + s} \right] = \frac{1}{1 + s},
\]

We consider now a test function \( f \) and invoking the density (17) we have \( (\nu = \frac{1}{\mu} > 1) \):

\[
E \left[ f \left( \frac{1}{1 + X} \right) \right] = \frac{\sin(\pi \mu)}{\pi \mu} \int_0^\infty \frac{dy}{y^2 + 2y \cos(\pi \mu) + 1} f \left( \frac{1}{1 + y^{\mu}} \right).
\]

Changing the variables \( z = \frac{1}{1+y^{\mu}} \), we deduce:

\[
E \left[ f (A) \right] = \frac{\sin(\pi \mu)}{\pi} \int_0^1 \frac{dz(1-z)^{\mu-1}}{z^{\mu+1}} f (z) \Delta(z),
\]

where:

\[
\Delta(z) = \frac{1}{(z^{-1}-1)^{2\mu} + 2(z^{-1}-1)^\mu \cos(\pi \mu) + 1} z^{2\mu}
\]

\[
= \frac{1}{(1-z)^{2\mu} + 2(1-z)^\mu z^\mu \cos(\pi \mu) + z^{2\mu}}.
\]

and (20) follows easily. \( \square \)

In Figure 1, we have plotted the density function \( g \) of \( A \), for several values of \( \mu \).

**Remark 3.2.** Similar discussions have been made in [4] in the framework of a skew Bessel process with dimension \( 2 - 2\alpha \) and skewness parameter \( p \). Formula (20) is a particular case of formula in [4] for the density of the time spent positive (called \( f_{p,\alpha} \) in [4]).
3.3 The case of many stable (1/2) variables

In this Subsection, we consider again $n$ iid stable (1/2) variables, i.e.: $T_1, \ldots, T_n$, and we will study the distribution of:

$$A^{(1)}_1 = \frac{T_1}{T_1 + \ldots + T_n}. \quad (21)$$

The following Theorem answers to an open question (and even in a more general sense) stated at the end of [9].

**Theorem 3.3.** The density function of the random variable $A^{(1)}_1$ is given by:

$$P \left( A^{(1)}_1 \in dz \right) = \frac{1}{\pi \sqrt{z(1-z)}} \frac{dz}{\left((n-1)z + \frac{1}{n-1}(1-z)\right)}, \quad z \in [0, 1]. \quad (22)$$

**Proof of Theorem 3.3.**

We first remark that, with $C$ denoting a standard Cauchy variable, using e.g. (2):

$$A^{(1)}_1 \overset{\text{(law)}}{=} \frac{T_1}{T_1 + (n-1)^2T_2} \overset{\text{(law)}}{=} \frac{1}{1 + (n-1)^2C^2}. \quad (23)$$

Hence, with $f$ standing again for a test function, and invoking the density of a standard
Figure 2: The density function $h$ of $A_1^{(1)}$, for several values of $n$.

Cauchy variable, that is: for every $x \in \mathbb{R}$, $g(x) = \frac{1}{\pi(1+x^2)}$ we have:

$$E[f(A_1^{(1)})] = E\left[f\left(\frac{1}{1+(n-1)^2z^2}\right)\right] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} f\left(\frac{1}{1+(n-1)^2x^2}\right)$$

$x^2 = y$:

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{dy}{2\sqrt{y}(1+y)} f\left(\frac{1}{1+(n-1)^2y}\right)$$

Changing the variables $z = \frac{1}{1+(n-1)^2y}$, we deduce:

$$E[f(A_1^{(1)})] = \frac{1}{\pi} \int_{0}^{1} \frac{dz}{(n-1)^2z^2} \frac{(n-1)\sqrt{z}}{\sqrt{z-1}\left(1+\frac{1}{(n-1)^2}\right)} f(z),$$

and (22) follows easily.

Figure 2 presents the plot of the density function $h$ of $A_1^{(1)}$, for several values of $n$.

**Corollary 3.4.** The following convergence in law holds:

$$n^2A_1^{(1)}(n) \xrightarrow{\text{(law)}} C^2.$$

(24)
Proof of Corollary 3.4.
It follows from Theorem 3.3 by simply remarking that \( C \overset{\text{(law)}}{=} C^{-1} \). Hence:
\[
n^2 A^{(1)}_1(n) = \frac{n^2}{1 + (n-1)^2 C^2} = \frac{1}{\frac{n}{\pi} + \left(\frac{n-1}{\pi}\right)^2 C^2} \xrightarrow{n \to \infty} \frac{1}{C^2} \overset{\text{(law)}}{=} C^2.
\]

4 Conclusion and comments

We end up this article with some comments: usually, a scaling argument is "one-dimensional", as it involves a time-change. Exceptionally (or so it seems to the authors), here we could apply a scaling argument in a multivariate framework. We insist that the scaling Lemma plays a key role in our proof. The curious reader should also look at the totally different proof of this Theorem in [2], which mixes excursion theory and the Feynman-Kac method.

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